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CLOSURE AND LOWER CLOSURE THEOREMS FOR MULTIDIMENSIONAL PROBLEMS

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8. Closure and Lower Closure Theorems for Multidimensional Problems

8.1. Closure Theorems for Orientor Fields

Here \( G \) denotes as in (7.1) an open bounded subset of the \( t \)-space \( E_t \), for every \( t \in \text{cl} \; G \) we denote by \( A(t) \) a given subset of the \( y \)-space \( E_y \), and by \( A \) the set of all \( (t,y) \) with \( t \in \text{cl} \; G \), \( y \in A(t) \). We assume \( A \) to be closed, and for every \( (t,y) \in A \) we denote by \( Q(t,y) \) a given subset of the \( z \)-space \( E_z \). We consider here pairs \( z(t) = (z^1, \ldots, z^n) \), \( y(t) = (y^1, \ldots, y^s) \), \( t \in G \), of vector functions \( z \in (L_1(G))^n \), \( y \in (L_1(G))^s \), such that

\[
y(t) \in A(t), \quad z(t) \in Q(t,y(t)) \quad \text{a.e. in } G.
\] (8.1.1)

We may say that the pair \((z,y)\) is an abstract solution of the orientor field (8.1.1). In applications there are suitable functional relations between \( z \) and \( y \), that here are not needed.

(8.1.i) A first closure theorem. Let \( G, A(t), A, Q(t,y) \) be as above, \( G \) open and bounded, \( A \) closed, and let us assume that the sets \( Q(t,y) \) satisfy property (Q) in \( A \). Let \( y(t), \; y_k(t), \; t \in G \), be \( s \)-vector functions, \( y, y_k \in (L_1(G))^s \), and \( z(t), z_k(t), \; t \in G \), be \( r \)-vector functions, \( z, z_k \in (L_1(G))^r \), with \( y_k(t) \in A(t), \; z_k(t) \in Q(t,y_k(t)) \) a.e. in \( G, \; k = 1, 2, \ldots \). If \( y_k \rightarrow y \) strongly in \((L_1(G))^s \), and \( z_k \rightarrow z \) weakly in \((L_1(G))^r \), then \( y(t) \in A(t), z(t) \in Q(t,y(t)) \) a.e. in \( G \).

This theorem is a particular case of (8.2.i) below.
8.2 A Closure Theorem for Trajectories With Singular Components

Let \( G \) be as above a bounded open subset of the \( t \)-space \( E_\nu \), and let \( I_o \) be an interval containing \( \text{cl } G, G \subset \text{cl } G \subset I_o \subset E_\nu \). It is not restrictive to assume \( I_o = [a,b] \), or \([a,\ldots,a,b,\ldots,b]\) for some \( b > a \). For every \( t \in I \), \( t = (t^1,\ldots,t^\nu) \), let \([o,t]\) denote the interval \([o,\ldots,o,t^1,\ldots,t^\nu]\). Then, given any real-valued \( L \)-integrable function \( z(t), t \in I_o \), we may consider the function \( F \) defined by

\[
F(t) = \int_{[o,t]} z(\tau) d\tau = \int_o^t z(\tau) d\tau, \quad t \in I_o, \quad (8.2.1)
\]

where the integration in \([o,t]\) will be often denoted simply by \( \int_o^t \), and where \( d\tau = d\tau^1 \ldots d\tau^\nu \) as usual. Note that \( F(t) \) is then a continuous function on \( I_o \) with \( F(t) = 0 \) whenever \( t = (t^1,\ldots,t^\nu) \in I_o, t^1 \ldots t^\nu = o \).

We shall consider the countable system \( \{ t_\rho \} \) of points \( t_\rho = (t^1_\rho,\ldots,t^\nu_\rho) \) with \( t^j_\rho = \rho_j b, j = 1,\ldots,\nu \), and \( \rho_j \) rational, and the countable system \( \{ I \} \) of all intervals \( I = [\alpha,\beta] \subset I_o, \alpha = (\alpha^1,\ldots,\alpha^\nu), \beta = (\beta^1,\ldots,\beta^\nu), \alpha^1 < \beta^1, \alpha,\beta \in \{ t_\rho \} \).

For any given \( F(t) \), \( t \in I_o \), we may then consider the usual differences \( \Delta_I F \) of order \( \nu \) relative to the \( 2^\nu \) vertices of \( I \); in other words

\[
\Delta_I F = F(\beta) - F(\alpha) \text{ if } \nu = 1,
\]

\[
\Delta_I F = F(\beta^1,\beta^2) - F(\alpha^1,\beta^2) - F(\beta^1,\alpha^2) + F(\alpha^1,\alpha^2) \text{ if } \nu = 2, \text{ etc.}
\]

If \( F \) is obtained by integration in \([o,t]\) of a function \( z \in L_1(I_o) \) as in (8.2.1), then

\[
\Delta_I F = \int_{[\alpha,\beta]} z(\tau) d\tau = \int_{[\alpha,\beta]} z(\tau) d\tau, \quad I = [\alpha,\beta] \subset I_o, \quad (8.2.2)
\]

\[
\Delta_I F = \int_{[\alpha,\beta]} z(\tau) d\tau = \int_{[\alpha,\beta]} z(\tau) d\tau, \quad I = [\alpha,\beta] \subset I_o,
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\]
with the usual conventions.

We recall here that for a function $F$ as in (8.2.2) the following holds:
For almost all $t_0 \in I_0$ and hypercube of $I, t_0 \in I \subset I_0$, of side length $h > 0$ we have $|I|^{-1} \Delta_i F + z(t_0)$ as $h \to 0^+$. 

An arbitrary function $S(t), t \in I_0$, is said to be singular in $I_0$ if for almost all $t_0 \in I_0$ we have $|I|^{-1} \Delta_i S \to o$ as $h \to 0^+$, where as above $I$ denotes any hypercube of side length $h$ with $t_0 \in I \subset I_0$.

Let $G, A(t), A$ be as in (8.1.1). For every $(t,y) \in A$ we shall consider subsets $\tilde{Q}(t,y)$ of the $\tilde{z}$-space $E_{r+\sigma}, \tilde{z} = (\tilde{z}^1, \ldots, \tilde{z}^{r+\sigma}), r \geq o, \sigma \geq o$, with the following properties:

1. There is an $L$-integrable function $\psi(t) \geq o, t \in G$, such that if $\tilde{z} = (\tilde{z}^1, \ldots, \tilde{z}^r, \tilde{z}^{r+1}, \ldots, \tilde{z}^{r+\sigma}) \in \tilde{Q}(t,y), (t,y) \in A$, then $z^i \geq \psi(t)$, $i = r + 1, \ldots, r + \sigma$.

2. If $\tilde{z} = (\tilde{z}^1, \ldots, \tilde{z}^r, -\tilde{z}^{r+1}, \ldots, -\tilde{z}^{r+\sigma}) \in \tilde{Q}(t,y)$ for some $(t,y) \in A$, then also every point $\tilde{z} = (\tilde{z}^1, \ldots, \tilde{z}^r, \tilde{z}^{r+1}, \ldots, \tilde{z}^{r+\sigma})$ with $\tilde{z}^i \geq \tilde{z}^i$, $i = r + 1, \ldots, r + \sigma$, belongs to $\tilde{Q}(t,y)$.

(8.2.1) A second closure theorem. Let $I_0, G, A(t), A$ as above, $A$ closed, and for every $(t,y) \in A$ let $\tilde{Q}(t,y)$ be a given subset of the $\tilde{z}$-space $E_{r+\sigma}$ satisfying properties 1 and 2 above, and in addition satisfying property (Q) in A. Let $y(t) = (y^1, \ldots, y^s), y_k(t) = (y_k^1, \ldots, y_k^s), t \in G$, be elements of $(L_1(G))^s$, let $\tilde{z}(t) = (\tilde{z}^1, \ldots, \tilde{z}^{r+\sigma}), \tilde{z}_k(t) = (\tilde{z}_k^1, \ldots, \tilde{z}_k^{r+\sigma}), t \in G$, be elements of $(L_1(G))^{r+\sigma}$, $k = 1, 2, \ldots$, and take $F(t) = \int_0^t z^i(\tau)d\tau, F_k(t) = \int_0^t z_k^i(\tau)d\tau$ for $t \in I_0$, with
\[ z^i(t) = z^i_k(t) = 0 \text{ for all } t \in I_0 - G, \quad k = 1, 2, \ldots. \] Let us assume that \( y^i_k(t) \in A(t), \quad z^i_k(t) \in \hat{Q}(t, y^i_k(t)) \) a.e. in \( G, \quad k = 1, 2, \ldots, \) that \( y^i_k \to y^i \) strongly in \( L_1(G), \) \( i = 1, \ldots, s, \) that \( z^i_k \to z^i \) weakly in \( L_1(G), \) \( i = 1, \ldots, s, \) and that \( f^i_k + f^i(t) + s^i(t) \) pointwise for all \( t \in \{ t \}_\rho, \) \( i = r + 1, \ldots, r + \sigma, \) as \( k \to \infty, \) where \( S^i(t), \) \( t \in I_0, \) are nonnegative singular functions on \( I_0. \) Then, \( y(t) \in A(t), \quad \tilde{z}(t) \in \hat{Q}(t, y(t)) \) a.e. in \( G. \)

**Proof.** We denote by \( N \) the sum \( N = r + \sigma, \) and by \( \tilde{z}(t) = (z^1_k, \ldots, z^{r+\sigma}_k), \)

\[ \tilde{z}_k(t) = (z^1_k, \ldots, z^{r+\sigma}_k), \quad t \in I_0, \] the same functions above with \( \tilde{z}(t) = \tilde{z}_k(t) = 0 \)

for \( t \in I_0 - G, \) \( k = 1, 2, \ldots, \) and we take also \( \psi(t) = 0 \) for \( t \in I_0 - G. \) Note that

\[ z^i_k(t) \geq - \psi(t), \quad t \in I_0, \quad i = r + 1, \ldots, r + \sigma, \] and that \( \tilde{z}_k(t) \in \hat{Q}(t, y^i_k(t)) \) a.e. in \( G, \) \( k = 1, 2, \ldots. \)

