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CONTROL PROBLEMS WITH DISTRIBUTED AND BOUNDARY CONTROLS

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7. Control Problems with Distributed and Boundary Controls

7.1. Problems of Optimization with Distributed Controls

We shall deal here with an independent real vector variable $t = (t^1,\dots,t^\nu)$, $\nu \geq 1$. We are interested in control problems in a fixed open bounded subset G of the t-space E_ν. In most cases the state variable x will be an n-vector $x = (x^1,\dots,x^n)$, whose components x^i are functions of t in G, and, in most cases, we shall require that each x^i belongs to some Sobolev space space $W_p^\ell(G)$ in G, $1 \leq p \leq \infty$, and thus $x \in (W_p^\ell(G))^n$. Indeed, Sobolev functions are the natural generalization for multidimensional problems of the AC (absolutely continuous) functions we have used in one-dimensional problems.

Nevertheless, we shall not need specify in advance the spaces W_p^ℓ (G), or $(W_p^\ell(G))^n$, which are all Banach spaces. All we need to state at the moment is that the state variable x is an element of a given Banach space S.

We shall assume that the control functions of the problems under consideration are vector valued measurable functions $u(t) = (u^1,\dots,u^m)$, $t \in G$, (distributed controls). The set of all measurable m-vector functions u in G will be denoted by T (set, or space, of distributed controls). We may further require restrictions on the allowable values of the control functions (unilateral constraints on the controls) as in the one-dimensional situation.

We shall denote by ∂G the boundary of G, and later we shall require certain regularity properties for the boundary ∂G, particularly in order to discuss boundary conditions and boundary controls, but we need not specify the properties of the boundary for the moment.

Thus, for the moment we shall denote by x any element of a given Banach space S, and we shall consider the two operators

$$\mathcal{L} : S \to Z \subset (L_p(G))^r, \mathcal{M} : S \to Y \subset (L_p(G))^s, \tag{7.1.1}$$

mapping S into two spaces Z and Y of vector functions in G. In other words, the images of any $x \in S$ under $\mathcal{L}$ or $\mathcal{M}$ are r-vector and s-vector functions in G, whose components are all L_p-integrable in G for some p, $1 \leq p \leq +\infty$:

$$z(t) = (z^1,\dots,z^r) = (\mathcal{L}x)(t), \quad t \in G,$$

$$y(t) = (y^1,\dots,y^s) = (\mathcal{M}x)(t), \quad t \in G.$$

We shall now consider problems of optimization concerning the minimum of a multiple integral.

$$I[x,u] = \int_G f_o(t,(\mathcal{M}x)(t),u(t))dt \tag{7.1.2}$$

with state equations of the form

$$(\mathcal{L}x)(t) = f(t,(\mathcal{M}x)(t),u(t)) \quad \text{a.e. in } G. \tag{7.1.3}$$

These state equations are often said to be written in the "usual" or "strong" form. We shall see at the end of section 9 suitable generalizations, or state equations in the "weak" form, as often considered in partial differential equations theory. We shall see that these generalizations can be also discussed in the frame of the present theory.

Here for every $t \in c\ell G$ we assume that a given subset $A(t)$ of the y-space E_s is given, and we denote by A the set of all $(t,y) \in E_{\nu+s}$ with $t \in c\ell G$, $y \in A(t)$. Also, for every (t,y) A we assume that a given subset $U(t,y)$ of the u-space E_m is given, and we denote by M the set of all $(t,y,u) \in E_{\nu+s+m}$ with $(t,y) \in A$, $u \in U(t,y)$. Above, $f_o(t,y,u)$ and $f(t,y,u) = (f_1,\dots,f_s)$ are given real-valued functions defined on M. We are now in a position to state the unilateral constraints. Indeed, we shall require that $y(t) \in A(t)$ and that $u(t) \in U(t,y(t))$ for almost all t of G. In other words, we require that

$$(\mathcal{M}x)(t) \in A(t) \quad \text{a.e. in } G, \tag{7.1.4}$$

$$u(t) \in U(t,(\mathcal{M}x)(t)) \quad \text{a.e. in } G. \tag{7.1.5}$$

Of course, we do not exclude the cases $A(t) = E_s$ and $U(t,y) = E_m$, when these constraints are always satisfied. For every $(t,y) \in A$ the set $U(t,y)$ is as usual the control space for the distributed control u.

We shall be concerned with the problem of the minimum of the functional (7.1.2) with state equations (7.1.3) and constraints (7.1.4), (7.1.5) in suitable classes Ω of pairs x, u, $x \in S$, $u \in T$.

Whenever S is a suitable space of functions in G, say a Sobolev space $(W_p^\ell(G))^n$, and $\mathcal{L}$ and $\mathcal{M}$ are differential operators, as usually is the case, then (7.1.3) reduces to a system of partial differential equations in G, and may then be defined as those elements $x \in S$ satisfying given boundary conditions on ∂G.

Examples will be given in (7.2) below.

Here we need mention that on $\mathcal{M}$ and $\mathcal{L}$ only general assumptions are required. Namely, we require that $\mathcal{L}$ and $\mathcal{M}$ are operators as in (7.1.1) satisfying the following main hypothesis.

(H) If x, x_k, $k = 1,2,\ldots$, are elements of S and $x_k \to x$ weakly in S, then $\mathcal{M}x_k \to \mathcal{M}x$ strongly in $(L_p(G))^s$ and $\mathcal{L}x_k \to \mathcal{L}x$ weakly in $L_p(G)$.

The examples in (7.2) will show that this hypothesis is easily verified.

Note that, if $y_k = \mathcal{M}x_k$, $y = \mathcal{M}x$, $z_k = \mathcal{L}x_k$, $z = \mathcal{L}x$, then (H) simply states that $x_k \to x$ weakly in S implies that $y_k \to y$ strongly in Y and $z_k \to z$ weakly in Z.

Finally, a pair x, u is said to be <u>admissible</u> provided (a) $x \in S$; (b) $u \in T$, that is, $u(t) = (u^1,\ldots,u^m)$, $t \in G$, with u^i measurable in G, $i = 1,\ldots,m$; (c) $y = \mathcal{M}x \in Y$, that is, $y(t) = (y^1,\ldots,y^s)$, $t \in G$, with $y^i \in L_p(G)$, $i = 1,\ldots,s$; (d) $z = \mathcal{L}x \in Z$, that is, $z(t) = (z^1,\ldots,z^r)$, $t \in G$, with $z^i \in L_p(G)$, $i = 1,\ldots,r$; (e) $y(t) \in A(t)$ a.e. in G; (f) $u(t) \in U(t,y(t))$ a.e. in G; (g) $z(t) = f(t,y(t),u(t))$ a.e. in G. Requirements (e) and (f) are only an equivalent form for (7.1.4) and (7.1.5). Requirement (g) is only an equivalent form for (7.1.3), an abstract functional relation.

If x, u is an admissible pair, we shall often say that x is an (admissible) trajectory, and that u is an (admissible) control function, or strategy.

Whenever we need consider the functional (7.1.2), then we shall further require, for x, u to be admissible, that (h) $f_o(t,y(t),u(t))$ is L-integrable in G. Nevertheless, we may often consider only the functional equation (7.1.3) and constraints (7.1.4) and (7.1.5), require only properties (a-g), express them in terms of orientor fields (7.3), and state and prove closure theorems for such a situation.

Remark. The same operator $\mathcal{M}$ appears in (7.1.2), (7.1.3), (7.1.4), and (7.1.5). Actually, it may occur that different components of $y = \mathcal{M}x = (y^1,\dots,y^s)$ appear in (7.1.2), (7.1.3), (7.1.4), (7.1.5), and thus we may really deal with different operators, say $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, $\mathcal{M}_4$. In this case, $y = \mathcal{M}x$ may simply denote the composition of the vectors $y_i = \mathcal{M}_i x$ of orders s_i, $i = 1,2,3,4$, $s_1 + s_2 + s_3 + s_4 = s$.

7.2 Examples

In the examples below G is a fixed domain in the $\zeta\eta$-plane E_2, $\nu = 2$.

