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CLASSICAL LAGRANGE AND MAYER PROBLEMS

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14.1 EULER EQUATIONS, WEIERSTRASS, AND LEGENDRE-TYPE CONDITIONS

Let us consider the classical Bolza problem of the minimum of the functional

$$I[x] = \int_{t_1}^{t_2} f_o(t, x(t), x'(t))\, dt + g_o(t_1, x(t_1), t_2, x(t_2)), \quad (14.1.1)$$

with side conditions in the form of m differential equations

$$G_j(t, x(t), x'(t)) = o, \; j = 1,\dots,m, \; m < n, \; x = (x^1,\dots,x^n), \quad (14.1.2)$$

with given boundary conditions $(t_1, x(t_1), t_2, x(t_2)) \in B$, and constraints of the form $(t, x(t)) \in A$ and $(t, x(t), x'(t)) \in M$. Here A is a given subset of the tx - space E_{n+1}, M a given subset of the txx' - space E_{2n+1} whose projection on the tx - space is A, B is a given subset of the $t_1\, x_1\, t_2\, x_2$ - space E_{2n+2}, $x_1 = (x_1^1, \dots, x_1^n)$, $x_2 = (x_2^2, \dots, x_2^n)$, and f_o, G_j, $j = 1,\dots,m$, are given functions of class C' defined on M, and g_o is a given function of class C' defined on B. We denote by Ω the class of all AC functions x, or curves C: $x = x(t)$, $t_1 \le t \le t_2$, satisfying the requirements above. Let C: $x = x(t)$, $t_1 \le t \le t_2$, denote a particular element of Ω which gives a minimum for I in Ω, that is, $I[x] \le I[y]$ for all $y \in \Omega$. The main assumption here—the one which makes the problem "classical"—is that C lies in the interior of M, that is, $(t, x(t), x'(t))$ is an interior point of M for every $t \in [t_1, t_2]$. Then (t, x(t) is an interior point of A for every $t \in [t_1, t_2]$, and hence there exists some $\delta_o > o$ such that all points $(t, x) \in E_{2n+1}$ with $t_1 - \delta_o \leq t \leq t_2 + \delta_o$ and $|x - x(t')| \leq \delta_o$ for some $t' \in [t_1, t_2]$, belong to A. We shall denote by A_o the set of points (t, x) now indicated, $A_o \subset A$.

Note that in (14.1.1), we have $g_o = o$ for Lagrange problems, and $f_o = o$ for Mayer problems.

In the classical theory one more hypothesis is made, namely it is assumed that the m x n matrix $[\partial G_j/\partial x^i, j = 1,\ldots,m, \ i = 1,\ldots,n]$ is of rank m along C, that is, for every $t \in [t_1, t_2]$ when the arguments of the partial derivatives G_{jx^i}, are t, $x(t)$, $x'(t)$. It is then proved that the m equations $G_j = o$, $j = 1,\ldots,m$, can be completed into a system of n equations

$$G_j(t,x,x') = o, \ j = 1,\ldots,m, \ G_j(t,x,x') = u_j, \ j = m + 1,\ldots,n, \tag{14.1.3}$$

with all G_j of class C', such that the n x n functional determinant $[\partial G_j/\partial x^i, j, i = 1,\ldots n]$ is different from zero along C, and hence there exists a neighborhood M_o of the set $[(t, x(t), x'(t)), t_1 \leq t \leq t_2]$, $M_o \subset M$, such that the n equations (14.1.3) can be inverted yielding normal form ordinary differential equations $x_i' = f_i(t,x,u)$, these functions f_i also of class C'. In particular, if $u_j(t) = G_j(t, x(t), x(t))$, $j = m + 1,\ldots,n$, then $x_i'(t) = f_i(t, x(t), u(t))$, $i = 1,\ldots,n$. We shall prove later that this deduction is valid under usual classical conditions concerning x. Nevertheless, we shall prefer here to assume all of this as an hypothesis, first for the sake of simplicity, secondly because it allows us to assume nothing more than absolute continuity on x here and in the sections below.

Thus, we assume here (a) that M_o is some open subset of M and $\delta_o > o$ a number such that all points (t,x,x') with $t_1 - \delta_o \leq t \leq t_2 + \delta$, and $|x-x(t')| \leq \delta_o$, $|x'-x'(t')| \leq \delta_o$ for some $t' \in [t_1, t_2]$ belong to M_o; (b) that there are n-m functions $G_j(t,x,x')$, $j = m + 1,\dots,n$, of class C' in M_o such that the functional determinant $[\partial G_j/\partial x^i$, $j, i = 1,\dots,n]$ is $\neq o$ for every (t,x,x') $\in M_o$; (c) that the subset M_1 of all $(t,x,x') \in M_o$ on which $G_j = o$, $j = 1,\dots,m$, is mapped by the transformation

$$T: \quad t = t, \ x = x, \ u_j = G_j(t,x,x'), \ j = m + 1,\dots,n, \qquad (14.1.4)$$

onto an open subset N_1 of the t x u-space $E_{1+n+(n-m)}$, and that the transformation T is one - one. Thus we denote by

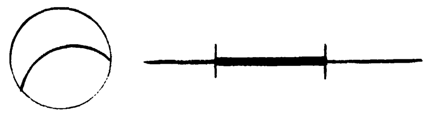
$$T^{-1}: \quad t = t, \ x = x, \ x_i' = f_i(t,x,u), \ i = 1,\dots,n, \qquad (14.1.5)$$

the inverse transformation with all f_i of class C' on N_1, and T^{-1} maps N_1 onto M_1, the set of all $(t,x,x') \in M_o$ with $G_j = o$, $j = 1,\dots,m$. Here we shall use the notations $x' = f(t,x,u)$ with $x' = (x_1',\dots,x_n')$, $f = (f_1',\dots,f_n)$, $u = (u_{m+1},\dots,u_n)$. Thus, if $u(t) = G_j(t, x(t), x'(t))$, $j = m + 1,\dots,n$, $u(t) = (u_{m+1},\dots,u_n)$, then $x'(t) = f(t, x(t), u(t))$ for all $t_1 \leq t \leq t_2$.

Note that, if for any given $(t,x) \in A$, $V(t,x)$ denotes the set of all x' with $(t,x,x') \in M_o$, then $x' \in V(t,x)$ and $V(t,x)$ is an open subset of the x'-space E_n. If $U(t,x)$ denotes the set of all $u \in E_{n-m}$ with $(t,x,u) \in N_1$, then $u \in U(t,x)$ and $U(t,x)$ is an open subset of the u-space E_{n-m}. For instance, for $n = 2$, $m = 1$, and given $(t,x) \in A$ (not indicated), the picture shows $V(t,x)$ as an open subset of the x'-space E_2, $U(t,x)$ as an open interval of the u-space

E_1, and then the image under f of (t,x) is the locus of the points $x' \epsilon V(t,x)$ where the only equation $G_1(t,x,x') = o$ is satisfied, (that is, the points $(t,x,x') \epsilon N_1$, the curve drawn in $V(t,x)$).

V(t,x) U(t,x)



x'-space n=2 u-space m=1

Since the relations (14.1.5) are simply the inverse relations (14.1.4), the following identities hold

$$G_j(t,x,\ f(t,x,u)) = o,\ j = 1,\dots,m,$$

$$G_j(t,x,\ f(t,x,u)) = u_j,\ j = m+1,\dots,n, \qquad (14.1.6)$$

for all $(t,x,u) \epsilon N_1$. We shall denote by $f_{n+1}(t,x,u)$ the function $f_o(t,x,f(t,x,u))$, $(t,x,u) \epsilon N_1$.

By the addition of the auxiliary variable x^{n+1} satisfying $dx^{n+1}/dt = f_o(t,x,x')$ and the boundary condition $x^{n+1}(t_1) = o$, the Bolza problem above is transformed into the Mayer type problem of the minimum of the functional

$$I[\hat{x},u] = g = g_o(t_1,\ x(t_1),\ t_2,\ x(t_2)) + x^{n+1}(t_2),$$

with side conditions

$$dx^i/dt = f_i(t,x,u),\ i = 1,\dots,n,$$

$$dn^{n+1}/dt = f_{n+1}(t,x,u) = f_o(t,x,f(t,x,u)), \qquad (14.1.7)$$

with the required boundary conditions relative to $x^1,\dots,x^n$, and the additional condition $x^{n+1}(t_1) = o$, where now the n-vector $x = (x^1,\dots,x^n)$ is replaced by the (n+1)-vector $\tilde{x} = (x^1,\dots,x^n,\ x^{n+1}) = (x,x^{n+1})$, and $u = (u^{m+1},\dots,u^n) \epsilon U$.