For every \( t \in G, \) \( t_o = (t^1_o, \ldots, t^\nu_o), \) let \( \delta_o = \delta_o(t_o) > 0 \) denote the distance of \( t_o \) from \( \partial G, \) and let \( Q = [\bar{t}, \bar{t} + h] \) denote any closed hypercube

\[ g = [t^j \leq t^j_o \leq \bar{t}^j + h, \quad j = 1, \ldots, \nu], \] with \( \bar{t}, \bar{t} + h \in \{ t \}_\rho, \) \( 0 < h < \delta / \nu, \)

\[ t^j \leq t^j_o \leq \bar{t}^j + h, \quad j = 1, \ldots, \nu. \] Then \( t_o \in q \subset G, \) and we denote here, for the sake of simplicity, by \( h \) also the \( \nu \)-vector \( (h, \ldots, h). \) Since \( y^i_k \to y \) strongly in \( Y, \) there is a subsequence which converges also pointwise a.e. in \( G. \) For the sake of simplicity, let us denote this sequence still \( [y^i_k]. \) Then, for almost all \( t_o \in G, \) we have \( y^i_k(t_o) \to y(t_o), \) \( (t_o, y^i_k(t_o)) \in A, \quad k = 1, 2, \ldots, \)

where \( A \) is a closed set. As \( k \to \infty \) we obtain \( (t_o, y(t_o)) \in A, \) or \( g_o(t_o) \in A(t_o), \)

and this relation holds for almost all \( t_o \in G. \)

Also, for almost all \( t_o \in G \) we have, as \( h \to 0^+, \)

\[ |q|^{-1} \int_q z^i(t)dt + z^i(t_o) \quad \text{, \quad } i = 1, \ldots, r + \sigma, \quad (8.2.3) \]
\[ |q|^{-1} \Delta_S q^i \to 0, \quad i = r + 1, \ldots, r + \sigma. \quad (8.2.4) \]

Because of the pointwise convergence \( y_k(t) \to y(t) \) a.e. in \( G \), we have

\( y_k(t) + y(t) \) is a subset \( G_0 \) of \( G \) with \( |G_0| = |G| \), and we know that there are

closed sets \( C_\lambda, \lambda = 1, 2, \ldots \), with \( C_\lambda \subset G_0 \), \( C_\lambda \subset C_{\lambda + 1} \), \(|C_\lambda| > |G_0| - \lambda^{-1}\), such

that \( y(t) \) is continuous on \( C_\lambda \) and \( y_k(t) \to y(t) \) uniformly on \( C_\lambda \) as \( k \to \infty \) for

each \( \lambda = 1, 2, \ldots \). Since \( G \) is bounded, and \( C_\lambda \subset G_0 \subset G \), each set \( C_\lambda \) is compact,

and hence \( y(t) \) is uniformly continuous on each \( C_\lambda \).

Let \( \lambda \) be any fixed integer with \( \lambda > |G|^{-1} \), hence \( |C_\lambda| > 0 \). Let \( \varepsilon > 0 \) be

an arbitrary number. There is some \( \delta_0' = \delta_0'(\varepsilon, \lambda) > 0 \) such that \( |t-t'| \leq \delta_0 \),

t, t' \in C_\lambda \), implies \( |y(t)-y(t')| \leq \varepsilon/2 \). Also, there is some \( k(\varepsilon, \lambda) > 0 \) such

that \( k \geq k(\varepsilon, \lambda) \), t \in C_\lambda \) implies \( |y_k(t)-y(t)| \leq \varepsilon/4 \). Then, for \( t, t' \in C_\lambda \),

\( |t-t'| \leq \delta_0'(\varepsilon, \lambda), k \geq k(\varepsilon, \lambda) \), we have also

\[
|y_k(t)-y_k(t')| \leq |y_k(t)-y(t)| + |y(t)-y(t')| + |y(t')-y_k(t')| \leq \varepsilon/4 + \varepsilon/2 + \varepsilon/4 = \varepsilon.
\]

Let \( X_\lambda(t), X_\lambda^*(t), t \in G \), be the characteristic functions of the sets \( C_\lambda \)

and \( G-C_\lambda \), so that \( X_\lambda + X_\lambda^* = 1 \) everywhere in \( G \). All functions \( X_\lambda(t) \) and

\( X_\lambda(t) z^j(t), j = 1, \ldots, \), are of class \( L_1(C_\lambda) \), and for every \( t_0 \in C_\lambda \) we have

\( X_\lambda(t_0) = 1, X_\lambda^*(t_0) z^j(t_0) = 0 \).

Then, for almost all \( t_0 \in C_\lambda \), we have also, as \( h \to 0^+ \),

\[
|q|^{-1} \int q X_\lambda(t) dt + X_\lambda(t_0) = 1, \quad (8.2.5)
\]

\[
|q|^{-1} \int q X_\lambda^*(t) z^i(t) dt + X_\lambda^*(t_0) z^i(t_0) = 0, \quad i = 1, \ldots, r + \sigma, \quad (8.2.6)
\]

\[
|q|^{-1} \int q X_\lambda^*(t) \psi(t) dt = o. \quad (8.2.7)
\]
Let \( C'_\lambda \) be the subset of \( C_\lambda \) where this occurs. Then \( C'_\lambda \) is measurable, \( C'_\lambda \subset C_\lambda \subset G_o \subset G \), and if \( G' \) denotes the set \( \bigcup \lambda C'_\lambda \), also

\[
|C'_\lambda| = |C_\lambda| > |G_o| > \lambda^{-1} > o, \quad |G'| = |\bigcup \lambda C'_\lambda| = |G_o| = |G|. \quad (8.1.12)
\]

Let \( H \) and \( H^* \) be the sets

\[
H = q \cap C_\lambda, \quad H^* = q - H = q - q \cap C_\lambda = q - C_\lambda.
\]

Let \( \eta > o \) be any positive number independent of \( \varepsilon \). Let \( t_0 \) be any point of \( C'_\lambda \), let \( y'_o = y(t_0) \), \( \tilde{z}_o = (z_0, z_0^1, \ldots, z_0^r) = \tilde{z}(t_0) \), and let

\[
M_{\lambda} = |\tilde{z}(t_0)| + 1. \quad \text{Let us fix } h > o \text{ so small that } h < \varepsilon/\nu, \ h < \delta_o/\nu, \ h < \delta'_o/\nu,
\]

where \( \delta_o = \delta_o(t_0) \), \( \delta'_o = \delta'_o(\varepsilon, \lambda) \), and also so small that

\[
|z^i(t_0) - |q|^{-1} \int_q z^i(t) \, dt| < \min \{ \eta N^{-1}, 1 \}, \ i = 1, \ldots, r + \sigma, \quad (5.2.8)
\]

\[
o \leq 1 - |q|/|H| \leq \min \{ \eta N^{-1} M_{\lambda}^{-1}, 1 \}, \quad (8.2.9)
\]

\[
||q|^{-1} \int_q x^*(t) \ z^i(t) \, dt| < \eta N^{-1}, \ i = 1, \ldots, r, \quad (8.2.10)
\]

\[
||q|^{-1} \int_q s^i \, dt| < \eta N^{-1}, \ i = r + 1, \ldots, r + \sigma, \quad (8.2.11)
\]

\[
|q|^{-1} \int_q \psi(t) \, dt < \eta N^{-1}. \quad (8.2.12)
\]

This is possible because of relations (8.2.3-7).

For \( t \in H \), and \( k \geq k(\varepsilon, \lambda) \) we have now

\[
|t - t_0| \leq \nu h \leq \min \{ \varepsilon, \delta_o, \delta'_o \},
\]

\[
|y_k(t) - y_o| = |y_k(t) - y_k(t_0)| + |y_k(t_0) - y(t_0)| \leq \varepsilon + \varepsilon = 2\varepsilon,
\]

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and hence \((t, y_k(t)) \in N_{\varepsilon}(t_0, y_0)\) for \(t \in H\) and \(k \geq k(\varepsilon, \lambda)\). Therefore, we have also

\[
\tilde{z}_k(t) \in \tilde{Q}(t_0, y(t_0); 3\varepsilon) \text{ for } t \in H \text{ and } k \geq k(\varepsilon, \lambda).
\]  

(8.2.13)

and finally

\[
|H|^{-1} \int_H \tilde{z}_k(t) dt \in \text{cl co } \tilde{Q}(t_0, y_0; 3\varepsilon), \quad k \geq k(\varepsilon, \lambda),
\]

(8.2.14)

since the last set is convex and closed.