1. Minimum of

$$I = \iint_G (x_\zeta^2 + x_\eta^2)\, d\zeta d\eta$$

with given boundary conditions on x.

This a free problem. It can be written as the Lagrange problem of the minimum of

$$I = \iint_G (u^2 + v^2)\, d\zeta d\eta,$$

with differential system

$$x_\zeta = u,\ x_\eta = v,$$

and the same boundary conditions on x. Here we have $f_o = u^2 + v^2$, $f = (f_1, f_2)$, $f_1 = u$, $f_2 = v$. We may take

$$S = W_2^1(G),\quad Y = L_2(G),\quad Z = (L_2(G))^2,$$

$$\mathcal{M}x = o \qquad \mathcal{L}x = (x_\zeta, x_\eta).$$

We know that $x_k \to x$ weakly in $W_2^1(G)$ implies that $x_k \to x$ strongly in $L_2(G)$, and $(x_k)_\zeta \to x_\zeta, (x_k)_\eta \to x_\eta$ weakly in $L_2(G)$. Thus, $x_k \to x$ weakly in S implies that $\mathcal{M}x_k \to \mathcal{M}x$ strongly in Y, and $\mathcal{L}x_k \to \mathcal{L}x$ weakly in Z. We have here m = 2, s = 1, r = 2, U = E_2.

2. Minimum of

$$I = \iint_G (x_\zeta^2 + x_\eta^2)\, d\zeta d\eta$$

with given boundary conditions on x and the differential equation

$$A(\zeta,\eta)x_\zeta + B(\zeta,\eta)x_\eta + C(\zeta,\eta)x + D(\zeta,\eta) = 0.$$

This problem can be reworded as the minimum of the integral

$$I = \iint_G (u^2 + v^2) d\zeta d\eta$$

with the system of partial differential equations

$$x_\zeta = u$$

$$x_\eta = v$$

$$A(\zeta,\eta)x_\zeta + B(\zeta,\eta)x_\eta = -C(\zeta,\eta)x - D(\zeta,\eta)$$

and the given boundary conditions on x. Here we have $f_o = u^2 + v^2$, $f = (f_1, f_2, f_3)$, $f_1 = u$, $f_2 = v$, $f_3 = -Cx - D$. We may take

$$S = W_2^1(G), \quad Y = L_2(G), \quad Z = (L_2(G))^3$$

$$\mathcal{M}x = x, \quad \mathcal{L}x = (x_\zeta, x_\eta, Ax_\zeta + Bx_\eta)$$

We assume that A, B, C, D are given bounded measurable function on G. Then $x_k \to x$ weakly in $S = W_2^1(G)$ implies that $\mathcal{M}x_k \to \mathcal{M}x$ strongly in Y and $\mathcal{L}x_k \to \mathcal{L}x$ weakly in Z. We have here $m = 2$, $s = 1$, $r = 3$, $U = E_2$.

3. Minimum of

$$I = \iint_G (\xi^2 + \eta^2 + x^2 + x_\xi^2 + x_\eta^2)\, d\xi d\eta$$

with given boundary conditions on x.

This is a free problem. It can be written as the Lagrange problem of the minimum of

$$I = \iint_G (\xi^2 + \eta^2 + x^2 + u^2 + v^2) d\xi d\eta$$

with differential system

$$x_\zeta = u, \quad x_\eta = v$$

and the same boundary conditions on x. Here we have

$$f_o = \zeta^2 + \eta^2 + x^2 + u^2 + v^2, \quad f = (f_1, f_2), \quad f_1 = u,\ f_2 = v.$$

We may take

$$S = W_2^1(G), \quad Y = L_2(G), \quad Z = (L_2(G))^2$$

$$\mathcal{M}x = x, \quad \mathcal{L}x = (x_\zeta, x_\eta)$$

Thus, $x_k \to x$ weakly in $S = W_2^1(G)$ implies that $\mathcal{M}x_k \to \mathcal{M}x$ strongly in $Y = L_2(G)$, and $\mathcal{L}x_k \to \mathcal{L}x$ weakly in $Z = (L_2(G))^2$. We have here $m = 2$, $s = 1$, $r = 2$, $U = E_2$.

4. Minimum of

$$I = \iint_G (\zeta^2 + \eta^2 + x^2 + x_\zeta^2 + x_\eta^2)\, d\zeta d\eta$$

with boundary conditions on x and the differential equation

$$A(\zeta,\eta)x_\zeta + B(\zeta,\eta)x_\eta + C(\zeta,\eta)x + D(\zeta,\eta) = 0$$

The problem can be reworded as the Lagrange problem of the minimum of

$$I = \iint_G (\zeta^2 + \eta^2 + x^2 + u^2 + v^2)\, d\zeta d\eta$$

with differential system

$$x_\zeta = u$$

$$x_\eta = v$$

$$A(\zeta,\eta)x_\zeta + B(\zeta,\eta)x_\eta = -C(\zeta,\eta)x - D(\zeta,\eta)$$

Here we have $f_o = \zeta^2 + \eta^2 + x^2 + u^2 + v^2$, $f = (f_1, f_2, f_3)$, $f_1 = u$, $f_2 = v$, $f_3 = -Cx - D$. We assume that A, B, C, D are measurable bounded functions in G. We may take

$$S = W_2^1(G), \quad Y = L_2(G), \quad Z = (L_2(G))^3,$$

$$\mathcal{M}x = x, \quad \mathcal{L}x = (x_\zeta, x_\eta, A(\zeta,\eta)x_\zeta + B(\zeta,\eta)x_\eta).$$

Then $x_k \to x$ weakly in $S = W_2^1(G)$ implies that $\mathcal{M}x_k \to \mathcal{M}x$ strongly in Y, and $\mathcal{L}x_k \to \mathcal{L}x$ weakly in Z. We have here $m = 2$, $s = 1$, $r = 3$, $U = E_2$.

5. Minimum of

$$I = \iint_G (x_\zeta^2 + x_\eta^2)\, d\zeta d\eta$$

with the differential equation

$$A(\zeta,\eta)x_{\zeta\zeta} + B(\zeta,\eta)x_{\zeta\eta} + C(\zeta,\eta)x_{\eta\eta} = D(\zeta,\eta)x_\zeta + E(\zeta,\eta)x_\eta + F(\zeta,\eta)x + H(\zeta,\eta)$$

and boundary conditions concerning the values of x, x_ζ, x_η on the boundary of G.

We assume that A, B, C, D, E, F, H are given measurable bounded functions on G. We may take

$$S = W_2^2(G), \quad Y = (L_2(G))^3, \quad Z = L_2(G),$$

$$\mathcal{M}x = (x, x_\zeta, x_\eta), \quad \mathcal{L}x = Ax_{\zeta\zeta} + Bx_{\zeta\eta} + Cx_{\eta\eta}.$$

Then $x_k \to x$ weakly in $S = W_2^2(G)$, implies $\mathcal{M}x_k \to$ x strongly in Y, and $\mathcal{L}x_k \to$ x weakly in Z. We have here $s = 3$, $r = 1$. We may take $m = 1$, $U = \{0\}$, and there is only one admissible control function $u = 0$. We assume above that the given boundary conditions on x, x_ζ, x_η do not determine x(t) x uniquely.

6. Minimum of

$$I = \iint_G (\zeta^2 + \eta^2 + x^2 + x_\zeta^2 + x_\eta^2)\, d\zeta d\eta$$

with the given boundary conditions on x and the system of partial differential equations

$$x_\zeta = u, \quad x_\eta = v.$$

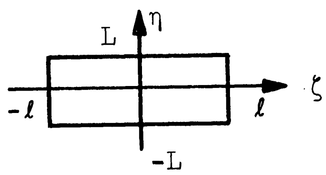
We assume that A, B, C are given measurable bounded functions on G. Here we have $f_o = Au^2 + Bv^2 + Cx^2$, $f = (f_1, f_2)$, $f_1 = u$, $f_2 = v$. We may take

$$S = W_2^1(G), \quad Y = L_2(G), \quad Z = (L_2(G))^2,$$

$$\mathcal{M}x = x, \quad \mathcal{L}x = (x_\zeta, x_\eta).$$

Thus $x_k \to x$ weakly in $S = W_2^1(G)$, implies $\mathcal{M}x_k \to \mathcal{M}x$ strongly in Y and $\mathcal{L}x_k \to \mathcal{L}x$ weakly in Z. We have here $m = 2$, $s = 1$, $r = 2$, $U = E_2$.