Then the Hamiltonian is

$$H(t,x,u,\ ) = \lambda_1 f_1 + \cdots + \lambda_n f_n + \lambda_{n+1} f_{n+1}$$

where $\lambda = (\lambda_1,\ldots,\lambda_{n+1})$, and by Pontryagin's principle there exists an AC vector function $\lambda(t) = (\lambda_1,\ldots,\lambda_{n+1})$, $t_1 \le t \le t_2$, and a measurable control function $u(t) = (u^{m+1},\ldots,u^n)$ such that $M(t) = H(t, x(t), \lambda(t))$ is also AC in $[t_1,t_2]$ and the equations hold

$$dx^i/dt = \partial H/\partial\lambda_i,\ d\lambda_i/dt = -\partial H/\partial x^i,\ i = 1,\ldots,n+1.$$

Moreover the relations hold

$$d\lambda_i/dt = -\partial H/\partial x_i = -(\lambda_1 f_{1x^i} + \cdots + \lambda_n f_{nx^i} + \lambda_{n+1} f_{n+1,x^i}),\ i = 1,\ldots,n,$$

$$d\lambda_{n+1}/dt = -\partial H/\partial x^{n+1} = o,$$

$$dM(t)/dt = \partial H/\partial t = -(\lambda_1 f_{1t} + \cdots + \lambda_n f_{nt} + \lambda_{n+1} f_{n+1,t}),$$

$$H(t, x(t), u(t), \lambda(t)) = \operatorname*{Min}_{u\in U} H(t, x(t)\ u, \lambda(t)) = M(t), \qquad (14.1.8)$$

a.e. in $[t_1,t_2]$. The equation for λ_{n+1} yields λ_{n+1} = constant in $[t_1,t_2]$. Since we know that $u = u(t)$ is interior to U and H has continuous first order partial derivatives in u, we conclude that these partial derivatives are all zero at t, x(t), u(t), λ(t), or

$$\partial H/\partial u^s = \lambda_1 f_{1u^s} + \cdots + \lambda_n f_{nu^s} + \lambda_{n+1} f_{n+1,u^s} = o,\ s = m+1,\ldots,n. \qquad (14.1.9)$$

First we shall deduce from (14.1.6) usual differential identities. Indeed, if we keep x fixed and all u^j fixed but u^s, $m+1 \le s \le n$, then (14.1.7) yields

$$\sum_{\alpha=1}^{n} G_{ix^{\alpha\prime}} f_{\alpha u^s} = o \quad \text{for } i \neq s,$$

$$\sum_{\alpha=1}^{n} G_{ix^{\alpha\prime}} f_{\alpha u^s} = 1 \quad \text{for } i = s,\ i = 1,\dots,n, \tag{14.1.10}$$

and then

$$f_{iu^s} = \Delta_{si}, \ i = 1,\dots,n,\ s = m+1,\dots,n, \tag{14.1.11}$$

where Δ_{ij}, i, j = 1,...,n, are the cosets of the elements of the m x n matrix $[\partial G_j/\partial x^i,\ j,\ i = 1,\dots,n]$. If we keep u fixed and all x^i fixed but x^j, $1 \leq j \leq n$, then (14.1.6) yields, by differentiation with respect to x^j,

$$G_{ix^j} + \sum_{\alpha=1}^{n} G_{ix^{\alpha\prime}} f_{\alpha x^j} = o, \quad i = 1,\dots,n, \tag{14.1.12}$$

and then

$$f_{ix^j} = - \sum_{\alpha=1}^{n} G_{\alpha x^j} \Delta_{\alpha i}, \ i,\ j = 1,\dots n. \tag{14.1.13}$$

Let us take

$$\lambda_i = - \ell_o f_{ox^{i\prime}} - \sum_{j=1}^{n} \ell_j G_{jx^{i\prime}}, \ i = 1,\dots,n, \tag{14.1.14}$$

where $\lambda_i = \lambda_i(t)$, all $f_{ox^{i\prime}}$, $G_{jx^{i\prime}}$ are taken along the optimal solution (that is, their arguments are t, x(t), x'(t)), and where $\ell_o, \ell_1,\dots,\ell_n$ are undetermined. Note that if we choose ℓ_o arbitrarily, then the remaining $\ell_1,\dots,\ell_n$ are uniquely determined by equations (14.1.14) since $\Delta = \det(G_{jx^{i\prime}}) \neq o$, and precisely we have

$$\ell_i = - \sum_{\alpha=1}^{n} (\lambda_\alpha - \ell_o f_{ox^\alpha}) \Delta_{\alpha i}, \ i = 1,\dots n. \tag{14.1.15}$$

By force of (14.1.14) the n-m identities (14.1.9) become successively

$$\lambda_{n+1} \sum_{\alpha=1}^{n} f_{ox}{}^{\alpha'} f_{\alpha u}{}^{s} - \sum_{i=1}^{n} f_{iu}{}^{s} (\ell_o f_{ox}{}^{i'} + \sum_{j=1}^{m} \ell_j G_{jx}{}^{i'}) = o,$$

$$(14.1.16)$$

$$(\lambda_{n+1} - \ell_o) \sum_{\alpha=1}^{n} f_{ox}{}^{\alpha'} f_{\alpha u}{}^{s} - \sum_{j=1}^{n} \ell_j (\sum_{i=1}^{n} f_{iu}{}^{s} G_{jx}{}^{i'}) = o, \quad s = m+1,\dots,n.$$

By (14.1.11), or $f_{iu}{}^{s} = \Delta_{si}$, the last parenthesis is zero for $j \neq s$, and $= 1$ for $j = s$, so that the n-m identities (14.1.16) yield

$$(\lambda_{n+1} - \ell_o) \sum_{\alpha=1}^{n} f_{ox}{}^{\alpha'} f_{\alpha u}{}^{s} - \ell_s = o, \quad s = m+1,\dots,n.$$

If we choose $\ell_o = \lambda_{n+1}$, then we obtain $\ell_s = o$, $s = m+1,\dots,n$, and relations (14.1.14), (14.1.15) become

$$\lambda_i = - \ell_o f_{ox}{}^{i'} - \sum_{j=1}^{m} \ell_j G_{jx}{}^{i'}, \quad i = 1,\dots,n, \qquad (14.1.17)$$

$$\ell_i = -\sum_{\alpha=1}^{n} (\lambda_\alpha - \ell_o f_{ox}{}^{\alpha'}) \Delta_{\alpha i}, \quad i = 1,\dots,m,$$

$$(14.1.18)$$

$$\ell_o = \lambda_{n+1}, \quad \ell_s = o, \quad s = m+1,\dots,n.$$

Now relations (14.1.8) yield, by force of (14.1.18) and (14.1.14)

$$d\lambda_i/dt = -\ell_o (f_{ox}{}^{i} + \sum_{\alpha=1}^{n} f_{ox}{}^{\alpha'} f_{\alpha x}{}^{i}) + \sum_{\alpha=1}^{n} f_{\alpha x}{}^{i} (\ell_o f_{ox}{}^{\alpha'} + \sum_{j=1}^{n} \ell_j G_{jx}{}^{\alpha'})$$

$$= -\ell_o f_{ox}{}^{i} + \sum_{j=1}^{m} \ell_j \sum_{\alpha=1}^{n} f_{\alpha x}{}^{i} G_{jx}{}^{\alpha'}$$

$$= -\ell_o f_{ox}{}^{i} - \sum_{j=1}^{m} \ell_j \sum_{\beta=1}^{n} G_{\beta x}{}^{i} (\sum_{\alpha=1}^{n} G_{jx}{}^{\alpha}, \Delta_{\beta\alpha})$$

$$= -\ell_o f_{ox}{}^{i} - \sum_{j=1}^{m} \ell_j G_{jx}{}^{i}, \quad i = 1,\dots,n. \qquad (14.1.19)$$

By comparing these relations with (14.1.14) we obtain

$$\ell_o f_{ox^i} + \sum_{j=1}^m \ell_j G_{jx^i} = (d/dt)(\ell_o f_{ox^{i'}} + \sum_{j=1}^m \ell_j G_{jx^{i'}}),\ i = 1,\ldots,n. \quad (14.1.20)$$

If we denote by $F(t,x,x',\ell)$ the function

$$F(t,x,x',\ell) = \ell_o f_o(t,x,x') + \sum_{j=1}^m \ell_j G_j(t,x,x'), \quad (14.1.21)$$

we see that (14.1.14) yields $\lambda_i = - F_{x^{i'}}$, $i = 1,\ldots,n$, and (14.1.20) becomes

$$F_{x^i} = (d/dt) F_{x^{i'}},\ i = 1,\ldots,n, \quad (14.1.22)$$

which are the analogous to the Euler equations for Bolza problems.

Note that $\Delta \neq o$ is bounded away from zero, all f_{ox^α}, $\Delta_{\alpha i}$ are measurable and bounded, and thus (14.1.18) shows that the multipliers $\ell_1(t),\ldots,\ell_m(t)$ are also measurable and bounded functions of t in $[t_1,t_2]$. These functions $\ell_j(t)$, $j = 1,\ldots,m$, are the classical multipliers for the Bolza (or Lagrange, or Mayer) problem, and their relation with the Pontryagin's multipliers λ_i, $i = 1,\ldots,n$, are displayed by relations (14.1.17) and (14.1.18), where $\ell_o = \lambda_{n+1}$ is a constant. Thus, Pontryagin's multipliers $\lambda_i(t)$, $i = 1,\ldots,n$, are AC in $[t_1, t_2]$, while the classical multipliers $\ell_j(t)$, $j = 1\ldots,m$, have been proved to be only measurable and bounded (Actual examples show that these ℓ_j can be actually discontinuous).