Because of the weak convergence of \(z_k(t) = (z_k^1, \ldots, z_k^r)\) to \(z(t) = (z^1, \ldots, z^r)\) in \((L_1(G))^r\), we can determine an integer \(k' = k'(t_0, \varepsilon, \lambda, \eta, \eta, h) \geq k(\varepsilon, \lambda)\) such that, for \(k \geq k'(t_0, \varepsilon, \lambda, \eta, h)\), we have

\[
|\int_H z^i_k(t) dt - \int_H z^i(t) dt| \leq \eta N^{-1}|H|, \quad i = 1, \ldots, r.
\]

(8.2.15)

Now, for \(k \geq k'(t_0, \varepsilon, \lambda, \eta, h)\) and \(i = 1, \ldots, r\), we have

\[
|z^i(t_0) - |H|^{-1} \int_H z^i_k(t) dt| \leq |z^i(t_0)| + |q|^{-1} \int_q |z^i(t) dt| + |q|^{-1}|H|^{-1} \int_q |z^i(t) dt|
\]

\[
+ |H|^{-1} \int_q (z^i(t) - z^i_k(t)) dt | + |q|^{-1}|H|^{-1} \int_q |z^i(t) dt|
\]

\[
= d_1 + d_2 + d_3 + d_4.
\]

(8.2.16)

By (8.2.8) we have \(d_1 \leq N^{-1}\eta\), by (8.2.15) we have \(d_2 \leq N^{-1}\eta\), and by (8.2.9), (8.2.10) we have \(d_4 \leq 2N^{-1}\eta\). Also, by (8.2.8), (8.2.9) and the definition of \(M_1\), we have

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\[ d_2 = |1 - |q| H| \cdot ||q|^{-1} \int_q z^i(t) dt | \leq \eta N^{-1} M^{-1} \cdot M_1 = N^{-1} \eta. \]

Thus, (8.2.16) yields, for \( k \geq k'(t_0, \varepsilon, \lambda, \eta, h) \),

\[ |z^i(t_0) - |H|^{-1} \int_H z^i_k(t) dt | \leq 5 N^{-1} \eta, \quad i = 1, \ldots, r. \quad (8.2.17) \]

We shall now obtain analogous estimates for \( i = r + 1, \ldots, r + \sigma \). We know that \( z^i_k(t) \geq - \psi(t) \), \( t \in G \), \( i = r + 1, \ldots, r + \sigma \), and hence, by force of (8.2.12), we have for all \( k \)

\[ |q|^{-1} \int_{H^*} z^i_k(t) dt \geq -|q|^{-1} \int_q X^*(t) \psi(t) dt \geq - N^{-1} \eta, \quad i = r + 1, \ldots, r + \sigma. \quad (8.2.18) \]

Let \( F^i_k(t) = F^i(t) + S^i(t) \), \( t \in I_0 \), \( i = r + 1, \ldots, r + \sigma \), so that \( F^i_k(t) \to F^i_\infty(t) \) as \( k \to \infty \) pointwise at all points \( t \in \{ t_0 \} \). By (8.2.2) and the definitions of \( F_k \) and \( F \) we have now

\[ |q|^{-1} \int_q z^i_k(t) dt = |q|^{-1} \Delta F^i_k, \quad |q|^{-1} \int_q z^i(t) dt = |q|^{-1} \Delta F^i, \quad i = r + 1, \ldots, r + \sigma. \quad (8.2.19) \]

On the other hand, by \( F^i_\infty = F^i + S^i \), we have

\[ |q|^{-1} \Delta F^i_\infty = |q|^{-1} \Delta F^i + |q|^{-1} \Delta S^i, \quad i = r + 1, \ldots, r + \sigma. \quad (8.2.20) \]

The \( 2^\gamma \) vertices of \( q \) are points of \( \{ t_0 \} \), hence \( \Delta F^i_k \to \Delta F^i_\infty \) as \( k \to \infty \), and we can determine the number \( k'(t_0, \varepsilon, \lambda, \eta, h) \) above so that, for \( k \geq k'(t_0, \varepsilon, \lambda, \eta, h) \) we have also

\[ | |q|^{-1} \Delta F^i_k - |q|^{-1} \Delta F^i_\infty | \leq N^{-1} \eta. \quad (8.2.21) \]

8
Finally, (8.2.19), (8.2.20), (8.2.21), and (8.2.11) yield

\[
\| q \|^{-1} \int | q \|^{-1} \int z^i_k(t) dt - | q \|^{-1} \int z^i(t) dt \\
= \| q \|^{-1} \Delta q \|^{-1} \int F^i_k - | q \|^{-1} \Delta q \|^{-1} \int F^i \\
\leq \| q \|^{-1} \Delta q \|^{-1} \int F^i_k - | q \|^{-1} \Delta q \|^{-1} \int F^i_o + | q \|^{-1} \Delta q \|^{-1} \int S^i \\
\leq N^{-1} \eta + N^{-1} \eta = 2N^{-1} \eta, \quad i = r + 1, \ldots, r + \sigma .
\]

(8.2.22)

We have now, for \( i = r + 1, \ldots, r + \sigma \) and \( k \geq k'(t_o, \varepsilon, \lambda, \eta, h) \),

\[
z^i(t_o) - \| H \|^{-1} \int H z^i(t) dt \\
\geq -z^i(t_o) - \| q \|^{-1} \int z^i(t) dt \\
\geq -\frac{\| \|}{\| H \|} \int z^i(t) dt \\
\geq -\frac{\| \|}{\| H \|} \int (z^i(t) - z^i_k(t)) dt \\
\geq -\frac{\| \|}{\| H \|} \int z^i_k(t) dt \\
= d_5 + d_6 + d_7 + d_8 .
\]

(8.2.23)

By (8.2.8) we have \( d_5 \geq -N^{-1} \eta \), by (8.2.9) and (8.2.22) we have \( d_7 \geq -4N^{-1} \eta \),

and by (8.2.10), (8.2.18) we have \( d_8 > -2N^{-1} \eta \). Finally, by (8.2.8), (8.2.9) and the definition of \( M_1 \), also

\[
d_6 = -(1 - \| q \|/\| H \|) \| q \|^{-1} \int z^i(t) dt \geq -N^{-1} \eta M_1^{-1} \cdot M_1 = -N^{-1} \eta .
\]

Thus, (8.2.23) yields, for \( k \geq k'(t_o, \varepsilon, \lambda, \eta, h) \),
\[ z^i(t_o) - |H|^{-1} \int_H z^i_k(t)dt \geq - 8N^{-1}\eta, \quad i = r+1, \ldots, r+\sigma. \quad (8.2.24) \]

Note that relations (8.2.17) and (8.2.24) can be written in vector form

\[ \tilde{z}(t_o) = \tilde{z}_k + \xi_k^+ + \xi_k^-, \quad z_k = |q|^{-1} \int_q z_k(t)dt, \quad k \geq k'(t_o, \varepsilon, \lambda, \eta, h), \quad (8.2.25) \]

where \( |\xi_k| \leq 8\eta, \xi_k^+ = (0, \ldots, 0, \xi_k^{r+1}, \ldots, \xi_k^{r+\sigma}) \), \( \xi_k^{r+1} > 0, \ldots, \xi_k^{r+\sigma} > 0. \)

By force of (8.2.14) we have then

\[ \tilde{z}_k \in \text{cl co} \tilde{Q}(t_o, y_o, 3\varepsilon), \]

hence, by property 2 of the sets \( \tilde{Q} \), also

\[ \tilde{z}_k + \xi_k^+ \in \text{cl co} \tilde{Q}(t_o, y_o, 3\varepsilon) \]

and finally

\[ \tilde{z}(t_o) = \tilde{z}_k + \xi_k^+ \in (\text{cl co} \tilde{Q}(t_o, y_o, 3\varepsilon))_{8\eta} \]

Since \( \eta \) is arbitrary and the set in parenthesis is closed, we have

\[ \tilde{z}(t_o) \in \text{cl co} \tilde{Q}(t_o, y_o, 3\varepsilon) \]

and this relation holds for every \( \varepsilon > 0. \) Then, by property (Q) we have also

\[ \tilde{z}(t_o) \in \cap_{\varepsilon > 0} \text{cl co} \tilde{Q}(t_o, y_o, 3\varepsilon) = \tilde{Q}(t_o, y_o), \]

for every point \( t_o \in C_\lambda. \) Thus, we have \( \tilde{z}(t) \in \tilde{Q}(t, y, t) \) for all \( t \in G' = UC_\lambda, \) that is, almost everywhere in \( G. \)
8.3. A Third Closure Theorem for Orientor Fields

We consider here the case where some or all of the $r$-components $z^i_k$, $i=1,\ldots,r$, of the vectors $z_k$ or $\tilde{z}_k$, of theorems (8.1.1), (8.2.1), converge strongly to $z^i$ instead of weakly. As we shall see below if all components $z^i_k$, $i=1,\ldots,r$, converge strongly, then instead of property (Q) we need only require property (U).

If, say, $\rho$ components $z^1_k,\ldots,z^\rho_k$ converge weakly and the remaining $r-\rho$ components $z^{\rho+1}_k,\ldots,z^r_k$ converge strongly, then the intermediate property (Q$^{\rho}$) can be required. We discussed this property (Q$^{\rho}$) in Appendix A, and we proved there that property (Q$^{0}$) is equivalent to property (U), that property Q$^{\rho}$ is equivalent to property (Q), and that property (Q$^{\rho}$) implies property (Q$^{\rho-1}$), $1 \leq \rho \leq r$.

Let us mention here that, if $\tilde{z}_o = (z^1_o,\ldots,z^{r+1}_o)$ is any point of the $\tilde{z}$-space $E_{r+\sigma}$, $r \geq 0$, $\sigma \geq 0$, if $\rho$ is any integer, $0 \leq \rho \leq r$, and $\beta$ any positive number, we denote by $N_{\beta}(\tilde{z}_o,\rho,r)$ and $N(\tilde{z}_o,\rho,r)$ the subsets of $E_{r+\sigma}$ defined by

$$N_{\beta}(\tilde{z}_o,\rho,r) = \{ z \in E_{r+\sigma} | |z^i - z^i_o| \leq \beta, i = \rho + 1,\ldots,r \},$$

$$N(\tilde{z}_o;\rho,r) = N_0(\tilde{z}_o,\rho,r) = \{ z \in E_{r+\sigma} | z^i = z^i_o, i = \rho + 1,\ldots,r \}.$$  

Equivalently, we could take $z'_o = (z^{\rho+1}_o,\ldots,z^r_o) \in E_{r-\rho}$ and then

$$N_{\beta}(\tilde{z}_o,\rho,r) = N_{\beta}(z'_o) \times E_{r+\sigma-\rho}, \quad N(\tilde{z}_o,\rho,r) = \{ z'_o \} \times E_{r+\sigma-\rho}.$$  

Given $G \subset E$, $A(t) \subset E^s$, $A \subset E^{v+s}$, $\tilde{Q}(t,y) \subset E_{r+\sigma}$ as in (8.2) we say that the sets $\tilde{Q}(t,y)$ satisfy property (Q$^{\rho}$) at the point $(t_o,y_o) \in A$ provided, for every point $\tilde{z}_o = (z^1_o,\ldots,z^{r+1}_o) \in E_{r+\sigma}$ we have
\[ \tilde{Q}(t_o, y_o) \cap N(\tilde{z}_o, \rho, r) = \cap_{\delta > 0} \cap_{\beta > 0} \tilde{Q}(t_o', y_o'; \delta) \cap N(\tilde{z}_o', \rho, r). \]

(8.3.i) A third closure theorem for orientor fields. The same as (8.2.i) where the sets \( \tilde{Q}(t, y) \subseteq E_{r+\sigma}, r \geq o, \sigma \geq o, \) satisfy property \( (Q_\rho) \) for some \( \rho, \)

\( o \leq \rho \leq r \leq r + \sigma, \) and where the \( n-\rho \) components \( \tilde{z}_k^i(t), i = \rho + 1, \ldots, r, \) of \( \tilde{z}_k(t), t \in G, k = 1, 2, \ldots, \) converge strongly to the components \( z_k^i(t) \) of \( \tilde{z}(t), t \in G, \) as \( k \to \infty. \) Then the same conclusion holds as in (8.2.i).