8. Lurie's magnetohydrodynamic channel problem (from K. A. Lurie, Topics in Optimization (ed. Leitman), Academic Press 1967), pp. 147-193.



Minimum of

$$I = \iint_G u(v^2 + w^2)\,d\zeta d\eta$$

with the system of four partial differential equations

$$x_\zeta = -uv \quad x_\eta = -uw + f(\zeta,\eta)$$

$$y_\zeta = w, \quad y_\eta = -v$$

and given boundary conditions concerning x and y on the boundary R of the given interval $G = [-\ell \leq \zeta \leq \ell, -L \leq \eta \leq L]$. The control variables v, w can take arbitrary values, the control variable u can take only values between two fixed positive numbers, $0 < a \leq u \leq b < +\infty$.

If E denotes the electric field, B the magnetic field, V the velocity of the mgh channel flow (in the ζ-direction), then I is the total Joule loss, $j = (v,w)$ denotes the density of the electric current, u is the specific resistance, $E = -\operatorname{grad} x$, $j = -\operatorname{curl} y$, $f = c^{-1}VB$, c a positive constant.

We have here $f_o = u(v^2 + w^2)$, $f = (f_1, f_2, f_3, f_4)$, $f_1 = -uv$, $f_2 = -uw + f(\zeta,\eta)$, $f_3 = w$, $f_4 = -v$. We may take

$$S = (W_2^1(G))^2, \quad Y = (L_2(G))^2, \quad Z = (L_2(G))^4,$$

$$\mathcal{M}(x,y) = (0,0), \quad \mathcal{L}(x,y) = (x_\zeta, x_\eta, y_\zeta, y_\eta).$$

Thus $x_k \to x$, $y_k \to y$ weakly in $W_2^1(G)$ implies $\mathcal{M}(x_k,y_k) \to \mathcal{M}(x,y)$ strongly in Y, and $\mathcal{L}(x_k,y_k) \to \mathcal{L}(x,y)$ weakly in Z. We have here $m = 3$, $s = 2$, $r = 4$, $U = [a,b] \times E_2$.

9. A problem proposed by A. G. Butkovsky, A. I. Egorov, and K. A. Lurie. (from SIAM J. Control, Vol. 6, 1968, pp. 437-477; particularly p. 457).

Minimum of

$$I = \int_0^T\int_0^a [A_o(t,\zeta,x,u) + A_1(t,\zeta,x,u) \frac{\partial x}{\partial \zeta}]dtd\zeta$$

with the initial condition $x(0,\zeta) = \varphi(\zeta)$, and the partial differential equation

$$\frac{\partial x}{\partial t} = B_o(t,\zeta,x,u) + B_1(t,\zeta,x,u)\frac{\partial x}{\partial \zeta}.$$

Here G is the interval $[0 \leq t \leq T,\ 0 \leq \zeta \leq a]$.

It can be written as the problem of the minimum of

$$I = \iint_G [A_o(t,\zeta,x,u) + A_1(t,\zeta,x,u)v]dt d\zeta$$

with data $x(0,\zeta) = \varphi(\zeta)$, and the partial differential equations

$$x_\zeta = v$$

$$x_t = B_o(t,\zeta,x,u) + B_1(t,\zeta,x,u)v.$$

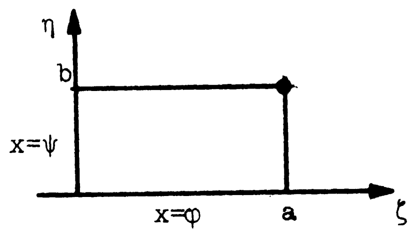
We have $f_o = A_o + A_1 v$, $f = (f_1, f_2)$, $f_1 = v$, $f_2 = B_o + B_1 v$. We may take

$$S = W_2^1(G), \quad Y = L_2(G), \quad Z = (L_2(G))^2,$$

$$\mathfrak{M}x = x, \quad \mathfrak{L}x = (x_\zeta, x_t).$$

Thus, $x_k \to x$ weakly in $S = W_2^1(G)$ implies $\mathfrak{M}x_k \to \mathfrak{M}x$ strongly in Y, and $\mathfrak{L}x_k \to \mathfrak{L}x$ weakly in Z. Here we have $m = 2$, $s = 1$, $r = 2$, $U = E_2$.

10. A problem proposed by Egorov, with physical interpretations. (A. I. Egorov, J. Appl. Math. Mech. 27, 1963, 1045-1058; Autom. Remote Control 25, 1964, 557-566).



Minimum of

$$I = x(a,b)$$

with the partial differential equation

$$\frac{\partial^2 x}{\partial\zeta\partial\eta} = f(\zeta,\eta,x,x_\zeta,x_\eta,w), \quad 0 \le \zeta \le a, \quad 0 \le \eta \le b,$$

and initial data

$$x(\zeta,0) = \varphi(\zeta), \quad x(0,\eta) = \psi(\eta),$$

where φ, ψ are given functions with $\varphi(0) = \psi(0)$.

By integration we have

$$x(\zeta,\eta) = \varphi(\zeta) + \psi(\eta) - \varphi(0) + \int_0^\zeta\int_0^\eta f(\alpha,\beta,x,x_\zeta,x_\eta,w)d\alpha d\beta,$$

$$0 \le \zeta \le a, \quad 0 \le \eta \le b,$$

and thus,

$$I = x(a,b) = \varphi(a) + \psi(b) - \varphi(0) + \int_0^a\int_0^b f(\zeta,\eta,x,x_\zeta,x_\eta,w)d\zeta d\eta.$$

The problem is equivalent to the problem of the minimum of

$$I = \iint_{00}^{ab} f(\zeta,\eta,x,x_\zeta,x_\eta,w)d\zeta d\eta$$

with data $x(\zeta,o) = \varphi(\zeta)$, $x(o,\eta) = \psi(\eta)$. This problem can be written as the Lagrange problem of the minimum of

$$I = \iint_{00}^{ab} f(\zeta,\eta,x,u,v,w)d\zeta d\eta$$

with data $x(\zeta,o) = \varphi(\zeta)$, $x(o,\eta) = \psi(\eta)$, and differential equations

$$x_\zeta = u, \quad x_\eta = v.$$

We may take

$$S = W_2^1(G), \quad Y = L_2(G), \quad Z = (L_2(G))^2$$

$$\mathcal{M}x = x, \quad \mathcal{L}x = (x_\zeta, x_\eta).$$

Thus, $x_k \to x$ weakly in S, implies $\mathcal{M}x_k \to \mathcal{M}x$ strongly in Y, and $\mathcal{L}x_k \to \mathcal{L}x$ weakly in Z. We have $m = 3$, $s = 1$, $r = 2$, $U = E_3$.

11. Free problems for multiple integrals. We are concerned here with the minimum of a multiple integral

$$I[x] = \int_G f_o(t,x(t),(\nabla x)(t))dt \qquad (8.5.1)$$

with the only constraint $(t,x(t)) \in A$, and possible boundary conditions concerning the boundary values of the vector-valued function $x(t) = (x^1,\dots,x^n)$, $t \in G \subset E_\nu$, on the boundary ∂G of G, and where $\nabla x = (\partial x^i/\partial t^s$, $i = 1,\dots,n$, $s = 1,\dots,\nu)$ denotes the system of $n\nu$ first order partial derivatives $\partial x^i/\partial t^s$. We shall take in (9.3) for S a Sobolev space of vector functions $x(t) = (x^1, \dots,x^n)$, $t \in G$, with each x^i possessing generalized partial derivative $\partial x^i/\partial t^s$, $i = 1,\dots n$, $s = 1,\dots,\nu$, we shall take $s = n$, $r = n\nu$, we shall take for $\mathcal{M}$: $S \to Y = L'$ the identity operator, or $\mathcal{M}x = x$, and for $\mathcal{L}$: $S \to V = L''$ the operator $\mathcal{L}x = \nabla x$. We shall take $m = n\nu$, $f = (f_{is}$, $i = 1,\dots,n$, $s = 1,\dots,\nu)$, $u = (u_{is}$, $i = 1,\dots,n$, $s = 1,\dots,\nu)$, with $f_{is} = u_{is}$, or $f = u$. Then the functional equation $\mathcal{L}x = f$ reduces to $\nabla x = u$. Finally, we shall take $U(t,y) = E_{n\nu}$, so that the constraint $u(t) \in U(t,y,(t))$ is always satisfied. The problem (8.1.1-3) reduces now to the problem of the minimum of (8.5.1) with the constraint $(t,x(t)) \in A$. Possible given boundary conditions will be used to defined the class Ω of elements $x \in S$ in which we seek the minimum of the integral (8.5.1). Details will be given in (10.3).