The first member of last relation (14.1.8) yields now an expression for $M(t)$:

$$M(t) = H(t, x(t), u(t), L(t)) = - \sum_{i=1}^n (\ell_o f_{ox^i} + \sum_{j=1}^m \ell_j G_{jx^{i'}}) f_i + \ell_o f_o,$$

and since the original differential equations are certainly satisfied along the optimal solution, that is, $G_j = o$, $j = 1,\dots,m$, we have, by force of (14.1.21),

$$M(t) = \ell_o f_o - \sum_{i=1}^m x^{i'} F_{x^{i'}} = \ell_o f_o + \sum_{j=1}^m \ell_j G_j - \sum_{i=1}^n x^{i'} F_{x^{i'}}$$

$$= F - \sum_{i=1}^n x^{i'} F_{x^{i'}}.$$

Since $\lambda_i(t)$, $i = 1,\dots n$, and $M(t)$ are AC in $[t_1, t_2]$, we conclude that $F_{x^{i'}}$, $i = 1,\dots,n$, and $F - \sum_{i=1}^n x^{i'} F_{x^{i'}}$ are AC functions of t along the optimal solution. In particular, they are continuous at every point $t_o \epsilon (t_1, t_2)$, or

$$[F - \sum_{i=1}^n x^{i'} F_{x^{i'}}]_o^- = [F - \sum_{i=1}^n x^{i'} F_{x^{i'}}]^+$$

$$[F_{x^{i'}}]_o^- = [F_{x^{i'}}]_o^+, \quad i = 1,\dots n, \qquad (14.1.23)$$

where the arguments in F and its derivatives are t, x(t), x'(t), $\ell(t)$, and the brackets denote limits at the left and at the right at t_o for every $t_o \epsilon (t_1,t_2)$. These relations are of interest of course at the points t_o of discontinuity of x'. In particular, at those points where $x'(t_o)$ has a discontinuity of the first kind, say $x'(t_o-o)$, $x'(t_o+o)$ exist and are distinct (corner points). Relations (14.1.23) are the corresponding of Weierstrass and Erdman corner conditions (7.1.ii).

Finally, if we assume as in (2.2) that B admits at the point $(t_1, x(t_1), t_2, x(t_2))$ of a tangent hyperplant B', then the transversality relations of Pantryagin's principle state that

$$\left[\sum_{i=1}^{n} \lambda_i(t)\, dx^i - M(t)\, dt\right]_1^2 = \lambda_o\, dg,$$

and this relation, using the expressions for λ_i, λ_{n+1}, $M(t)$ above, give

$$\left[\left(F - \sum_{i=1}^{n} x^{i'} F_{x^{i'}}\right) dt + \sum_{i=1}^{n} F_{x_i'}\, dx^i\right]_1^2 = \ell_o\, dg$$

which is the usual transversality relation for classical Bolza (or Lagrange, or Mayer) problems. For Lagrange and Mayer problems we have $g_o = o$, or $f_o = o$ respectively.

Along the optimal solution (t,x,x') we have $G_j(t,x,x') = o$, $j = 1,\dots,m$. We shall now consider for every $t \in [t_1, t_2]$ arbitrary vectors $X' = (X^{1'},\dots,X^{n'})$ satisfying $G_j(t, x(t), X') = o$, $j = 1,\dots,m$. According to the definitions and assumptions above we can obtain any vector X' as above by taking $X' = f(t, x,v)$ with v an arbitrary vector in N_1, and we have $v^s = G_s(t,x,X')$, $s = m + 1, \dots,n$, as well as $X' = f(t,x,v)$, with $v = (v^{m+1},\dots,v^n) \in N_1$. Then statement (P2) of Pontryagin's necessary condition yields

$$H(t,x,u,\lambda) \leq H(t,x,v,\lambda)$$

where $x = x(t)$, $u = u(t)$, $\lambda = \lambda(t)$, and $H = \lambda_1 f_1 + + \lambda_{n+1} f_{n+1}$. Hence

$$\ell_o f_o(t,x,x') - \sum_{i=1}^{n} x^{i'} \left(\ell_o f_{ox^{i'}}(t,x,x') + \sum_{j=1}^{m} \ell_j G_{jx^i}(t,x,x')\right)$$

$$\leq \ell_o f_o(t,x,X') - \sum_{i=1}^{n} X^{i'} \left(\ell_o f_{ox^{i'}}(t,x,x') + \sum_{j=1}^{m} \ell_j G_{jx^i}(t,x,x')\right). \qquad (14.1.24)$$

Since $G_j(t,x,x') = o$, $G_j(t,x,X') = o$, $j = 1,\dots m$, we have also

$$\sum \ell_j G_j(t,x,x') = o, \quad \sum \ell_j G_j(t,x,X') = o.$$

Adding these relations to (14.1.24) and using the function F defined in (14.1.21), we obtain by obvious manipulations

$$F(t,x,X',\ell) - F(t,x,x',\ell) - \sum_{i=1}^{n} (X^{i'} - x^{i'}) F_{x^{i'}} (t,x,x',\ell) = o, \qquad (14.1.25)$$

or

$$E(t,x,x', X', \ell) \geq o,$$

where E is the function defined by the first member of (14.1.25) and this function is analogous to the one introduced for free problems (7.1). The expression of E is the same, only written in terms of F instead of f. On the other hand, the Weierstrass function E above depends explicitly upon the m multipliers $\ell = (\ell_1,\dots,\ell_m)$ since these multipliers enter in the expression of F. Thus, C: $x = x(t)$, $t_1 \leq t \leq t_2$, being optimal, and $\ell(t)$ being the corresponding multipliers, we have

$$E(t, x(t), x'(t), X', \ell(t)) \geq o \qquad (14.1.26)$$

for all n-vectors X' satisfying $G_j(t, x(t), X') = o$, $j = 1,\dots,m$, and almost all t. This is the Weierstrass condition for Bolza problems.

Finally, if f and all G_j are of class C^2, then F is of class C^2, and the expression of E given by (14.1.25) can be written as the second remainder of the Taylor formula of F at x'(t). As for free problems (7.1), we shall then consider the quadratic form in $\xi = (\xi_1,\dots,\xi_n)$:

$$Q(t; \xi) = \sum_{ij} F_{x^{i'}x^{j'}} (t, x(t), x'(t), \ell(t)) \xi_i \xi_j.$$

As for free problems, we prove immediately that (14.1.26) implies that $Q(t; \xi)$

is semidefinite positive, in other words: if C: $x = x(t)$, $t_1, \leq t \leq t_2$, is optimal, and $\ell(t)$ are the corresponding multipliers, then

$$Q(t;\ \xi) = \sum_{ij} F_{x^{i'}x^{j'}}\ (t,\ x(t),\ x'(t),\ \ell(t))\ \xi_i\ \xi_j \geq o \qquad (14.1.27)$$

for all ξ, and almost all $t \in [t_1, t_2]$. In particular for $n = 1$, this simply means that

$$F_{x'x'}\ (t,\ x(t),\ x'(t),\ \ell(t)) \geq o$$

for almost all t. This is the Legandre condition for Bolza problems (or Lagrange, or Mayer).

14.2 JACOBI'S NECESSARY CONDITION

We shall now assume that all f_o, G_j are of class C^4. It is possible to define, for classical Bolza, or Lagrange, or Mayer problems, exactly as it was done for free problems, the concepts of first and second variations, under the general frame-work and assumptions of the previous section. Then it is possible to derive the accessory problem and to prove the corresponding Jacobi's necessary condition as for free problems. We leave the details to the reader. We only point out that the accessory equation have the same form as for free problems but are written in terms of F instead of f. Thus they are

$$\sum_{j=1}^{n} \{F_{x^i x^j} - (d/dt)F_{x^i x^j})\ \eta^j + (F_{x^i x^{j'}} - F_{x^{i'} x^j} - (d/dt)F_{x^i x^{j'}})\ \eta^{j'}$$

$$- F_{x^{i'} x^{j'}}\ \eta^{j''} = o,\ j = 1,\dots,n. \qquad (14.2.1)$$

Hence, if $\det(f_{x^{i'} x^{j'}}) \neq o$ along the optimal solution, we can always write these equations in normal form

$$\eta_i' = \sum_{j=1}^{n} (A_{ij}(t)\,\eta_j + B_{ij}(t)\,\eta_j'),\ i=1,\dots,n.$$

Note that in (14.2.1) the arguments of F and its derivatives are t, x(t), x'(t), $\ell(t)$. For n = 1, equations (14.2.1) reduce to

$$(d/dt)(F_{x'x'}\, d\eta/dt) + ((d/dt)F_{xx'} - F_{xx})\,\eta = o. \qquad (14.2.2)$$

Given an extremal arc of class C^2, say $E_{12}: x = x(t),\ t_1 \le t \le t_2$, we shall say that a point $3 = (t_3, x(t_3))$ is a right focus to $1 = (t_1, x(t_1))$ on E_{12} if there is a nonzero solution $\eta(t)$, $t_1 \le t \le t_2$, $\eta = (\eta^1,\dots,\eta^n)$, to system (14.2.1) with $\eta(t_1) = o$, $\eta(t_3) = o$. Analogously, we define a left focus to $2 = (t_2, x(t_2))$ on E_{12}.