**Proof.** The initial steps are the same as for the proof of (8.2.i). Now \( y_k + y \) strongly in \( (L_1(G))^s, z_k^i + z^i \) strongly in \( L_1(G), i = \rho + 1, \ldots, r; \) hence, there is a subsequence, say still \([k]\) for the sake of simplicity, such that

\[ y_k + y, z_k^i + z^i, i = \rho + 1, \ldots, r, \] pointwise a.e. in \( G, \) say in a subset \( G \) of \( G \) with \( |G| = |G|. \) There are now closed subsets \( C_\lambda,\lambda = 1, 2, \ldots, \) with

\[ C_\lambda \subseteq G, C_\lambda \subseteq C_{\lambda+1}, |C_\lambda| > |G| - \lambda^{-1}, \] such that all \( y(t) = (y^1, \ldots, y^s), z^i(t), i = \rho + 1, \ldots, r, \) are continuous on \( C_\lambda, \) and \( y_k(t) \to y(t), z_k^i(t) \to z^i(t), i = \rho + 1, \ldots, r, \) uniformly on \( C_\lambda \) as \( k \to \infty. \) As in the proof of (8.2.i) we take \( \lambda > |G|^{-1}, \) so that \( |C_\lambda| > o, \) and two numbers \( \varepsilon > o, \beta > o \) arbitrary. Then there is some \( \delta' = \delta'(\varepsilon, \lambda, \beta) > 0 \) so that \( |t-t'| \leq \delta', t, t' \in C_\lambda \) implies

\[ (|y(t) - y(t')| \leq \varepsilon/2, |z^i(t) - z^i(t')| \leq \beta/2, i = \rho + 1, \ldots, r). \] Also, we choose \( k(\varepsilon, \lambda, \beta) > o \) such that \( k \geq k(\varepsilon, \lambda, \beta), t \in C_\lambda \) implies \( |y_k(t) - y(t)| \leq \varepsilon/4, |z_k^i(t) - z^i(t)| \leq \beta/4. \) Then, for \( t, t' \in C_\lambda, |t-t'| \leq \delta(\varepsilon, \lambda, \beta), k \geq k(\varepsilon, \lambda, \beta) \) we have, as in the proof of (5.2.i),

\[ |y_k(t) - y_k(t')| \leq \varepsilon, \ |z_k^i(t) - z_k^i(t')| \leq \beta, i = \rho + 1, \ldots, r. \]
We proceed now as for (8.2.1): we define the sets $C'_\lambda$, we take any

$t_0 \in C'_\lambda$, and we fix $h > 0$ so that $h < \varepsilon / \nu$, $h < \delta_0 / \nu$, $h < \delta'_0 / \nu$, where $\delta_0 = \delta_0(t_0)$, $\delta'_0 = \delta'_0(\varepsilon, \lambda, \beta)$, and so that the corresponding hypercubes $q$ of side length $h$ and the sets $H = q \cap C'_\lambda$ satisfy relations (8.2.8-12). Thus, relations (8.2.13) holds now for all $t \in H$ and $k \geq k(\varepsilon, \lambda, \beta)$. On the other hand, for $t \in H = q \cap C'_\lambda$, $k \geq k(\varepsilon, \lambda, \beta)$, we certainly have

$$|z^i_k(t) - z^i(t)| \leq \beta, \quad |z^i_k(t) - z^i(t_0)| \leq \beta, \quad i = 0, 1, \ldots, r,$$

and hence

$$|z^i_k(t) - z^i(t_0)| \leq 2\beta, \quad i = 0, 1, \ldots, r.$$

Thus, relation (8.2.13) can now be rewritten in the stronger form

$$z^i_k(t) \in \tilde{Q}(t_0, \gamma_0; 3\varepsilon) \cap N_{2\beta}(\tilde{z}(t_0), r, \rho),$$

and consequently (8.2.14) becomes

$$|H|^{-1} \int_H \tilde{z}_k(t) dt \in \text{cl } \text{co } \left[ \tilde{Q}(t_0, \gamma_0; 3\varepsilon) \cap N_{2\beta}(\tilde{z}(t_0), r, \rho) \right] \quad (8.2.27)$$

for all $k \geq k(\varepsilon, \lambda, \beta)$.

Note that strong convergence in $L_1(G)$ implies weak convergence. Thus the arguments of (8.2.1) concerning the components $z^i_k$, $i = 1, \ldots, r + \sigma$, hold without change. In other words, we can determine some number $k'(t_0, \varepsilon, \lambda, \eta, h) \geq k(\varepsilon, \lambda, \beta)$ such that, for $k \geq k'(t_0, \varepsilon, \lambda, \beta, \eta, h)$, we have

$$|z^i(t_0) - |H|^{-1} \int_H z^i_k(t) dt | \leq 5N^{-1} \eta, \quad i = 1, \ldots, r,$$
\[
\tilde{z}(t_0) - |H|^{-1} \int_{H} z^i_k(t) dt \geq \beta N^{-1} \eta, \; i = r + 1, \ldots, r + \sigma.
\]

Thus, the same relation (8.2.25) holds

\[
\tilde{z}(t_0) = \tilde{z}_k + \zeta_k + \zeta_k^+, \quad \tilde{z}_k = |H|^{-1} \int_{H} \tilde{z}(t) dt,
\]

as in the proof of (8.2.1), where \(|\zeta_k| > \beta \eta, \zeta_k^+ \geq \eta, i = r + 1, \ldots, r + \sigma, k \geq k'(t_0, \varepsilon, \lambda, \beta, \eta, h)\), and where the point \(z_k\) is now in the right hand set (8.2.27). Thus, instead of (8.2.26) we have

\[
\tilde{z}(t_0) \in (\text{cl co } [\tilde{Q}(t_0, y_0; 3\varepsilon) \cap N_{2\beta}(\tilde{z}(t_0), r, \rho)]) \beta \eta
\]

where \(\varepsilon, \beta, \eta\) are arbitrary positive numbers. Finally

\[
\tilde{z}(t_0) \in \text{cl co } [\tilde{Q}(t_0, y_0; 3\varepsilon) \cap N_{2\beta}(\tilde{z}(t_0), r, \rho)]
\]

where \(\varepsilon\) and \(\beta\) are arbitrary. By property (Q) we have

\[
\tilde{z}(t_0) \in \cap_{\varepsilon > 0} \cap_{\beta > 0} \text{cl co } [\tilde{Q}(t_0, y_0; 3\varepsilon) \cap N_{2\beta}(\tilde{z}(t_0), r, \rho)]
\]

\[
\tilde{z}(t_0) \in \tilde{Q}(t_0, y_0) \cap N(\tilde{z}(t_0), r, \rho).
\]

Thus, \(\tilde{z}(t_0) \in Q(t_0, y(t_0))\) for every \(t_0 \in C_\lambda\) where \(\lambda\) is arbitrary, and finally \(\tilde{z}(t) \in \tilde{Q}(t, y(t))\) a.e. in \(G\).

### 8.4 Closure Theorems for Control Problems

We shall use here again the notation of (7.1). Thus, \(G\) is a bounded open subset of the \(t\)-space \(E_\nu\), \(t = (t^1, \ldots, t^\nu)\), and for every \(t \in \text{cl co } G\) a subset \(A(t)\) of the \(y\)-space \(E_\xi\) is assigned. Let \(A\) denote the set of all \((t, y)\) with \(t \in \text{cl } G, y \in A(t)\). For every \((t, y) \in A\) a subset \(U(t, y)\) of the \(u\)-space
\( E_m \) is assigned, and \( M \) denotes the set of all \((t,y,u)\) with \((t,y) \in A, u \in U(t,y)\). Let \( f(t,y,u) = (f_1, \ldots, f_r) \) be a given vector function on \( M \). For every \((t,y) \in A\) let \( Q(t,y) \) denote the set

\[
Q(t,y) = \{ z \in E_r \mid z = f(t,y,u), u \in U(t,y) \} \subseteq E_r.
\]

We first state and prove the implicit function theorem for orientor fields which we need below and which is similar to statement (1.6.i).

(8.4.i) An implicit function theorem for orientor fields. Let \( G, A(t), A, Q(t,y) \) be as above, \( G \) open and bounded, \( A \) and \( M \) closed, \( f \) continuous on \( M \). If \( z(t) = (z^1, \ldots, z^r), y(t) = (y^1, \ldots, y^s), t \in G \), are any two measurable vector functions such that \( y(t) \in A(t), z(t) \in Q(t,y(t)) \) a.e. in \( G \), then there is some measurable vector function \( u(t) = (u^1, \ldots, u^m), t \in G \), such that

\[
y(t) \in A(t), u(t) \in U(t,y(t)), z(t) = f(t,y(t), u(t)) \text{ a.e. in } G.
\]

Proof. The proof is only a modification of the one for (1.6.1) in Section 1. First, \( M \) is a (closed) subset of the Euclidean space \( E_{v+s+m} \); hence, a metric space which is the union of countably many compact subsets. Let \( F: M \to N \) be the continuous mapping defined by \((t,y,u) \to (t,y,f(t,y,u))\) so that \( N = F(M) \) is the subset of all \((t,y,z) \in E_{v+s+r}\) with \((t,y) \in A, z = f(t,y,u), u \in U(t,y),\) or \((t,y) \in A, z \in Q(t,y)\). Finally, let \( \sigma: G \to N \) be the measurable map defined by \( t \to (t,y(t),z(t)) \). If \( \varphi: N \to M \) denotes the partial inverse of \( F \) defined in (1.7.iii) such that \( F\varphi \) is the identity map on \( N \), then the map \( \psi: G \to M \) defined by \( \psi = \varphi \sigma \) is a mapping \( t \to (t,y(t),u(t)) \) such that \( z(t) = f(t,y(t),u(t)) \) a.e.
in $G$, $u(t), t \in G$, is measurable, and $u(t) \in U(t,y(t))$ a.e. in $G$.

\[
\begin{array}{c}
M \quad \Psi \\
F \\
G \quad \psi \\
\end{array}
\]

Let $Y = (L_1(G))^s$, $Z = (L_1(G))^r$, and let $T$ denote the set of all measurable vector functions $u(t) = (u_1, \ldots, u^m)$, $t \in G$.