Examples 1, 3, and 7 can be thought of as particular cases of the present example 11.

12. Lagrange problems of optimization with a total differential system of partial differential equations. We are concerned here with the minimum of a multiple integral

$$I[x,u] = \int_G f_o(t,x(t),u(t))dt, \tag{8.5.2}$$

where the functions $x(t) = (x^1,\dots,x^n)$, $u(t) = (u^1,\dots,u^m)$, $t \in G$, satisfying a total differential system of partial differential equations (Dieudonné-Rashevski differential system)

$$\partial x^i/\partial t^s = f_{is}(t,x(t),u(t)) \text{ a.e. in } G,\ i = 1,\dots,n,\ s = 1,\dots,\nu, \tag{8.5.3}$$

constraints

$$(t,x(t)) \in A,\ u(t) \in U(t,x(t)) \text{ a.e. in } G, \tag{8.5.4}$$

and possible boundary conditions on the boundary values of the vector valued functions $x(t) = (x^1,\dots,x^n)$, $t \in G \subset E_\nu$, on the boundary ∂G of G. We shall take in (10.4) for S a Sobolev space of vector functions $x(t) = (x^1,\dots,x^n)$, $t \in G$, with each x^i possessing generalized partial derivatives $\partial x^i/\partial t^s$, we shall take $s = n$, $r = n\nu$, we shall take for $\mathcal{M}$: $S \to Y$ the identity operator, or $x = x$, and for $\mathcal{L}$: $S \to Z$ the operator $x = \nabla x$. Here m is any integer, and $u(t) = (u^1,\dots,u^m) \in T$. The problem (8.1.1-3) reduces now to the problem (8.5.2-4). Possible given boundary conditions will be used to restrict the elements $x \in S$ which enter in pairs x, u, $x \in S$, $u \in T$, satisfying (8.5.2), (8.5.3) of the class Ω. Details will be given in (9.3).

Examples 1, 3, 7, and 8 can be thought of as particular cases of the present example 12.

13. Abstract free problems. We are concerned with the minimum of a multiple integral

$$I[x] = \int_G f_o(t,(\mathcal{M}x)(t),(\mathcal{L}x)(t))dt \qquad (8.5)$$

with the only constraint $(t,(\mathcal{M}x)(t)) \in A$, and possible other restrictions on $x \in S$. Here $\mathcal{M}$: $S \to Y$, $\mathcal{L}$: $S \to Z$ are operators as in (8.1-2), we take $m = r$, $f = (f_1,\dots,f_r)$, $u = (u^1,\dots,u^r)$, with $f_i = u^i$, $i = 1,\dots,r$, or $f = u$, and we take $U = E_r$. The equations $\mathcal{L}x = f$ reduces now to $\mathcal{L}x = u$, hence $x \in S$ determines $u \in T$ uniquely. The problem (8.1.1-3) reduces now to the problem of the minimum of the functional

$$I[x,u] = \int_G f_o(t,(\mathcal{M}x)(t),u(t))dt$$

with functional equation

$$(\mathcal{L}x)(t) = u(t) \quad \text{a.e. in } G,$$

and the only constraint $(t,(\mathcal{M}x)(t)) \in A$ a.e. in G. The constraint on the control variable reduces here to $u(t) \in U = E_r$, $u(t) = (u^1,\dots,u^r)$, and is always satisfied. We have here a problem as discussed in (8.1-2). We may expect that further restrictions are needed to define the classes Ω of pairs x, u, $x \in S$, $u \in T$, on which we seek the minimum of (8.5).

14. A problem of optimization with equations of evolution. We are concerned with the minimum of a multiple integral

$$I[x,u] = \int_0^T \int_{G_0} f_0(t,\tau,x(t,\tau),u(t,\tau))\,dt\,d\tau,$$

where the functions $x(t,\tau) = (x^1,\dots,x^n)$, $u(t,\tau) = (u^1,\dots,u^m)$, $(t,\tau) \in G = (0,T) \times G_0$, $G_0 \subset E_\nu$, satisfy an evolution type equation

$$\partial x/\partial t = \mathcal{A}x + f(t,\tau,x(t,\tau),u(t,\tau)), \text{ a.e. in } G,$$

and constraints

$$x(t,\tau) \in A,\ u(t,\tau) \in U, \text{ a.e. in } G,$$

and possible boundary conditions on the boundary values of the vector valued function x on the boundary ∂G of G. Here G_0 is a bounded open set in the τ-space E_ν, and $\mathcal{A}$ is a differential operator of order h in the fixed set G_0 of the τ-space E_ν. We shall take in (9.7) for S a suitable space of vector functions $x(t,\tau) = (x^1,\dots,x^n)$, $(t,\tau) \in G = (0,T) \times G_0$, possessing generalized partial derivatives $\partial x/\partial t$ and $D_\tau^\alpha x$, $\alpha = (\alpha_1,\dots,\alpha_\nu)$, $0 \le |\alpha| \le h$, $|\alpha| = \alpha_1+\dots+\alpha_m$. We shall take for $\mathcal{M}$ the identity operator as usual, $\mathcal{M}x = x$, and for $\mathcal{L}$ the operator $\mathcal{L}x = \partial x/\partial t - \mathcal{A}x$. Details will be given in (9.7).

7.3 A Formulation in Terms of Orientor Fields

The functional equation (8.1.1) can be written in terms of orientor fields, and this makes it possible to simplify the formulation of the same functional equation. With the same notations of (8.1-2), let us denote by $Q(t,y)$ the usual subset of E_r

$$Q(t,y) \;=\; f(t,y,U(t,y)) \;=\; [z \in E_r \,|\, z = f(t,y,u),\, u \in U(t,y)], \tag{7.3.1}$$

where $(t,y) \in A$. We say now that an element $x \in X$ is admissible provided (a') $x \in S$, (b') $y = \mathcal{M}x \in Y$; hence, $y(t) = (y^1,\dots,y^s)$, $t \in G$, $y^i \in L_p(G)$, $i = 1,\dots,s$; (c') $z = \mathcal{L}x \in Z$; hence, $z(t) = (z^1,\dots,z^r)$, $t \in G$, $z^j \in L_p(G)$, $j = 1,\dots,r$; (d') $(t,y(t)) \in A$ a.e. in G; (e') $z(t) \in Q(t,y(t))$ a.e. in G.

Here (e') can be written in the form

$$(\mathcal{L}x)(t) \in Q(t,(\mathcal{M}x)(t)) \text{ a. . in } G. \tag{7.3.2}$$

If a pair x, u, $x \in S$, $u \in T$, is admissible in the sense of (8.2), then obviously x satisfies the properties (a'-e') above. The converse is also true under very mild hypotheses. Indeed, if we assume that A and M are closed sets, and that $f(t,y,u) = (f_1,\dots,f_s)$ is continuous on M, then, by the use of the implicit function theorem, we can prove that for any element $x \in S$ satisfying (a'-e') above, there is an element $u \in T$ such that x, u is admissible. In other words, there is some $u(t) = (u^1,\ldots,u^m)$, $t \in G$, with u^j measurable in G, $j = 1,\ldots,m$, such that $u(t) \in U(t,y(t))$, $z(t) = f(t,y(t),u(t))$ a.e. in G. The details of the proof are the same as in (1.6), and are not repeated here.