(Jacobi necessary condition). Let f_o, G_j, j = 1,...n, all of class C^4, and let E_{12} be an extremal $x = x(t)$, $t_1 \le t \le t_2$, satisfying equations $G_j = o$, whose points (t, x(t)) are all interior to A, and which gives a minimum of I[x] in the class Ω of all curves C: $x = x(t)$, $t_1 \le t \le t_2$, having the same end points $1 = (t_1, x(t_1))$, $2 = (t_2, x(t_2))$ of E_{12}. Assume that x(t) is of class C^2, that $\ell(t)$ is of class C^1, that $\det(F_{x^{i\prime} x^{j\prime}}) \ne o$ for all $t_1 \le t \le t_2$. Then there cannot be any right focus to 1 as well as any left focus to 2, between 1 and 2 on E_{12}, say $3 = (t_3, x(t_3))$, $t_1 < t_3 < t_2$. Again, we leave the proof as an exercise to the reader.

A Lemma On Extremal Arcs

As in (8.3) we shall consider two parametric curves

$$C: \quad t = t_3(a), \; x = x_3(a), \; a' \le a \le a'',$$

$$D: \quad t = t_4(a), \; x = x_4(a), \; a' \le a \le a'',$$

in the tx-space E_{n+1}, $x_3 = (x_3^1, \dots, x_3^n)$, $x_4 = (x_4^1, \dots, x_4^n)$, and a family of vector functions $\ell(t, a) = (\ell_1, \dots, \ell_m)$, $a' \le a \le a''$, and of curves

$$E_a: \quad x = x(t, a), \; t_3(a) \le t \le t_4(a), \; x = (x', \dots, x'')$$

for $a' \le a \le a''$, such that (a) each E_a joins the joints $3 = [t_3(a), x_3(a)]$ and $4 = [t_4(a), x_4(a)]$; (b) each curve E_a satisfies the equations

$$G_j(t, x(t, a), x'(t, a)) = o, \; t_3(a) \le t \le t_4(a), \qquad (14.2.3)$$

$$x(t_3(a), a) = x_3(a), \; x(t_4(a), a) = x_4(a),$$

for every $a' \le a \le a''$; (c) $t_3(a)$, $x_3(a)$, $t_4(a)$, $x_4(a)$ are of class C^1, and $x(t, a)$ is continuous with $x' = \partial x/\partial t$, $x_a = \partial x/\partial a$, $x'_a = \partial^2 x/\partial t \, \partial a$ for $t_3(a) \le t \le t_4(a)$, $a' \le a \le a''$.

We denote by J and J* the line integrals

$$J = \int F dt = \int F(t, x(t), x'(t), \ell(t)) \, dt,$$

$$J^* = \int [F dt + \textstyle\sum_i (dx_i - x'_i \, dt) \, F_{x^i{}'}],$$

where F is the usual function defined in (14.1.21) with $\ell_o = 1$ or

$$F(t,x,x',\ell) = f_o(t,x,x') + \textstyle\sum_{j=1}^m \ell_j \, G_j(t,x,x').$$

The integral J taken along the curve E_a with values $\ell = \ell(t,a) = (\ell_1, \dots, \ell_m)$, or

$$J(a) = J[E_a] = \int_{t_3(a)}^{t_4(a)} F(t, x(t,a), x'(t,a), \ell(t,a))\, dt,$$

is now a differentiable function of a for $a' \leq a \leq a''$, and

$$dI/da = [F\, dt/da]_3^4 + \int_{t_3(a)}^{t_4(a)} \sum_i (F_{x^i} x_a^i + F_{x^{i'}} x_a^{i'})\, dt,$$

since the m terms which are obtained by differentiating F with respect to ℓ_j, $j = 1,\ldots,m$, multiply the corresponding expressions G_j which are all zero along E_a because of (b).

(i) For every $\underline{\underline{a}}$ such that E_a satisfies the Euler equations $F_{x^i} = (d/dt) F_{x^{i'}}$, $i = 1,\ldots,n$, we have

$$d\, J = [F\, dt + \sum_i (dx^i - x^{i'} dt)\, F_{x^{i'}}]_3^4 \qquad (14.2.4)$$

The proof is the same as in (8.3).

(ii) Under the same conditions above, if along every arc E_a: $x = x(t,a)$, $t_3(a) \leq t \leq t_4(a)$, $a' \leq a \leq a''$, the n+m equations $F_{x^i} = (d/dt) F_{x^{i'}}$, $i = 1,\ldots n$, $G_j = o$, $j = 1,\ldots,m$, are satisfied, and say $E_{56} = E_{a'}$, $E_{78} = E_{a*}$, then

$$J[E_{78}] - J[E_{56}] = J^*[D_{68}] = J^*[C_{57}]. \qquad (14.2.5)$$

The proof is the same as in (8.3).

15. FIELDS AND SUFFICIENT CONDITIONS

15.1 EQUATIONS OF A FIELD

We shall assume that $f_o(t,x,x')$, $G_j(t,x,x')$, $j = 1,\dots,m$, are of class C^3. Let R be a simply connected region of the tx-space E_{n+1}, $x = (x^1,\dots,x^n)$. Assume that the n+m functions $p(t,x) = (p_1,\dots,p_n)$, $\ell(t,x) = (\ell_1,\dots,\ell_m)$, $(t,x) \in R$, are such that

(a) $p_i(t,x)$, $i = 1,\dots,n$, $\ell_j(t,x)$, $j = 1,\dots,m$, are of class C^1 in R;

(b) $G_j(t,x, p(t,x)) = o$, $j = 1,\dots,n$, for every $(t,x) \in R$,

(c) the line integral

$$J^*[C] = \int_C [F(t,x, p(t,x), \ell(t,x))\, dt + \sum_{i=1}^n (dx^i - p_i(t,x)\, dt)\, F_{x^{i\prime}}(t, x, p(t,x), \ell(t,x))]$$

depends only upon the end points of C. Then we say that R is a field of slope function p(t,x) and multipliers $\ell(t,x)$. The line integral $J^*[C]$ is said to be the Hilbert integral relative to f_o, the equations $G_j = o$, the slope function p(t,x) and the multipliers $\ell(t,x)$. Note that $J^*[C]$ is defined for every parametric curve C in R. Also, J^* can be written in the equivalent form

$$J^*[C] = \int_C A_o\, dt + A_1\, dx^1 + \dots + A_n\, dx^n$$

$$A_o = A_o(t,x) = F(t,x, p(t,x), \ell(t,x)) - \sum_i F_{x^{i\prime}}(t,x, p(t,x), \ell(t,x))\, p_i(t,x)$$

$$A_i = A_i(t,x) = F_{x^{i\prime}}(t,x, p(t,x), \ell(t,x)),\ i = 1,\dots,n.$$

The differential system $dx/dt = p(t,x)$, or $dx^i/dt = p_i(t, x(t))$, $i = 1,\dots,n$, (15.1.1)

are said to be the differential equations of the field. Having assumed $p(t,x)$ of class C^1 in R, system (15.1.1) satisfies a local existence and uniqueness theorem in R. Thus, through every point (t_o,x_o) in the interior G of R there passes one and only one solution E: $x = x(t)$, $t' < t < t''$, $(t' < t_o < t''$, $x_o = x(t_o))$, and because of (b) we have

$$dx^i/dt = p_i(t, x(t)),\ i = 1,\dots,n,\ G_j(t, x(t), x'(t)) = o,\ j = 1,\dots,\ell,$$

that is, each E satisfies the equations of the field as well as the given equations $G_j = o$.

The solutions E: $x = x(t)$, $t' < t < t''$, of equation (15.1.1) in G = int R are said to be the extremals of the field **R**, and R is said to be a field of extremals. We shall see that these solutions E: $x = x(t)$ are actually extremals of the integral $I[C] = \int f_o\, dt$ with side conditions $G_j = o$, $j = 1,\dots,m$.

(i) On any extremal E_{12}: $x = x(t)$, $t_1 \le t \le t_2$, of the field R we have $I[E_{12}] = J[E_{12}] = J^*[E_{12}]$. Indeed, $dx^i/dt = p_i(t,x)$, $i = 1,\dots,n$, along E_{12}, hence $J[E_{12}] = J^*[E_{12}]$, and $G_j = o$, $j = 1,\dots,m$, hence $I[E_{12}] = J[E_{12}]$.

(ii) The Hilbert integral $J^*[C]$ depends only on the end points of C in R, and $J^*[C] = o$ on every closed curve C in R.