(8.4.ii) A closure theorem in $G$. Let $G$ be open and bounded, $A$ and $M$ closed, $f$ continuous on $M$, and let us assume that the sets $Q(t,y)$ satisfy property (Q) in $A$. If $y_k^r, z_k^r, u_k^s, k = 1,2,\ldots$, is a sequence of elements $y_k^r \in Y$, $z_k^r \in Z$, $u_k^s \in T$, with $y_k^r(t) \in A(t)$, $u_k^s(t) \in U(t,y_k^r(t))$, $z_k^r(t) = f(t,y_k^r(t), u_k^s(t))$ a.e. in $G$, $k = 1,2,\ldots$, and $y_k^r \rightarrow y$ strongly in $Y$ and $z_k^r \rightarrow z$ weakly in $Z$, then there is at least one element $u \in T$ such that $y(t) \in A(t)$, $u(t) \in U(t,y(t))$, $z(t) = f(t,y(t), u(t))$ a.e. in $G$.

Proof. First the elements $y_k^r, z_k^r, k = 1,2,\ldots$, satisfy the orientor field relations $y_k^r(t) \in A(t)$, $z_k^r(t) \in Q(t,y_k^r(t))$ a.e. in $G$, $k = 1,2,\ldots$. Hence, by force of closure theorem (8.1.1) we have also $y(t) \in A(t)$, $z(t) \in Q(t,y(t))$ a.e. in $G$. Finally, by force of the implicit function theorem (8.4.ii) there is a measurable vector function $u: G \rightarrow E_{m'}$, or $u \in T$, such that $u(t) \in U(t,y(t))$ and $z(t) = f(t,y(t), u(t))$ a.e. in $G$.

Closure theorem (8.4.ii) has an analogous version for boundary controls.

Let $\Gamma \subset \partial G$ be a set which can be decomposed into finitely many nonoverlapping parts $\Gamma_1, \ldots, \Gamma_n$, each $\Gamma_j$ being the image under a transformation $\zeta: I \rightarrow \Gamma_1$ of class $K$ of the unit interval $I$ of dimension $n-1$, so that a natural area measure
function σ is defined on Γ.

As usual we assume that for any t ∈ Γ a subset B(t) of the j-space $E'_v$ is assigned, and we denote by B the subset of $E'_v$ of all $(t, j)$ with $t ∈ Γ$, $j ∈ B(t)$. For every $(t, j) ∈ B$ let $V(t, j)$ be a given subset of the v-space $E'_m$, and let M be the subset of $E'_v + M'$ of all $(t, j, v)$ with $(t, j) ∈ B$, $v ∈ V(t, j)$.

Let $g(t, j, v) = (g_1, ..., g_r)$ be a given function on M, and for every $(t, j) ∈ B$ let $R(t, j)$ denote the set of all $\tilde{z} = (\tilde{z}_1, ..., \tilde{z}_r)$ with $\tilde{z} = g(t, j, v)$, $v ∈ V(t, j)$.

Let $\tilde{Y} = (L_1(Γ))^S$, $\tilde{Z} = (L_1(Γ))^R$, and let $\tilde{T}$ be the set of all σ-measurable vector functions $v(t) = (v^1, ..., v^m)$, $t ∈ Γ$.

(8.4.iii) A closure theorem on Γ. Let $Γ$, $B$, $\tilde{M}$ be as above, $Γ$, $B$, $\tilde{M}$ closed, $g$ continuous on $\tilde{M}$, and let us assume that the sets $R(t, y)$ satisfy property (Q) in B. If $\hat{v}_k, \hat{z}_k, v_k, k = 1, 2, ..., \hat{v}_k, v_k ∈ \tilde{Y}$, $\hat{z}_k, v_k ∈ \tilde{T}$, with $\hat{v}_k(t) ∈ B(t), v_k(t) ∈ V(t, j_k(t)), \hat{z}_k(t) = g(t, j_k(t), v_k(t))$ σ-a.e. in $Γ$, $k = 1, 2, ..., $ and $\hat{v}_k + \hat{v}$ strongly in $\tilde{Y}$ and $\hat{z}_k + \hat{z}$ weakly in $\tilde{Z}$, then there is at least one element $v ∈ \tilde{T}$ such that $\hat{v}(t) ∈ B(t), v(t) ∈ V(t, j(t)), \hat{z}(t) = g(t, j(t), v(t))$ σ-a.e. in $Γ$.

Proof. Here $Γ$ is the finite union of sets $Γ_i$. We can limit ourselves to one set $Γ_i$. Note that $Γ_i$ is the image of an interval $I ⊆ E_{v-1}$ under a map $\gamma_i: I → Γ_i$, say $t = t(τ)$, $τ ∈ I$, of class K. Thus, sets of σ-measure zero on $Γ_i$ and sets of measure zero on I correspond. Note that $\hat{v}_k(t) + \hat{v}(t)$ strongly
in $L_1(\Gamma)$ (with respect to the measure $\sigma$) if and only if $\mathcal{F}_k(t(\tau)) \rightarrow \mathcal{F}(t(\tau))$ strongly in $L_1(\Gamma)$; analogously, $\mathcal{Z}_k(t) \rightarrow \mathcal{Z}(t)$ weakly in $L_1(\Gamma)$ if and only if $\mathcal{Z}_k(t(\tau)) \rightarrow \mathcal{Z}(t(\tau))$ weakly in $L_1(\Gamma)$. The problem under consideration, which concerns the orientor field

$$\mathcal{F}(t) \in \mathcal{B}(t), \mathcal{Z}(t) \in \mathcal{R}(t,\mathcal{F}(t)) \quad \sigma\text{-a.e. in } \Gamma,$$

where $\mathcal{R}(t,\mathcal{F}) = g(t,\mathcal{F},V(t,\mathcal{F}))$, is now transformed into a problem concerning the orientor field

$$\mathcal{F}(t(\tau)) \in \mathcal{B}(t(\tau)), \mathcal{Z}(t(\tau)) \in \mathcal{R}(t(\tau)), \mathcal{F}(t(\tau)) \quad \text{a.e. in } I.$$

We have proved in (App. A) that, the sets $\mathcal{R}(t,\mathcal{F})$ have property $(Q)$ in $I$ if and only if the sets $\mathcal{R}(t(t),\mathcal{F})$ have property $(Q)$ in $I$. We can now, therefore, apply theorem (§4.1), or, equivalently, first closure theorem (§1.1) for orientor fields, and then implicit function theorem (§4.1).

### 8.5. Lower Closure Theorems

Let $G, A(t), A, U(t,y), M$ be as in (§4), and let $f(t,y,u) = (f_1, \ldots, f_r)$, $f_0(t,y,u)$ be continuous functions on $M$. For every $(t,y) \in A$ let $\overline{Q}(t,y)$ denote the set

$$\overline{Q}(t,y) = \{(z_0, z) \in E_{r+1}^c | z_0 ^c \geq f_0(t,y,u), z = f(t,y,u), u \in U(t,y) \} \subset E_{r+1}^c.$$

As before let $Y = (L_1(G))^S, Z = (L_1(G))^R$, and let $T$ be the set of all measurable vector-functions $u(t) = (u_1, \ldots, u_m), t \in G$.

We say that the property of lower closure holds in $G$ provided the following occurs: For every sequence $y_{k}', z_{k}', u_{k}', k = 1, 2, \ldots$, of elements $y_k \in Y, z_k \in Z,$
$u_k \in T$, with $y_k(t) \in A(t), u_k(t) \in U(t, y_k(t)), z_k(t) = f(t, y_k(t), u_k(t))$ a.e. in $G$, with $z^O_k(t) = f^O(t, y_k(t), u_k(t)) \in L_1(G)$, and $y_k \rightarrow y$ strongly in $Y$, $z_k \rightarrow z$ weakly in $Z$, and $\lim \int_G z^O_k(t) dt < +\infty$, then there is at least one element $u \in T$ such that $y(t) \in A(t), u(t) \in U(t, y(t)), z(t) = f(t, y(t), u(t))$ a.e. in $G$, $z^O(t) = f^O(t, y(t), u(t)) \in L_1(G)$, and $\int_G z^O(t) dt \leq \lim \int_G z^O_k(t) dt$.

(8.5.1) A lower closure theorem in $G$. Let $G$ be open and bounded, let $A(t), A, U(t, y), M$ as in (8.4), $A$ and $M$ closed, let $f, f^O$ be continuous on $M$, and let us assume that the sets $\tilde{Q}(t, y)$ have property $(\Omega)$ in $A$, and that $f^O(t, y, u) \geq -\psi(t)$ for all $(t, y, u) \in M$ and some $\psi \geq 0, \psi \in L_1(G)$. Then, the lower closure property holds.

Proof. First, let us consider an interval $I_0$ containing $G$. It is not restrictive to assume $I_0 = [0 \leq t \leq b, j = 1, \ldots, n]$, or $I_0 = [0, b]$, where $0$ and $b$ denote the $n$-vectors $(0, \ldots, 0), (b, \ldots, b)$.