7.4 Comments on Condition (H) and the Graph Norm

If S^* denotes the topological dual of S, we denote by (x',x) the application of $x^* \in S^*$ to $x \in S$. A sequence x_k, $k = 1, 2,\ldots,$ of elements $x_k \in S$ is said to converge weakly to an element $x \in S$ provided $(x^*,x_k) \to (x^*,x)$ for every $x^* \in S^*$.

With the notations of (7.2), let $Y = (L_p(G))^s$ and $Z = (L_p(G))^r$. If $1 \le p < +\infty$, then $Y^* = (L_q(G))^s$, $Z^* = (L_q(G))^r$, $q^{-1} + p^{-1} = 1$, and then

$$(y,\varphi) = \sum_{j=1}^{s} \int_G y^i \varphi^i \, dt \quad \text{for } y \in Y,\ \varphi \in Y^*,$$

$$(z,\psi) = \sum_{j=1}^{r} \int_G z^j \psi^j \, dt \quad \text{for } z \in Z,\ \psi \in Z^*.$$

Here $\varphi = (\varphi_1,\ldots,\varphi_s)$, $\varphi_i \in L_q(G)$, $i = 1,\ldots,s$, and analogously for $\psi = (\psi_1,\ldots,\psi_r)$. Thus $y_k \to y$ weakly in Y means that $(y_k,\varphi) \to (y,\varphi)$ as $k \to \infty$ for every $\varphi \in Y^*$; analogously, $z_k \to z$ weakly in Z means that $(z_k,\psi) \to (z,\psi)$ as $k \to \infty$ for every $\psi \in Z^*$.

Condition (H) can be drastically simplified in the following situation which is often encountered. Let X be a Banach space of elements x and norm $\|x\|$, let X_o be a linear subspace of X, let $\mathfrak{M}: X_o \to Y$, $\mathfrak{K}: X_o \to \mathbf{Z}$ be linear operators, and let S be the completion of X_o by means of the norm

$$|||x||| = (\|x\|^2 + \|\mathfrak{M}x\|^2 + \|\mathfrak{L}x\|^2)^{1/2},$$

where the norms within the parentheses are norms in X, Y, Z respectively (see App. B). Then, S is a Banach space with norm $|||x|||$, and $|||x|||$ is said to be the graph norm. As mentioned in App. , to each element $x \in S$ there corresponds a unique element $x_o \in X$, which we may denote simply by x, a vector function $y(t) = (y_o^1,\dots,y_o^s)$, $t \in G$, with $y_o^i \in L_p(G)$, a vector function $z(t) = (z_o^1,\dots,z_o^r)$, $t \in G$, with $z_o^i \in L_p(G)$, sequences $[x_k]$ of elements $x_k \in X_o$, $k = 1, 2,\dots$, such that, if $y_k = (y_k^1,\dots,y_k^s) = \mathfrak{M}x_k$, $z_k = (z_k^1,\dots,z_k^r) = \mathfrak{L}x_k$, $k = 1, 2,\dots$, then $\|x_k - x\| \to o$, $\|y_k^i - y_o^i\|_{L_p} \to o$, $i = 1,\dots,s$, $\|z_k^j - z_o^j\|_{L_p} \to o$, $j = 1,\dots,r$, as $k \to \infty$. We define $\mathfrak{M}$: $S \to Y$, $\mathfrak{L}$: $S \to Z$, by taking $\mathfrak{M}x = y_o$, $\mathfrak{L}x = z_o$. We know from (App. B) that $\mathfrak{M}$ and $\mathfrak{L}$ are thereby uniquely defined, and are linear operators from the Banach space S into Y and Z respectively.

Let X* denote the topological dual of X, and (x,x*) the application of $x^* \in X^*$ to $x \in X$. A sequence $[x_k]$ of elements $x_k \in S$, $k = 1, 2,\dots$, is said to converge weakly in S to an element $x \in S$, provided

$$(x_k,x^*) \to (x,x^*) \text{ for every element } x^* \in X^*,$$

$$\int_G y_k^i\, \varphi\, dt \to \int_G y^i\, \varphi\, dt \text{ for every } \varphi \in L_q(G),\quad i = 1,\dots,s,$$

$$\int_G z_k^j\, \psi\, dt \to \int_G z^j\, \psi\, dt \text{ for every } \psi \in L_q(G),\quad j = 1,\dots,r,$$

where $y_k = \mathfrak{M}x_k$, $z_k = \mathfrak{L}x_k$, $y = \mathfrak{M}x$, $z = \mathfrak{L}x$, $y_k(t) = (y_k^1,\dots,y_k^s)$, $z_k(t) = (z_k^1,\dots,$ $z_k^r)$, $y(t) = (y^1,\dots,y^s)$, $z(t) = (z^1,\dots,z^r)$, $t \in G$, and $q^{-1} + p^{-1} = 1$, and the usual conventions.

In this situation $x_k \to x$ weakly certainly implies that $y_k \to y$ weakly in Y and $v_k \to v$ weakly in V. The part of (H) concerning $\mathcal{L}$ is thus trivial, and all we have to require is that (H_o) if $x_k \to x$ weakly in S as $k \to \infty$, then $y_k \to y$ strongly in Y as $k \to \infty$, where $y_k = \mathfrak{M}x_k$, $y = \mathfrak{M}x$, $k = 1, 2, \ldots$.

7.5 Problems of Optimization with Distributed and Boundary Controls: Bolza-Type Problems

We are interested, as in (7.2), in control problems in an open bounded subset G of E_ν, and in most cases the state variable x will be an element of a Sobolev space $W_p^\ell(G)$, $1 \leq p \leq +\infty$, $\ell \geq 1$. Nevertheless, we need not specify in advance this space, and therefore we shall only state at the moment that the state variable x is an element of a given Banach space S.

We shall assume that, beside G, another set Γ is given, say of dimension ν-1, and we may think of Γ as a subset of ∂G. We shall assume that Γ is the union of finitely many parts $\Gamma_1,\ldots,\Gamma_N$, each Γ_i being the image of a fixed interval $I \subset E_{\nu-1}$ under a transformation of class K (see App. B), so that Γ can be thought of as a part of the boundary ∂G of Γ on which a natural area measure function σ is defined.

We shall assume that the control functions are a vector-valued measurable function $u(t) = (u^1,\ldots,u^m)$, $t \in G$ (distributed control), and a vector-valued σ-measurable function $v(t) = (v^1,\ldots,v^{m'})$, $t \in \Gamma$ (boundary control). We shall denote by T the set of all measurable u on G, and by $\dot{T}$ the set of all σ-measurable v on Γ. We may further require restrictions on the values of the control functions u and v as usual (unilateral constraints).

We shall consider operators $\mathcal{L}$, $\mathcal{J}$, $\mathcal{M}$, $\mathcal{K}$,

$$\mathcal{L}: \quad S \to (L_p(G))^r, \qquad \mathcal{J}: \quad S \to (L_p(\Gamma))^{r'}$$

$$\mathcal{M}: \quad S \to (L_p(G))^s, \qquad \mathcal{K}: \quad S \to (L_p(\Gamma))^{s'} \tag{7.5.1}$$

and we shall denote the images of any $x \in S$ under $\mathcal{L}$, $\mathcal{J}$, $\mathcal{M}$ $\mathcal{K}$ by the notations

$$z(t) = (z^1,\dots,z^r) = (\mathcal{L}x)(t), \ t \in G,$$

$$y(t) = (y^1,\dots,y^s) = (\mathcal{M}x)(t), \ t \in G,$$

$$\dot{z}(t) = (\dot{z}^1,\dots,\dot{z}^{r'}) = (\mathcal{J}x)(t), \ t \in \Gamma,$$

$$\dot{y}(t) = (\dot{y}^1,\dots,\dot{y}^{s'}) = (\mathcal{K}x)(t), \ t \in \Gamma.$$

We shall now consider problems of optimization concerning the minimum of the functional

$$I[x,u,v] = \int_G f_o(t,(\mathcal{M}x)(t),u(t))dt + \int_\Gamma g_o(t,(\mathcal{J}x)(t),v(t))d\sigma \tag{7.5.2}$$

with state equations of the form

$$(\mathcal{L}x)(t) = f(t,(\mathcal{M}x)(t),u(t)) \text{ a.e. in } G, \tag{7.5.3}$$

$$(\mathcal{J}x)(t) = g(t,(\mathcal{K}x)(t),v(t)) \ \sigma\text{-a.e. in } \Gamma. \tag{7.5.4}$$

These are again state equations written in the strong form (see (7.1)). We shall discuss at the end of section 9 corresponding state equations written in generalized, or weak forms.