(iii) If E_{12}: $x = x(t)$, $t_1 \le t \le t_2$, $x = (x^1,\dots,x^n)$, is an arc of an extremal of a field R of end points $1 = (t_1, x_1)$, $2 = (t_2, x_2)$, if C_{12}: $x = X(t)$, $t_1 \le t \le t_2$, $X = (X^1,\dots,X^n)$ is any other curve lying in R joining 1 and 2, then

$$I[C_{12}] - I[E_{12}] = \int_{t_1}^{t_2} E(t,X,p,\ell,X')\, dt =$$

$$\int_{t_1}^{t_2} E(t,\ X(t),\ p(t,\ X(t)),\ \ell(t,\ X(t)),\ X'(t))\ dt, \qquad (15.1.2)$$

where E is the Weierstrass function defined in

<u>Proof of (iii)</u>. By force of (i) we have $I[E_{12}] = J[E_{12}] = J^*[E_{12}]$ since E_{12} is an extremal of the field, and by force of (ii) we have $J^*[E_{12}] = J^*[C_{12}]$ since E_{12} and C_{12} are in R and have the same end points. Finally, by the use of the definition of E in (14.1.25) we have

$$I[C_{12}] - I[E_{12}] = J[C_{12}] - J^*[E_{12}] = I[C_{12}] - J^*[C_{12}]$$

$$= \int_{t_1}^{t_2} E(t,\ X(t),\ p(t,\ X(t)),\ \ell(t,\ X(t),\ X'(t))\ dt. \qquad (15.1.3)$$

15.2 CHARACTERIZATION OF A FIELD

Since R is simply connected, requirement (c) of (15.1) for a field is equivalent to the requirement that J* is the integral of an exact differential, or that the n(n+1)/2 equations hold everywhere in R:

$$\partial A_o/\partial x^i = \partial A_i/\partial t,\ \partial A_i/\partial x^j = \partial A_j/\partial_x i,\ i \neq j,\ i,\ j = 1,\ldots,n. \qquad (15.2.1)$$

We shall write these equations in a suitable form. To this purpose we note the identities below which are similar to those in (12.4):

$$\partial A_i/\partial x^i - \partial A_j/\partial x^i = \partial F_x i'/\partial x^j - \partial F_x j'/\partial x^i = F_x i'_x j - F_x j'_x i + \textstyle\sum_h (F_x i'_x h' \partial p_h/\partial x^j$$

$$- F_x j'_x h' \partial' p_h/\partial x^i) + \textstyle\sum_h (F_x i'_{\ell_h} \partial \ell_h/\partial x^j - F_x j'_{\ell_h} \partial \ell h/\partial x^i), \qquad (15.2.2)$$

where $i \neq j$, $i, j = 1,\dots,n$. Also we have

$$\partial A_0/\partial x^i - \partial A_i/\partial t = (\partial/\partial x^i)(F - \textstyle\sum_j F_{x^{j\prime}}\, p_j) - \partial F_{x^{i\prime}}/\partial t =$$

$$= F_{x^i} + \textstyle\sum_j F_{x^{j\prime}}\, \partial p_j/\partial x^i + \sum_j F_{\ell_j}\, \partial \ell_j/\partial x^i - F_{x^{i\prime}t} - \sum_j F_{x^{i\prime}x^{j\prime}}\partial p_j/\partial t$$

$$- \textstyle\sum_j F_{x^{i\prime}\ell_j}\, \partial \ell_j/\partial t - \sum_j F_{x^{j\prime}x^i}\, p_j - \sum_j \sum_h F_{x^{j\prime}x^{h\prime}}\, p_j\, \partial p_h/\partial x^i$$

$$- \textstyle\sum_j \sum_h F_{x^{j\prime}\ell_h}\, p_j\, \partial \ell_h/\partial x^i - \sum_j F_{x^{j\prime}}\, \partial p_j/\partial x^i \qquad (15.2.2)$$

By elimination of the second and the tenth terms, by writing h for j in the fifth term, and adding and subtracting terms, we have the identities

$$\partial A_0/\partial x^i - \partial A_i/\partial t = F_{x^i} + \textstyle\sum_j F_{\ell_j}\partial \ell_j/\partial x^i - F_{x^{i\prime}t} - \sum_j F_{x^{i\prime}x^j}\, p_j$$

$$- \textstyle\sum_j F_{x^{i\prime}x^{j\prime}}(\partial p_j/\partial t + \sum_h p_h \partial p_j/\partial x_h) - \sum_j F_{x^{i\prime}\ell_j}(\partial \ell_j/\partial t + \sum_h p_h \partial \ell_j/\partial x^h)$$

$$+ \textstyle\sum_j [F_{x^{i\prime}x^j} - F_{x^{j\prime}x^i} + \sum_h(F_{x^{i\prime}x^{h\prime}}\partial p_h/\partial x^j - F_{x^{j\prime}x^{h\prime}}\partial p_h/\partial x^i)$$

$$+ \textstyle\sum_h (F_{x^{i\prime}\ell_h}\partial \ell_h/\partial x^j - F_{x^{j\prime}\ell_h}\partial p_h/\partial x^i)]\, p_j,$$

or

$$\partial A_0/\partial x^i - \partial A_i/\partial t = F_{x^i} - (d/dt)F_{x^{i\prime}} + \textstyle\sum_j p_j\, (\partial A_i/\partial x^j - \partial A_j/\partial x^i), \qquad (15.2.3)$$

where d/dt denotes the directional derivative $\partial/\partial t + \sum_j p_j\, \partial/\partial x^j$, that is, in the direction of the field satisfying the differential equations $dx^i/dt = p_i(t,x)$, $i = 1,\dots,n$, or $dx/dt = p(t,x)$. Note that in the equations above $F_{\ell_j} = G_j$, $j = 1,\dots,m$.

(15.2.i) If R is a simply connected region and $p(t,x) = (p_1,\dots,p_n)$, $\ell(t,x) = (\ell_1,\dots,\ell_m)$ are of class C^1 in R, and $G_j(t,x, p(t,x)) = 0$ $j = 1,\dots,m$, everywhere in R, then $I[\overset{*}{C}]$ depends only upon the end points of C for curves C in R if and only if the solutions $x = x(t)$ of the system $dx/dt = p(t,x)$ are extremals of I(C) satisfying $G_j = o$, $j = 1,\ldots,m$, and multipliers $\ell(t, x(t)) = (\ell_1,\ldots,\ell m)$ and the $n(n-1)/2$ equations hold $\partial A_i/\partial x^j - \partial A_j/\partial x^i = o$, $i \neq j$, $i, j = 1,\ldots,n$, $(t,x) \in R$.

These $n(n-1)/2$ equations can be written explicitly using identities (15.2.2). For $n = 1$ there are no such equations.

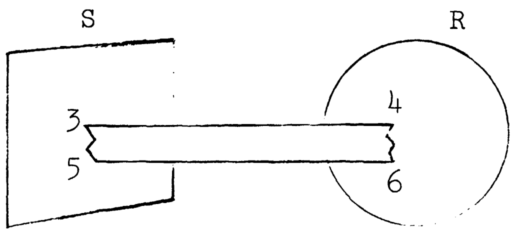
<u>Proof</u>. Along the solutions E of the system $dx/dt = p(t,x)$ we have

$$dx^i/dt = p_i,\ d^2x^i/dt^2 = (d/dt)\ p_i(t, x(t)) = \partial p_i/\partial t + \textstyle\sum_h p_h\ \partial p_i/\partial x^h.$$

Thus, if (c) holds, then equations (15.2.1.) are satisfied, hence $A_{ix^j} - A_{jx^i} = o$, $i \neq j$, $i, j = 1,\ldots,n$, and by force of (15.2.3), also $F_{x^i} - (d/dt)\ F_{x^{i'}} = o$, that is, the solutions E of the field are extremals, with multipliers $\ell(t, x(t))$. Conversely, if the equations $A_{ix^j} - A_{jx^i} = o$, $i \neq j$, $i, j = 1,\ldots,n$, are satisfied, and $F_{x^i} - (d/dt)\ F_{x^{i'}} = o$, $i = 1,\ldots,n$, then all $n(n+1)/2$ equations (15.2.1) are satisfied and (c) holds.

Equations (15.2.1) are not easy to verify, and, therefore, geometric processes have been devised for the construction of a field. The one we described here is the generalization of the process indicated in (12.5).

Let E_a be an n-parameter family of extremals of I[C], say E_a: $x = x(t\ a)$, $a = (a_1,\ldots,a_n)$, simply covering a simply connected region R, and let $\ell(t, a) = (\ell_1,\ldots,\ell_n)$ be the corresponding multipliers. Thus, there is a region R of the auxiliary $ta_1 \ldots a_n$- space which is mapped one-one and onto R by the relation $t = t$, $x = x(t, a)$ (and thus there is an inverse map $t = t$, $a = a(t, x)$). Let us assume that $x(t, a)$, $\ell(t, a)$ are continuous with the partial derivatives x', x'', x_{a_i}, ℓ_t, ℓ_{a_i}, $i = 1, \ldots, n$. Then, the slope function $p = p(t, x) = (p_1, \ldots, p_n)$, $(t, x) \in R$ is given by $p(t, x) = x'(t, a(t, x))$, $i = 1, \ldots, n$, and along E_a the slope and the multipliers are $p(t, x) = p(t, x(t, a))$, $\ell(t, x) = \ell(t, x(t, a))$. The conditions (a) and (b) for a field are thus satisfied. Let us assume that every extremal E_a of the family $x=x(t,a)$ cuts a given hypersurface S at some point (t_o, x_o), $t_o = t_o(a)$, $x_o = x(t_o, a)$ and assume that $t_o(a)$ is a function of class C^1. Then also $x_o = x(t_o(a), a)$, $\ell = \ell(t_o(a), a))$ are functions of class C^1 in a. For every point $(t, x) \in R$ there passes one and only one extremal E_a with $a = a(t, x)$.



The corresponding point (t_o, x_o) that E_a has on S then depends on (t, x). Precisely $t_o = t_o(a(t, x))$, $x_o = x_o[t_o(a(t, x)), a(t, x)]$. These functions too are of class C^1 in R.

Now if (t, x) describes a parametric curve, say D_{46}: $t = t(\alpha)$, $x = x(\alpha)$, $\alpha' \leq \alpha \leq \alpha''$, in R and of class C^1, then the corresponding point (t_o, x_o) describes a curve C_{35} also of class C^1 on S. By using the functions $p(t, x)$, $\ell(t, x)$ we can still define $J^*[C_{35}]$. We have

(15.2.ii) If the extremals E_a cut a given hypersurface S and, for every curve D_{46} in R, the corresponding integral $J[C_{35}]$ depends only upon the end points.

of C_{35}, then the field condition (c) is satisfied in R.