We shall now introduce the auxiliary variables $z^O$ and $u^O$, the vector $\tilde{u} = (u^O_1, u^O_2, \ldots, u^O_n) = (u^O, u)$, the vector function $\tilde{f}(t, y, \tilde{u}) = (\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_r)$ with $\tilde{f}^O = u^O$, the control space $\tilde{U}(t, y) = \{\tilde{u} = (u^O, u) | u^O \geq f^O(t, y, u), u \in U(t, y)\}$, and the functional relations

$$z(t) = f(t, y(t), u(t)) \quad z^O(t) = f^O = u^O(t) \quad \text{a.e. in } G,$$

where $y(t) = (y^1, \ldots, y^S)$, $z(t) = (z^1, \ldots, z^r)$, $t \in G$,

and we shall take

$$F^O(t) = \int_0^t z^O(\tau) d\tau,$$
where the integration ranges over the interval \( [0,t] \subseteq I_o \), and \( z^o(\tau) = 0 \) for \( \tau \in I_o - G \).

In the new notations the usual sets \( Q(t,y) \subseteq E_r \), \( \tilde{Q}(t,y) \subseteq E_{r+1} \) have now the interpretation

\[
\tilde{Q}(t,y) = \{ z = (z^0, z) \mid z^0 \geq f_o(t,y,u), \ z = f(t,y,u), \ u \in U(t,y) \}
\]

\[
= f(t,y, \tilde{U}(t,y)) \subseteq E_{r+1} ,
\]

\[
Q(t,y) = \{ z \mid z = f(t,y,u), \ u \in U(t,y) \} = f(t,y, U(t,y)) \subseteq E_r .
\]

It is convenient to define \( f_o \) and \( \tilde{f}_o \) outside \( G \) by taking \( f_o = \tilde{f}_o = 0 \) for \( t \in I_o - G \).

Note that \( f_o(t,y,u) \geq -\psi(t) \) for all \( (t,y,u) \in M \); hence

\[
\int_G z^o(t)dt = \int_G f_o(t,y(t),u(t))dt \geq -\int_G \psi(t)dt = -L,
\]

where \( L \geq 0 \) is a fixed number. Hence, if \( x_k, u_k, k = 1,2,\ldots, \) is any sequence of admissible pairs with \( x_k \rightarrow x \) weakly in \( S \), \( \lim \int_G z^o(t)dt < +\infty \) as \( k \rightarrow \infty \), and we denote by \( i \) this \( \lim \), then \( -L \leq i < +\infty \), that is, \( i \) is finite. We take a subsequence, say still [\( k \)] for the sake of simplicity, so that \( \int_G z^o_k(t)dt \rightarrow i \) as \( k \rightarrow \infty \), and we can even assume that \( \int_G z^o_k(t)dt \leq i + k^{-1} \leq i + 1 \) for all \( k \).

We take \( z^o_k(t) = \tilde{f}_o = u^o_k(t) = f_o(t,y_k(t),u_k(t)) \), \( t \in G \), and

\[
F^o_k(t) = \int_0^t u^o_k(\tau)d\tau = \int_{\tau=0}^{\tau=t} f_o(\tau,y_k(\tau),u_k(\tau))d\tau, \ t \in I_o , \ k = 1,2,\ldots .
\]

Then \( z_k \rightarrow z \) weakly in \( Z \), \( y_k \rightarrow y \) strongly in \( Y \), and
\( y_1(t) \in A(t), (z_{k(t)}^0(t), z_{k(t)}) \in \bar{\mathcal{Q}}(t, y_k(t)) \subset E_{r+1} \text{ a.e. in } G, \ k = 1, 2, \ldots \).

Let \( z_{k(t)}^{0+} = f_0(t, y_k(t), u_k(t)) + \psi(t) \) for \( t \in G \), \( z_{k(t)}^{0+} = 0 \) for \( t \in I_0^t - G \), and note that \( z_{k(t)}^{0+} \geq 0 \) in \( I_0^t \), and \( \int_{I_0^t} z_{k(t)}^{0+} dt \leq L_i + 1 \). Let \( F_k^+(t), t \in I_0^t \), be defined by \( F_k^+(t) = \int_{I_0^t} z_{k(t)}^{0+}(\tau) d\tau, t \in I_0^t \). Let \( z^-(t) = \psi(t) \) for \( t \in G \), \( z^-(t) = 0 \) for \( t \in I_0^t - G \), and let \( F^-(t), t \in I_0^t \), be defined by \( F^-(t) = \int_{I_0^t} z^-(\tau) d\tau \).

Then

\[
F_k^0(t) = \int_{I_0^t} f_0(\tau, y_k(\tau), u_k(\tau)) d\tau = F_k^+(t) - F^-(t), t \in I_0^t.
\]

Here \( F^-(t) \) is a fixed nonempty continuous function of \( t \) in \( I_0^t \), while the functions \( F_k^+(t) \) are nonnegative continuous functions of \( t \) in \( I_0^t \) with a common bound

\[
o \leq F_k^+(t) \leq F_k^+(b) = \int_{I_0^t} z_{k(t)}^{0+} dt \leq L_i + 1 \leq \infty.
\]

For every interval \( I \subset \mathbb{R} \) we denote as usual by \( \Delta_{I_k}^+ \) the differences of order \( v \) of \( F_k^+ \) relative to the \( 2^v \) vertices of \( I \); hence

\[
\Delta_{I_k}^+ = \int_{I_0^t} z_{k(t)}^{0+} dt \geq 0, \quad \Delta_{I_k}^- = \int \psi(t) dt \geq 0.
\]

Of course, each of the interval functions \( \Delta_{I_k}^+ \) is also absolutely continuous, but this does not play an essential part below. It interests here to know that the nonnegative interval functions \( \Delta_{I_k}^+ \) are nonnegative, additive, and of uniform bounded variations, the total variations being all \( \leq L_i + 1 \).

Let us consider the countable lattice point \( \{t_\rho\} \), or \( t = b\rho \in I_0^t \), \( \rho = (\rho_1, \ldots, \rho_v) \), where \( \rho_1, \ldots, \rho_v \) are arbitrary rational numbers, \( 0 \leq \rho_j \leq 1 \), \( j = 1, \ldots, v \). Let \( \{I\} \) be the countable system of intervals \( I \subset \mathbb{R} \) whose vertices
are points $t \in \{t_0\}$. We may order the points $t_0$ into a sequence. Since the functions $z^+_k(t)$, $t \in \mathbb{R}$, are uniformly bounded in $\mathbb{R}$, (and, hence, at each $t = t_0$), we can successively select subsequences which are convergent at $t = t_0$, and then, by the diagonal process, we can select a unique subsequence, say $[k_s]$, of integers $k_s$, such that the limits $z^+_{k_s}(t) \to F(t)$ as $s \to \infty$ exist for every $t \in \{t_0\}$. Then, the interval functions $\Delta_\mathcal{I} z^+_k \to \Delta_\mathcal{I} F^+$ also have limits as $s \to \infty$ for every interval $\mathcal{I}$ of the countable collection $\{I\}$. The limits $\Delta_\mathcal{I} z^+$ is a nonnegative additive interval function for $I \in \{I\}$, and $\Delta_\mathcal{I} z^+$ has bounded variation, namely a total variation $\leq L + \mathcal{I} \lambda$ (for $\mathcal{I}$ describing the collection $\{I\}$).

For every $t_0 \in \mathcal{I}_o$ and interval $I \in \{I\}$ with $t_0 \in I$ we may consider the quotients $\Delta_\mathcal{I} F^+ / \text{meas } I$. We know that for $I$ a hypercube, $I \in \{I\}$, $t_0 \in I$, and almost all $t_0 \in \mathcal{I}_o$, the limit $\Delta_\mathcal{I} F^+ / \text{meas } I \to z(t_0)$ exists as diam $I \to 0$ [App. B.6.1, in conjunction with B.1. Remark 2, case 3]. Moreover, $z(t)$ is finite almost everywhere in $\mathcal{I}_o$, nonnegative, L-integrable in $\mathcal{I}_o$, and zero in $I^c$.

Furthermore, if

$$F(t) = \int_0^t z(\tau)d\tau, \quad t \in \mathcal{I}_o,$$

hence

$$\Delta_\mathcal{I} F = \int_\mathcal{I} z(t)dt, \quad I \subset \mathcal{I}_o.$$ 

then $0 \leq \Delta_\mathcal{I} F \leq \Delta_\mathcal{I} F^+$ for every $I \in \{I\}$, and the difference $\Delta_\mathcal{I} S = \Delta_\mathcal{I} F^+ - \Delta_\mathcal{I} F$ is a nonnegative interval function for $I \in \{I\}$, of bounded variation, and singular, that is, $\Delta_\mathcal{I} S / |I| \to 0$ as diam $I \to 0$ for almost all $t_0 \in \mathcal{I}_o$ (with $t_0 \in I \subset \mathcal{I}_o$, $I \in \{I\}$, I a hypercube). Equivalently, $F^+$ possesses a Lebesgue decomposition.
\( F^+(t) = F(t) + S(t) \), \( F \) an integral function, \( S \) singular, \( S \geq 0, \Delta_1 S \geq 0 \).

By subtracting the fixed function \( F^- \) we have

\[
F^0(t) = F^+(t) - F^-(t) = (F^+(t) - F^-(t)) + S(t),
\]

or

\[
F^0(t) = F(t) + S(t),
\]

where \( F = F^+ - F^- \) is an integral function, \( S \) is singular, \( S \geq 0, \Delta_1 S \geq 0 \)

for \( I \in \{I\} \), and

\[
F(t) = \int_0^t (z(\tau) - \psi(\tau)) d\tau, \quad t \in I_0.
\]

We can now apply closure theorem (8.2.1) with \( \sigma = 1, N = r + 1 \). We conclude that

\[
y(t) \in A(t), (z^0(t), z(t)) \in \tilde{Q}(t, y(t)) \text{ a.e. in } G.
\]

Then, by implicit function theorem (8.4.i), there is a measurable function

\[
\tilde{u}(t) = (u^0, u) = (u^0, u^1, \ldots, u^m), \quad t \in G,
\]

such that

\[
y(t) \in A(t), \quad u(t) \in U(t, y(t)),
\]

\[
u^0(t) \geq z^0(t) \geq f^0(t, y(t), u(t)) \geq - \psi(t),
\]

\[
z(t) = f(t, y(t), u(t)).
\]

Note that \( z^0 \) is certainly integrable in \( G \), and so is \( f^0(t, y(t), u(t)) \) and

\[
\int_G z^0(t) dt \leq \int_G f^0(t, y(t), u(t)) dt \leq \int_G u^0(t) dt = \int_{I_0} \tilde{f}^0 dt
\]

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\[ F(b) = F^0(b) - S(b) = \lim_{s \to \infty} F^0_k(b) = S(b) \leq 1 - S(b) \leq i. \]

Theorem (8.5.1) is thereby proved.