Here for every $t \in c\ell G$ we assume that a given subset $A(t)$ of the y-space E_s is given, and we denote by A the set of all $(t,y) \in E_{\nu+s}$ with $t \in c\ell G$ $y \in A(t)$. For every $(t,y) \in A$ we assume that a given subset $U(t,y)$ of the u-space E_m is given, and we denote by M the set of all $(t,y,u) \in E_{\nu+s+m}$ with $(t,y) \in A$, $u \in U(t,y)$.

Analogously, for every $t \in \Gamma$ we assume that a given subset $B(t)$ of the $\dot{y}$-space $E_{s'}$ is given, and we denote by B the set of all $(t,\dot{y}) \in E_{\nu+s'}$ with $t \in \Gamma$, $\dot{y} \in B(t)$. For every $(t,\dot{y}) \in B$ we assdume that a given subset $V(t,\dot{y})$ of the v-space $E_{m'}$ is given, and we denote by $\dot{M}$ the set of all $(t,\dot{y},v) \in E_{\nu+s'+m'}$ with $(t,\dot{y}) \in B$, $v \in V(t,\dot{y})$.

Above, $f_o(t,y,u)$ and $f(t,y,u) = (f_1,\dots,f_r)$ are given real-valued functions defined on M; $g_o(t,\dot{y},v)$ and $g(t,y,v) = (g_1,\dots,g_{r'})$ are given real-valued functions defined on M'.

Finally, we can state the unilateral constraints. Indeed, we shall require $y(t) \in A(t)$, $u(t) \in U(t,y(t))$ for almost all $t \in G$, and $\dot{y}(t) \in B(t)$, $v(t) \in V(t,\dot{y}(t))$ σ-a.e. on Γ. In other words, we require

$$(\mathcal{M}x)(t) \in A(t) \text{ a.e. in } G,\ (Kx)(t) \in B(t)\ \sigma\text{-a.e. on } \Gamma, \tag{7.5.5}$$

$$u(t) \in U(t,(\mathcal{M}x)(t)) \text{ a.e. in } G,\ v(t) \in V(t,(Kx)(t))\ \sigma\text{-a.e. on } \Gamma. \tag{7.5.6}$$

Of course, we do not exclude the casea $A(t) = E_s$, $B(t) = E_{s'}$, $U(t,y) = E_m$, $V(t,\dot{y}) = E_{m'}$ when these constraints are always satisfied. As usual we say that $U(t,y)$ is the distributed control space, and that $V(t,\dot{y})$ is the boundary control space.

We shall be concerned with the problem of the minimum of the functional (7.5.2), with state equations (7.5.3), (7.5.4) and constraints (7.5.5) and (7.5.6), in suitable classes Ω of triples x, u, v, $x \in S$, $u \in T$, $v \in \dot{T}$.

In most cases, S is a Sobolev space on G, $\mathcal{L}$ and $\mathcal{M}$ are differential operators, and thus (7.5.3) is a system of partial differential equations in G. Relation (7.5.4) may be reduced to a system of boundary conditions as in (7.1), or to a system of differential equations on ∂G.

Examples will be given in (7.6) below.

Here we need mention that on $\mathcal{M}$, $\mathcal{K}$, $\mathcal{L}$, $\mathcal{J}$ only general assumptions are required. Namely, we require that $\mathcal{M}$, $\mathcal{K}$, $\mathcal{L}$, $\mathcal{J}$ are operators as in (7.1) satisfying the following main hypothesis:

(H_1) If x, x_k, k = 1, 2,..., are elements of S and $x_k \to x$ weakly in S, then $\mathcal{M}x_k \to \mathcal{M}x$ strongly in $L_p(G)^s$, $\mathcal{K}x_k \to \mathcal{K}x$ strongly in $(L_p(\Gamma))^{s'}$, $\mathcal{L}x_k \to \mathcal{L}x$ weakly in $(L_p(G))^r$, $\mathcal{J}x_k \to \mathcal{J}x$ weakly in $(L_p(\Gamma))^{r'}$.

As an alternate hypothesis we shall consider the case

(H_1') Same as in (H_1) with $\mathcal{J}x_k \to \mathcal{J}x$ strongly in $(L_p(\Gamma))^{r'}$.

Finally, a triple x, u, v is said to be <u>admissible</u> provided (a) $x \in S$, (b) $u \in T$, $v \in \dot{T}$, that is, $u(t) = (u^1,\dots,u^m)$, $t \in G$, with u^j measurable in G, $j = 1,\dots,m$, and $v(t) = (v^1,\dots,v^{m'})$, $t \in \Gamma$, with v^j σ-measurable in Γ, $j = 1, \dots,m'$; (c) $y = \mathcal{M}x \in (L_p(G))^s$ and $\dot{y} = \mathcal{K}x \in (L_p(\Gamma))^{s'}$, that is, $y(t) = (y^1,\dots, y^s)$, $t \in G$, with $y^i \in L_p(G)$, $i = 1,\dots,s$, and $\dot{y}(t) = (y^1,\dots,y^{s'})$, $t \in \Gamma$, with $y^i \in L_p(\Gamma)$ (with respect to the measure σ on Γ); (d) $z = \mathcal{L}x \in (L_p(G))^r$, and $\dot{z} = \mathcal{J}x \in (L_p(\Gamma))^{r'}$, that is, $z(t) = (z^1,\dots,z^r)$, $t \in G$, with $z^j \in L_p(G)$, $j = 1,\dots,r$, and $\dot{z}(t) = (\dot{z}^1,\dots,\dot{z}^{r'})$, $t \in \Gamma$, with $\dot{z}^j \in L_p(\Gamma)$, $j = 1,\dots,r'$; (e) $(t,y(t)) \in A$ a.e. in G, and $(t,\dot{y}(t)) \in B$ σ-a.e. in Γ; (f) $u(t) \in U(t,y(t))$ a.e. in G, and $v(t) \in V(t,\dot{y}(t))$ σ-a.e. on Γ; (g) $z(t) = f(t,y(t),u(t))$ a.e. in G, and $\dot{z}(t) = g(t,\dot{y}(t),v(t))$ σ-a.e. on Γ. Requirements (e) and (f) are only an equivalent form for (7.5.5) and (7.5.6). Requirements (g) are only an equivalent form for abstract functional relations (7.5.3), (7.5.4).

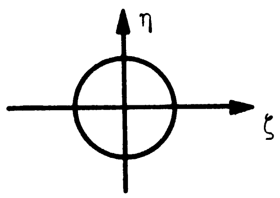
If x, u, v is an admissible triple, we shall often say that x is an (admissible) trajectory, that u is an (admissible) distributed control, and that v is an (admissible) boundary control.

Whenever we need to consider the funtional (7.5.1), we shall of course require, in addition to (a-g) that (h) $f_o(t,(\mathcal{M}x)(t),u(t)) \in L_1(G)$ and that $g_o(t,(Kx)(t),u(t)) \in L_1(\Gamma)$.