Indeed, by force of (14.2.5), we have as in (12.5)

$$J^*[D_{46}] = J^*[C_{35}] + J[E_{56}] - J[E_{34}].$$

Statement (15.2.ii) applies particularly well to the case where all extremals E_a pass through a fixed point $P = (t_o, x_o)$ and then $S = (t_o, x_o)$.

15.3 SUFFICIENT CONDITIONS FOR AN EXTREMUM

<u>Theorem I</u> (Sufficient condition for a strong relative minimum). If the extremal E_{12} with multipliers $\ell(t)$ is contained in a field R with slope function $p(t, x) = (p_1,\dots,p_n)$ and multipliers $\ell(t, x) = (\ell_1,\dots,\ell_m)$, if $E(t, x, p(t, x), \ell(t, x), X') \geq o$ for all vectors X' and all points $(t, x) \in R$, then for every curve C_{12} lying in R and having the same end points 1 and 2 of E_{12} we have $I[C_{12}] \geq I[E_{12}]$. If $E > o$ for all $(t, x) \in R$ and $X' \neq p(t, x)$, then $I[C_{12}] > I[E_{12}]$ for all curves C_{12} not identical to E_{12}.

<u>Theorem II</u> (Sufficient condition for a week relative minimum). Suppose that an extremal E_{12}: $x = x(t)$, $t_1 \leq t \leq t_2$, with multipliers $\ell(t)$ has the properties (a) the quadratic form Q of (14.1.27) taken along E_{12} is definite positive for every $t_1 \leq t \leq t_2$; (b) E_{12} is contained in a field R of extremals. Then there is an $\epsilon > o$ such that, for any curve C_{12}: $x = X(t)$, $t_1 \leq t \leq t_2$, having the same end points 1 and 2 as E_{12} with $|X(t) - x(t)| \leq \epsilon$, $|X'(t) - x'(t)| \leq \epsilon$, $t_1 \leq t \leq t_2$, we have $I[C_{12}], \geq I[E_{12}]$, and = sign holds only if C_{12} is identical with E_{12}. For n = 1 assumption (a) is replaced by the assumption $F_{x'x'}(t, x(t), x'(t), \ell(t) > o$ for $t_1 \leq t \leq t_2$. Condition (b) can be replaced by the condition (b') no point 3 between 1 and 2 on E_{12}, nor

2, is conjugate to 1.

Theorem III. (Sufficient condition for a strong relative minimum). Suppose that an extremal E_{12}: $x = x(t)$, $t_1 \leq t \leq t_2$, with multipliers $\ell(t)$ has the properties (a) $E(t, x, x', \ell, X') \geq o$ for all $t_1 \leq t \leq t_2$, $|x-x(t)| \leq \delta$, and all vectors X'; (b) E_{12} is contained in a field R of extremals. Then there is an $\epsilon > 0$ such that for any curve C_{12}: $x = x(t)$, $t_1 \leq t \leq t_2$, having the same end points 1 and 2 as E_{12} with $|X(t) - x(t)| \leq \epsilon$ for $t_1 \leq t \leq t_2$, we have $I[C_{12} \geq I[E_{12}]$. If $E > o$ for all t,x,x',X' as above with $X' \neq x'$, then $I[C_{12}]$ - $I[E_{12}]$ for every curve C_{12} as above distinct from E_{12}. Assumption (b) can be replaced by (b') no point 3 between 1 and 2 on E_{12}, nor 2, is conjugate to 1.

The proofs of Theorems I, II, III are the same as in no. 12.6.

15.4 HAMILTON - JACOBI EQUATION

We shall assume as above that f_o and all G_j, $j = 1,\ldots,m$, are of class C^2. Let us consider an arbitrary field R in the tx-space E_n, $x = (x^1,\ldots,x^n)$, with slope function $p(t,x) = (p_1,\ldots,p_n)$, and multipliers $\ell(t,x) = (\ell_1,\ldots,\ell_m)$ all of class C^1 in R. Then the relative Hilbert integral $J^*[C]$ depends only on the end points of the curve C in the simply connected region R. If (t_o, x_o) is a fixed point of R, and (t,x) denotes any point of R, then

$$W(t,x) = \int_{(t_o,x_o)}^{(t,x)} F(t,x, p(t,x), \ell(t,x))$$

$$+ \sum_i (dx_i - p_i(t,x)\, dt)\, F_{x^{i'}}(t,x, p(t,x), \ell(t,x)) \qquad (15.4.1)$$

defines a function of (t,x) in R independent of the arbitrary curve C joining (t_o,x_o) to (t,x) in R. We have also

$$W(t,x) = \int_{(t_o,x_o)}^{(t,x)} A_o dt + A_1 dx^1 + \ldots + A_n dx^n,$$

and for every (t,x) in the open set $G = \text{int } R$ the partial derivatives

$$W_t(t,x) = A_o = F(t,x,\ p(t,x),\ \ell(t,x) - \textstyle\sum_j F_{x^{j'}}(t,x,\ p(t,x),\ \ell(t,x))p_j(t,x)$$

$$W_{x^i}(t,x) = A_i = F_{x^{i'}}(t,x,\ p(t,x),\ \ell(t,x)),\ i = 1,\ldots,n, \qquad (15.4.2)$$

exist and are continuous in G. We shall use the notation $W_x = (W_{x^1},\ldots,W_{x^n})$. In (14.1) we considered the Hamilton function

$$H(t,\ x,\ \lambda,\ u) = \lambda_1 f_1 + \ldots + \lambda_n f_n + \lambda_{n+1} f_{n+1}$$

which we now shall transform as in (14.1) by the substitutions

$$\lambda_{n+1} = \ell_o = 1,\ \lambda_i = -\ell_o f_{ox^{i'}} - \textstyle\sum_{j=1}^{n} \ell_j G_{jx^{i'}},\ i = 1,\ldots,n,\ x^{i'} = f_i(t,x,u),$$

the last ones begin equivalent to the relations $G_j(t,x,x') = o$, $j = 1,\ldots,n$, $u_j = G_j(t,x,x')$, $j = m+1,\ldots,n$. Then H becomes a function of t, x, x', ℓ, say

$$H(t,x,x',\ell) = f_o(t,x,x') - \textstyle\sum_{i=1}^{n} x^{i'}(f_{ox^i} - \sum_{j=1}^{m} \ell_j G_{jx^{i'}}).$$

For $x' = p(t,x)$, $\ell = \ell(t,x)$, we have $G_j(t,x,\ p(t,x)) = o$, $j = 1,\ldots,m$, and hence

$$H(t,x,\ p(t,x),\ \ell(t,x)) = F(t,x,\ p(t,x),\ \ell(t,x)) - \textstyle\sum_{i=1}^{n} p_i(t,x)\, F_{x^{i'}}(t,x,\ p(t,x),\ \ell(t,x)) \qquad (15.4.3)$$

Relations (15.4.2) show now that W satisfies

$$W_t(t,x) = F(t,x,\ p(t,x)) - \Sigma_{i=1}^{n} p_i(t,x)\ W_{x^i}(t,x), \qquad (15.4.4)$$

or

$$W_t(t,x) = H(t,x,\ W_x(t,x),\ \ell(t,x))$$

which is a partial differential equation for W where still the functions $p(t,x)$, $\ell(t,x)$ defining the field R appear.

Nevertheless, we may be able to eliminate p and ℓ. To this purpose we may assume, in addition to the general hypotheses of a field, that

(d) For every $(t,x) \in R$ the n equations

$$\Lambda_i = F_{x_i'} = f_o(t,x,\ f(t,x,u)) - \Sigma_{j=1}^{m} \ell_j\ G_{x^{i'}}(t,x,\ f(t,x,u)),\ i = 1,\dots,n, \qquad (15.4.5)$$

can be inverted so that

$$u_j = Z_s(t,x,\Lambda),\ s = m+1,\dots,n,\ \ell_j = L_j(t,x,\Lambda),\ j = 1,\dots,m. \qquad (15.4.6)$$

We shall use the notation $u = Z(t,x,\Lambda)$, $\ell = L(t,x,\Lambda)$ with $\Lambda = (\Lambda_1,\dots,\Lambda_n)$. Then the relations $x^{i'} = f_i(t,x,u)$ yield $x^{i'} = f_i(t,x,\ Z(t,x,\Lambda))$, or

$$x' = P(t,x,\Lambda) = f(t,x,\ Z(t,x,\Lambda)),\ \ell = L(t,x,\Lambda).$$

Actually, in (d) we assume that a region $\mathcal{R}$ of the $txu\ell$-space is transformed by (15.4.5) one-one onto a region L' of the $tx\Lambda$-space E_{2n+1} so that both the direct and inverse transformations (15.4.5) and (15.4.6) are of class C^1, and that $(t,x,u,\ell) \in \mathcal{R}$ for all $(t,x,u) \in R_o$, $\ell = \ell(t,x)$.