Statement (8.5.1) has an analogous version for boundary controls. Let
\[ \Gamma \subset \partial G, B(t), B, V(t, \tilde{y}), \tilde{M}, \tilde{y}, \tilde{Z}, \tilde{T} \text{ as in (8.4.iii).} \] Let \( g(t, \tilde{y}, v) = (g_1, \ldots, g_r), \) \( g_0(t, \tilde{y}, v) \) be continuous functions on \( M. \) For every \( (t, \tilde{y}) \in B \) let \( \tilde{R}(t, \tilde{y}) \) denote the sets
\[ \tilde{R}(t, \tilde{y}) = [(z^0, z) \in \tilde{E}_{r+1} | z^0 \geq g_0(t, \tilde{y}, v), z = g(t, \tilde{y}, v), v \in V(t, \tilde{y})] \subseteq \tilde{E}_{r+1} \]

We say that the property of lower closure holds in \( \Gamma \) provided the following occurs: For every sequence \( y_k^0, z_k^0, v_k, k = 1, 2, \ldots, \) of elements \( y_k^0 \in \tilde{Y}, \) \( z_k^0 \in \tilde{Z}, v_k \in \tilde{T}, \) with \( y_k^0(t) \in B(t), v_k(t) \in V(t, y_k^0(t)), z_k^0(t) = g(t, y_k^0(t)), \) \( v_k(t) \) \( \sigma \)-a.e. in \( \Gamma, \) with \( z_k^0(t) = g_0(t, y_k^0(t), v_k(t)) \in L_1(\Gamma), \) and \( \tilde{y}_k \to \tilde{y} \) strongly in \( \tilde{Y}, \) \( z_k^0 \to z \) weakly in \( \tilde{Z}, \) and \( \lim \int_{\Gamma} z_k^0(t) \, d\sigma < +\infty, \) then there is at least one element \( v \in \tilde{T} \) such that \( \tilde{y}(t) \in B(t), v(t) \in V(t, \tilde{y}(t)), \tilde{z}(t) = g(t, \tilde{y}(t), v(t)) \) \( \sigma \)-a.e. in \( \Gamma, \) \( z^0(t) = g(t, \tilde{y}(t), v(t)) \in L_1(\tilde{G}), \) and
\[ \int z^0(t) d\sigma \leq \lim \int z_k^0(t) d\sigma. \]

(8.5ii) A lower closure theorem on \( \partial G. \) Let \( \Gamma, B(t), B, V(t, \tilde{y}), \tilde{M}, \tilde{y} \) as in (8.4.iii), \( B, M \) closed, let \( g, g_0 \) be continuous on \( \tilde{M}, \) and let us assume that the sets \( \tilde{R}(t, \tilde{y}) \) have property \( (Q) \) in \( \Gamma, \) and that the following condition holds \( (\psi): \)
\[ g_0(t, \tilde{y}, v) \geq - \psi(t) \text{ for all } (t, \tilde{y}, v) \in \tilde{M} \text{ and some } \psi \geq 0, \psi \in L_1(\Gamma). \] Then, the lower closure property on \( \Gamma \) holds.

Proof. As in the proof of (8.4.iii) we consider only one set \( \Gamma_i \) as the image of an interval \( I \subset E_{r+1} \) under a transformation \( T_i \) of class \( K, \) or \( t = t(\tau), \)

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\[ \tau \in I. \text{ Then} \]
\[ \int_\Gamma z^0(t) d\sigma = \int_I z^0(t(\tau)) J(\tau) d\tau, \]
where \( J(\tau) \) (Jacobian) is a measurable function on \( I \) with \( 0 < K^{-1} \leq J(\tau) \leq K < +\infty \) for some constant \( K \geq 1 \). Thus, the given problem on \( \Gamma \) can be transformed into an analogous problem on \( I \) concerning the integral (7.1.1) and the functional relations

\[ \hat{y}(t(\tau)) \in B(t, \tau), \ z^0(t(\tau))J(\tau) = f(t(\tau), \hat{y}(t(\tau))), \ v(t(\tau))J(\tau), \]

\[ v(t(\tau)) \in V(t(\tau), \hat{y}(t(\tau))) \text{ a.e. in } I. \]

Now, sets of \( \sigma \)-measure zero in \( \Gamma \) and sets of measure zero on \( I \) correspond. Also, \( \hat{y}_k(t(\tau)) \to \hat{y}(t(\tau)) \) strongly in \( L^1(\Gamma) \) (with respect to the measure \( \sigma \)) if and only if \( \hat{y}_k(t(\tau)) \to \hat{y}(t(\tau)) \) strongly in \( L^1(I) \); and \( z^0_k(t) \to z^0(t) \) weakly in \( L^1(\Gamma) \) if and only if \( z^0_k(t(\tau))J(\tau) \to z^0(t(\tau))J(\tau) \) weakly on \( L^1(I) \). The problem under consideration, which concerns the orientor field

\[ \hat{y}(t) \in B(t), \ (z^0(t), z(t)) \in \tilde{R}(t, \hat{y}(t)) \text{ a.e. in } \Gamma, \]
where \( \tilde{R}(t, \hat{y}) = [z^0 \geq g_0(t, \hat{y}, v), z = g(t, \hat{y}, v), v \in V(t, \hat{y})] \), is now transformed into a problem concerning the orientor field

\[ \hat{y}(t(\tau)) \in B(t(\tau)), (z^0(t(\tau)))J(\tau), z(t(\tau))J(\tau) \in R^*(\tau, \hat{y}(t(\tau))), \text{ a.e. in } I, \]
where \( R^*(\tau, \hat{y}(t(\tau))) = [z^0 \geq g(t(\tau), \hat{y}, v) J(\tau), z = g(t(\tau), \hat{y}, v)J(\tau), v \in V(t(\tau), \hat{y})] \). We have proved in (App.A) that the sets \( \tilde{R}(t, \hat{y}) \) have property (Q) in \( \Gamma \) if and only if the sets \( \tilde{R}(t(\tau), \hat{y}) \) have property (Q) in \( I \), and this in turn if and only if the sets \( R^*(\tau, \hat{y}) \) have property (Q) in \( I \). We can now, therefore, apply theorem (8.5.1).
Remark. Under the same hypotheses of (8.3.i) the following is true: If \( x_k, u_k, k = 1,2,\ldots \), is any sequence of admissible pairs with \( x_k \to x \) weakly in \( S \) and \( \lim I[x_k, u_k] < +\infty \), then there is some \( u \in T \) such that the pair \( x, u \) is admissible, and for every open set \( G_o \subset G \) we have

\[
\int_{G_o} f_o(t, y_k(t), u(t)) dt \leq \lim_{k \to \infty} \int_{G_o} f_o(t, y_k(t), u_k(t)) dt.
\]

In particular, for \( G_o = G \) this relation reduces to \( I[x, u] \leq \lim I[x_k, u_k] \).

To prove this stronger form of (9.3.i) we must first consider the family \([J]\) of all cubes \( J \subset G, J = [b_\rho, b_\sigma], \rho = (\rho_1, \ldots, \rho_v), \sigma = (\sigma_1, \ldots, \sigma_v), \rho_j < \sigma_j, \rho_j, \sigma_j \text{ rational} \). This collection is countable. Then, by the diagonal process, we can extract the subsequence, say now \([k_s]\), in the proof of (8.3.i) in such a way that

\[
\lim_{s \to \infty} \int_J f_0(t, y_{k_s}(t), u_s(t)) dt = \lim_{k \to \infty} \int_J f_0(t, y_k(t), u_k(t)) dt,
\]

as well as

\[
\lim_{s \to \infty} \int_{G} f_0(t, y_{k_s}(t), u_s(t)) dt = \lim_{k \to \infty} \int_{G} f_0(t, y_k(t), u_k(t)) dt.
\]

We can proceed now as in the proof of (9.3.i), and obtain an element \( u \in T \) such that \( x, u \) is admissible, and \( I[x, u] \leq \lim_{s \to \infty} I[x_{k_s}, u_s] = \lim_{k \to \infty} I[x_k, u_k] \). If \( G_o \) is any open subset of \( G \), then \( G_o \) is the union of countably many disjoint (closed) intervals \( J \in [J] \). Given \( \varepsilon > 0 \) there is a finite set of these intervals say \( J_1, \ldots, J_N \), such that
\[ \int_{G \cup \cup J_s} |f_o(t, y(t), u(t))| \, dt < \varepsilon , \quad \int_{G \cup \cup J_s} \psi(t) \, dt < \varepsilon . \]

Hence

\[
\int_{G_o} f_o(t, y(t), u(t)) \, dt \leq \sum_{s} \int_{J_s} f_o \, dt + \varepsilon = \\
= \lim_{s \to \infty} \int_{U \cup J_s} f_o \, dt + \lim_{s \to \infty} \int_{G_o \cup \cup J_s} (f_o + \psi) \, dt + \varepsilon \\
\leq \lim_{k \to \infty} \int_{G_o} f_o(t, y_k(t), u_k(t)) \, dt + 2\varepsilon,
\]

where \( \varepsilon > 0 \) is arbitrary. This proves the statement.

Remark. Condition \((\psi)\) in statement (9.3.i) is certainly satisfied if, for instance, \( f_o(t, y, u) \geq 0 \) for all \( (t, y, u) \in M \), or \( f_o(t, y, u) > \nu \) for all \( (t, y, u) \in M \), where \( \nu \) is some real constant. Nevertheless, condition \((\psi)\) in (9.3.i) can be reduced. A generalization of (9.3.i) will be given below (9.3.ii), where we shall use the following weaker assumption.