7.6 Examples

1. Let G be the unit circle $\zeta^2 + \eta^2 < 1$, and $\Gamma = \partial G$ the boundary of G. Thus, Γ is described in polar coordinates ρ, θ by $\rho = 1$, $0 \le \theta \le 2\pi$. Consider the problem of minimum of

$$I = \iint_G (x_\zeta^2 + x_\eta^2)\,d\zeta\,d\eta + \int_\Gamma x^2\,d\theta$$



with the constraint

$$x(\zeta,\eta) \ge (1-\zeta^2-\eta^2)^{1/2}, \quad (\zeta,\eta) \in G.$$

This problem can be written as the Bolza problem of the minimum of the functional

$$I = \iint_G (u^2+v^2)d\zeta d\eta + \int_\Gamma x^2 d\theta$$

with differential equations in G

$$x_\zeta = u, \quad x_\eta = v.$$

Here we have $f_o = u^2 + v^2$, $g_o = x^2$, $f = (f_1, f_2)$, $f_1 = u$, $f_2 = v$, and we may take $S = W_2^1(G)$, $Y = L_2(G)$, $Z = (L_2(G))^2$, $\dot{Y} = L_2(\Gamma)$, $\dot{Z} = \{o\}$, $\mathcal{L}x = (x_\zeta, x_\eta)$ $\mathcal{M}x = o$, $\mathcal{J}x = o$, $\mathcal{K}x = x|\Gamma$, $g = o$.

We know that $x_k \to x$ weakly in $W_2^1(G)$ implies that $x_k \to x$ strongly in $L_2(G)$, that $(x_k)_\zeta \to x_\zeta$, $(x_k)_\eta \to x_\eta$ weakly in $L_2(G)$, and that $x_k|\Gamma \to x|\Gamma$ strongly in $L_2(\Gamma)$. Thus, $x_k \to x$ weakly in S implies that $\mathcal{M}x_k \to \mathcal{M}x$ strongly in Y, that $\mathcal{L}x_k \to \mathcal{L}x$ weakly in Z, that $Kx_k \to Kx$ strongly in $\dot{Y}$, and trivially $\mathcal{J}x_k \to \mathcal{J}x$ since $\mathcal{J}x_k = \mathcal{J}x = o$. We have here $m = 2$, $s = 1$, $r = 2$, $s' = 1$, $r' = 1$, $U = E_2$, and trivially $V = \{o\}$.

Concerning the given constraint, we take for $A(\zeta,\eta)$ the set of all real numbers $y \geq (1-\zeta^2-\eta^2)^{1/2}$, $(\zeta,\eta) \in c\ell G$.

2. Let G be the unit circle $\zeta^2 + \eta^2 < 1$, let $\Gamma = \partial G$ so that Γ is described in polar coordinates ρ, θ by $\rho = 1$, $o \leq \theta \leq 2\pi$. Let us consider the problem of minimum of

$$I = \iint_G(\zeta^2 + \eta^2 + x^2 + x_\zeta^2 + x_\eta^2)d\zeta d\eta + \int_\Gamma(x_\theta^2 + x^2)d\theta$$

with differential equations

$$x_{\zeta\zeta} + x_{\eta\eta} = f(\zeta,\eta,x,x_\zeta,x_\eta,u) \text{ in } G,$$

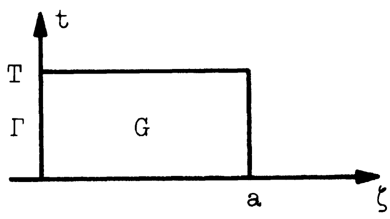
$$x_\theta = g(\theta,x,v) \text{ on } \Gamma.$$

Here we have $f_o = \zeta^2 + \eta^2 + x^2 + x_\zeta^2 + x_\eta^2$, $g_o = g^2(\theta,x,v)$, and f, g as indicated. We may take $S = W_2^2(G)$, $Y = (L_2(G))^3$, $Z = L_2(G)$, $\dot{Y} = L_2(\Gamma)$, $\dot{Z} = L_2(\Gamma)$. The controls are here u in G and v on Γ. We may take $\mathcal{L}x = x_{\zeta\zeta} + x_{\eta\eta}$, $\mathcal{M}x = (x, x_\zeta, x_\eta)$, $\mathcal{J}x = x_\theta = -x_\zeta \sin\theta + x_\eta \cos\theta$, $\mathcal{K}x = x$ on Γ.

We know that $x_k \to x$ weakly in $W_2^2(G)$ implies $x_k \to x$, $(x_k)_\zeta \to x_\zeta$, $(x_k)_\eta \to x_\eta$ strongly in $L_2(G)$, $(x_k)_{\zeta\zeta} \to x_{\zeta\zeta}$, $(x_k)_{\eta\eta} \to x_{\eta\eta}$ weakly in G, and $(x_k)_\zeta \to x_\zeta$, $(x_k)_\eta \to x_\eta$ strongly on Γ. Thus, $x_k \to x$ weakly implies $\mathcal{L}x_k \to \mathcal{L}x$ weakly in Z, $\mathcal{M}x_k \to \mathcal{M}x$ strongly in Y, $\mathcal{J}x_k \to \mathcal{J}x$ strongly in $\dot{Z}$, $\mathcal{K}x_k \to \mathcal{K}x$ strongly in $\dot{Y}$. Here we have m = 1, m' = 1, s = 3, s' = 1, r = 1, r' = 1. Condition (H') is satisfied.

3. Minimum of

$$I = \int_o^T \int_o^a [A_o(t,\zeta,x,u) + A_1(t,\zeta,x,u) \frac{\partial x}{\partial \zeta}]dtd\zeta$$



with the initial condition $x(o,\zeta) = \varphi(\zeta)$, the partial differential equation

$$\frac{\partial x}{\partial t} = B_o(t,\zeta,x,u) + B_1(t,\zeta,x,u) \frac{\partial x}{\partial \zeta}, \quad (t,\zeta) \in G,$$

and the side condition

$$\frac{\partial x(t,o)}{\partial \zeta} = C_o(t,x,w) + C_1(t,x)\frac{\partial x(t,o)}{\partial t}, \quad o \leq t \leq T,$$

where $G = [o \leq t \leq T,\ o \leq \zeta \leq a]$, $A_o, \ldots, C_1$ are given functions, and $u(t,\zeta)$, $w(t,\zeta)$ are controls.

It can be written as the problem of minimum of

$$I = \iint_G [A_o(t,\zeta,x,u) + A_1(t,\zeta,x,u)v]\,dt\,d\zeta$$

with data $x(o,\zeta) = \varphi(\zeta)$, and equations

$$x_\zeta = v \quad \text{in } G,$$

$$x_t = B_o(t,\zeta,x,u) + B_1(t,\zeta,x,u)v \quad \text{in } G,$$

$$x_\zeta - C_1(t)\,x_t = C_o(t,x,w) \quad \text{in } \Gamma,$$

where $\Gamma = [\zeta = o,\ o \leq t \leq a]$. We have here $f_o = A_o + A_1 v$, $g_o = o$, $f = (f_1, f_2)$, $f_1 = v$, $f_2 = B_o + B_1 v$, $g = C_o(t,x,w)$ on Γ. We may take

$$S = W_2^1(G), \quad Y = L_2(G), \quad Z = (L_2(G))^2,$$

$$\dot{Y} = L_2(\Gamma), \quad \dot{Z} = L_2(\Gamma),$$

$$\mathcal{L}x = (x_\zeta, x_t),\ \mathcal{M}x = x \text{ in } G, \text{ and } \mathcal{J}x = x_\zeta - \rho_1 x_t,\ \mathcal{K}x = x \text{ on } \Gamma.$$

We assume here that C_1 is a measurable bounded function on Γ. The control functions are here u, v in G, and w on Γ.

Thus, $x_k \to x$ weakly in $S = W_2^1(G)$ implies $\mathfrak{M}x_k \to \mathfrak{M}x$ strongly in Y, $\mathfrak{L}x_k \to \mathfrak{L}x$ weakly in Z, and $\mathfrak{K}x_k \to \mathfrak{K}x$ strongly on $\dot{Y}$, $\mathfrak{J}x_k \to \mathfrak{J}x$ weakly on Z. Here we have $m = 2$, $m' = 1$, $s = 1$, $s' = 1$, $r = 2$, $r' = 1$, $U = E_2$, $V = E_1$.