If we replace $p = x'$ by $P(t,x,\Lambda)$ and ℓ by $L(t,x,\Lambda)$ in the expression (15.4.3) for H, we transform H into a function

$$H_*(t,x,\Lambda) = F(t,x,\ P(t,x,\Lambda)) - \Sigma_{i=1}^{n}\ P_i(t,x,\Lambda)\Lambda_i,$$

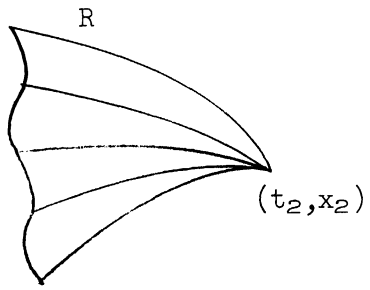
and equation (15.4.4) becomes

$$W_t(t,x) = H_*(t,x,W_x). \qquad (15.4.7)$$

This is the Hamilton-Jacobi first order partial differential equation of a field, and (15.4.7) is independent of the field.

This last statement has to be understood in the sense that, given the simply connected region R in E_{n+1}, for any pair $p(t,x)$, $\ell(t,x)$ of functions of class C^1 in $\mathcal{R}$ which make R a field and $(t,x,\ p(t,x),\ \ell(t,x)) \in \mathcal{R}$, then equation (15.4.7) holds.

There are extensions to the statements above for fields R with singularities. We shall consider here only the case of one-point singularity, $S = (t_2,\ x_2)$. Besides the assumptions and conventions of (15.2), let us assume that every point $(t,x) \in R$ belongs to one and only one extremal of the field $E(t,x)$: $x = X(t)$, $t \leq t \leq t_2$, joining (t,x) to $(t_2,\ x_2)$, so that $(t,\ X(t))$



$\in$ R-S for $t \leq t < t_2$, $X(t) = x$, $X(t_2) = x_2$. Then instead of (15.4.1) we shall consider the function

$$W_o(t,x) = \int_{(t,\ x)}^{(t_2,\ x_2)} F\ dt + \Sigma_i (dx_i - p_i(t,x))\ F_{x_i}{}'$$

where the integration is taken along any curve C from (t,x) to $(t_2,\ x_2)$, in particular along $E(t,x)$. Thus

$$W_o(t,x) = \int_{E(t,x)} F\ dt = \int_{E(t,x)} f_o\ dt = I[E(t,x)],$$

that is, $W_o(t,x)$ is the value of the functional I along the extremal $E(t,x)$ from (t,x) to the "target" (t_2, x_2). As above, for $(t,x) \in G = \text{int } R$, we have

$$W_{ot} = -A_o = -F - \sum_j F_{x^i}' p_j, \quad W_{ox^i} = -A_i = -F_{x^i}, \quad i = 1,\ldots,n.$$

Then, as above W satisfies the Hamilton-Jacobi first order partial differential equation in G.

16. DEDUCTION OF NECESSARY CONDITIONS FOR ISOPERIMETRIC PROBLEMS

16.1 CONSTANCY OF THE CLASSICAL MULTIPLIERS

We consider here the classical isoperimetric problem

$$I[x] = \int_{t_1}^{t_2} f_o\ (t,x,x')\ dt = \text{minimum},$$

$$J_j[x] = \int_{t_1}^{t_2} g_j\ (t,x,x')\ dt = C_j,\ j = 1,\ldots,m,$$

with usual boundary conditions and constraint $(t, x(t)) \in A$. By the introduction of an auxiliary variables $y = (y^1,\ldots,y^m)$ the problem above can be written as a Lagrange problem with m differential equations, and n+m variables, or $\tilde{x} = (x,y) = (x^1,\ldots,x^n, y^1,\ldots,y^m)$:

$$I[x,y] = \int_{t_1}^{t_2} f_o\ (t,x,x')\ dt = \text{minimum},$$

$$G_j\ (t,x, y,x',y') = g_j\ (t,x,x') - y^{j\,'} = o,\ j = 1,\ldots,m,$$

$$y^j(t_1) = o,\ y^j(t_2) = C_j,\ j = 1,\ldots,m,$$

and the remaining boundary conditions on the variables $x^1,\ldots,x^n$.

The n x m matrix of the partial derivatives $[\partial G_s/\partial x^{i\,'},\ \partial G_s/\partial y^j,\ y = 1,\ldots,m,\ i = 1,\ldots,n,\ j = 1,\ldots,m]$ has certainly maximum rank m since $\partial G_s/\partial j^j = -\delta_{sj} = -1$ for $s = j$; $= o$ for $s \neq j$. Whenever the remaining assumptions mentioned in (14.1) are satisfied, we know that there is a constant ℓ_o and m functions $\ell_j(t)$, $j = 1,\ldots,m$, such that, by taking

$$\bar{F} = \ell_o f_o + \Sigma_j \ell_j (g_j - y^{s'}), \quad F = \ell_o f_o + \Sigma_j \ell_j g_j,$$

the partial derivatives $\bar{F}_{x^{i'}}$, $\bar{F}_{y^{j'}}$, taken along the optimal solution coincide (a.e. in $[t_1, t_2]$) with AC functions of t, and that

$$\bar{F}_{x^i} = (d/dt)\, \bar{F}_{x^{i'}}, \ i = 1,\ldots,n, \quad \bar{F}_{y^j} = (d/dt)\, \bar{F}_{y^{j'}}, \ j = 1,\ldots,m,$$

or

$$F_{x^i} = (d/dt)\, F_{x^i}, \ i = 1,\ldots,n, \quad o = (d/dt)\, \ell_j, \ j = 1,\ldots,m.$$

Hence, all classical multipliers ℓ_j, $j = 1,\ldots,m$, can be taken as constants in $[t_1, t_2]$. This proves the multiplier rule in the usual form stated for isoperimetric problems with constant multipliers.

The Weierstrass condition for $\bar{F}$, or

$$\bar{F}(t,x,X',y,Y') - \bar{F}(t,x,x',y,y') - \Sigma_i (X^{i'} - x^{i'})\, \bar{F}_{x_i'}(t,x,x',y,y')$$

$$- \Sigma_j (Y^{j'} - y^{j'})\, \bar{F}_{y^{j'}}(t,x,x',y,y') \geq o$$

reduces to

$$(F(t,x,X') - \Sigma \ell_j Y^{j'}) - (F(t,x,x') - \Sigma \ell_j y^{j'}) - \Sigma_i (X^{i'} - x^{i'}) F_{x^{i'}}(t,x,x')$$

$$+ \Sigma \ell_j (Y^{j'} - y^{j'}) \geq o,$$

that is, $E(t,x,x',X') \geq o$, where E is the Weierstrass function in terms of F, with no trace of the auxiliary variables.

Finally, the transversality relation in terms of $\bar{F}$ is

$$[(\bar{F} - \textstyle\sum_i x^{i'} \bar{F}_x{}_{i'} - \sum_j y^{j'} \bar{F}_y{}_{j'})\, dt + \sum_i \bar{F}_x{}_{i'}\, dx^i + \sum_j \bar{F}_y{}_{j'}\, dy^j]_1^2 = o$$

where $dy_1^j = o$, $dy_2^j = o$ since the auxiliary variables y^j have fixed values at t_1 and t_2, and the first parenthesis coincides with $F - \sum x^{i'} F_x{}_{i'}$. Thus we obtain the transversality relation in the usual form for isoperimetric problems in terms of F:

$$[(F - \textstyle\sum_i x_i' F_x{}_{i'})\, dt + \sum_i F_x{}_{i'}\, dx^i]_1^2 = o.$$

For the isoperimetric problem of no. (   ), concerning the arc C of given length L joining the points 1 = (t_1, o), 2 = (t_2, o), lying above the t-axis, and such that the area between C and the segment$_{12}$ is maximum, we have

$$I[x,u] = \int_{t_1}^{t_2} x dt = \max$$

$$J[x,u] = \int_{t_1}^{t_2} (1+x'^2)^{1/2}\, dt = L, x(t_1) = o,\ x(t_2) = o.$$

For an optional solution there must be constants ℓ_o, ℓ_1 such that if

$$F = \ell_o\, x + \ell_1 (1+x'^2)^{1/2},$$

then

$$F_x = (d/dt) F_{x'},$$

or

$$\ell_o = \ell_1 (d/dt)[x' (1+x'^2)^{-1/2}]$$

The problem

$$(d/dt)[x' (1+x'^2)^{-1/2}) = 1/c$$

yields

$$x'(1+x'^2)^{-1/2} = \frac{1}{c}(t - \alpha) \qquad \frac{x'^2}{1+x'^2} = \frac{1}{c^2}(t - \alpha)^2,$$

$$dx/dt = (t - \alpha)[c^2 - (t - \alpha)^2]^{-1/2},$$

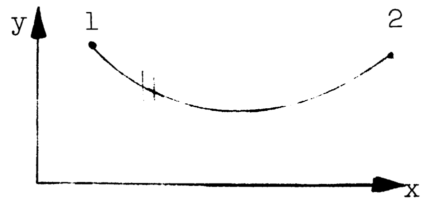
$$x - \beta = (c^2 - (t - \alpha)^2)^{1/2},$$

$$(t - \alpha)^2 + (x - \beta)^2 = c^2.$$

Thus the optional solution (if any) must be an arc of circle, representable in the form $x = x(t)$, $t_1 \leq t \leq t_2$. This is possible only if $t_2 - t_1 \leq L \leq (\pi/2)(t_2 - t_1)$, and then the arc C is uniquely determined.