Condition \((\psi^*)\). For every \( \tilde{t} \in \text{cl} \, G \) there are a neighborhood \( N(\tilde{t}) \) of \( \tilde{t} \) in \( \text{cl} \, G \), an \( L \)-integrable function \( \psi(t) \geq 0 \), and real numbers \( b_1, \ldots, b_r \) (all \( b_1, \ldots, b_r \) and \( \psi \) may depend on \( \tilde{t} \) and \( N(\tilde{t}) \)) such that

\[
\tilde{f}(t, y, u) = f_o(t, y, u) - \sum_{j=1}^{r} \tilde{t}_j f_j(t, y, u) \geq -\psi(t) \tag{8.5.2}
\]

for all \( (t, y, u) \in M \) with \( t \in N(\tilde{t}) \), with exception perhaps of a set of points \( (t, y, u) \) whose \( t \)-coordinate lies on a set of measure zero in \( G \).
We shall note that, under condition \((\psi_t^*)\), it is natural to consider the sets

\[
\tilde{Q}(t,y) = \{ (z^0, z) \mid z^0 \geq f_0(t,y,u), z = f(t,y,u), u \in U(t,y) \},
\]
or the analogous sets

\[
\tilde{Q}^*(t,y) = \{ (z^0, z) \mid z^0 \geq \tilde{f}_0(t,y,u), z = f(t,y,u), u \in U(t,y) \}.
\]

It is easy to see that the sets \(\tilde{Q}\) are closed, or convex, or satisfy property \((Q)\) if and only if the same occurs for the sets \(\tilde{Q}^*\). Indeed, the sets above are transformed into one another by the fixed affine transformation

\[
Z^0 = z^0 - b \cdot z, \ Z = z.
\]

(8.5.ii) The same as (8.3.i) with hypothesis \((\psi_t^*)\) replacing \((\psi_t)\).

Proof. The set \(\text{cl } G \subseteq E\) is closed and bounded, hence compact. For every \(\bar{t} \in \text{cl } G\) there is a neighborhood \(N_{\delta}(\bar{t})\) and constants \(b = (b_1, \ldots, b_r)\) as stated in assumption \((\psi_t^*)\). These neighborhoods form a covering of \(\text{cl } G\). Thus, finitely many of the same neighborhoods cover \(G\), say \(N_{\delta(\gamma)}(t_\gamma), \gamma = 1, \ldots, N\).

Let \(b_{\gamma 1}, \ldots, b_{\gamma r}\) be the corresponding constants, so that

\[
\tilde{f}_{0 \gamma}(t,y,u) = f_0(t,y,u) = -\sum_{j=1}^{r} b_{\gamma j} f_j(t,y,u) \geq \psi(t)
\]

for all \((t,y,u) \in M, t \in N_{\delta(\gamma)}(t_\gamma)\), where \(\psi(t)\) is \(L\)-integrable in \(N_{\delta(\gamma)}(t_\gamma)\), and

\[
\bigcup_{\gamma=1}^{N} N_{\delta(\gamma)}(t_\gamma) \supseteq \text{cl } G.
\]
The set $H = \{ t | t-t_\gamma = \delta_\gamma, \gamma = 1, \ldots, N \}$ is closed and has measure zero in $E$. Hence, $G-H$ is open, $\text{meas}(G-H) = \text{meas} G$, and each component of $G-H$ is also open. If we now take

$$G_1 = (G-H) \cap (\text{int } N_\delta(t))$$

$$G_\gamma = (G-H) \cap (\text{int } N_\delta(t)) - G_1 - \ldots - G_{\gamma-1}, \gamma = 2, \ldots, N,$$

we see that $G-H$ has been divided into $\Gamma$ disjoint open sets $G_\gamma, \gamma = 1, \ldots, N,$ $G_\gamma \subset N_\delta(t),$, and each component of $G-H$ belongs entirely to one and only one set $G_\gamma$. Possibly empty sets $G_\gamma$ above can be eliminated by suitable reindexing.

Since $\text{meas } H = 0$ we have now

$$\int_G f_k^0(t) dt = \int_G f_\gamma(t,y_k(t)) dt = \int_{G-H} f_\gamma dt = \sum_{\gamma=1}^{N} \int_{G_\gamma} f_\gamma dt$$

$$= \sum_{\gamma=1}^{N} \int_{G_\gamma} [f_\gamma^0(t,y_k(t),u_k(t)) dt + \sum_{j=1}^{r} b_j f_j(t,y_k(t),u_k(t))] dt$$

(8.5.3)

Let $L = \lim_{k \to \infty} \int_G z_k^0(t) dt$ as $k \to \infty$, $L < +\infty$.

On each open set $G_\gamma \subset N_\delta(t)$ we have $\bar{f}_\gamma(t,y,u) > \Psi(t)$ for all $(t,y,u) \in M$, $t \in G_\gamma, \Psi \in L(G_\gamma)$, and we denote by $L_\gamma$ the constant $L_\gamma = \int_G \Psi dt, \gamma = 1, \ldots, N.$ Also, $v_k \rightharpoonup v$ weakly in $V$ as $k \to \infty$. Thus, $v \in L(G)$, and if $X_\gamma(t), t \in G$, denotes the characteristic function of $G_\gamma$ in $G$, then

$$\int_G z_k^0(t) dt \to \int_G z^0(t) dt$$

$$\int_G X_\gamma(t) z_k(t) dt \to \int_G X_\gamma(t) z(t) dt$$

$\gamma = 1, \ldots, \Gamma$, as $k \to \infty$, where $z_k(t) = f(t,y_k(t),u_k(t))$ a.e. in $G$. Hence

$$\int_{G_\gamma} f(t,y_k(t),u_k(t)) dt \to \int_{G_\gamma} f(t,y(t),u(t)) dt, \gamma = 1, \ldots, N$$

(8.5.4)
Again, for every $\gamma = 1, \ldots, N$, we have

$$\int_G \overline{P}_o(t, y_k(t), u_k(t)) dt = \int_G (f_o - b f) dt$$

$$= \int_G f_o dt - \sum_{s \notin \gamma} \int_G f_s dt - \sum_{j} b_j \int_G f_j dt$$

$$\leq \int_G f_o (t, y_k(t), u_k(t)) dt + \sum_{s \notin \gamma} \int_G \psi_s dt - \sum_{j} b_j \int_G f_j (t, y_k(t), u_k(t)) dt.$$

As $k \to \infty$, we have now

$$L_\gamma \leq \lim \int_G \overline{P}_o (t, y_k(t), u_k(t)) dt \leq L + \sum_{s \notin \gamma} L_s - \sum_{j} b_j \int_G z_j (t) dt,$$

for every $\gamma = 1, \ldots, N$. Since $L < + \infty$, each of these $\lim$ is finite. In addition certainly $L > - \infty$, and thus $L$ is finite.

The sets $\overline{Q}^*(t, y)$ are closed, convex, and satisfy property (Q) in $A_\gamma$, as pointed out in Remark before (8.5.1i), and where $A_\gamma = \{(t, y) | t \in G, (t, y) \in A\}$.

We can now apply (8.5.1) to the integrals $\int_G \overline{P}_o dt$, $\gamma = 1, \ldots, N$. By $N$ successive applications of (8.5.1) and successive extractions, we obtain a subsequence which we shall still call $[k]$ for the sake of simplicity, and measurable functions $\overline{u}_\gamma (t)$, $t \in G$, such that

$$\overline{u}_\gamma (t) \in U(t, y(t)), z(t) = f(t, y(t), \overline{u}_\gamma (t)), \text{ a.e. in } G,$$

$$\int_G \overline{P}_o (t, y(t), \overline{u}_\gamma (t)) dt \leq \lim \int_G \overline{P}_o (t, y(t), u_k(t)) dt, \gamma = 1, \ldots, N. \quad (8.5.5)$$

If now $u(t)$, $t \in G$, is defined by taking $u = u_\gamma$ in $G_\gamma$, and $u$ arbitrary in $H \cap G$, then
\[ u(t) \in U(t,y(t)), z(t) = f(t,y(t),u(t)) \text{ a.e. in } G, \]

\[ \int_{\gamma} f_{o}(t,y(t),u(t))dt \leq \lim_{\gamma} \int_{\gamma} f_{o}(t,y_{k}(t),u_{k}(t))dt, \gamma = 1, \ldots, N. \]  

(8.5.6)

Now relation (8.5.4) becomes

\[ \int_{\gamma} f(t,y(t),u(t))dt = \lim_{\gamma} \int_{\gamma} f(t,y_{k}(t),u_{k}(t))dt, \gamma = 1, \ldots, N. \]  

(8.5.7)

By relations (8.5.3), (8.5.5), and (8.5.7), we have

\[ \int_{\gamma} z_{o}^{\gamma}(t)dt = \int_{\gamma} f_{o}(t,y(t),u(t))dt = \sum_{\gamma} \int_{\gamma} f_{o} dt \]

\[ = \sum_{\gamma} [\int_{\gamma} f_{o}(t,y(t),u(t))dt + \sum_{j} b_{\gamma j} \int_{\gamma} f_{j}(t,y(t),u(t))dt] \]

\[ \leq \sum_{\gamma} [\lim_{\gamma} \int_{\gamma} f_{o}(t,y_{k}(t),u_{k}(t))dt + \sum_{j} b_{\gamma j} \lim_{\gamma} \int_{\gamma} f_{j}(t,y_{k}(t),u_{k}(t))dt] \]

\[ \leq \lim_{\gamma} [\sum_{\gamma} \int_{\gamma} f_{o} dt + \sum_{j} b_{\gamma j} \int_{\gamma} f_{j} dt] \]

\[ = \lim_{\gamma} \int_{\gamma} f_{o}(t,y_{k}(t),u_{k}(t))dt = \lim_{\gamma} \int_{\gamma} z_{k}^{\gamma}(t)dt. \]

This proves statement (8.5.ii).