17. EXAMPLES AND APPLICATIONS

17.1. SHAPE OF A HANGING ROPE



A flexible rope of uniform density hangs at rest with its end points fixed. Its shape can be determined by using the principle that a mechanical system is in a state of stable equilibrium when its potential energy consistent with the constraints is minimum. Here the center of masses will be as low as possible. If L is the length of the rope, δ the density of the rope (per unit length, C: $y = y(x)$, $x_1 \leq x \leq x_2$, the shape of the rope, $1 = (x_1,y_1)$, $2 = (x_2,y_2)$ the fixed end points, then ds is the element of length, δds the element of area, $M = \int_{x_1}^{x_2} \delta ds = \delta L$ the total mass, $\mu = \int_{x_1}^{x_2} \delta y\, dx$ the moment of the rope with respect to the x-axis. The ordinate of the center of masses is $\mu/M = L^{-1} \int_{x_1}^{x_2} y\, ds$. Thus, we have to minimize the integral

$$I[y] = \int_{x_1}^{x_2} y\, ds = \int_{x_1}^{x_2} y(1 + y'^2)^{1/2}\, dx \tag{17.1.1}$$

with the constraint that the length of the rope is L, that is,

$$I[y] = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} (1 + y'^2)^{1/2}\, dx = L. \tag{17.1.2}$$

We shall denote by K the collection of all curves C: $y = y(x)$, $x_1 \leq x \leq x_2$, with $y(x_1) = y_1$, $y(x_2) = y_2$, $y(x) \in AC$, and by K_L the subcollection of these curves of length L. We assume $x_1 < x_2$, and $|12| < L$, that is

$$|12|^2 = (x_2 - x_1)^2 + (y_2 - y_1) < L^2. \qquad (17.1.3)$$

We have to determine the minimum of I[y] in the collection K_L. Here we have $f = y(1 + y'^2)^{1/2}$, $g = (1 + y'^2)^{1/2}$, and according to (14.1), we have to study the integrand

$$F = f + \lambda g = (y + \lambda)(1 + y'^2)^{1/2}.$$

with

$$F_{y'} = (y + \lambda)(1 + y^{12})^{-1/2}, \; F_{y'y'} = (y + \lambda)(1 + y'^2)^{-3/2}. \qquad (17.1.4)$$

The Euler equation (14.1.22) yields

$$y + \lambda = c(1 + y'^2)^{1/2}$$

which is the same equation (21.11) up to a displacement parallel to the x-axis. Thus we have

$$y + \lambda = b \text{ ch } \frac{x - a}{b}, \qquad (17.1.5)$$

where ab are constants of integration, $b \neq 0$, and λ is still undetermined.

If there is a curve C in the class K_L which gives a minimum of I[y] in K_L, then this curve must be made up of arcs of catenaries (17.1.5), and thus $y + \lambda \neq 0$ on each of these arcs. Thus we should have $y + \lambda \neq 0$ also at any of the possible corner points, and this makes $F_{y'}$ an always increasing or decreasing function of y'. By Erdman corner condition this rules out the possibility of corner points. Thus, if a curve C of K_L gives a minimum for I[y] in K_L then C is a single arc of a catenary (17.1.5). We have now to determine λ, a, $b \neq 0$ in such a way that $y(x_1) = x_1$, $y(x_2) = x_2$, $J(y) = L$. By computation of the integral J on the curves (17.1.5) we obtain the system of finite equations,

$$y_1 + \lambda = b \text{ ch } \frac{x_1 - a}{b}, \; y_2 + \lambda = b \text{ ch } \frac{x_2 - a}{b},$$

$$b\left(\text{sh } \frac{x_2 - a}{b} - \text{sh } \frac{x_1 - a}{b}\right) = L.$$

By taking $x - a = bt$, we have to determine t_1, t_2, λ, a, b in such a way that

$$x_1 = a + bt_1, \; x_2 = a + bt_2, \; y_1 + \lambda = b \text{ ch } t_1, \; y_2 + \lambda = b \text{ ch } t_2,$$

$$b(\text{sh } t_2 - \text{sh } t_1) = L.$$

It is not restrictive to assume $b > 0$, $t_1 < t_2$, since the opposite case can be reduced to this one by interchanging b with $-b$, t_1 with t_2, λ with $-\lambda - y_1 - \lambda_2$. By taking $\mu = 2^{-1}(t_2 + t_1)$ and $\nu = 2^{-1}(t_2 - t_1)$, hence $t_1 = \mu - \nu$, $t_2 = \mu + \nu$, these equations yield

$$\begin{gathered} \mu = (2b)^{-1}(x_1 + x_2 - 2a), \; \nu = (2b)^{-1}(x_2 - x_1) \\ y_2 - y_1 = 2b \text{ sh } \mu \text{ sh } \nu, \; L = 2b \text{ ch } \mu \text{ sh } \nu \end{gathered} \tag{17.1.6}$$

and hence

$$\text{th } \mu = \frac{y_2 - y_1}{L}$$

Since $|y_2 - y_1|/L < 1$, this equation has one and only one solution, and μ has the sign of $y_2 - y_1$. Thus

$$\text{sh } \mu = (y_2 - y_1)[L^2 - (y_2 - y_1)^2]^{-1/2}, \tag{17.1.7}$$

and by using second and third equations (17.1.6), we obtain

$$\frac{\text{sh } \nu}{\nu} = \frac{[L^2 - (y_2 - y_1)^2]^{1/2}}{x_2 - x_1} \equiv K \tag{17.1.8}$$

and relation (17.1.8) implies $K > 1$. Since $\text{sh } \nu/\nu$ is an increasing continuous function of $\nu > 0$ with limits 1 and $+\infty$ for $\nu \to 0$ and $\nu \to \pm\infty$, respectively, we conclude that there is one and only one value of $\nu > 0$ satisfying (17.1.8), and thus (17.1.8) has the two solutions $\pm \nu$. By the equations above we can now determine b, a, λ, and b will have the same sign of ν. Now for a minimum we must have $F_{y'y'} \geq 0$, hence $y + \lambda \geq 0$. Since $y + \lambda$ has a constant sign, and the same sign of b, we conclude that we must choose above $\nu > 0$, hence $b > 0$, and $y + \lambda > 0$ along the uniquely determined arc of catenary (17.1.5) so determined of length L through 1 and 2.

If we note that on E we have

$$F_{y'y'} = (y + \lambda)(1 + y'^2)^{-3/2} = b\ ch^{-2}\frac{x - a}{b},\ F_{yy} = 0,$$

$$F_{yy'} = (1 + y'^2)^{-1/2} = ch^{-1}\frac{x - a}{b},\ \frac{d}{dx}F_{yy'} = b^{-1}ch^{-2}\frac{x - a}{b},$$

the accessory equation (14.2.2) for F relatively to the extremal E becomes

$$\frac{d}{dx}\left(b\ ch^{-2}\frac{x - a}{b}\eta\right) - b^{-1}\eta\ ch^{-2}\frac{x - a}{b} = 0,\ x_1 \leq x \leq x_2. \qquad (17.1.9)$$

In the variable t as above and the unknown function $u(t) = \eta(a+bt)$, we have the equation

$$\frac{d}{dt}\left(ch^{-2}t\ \frac{du}{dt}\right) - u\ ch^{-2}t = 0,\ t_1 \leq t \leq t_2. \qquad (17.1.10)$$

Equations of the form $(\varphi u')' = \psi u$, with $\varphi > 0$, $\psi > 0$, are known to have the property that every solution $u(t)$ with $u(t_1) = 0$, $u'(t_1) = \mu > 0$, are monotone increasing for $\mu > 0$, monotone decreasing for $\mu < 0$. Indeed, in a right neighborhood of t_1 we must have $u' > 0$ hence $(\varphi u)' > 0$, $\varphi u'$ increasing hence $\varphi u' > 0$, $u' > 0$, and u' increasing. Since $\varphi u'$ is the integral from t_1 to t of ψu, we conclude that $\varphi u' > 0$, hence $u' > 0$ for every $t \geq t_1$. Analogously, if $\mu < 0$, then $u < 0$, $u' < 0$ for every $t \geq t_1$. We conclude that equation (17.1.10) has no nonzero solution which is zero at t_1 and t_2, and the same conclusion holds for equation (17.1.9) and x_1, x_2. Thus, no point of E_{12} (not even 2) can be a focus of 1. By statement of (15.3) we conclude that E_{12} is at least a weak relative minimum.

17.2 EXERCISES

1. Problem. It is required to extremize

$$I = \int_{t_1}^{t_2} f(t,x,x')dt + \varphi(w)$$

with respect to the function $x(t)$ and the constant w in such a way that

$$J = \int_{t_1}^{t_2} g(t,x,x')dt + \psi(w)$$

has a constant value L, and $x(t)$ has prescribed values x_1 and x_2 at the fixed points t_1 and t_2. Show that the required extremum satisfies the equations

$$f^*_x - (d/dt)f^*_{x'} = 0, \quad \varphi^{*\prime}(w) = 0,$$

where $f^* = f + \lambda\, g$, $\varphi^* = \varphi + \lambda\psi^*$, λ a constant.

2. Problem. A perfectly flexible uniform rope of length L hangs in (unstable equilibrium with one end fixed at (t_1, x_1), and passing over a frictionless pin at (t_2, x_2). Find the position of the free end of the rope. It is clear that the portion of the rope extended between the given points hangs in form of a catenary. Use the conclusions of problem 1.