4. Let G and Γ as before. Consider the problem of minimum of

$$I = \iint_G (x_\zeta^2 + x_\eta^2) d\zeta d\eta + \int_\Gamma x^2 \, d\theta$$

with the constraint

$$x(\zeta,\eta) \geq (1 - \zeta^2 - \eta^2)^{1/2}, \quad (\zeta,\eta) \in G,$$

and

$$|dx/d\theta| \leq 1 \text{ on } \Gamma.$$

This problem can be written as the Bolza problem

$$I = \int_G (u^2 + v^2) d\zeta d\eta + \int_\Gamma x^2 d\theta$$

with differential equations

$$x_\zeta = u, \quad x_\eta = v \quad \text{in } G,$$

$$x_\theta = w \quad \text{on } \Gamma.$$

The (distributed) controls are u, v in G, and the (boundary) control is w on Γ, with control spaces

$$(u,v) \in U = E_2, \quad w \in V = [-1 \leq w \leq 1].$$

We have $f_o = u^2 + v^2$, $g_o = x^2$, $f = (f_1, f_2)$, $f_1 = u$, $f_2 = v$, $g = w$, and we take $S = W_2^1(G)$, $Y = L_2(G)$, $Z = (L_2(G))^2$, $\dot{Y} = L_2(\Gamma)$, $\dot{Z} = L_\infty(\Gamma)$, $\mathcal{L}x = (x_\zeta, x_\eta)$, $\mathcal{M}x = o$ in G, and $\mathcal{J}x = x_o$, $\mathcal{K}x = x$ on Γ. We have $m = 2$, $m' = 1$, $s = 1$, $s' = 1$, $r = 2$, $r' = 1$. We take for Ω the class of all admissible systems x, u, v, w with x Lipschitz on Γ of constant one. If $x_k \to x$ weakly in $S = W_2^1(G)$, then $\mathcal{L}x_k \to \mathcal{L}x$ weakly in Z, $\mathcal{M}x_k \to \mathcal{M}x$ strongly in Y (trivially), and $\mathcal{K}x_k \to \mathcal{K}x$ strongly in $\dot{Y}$. It remains to prove that for some subsequence $[k_s]$ we have $\mathcal{J}x_{k_s} \to \mathcal{J}x$ weakly in $L_\infty(\Gamma)$ as $k \to \infty$.

Note that the functions $(x_k)_\theta$ are uniformly bounded, hence there exist a subsequence which converges weakly in $L_\infty(\Gamma)$. Also, $x_k \to x$ strongly in $L_2(\Gamma)$, hence strongly in $L_1(\Gamma)$, in measure on Γ, and there is a subsequence which convergent pointwise a.e. on Γ. There is a subsequence $[x_{k_s}]$ with $(x_{k_s})_\theta \to w(\theta)$ weakly in Γ, $-1 \leq w(\theta) \leq 1$, and $x_{k_s} \to x$ pointwise a.e. on Γ. Thus, x is Lipschitz on Γ (relatively to the set of measure 2π where this pointwise convergence occurs). Also, for every $\varepsilon > 0$ there are N points $o < \theta_1 < \theta_2 < \ldots < \theta_N < 2\pi$ such that $x_{k_s} \to x$ at each of these points and $\theta_{i+1} - \theta_i < \varepsilon$, $i = 1,\ldots,N-1$, $\theta_1 + 2\pi - \theta_N \leq \varepsilon$. There is also some $\bar{s}$ such that $|x_{k_s}(\theta_i) - x(\theta_i)| \leq \varepsilon$ for all $s \geq \bar{s}$. For every $s \geq \bar{s}$ and $\theta_i \leq \theta \leq \theta_{i+1}$ we have now

$$|x_{k_s}(\theta) - x(\theta)| \leq |x_{k_s}(\theta) - x_{k_s}(\theta_i)| + |x_{k_s}(\theta_i) - x(\theta_i)|$$

$$+ |x(\theta_i) - x(\theta)| \leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.$$

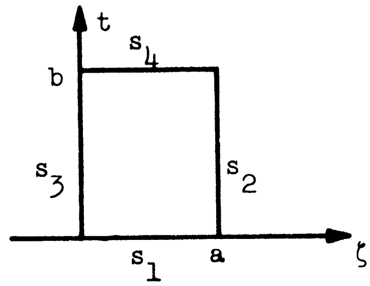
An analogous argument holds for $\theta_N \leq \theta \leq \theta_1 + 2\pi$. Thus $x_k \to x$ uniformly on Γ, and we may well assume that x is continuous and Lipschitz on Γ. From the weak convergence $(x_{k_s})_\theta \to w$ and for any two α, β in Γ we have now

$$x_{k_s}(\beta) - x_{k_s}(\alpha) = \int_\alpha^\beta x_{k_s}(\tau)d\tau \to \int_\alpha^\beta w(\tau)d\tau, \qquad \text{as } s \to \infty,$$

$$x_{k_s}(\alpha) \to x(\alpha),\ x_{k_s}(\beta) \to x(\beta), \qquad \text{as } s \to \infty,$$

and hence $x(\beta) - x(\alpha) = \int_\alpha^\beta w(\tau)d)$. Thus, $w = x_\theta$ a.e. on Γ, and $\mathcal{J}x_{k_s} \to \mathcal{J}x$ weakly on Γ.

5. Let G be the interval $G = [o < \zeta < a,\ o < t < b]$, and let $s_1 = [t = o,\ o \le \zeta \le a]$, $s_2 = [\zeta = a,\ o \le t \le b]$, $s_3 = [\zeta = o,\ o \le t \le b]$, $s_4 = [t = b,\ o \le \zeta \le a]$ be the side of G. Let us consider the problem of the minimum of



$$I = \int_o^a [h(\zeta) - x(\zeta,b)]^2 d\zeta$$

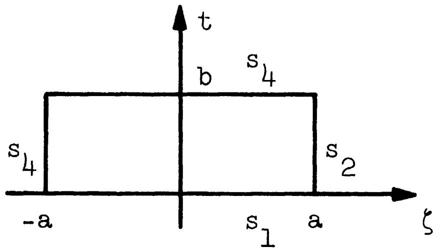
with differential equation

$$B_1(\zeta,t)x_t + B_2(\zeta,t)x_\zeta + x = u \text{ in } G$$

and given boundary data on s_1, s_2, s_3. Here B_1, B_2 are given measurable bounded functions on G, and h a given function on s_4. We have only the distributed control u with control space $U = [\alpha \le u \le \beta]$, α, β given positive constants. Here we have $f_o = o$, $g_o = h(\zeta) - x(\zeta,b)$ on s_4, $f = -x + u$ in G, $g = o$ on s_4.

We may take $S = W_2^1(G)$, $Z = L_2(G)$, $Y = L_2(G)$, $\dot{Z} = o$ on s_4, $\dot{Y} = L_2(s_4)$, $\mathcal{L} = B_1 x_t + B_2 x_\zeta$, $\mathfrak{M}x = x$ in G, and $\mathcal{J}x = o$, $\mathfrak{K}x = x$ on s_4. Then, $x_k \to x$ weakly in S implies $\mathcal{L}x_k \to \mathcal{L}x$ weakly in Z, $\mathfrak{M}x_k \to \mathfrak{M}x$ strongly in Y, $\mathcal{J}x_k \to \mathcal{J}x$ strongly in $\dot{Z}$ (trivial), $\mathfrak{K}x_k \to \mathfrak{K}x$ strongly in $\dot{Y}$.

6. Let G be the interval $[-a < \zeta < a,\ o < t < b]$, and let $s \ = [t = o,$ $-a \leq \zeta \leq a]$, $s_2 = [\zeta = a,\ o \leq t \leq b]$, $s_3 = [\zeta = -a,\ o \leq t \leq b]$, $s_4 = [t = b,\ -a \leq \zeta \leq a]$ be the side of G. Let us consider the problem of minimum of the functional



$$I = \int_{-o}^{a} [h(\zeta) - x(\zeta,b)]^2 d\zeta$$

with differential equations

$$x_t - Ax_{\zeta\zeta} = o \text{ on } G,$$

$$\lambda x_\zeta(a,t) = B_1[v_1(t) - x(a,t)] \text{ on } s_2$$

$$\lambda x_\zeta(-a,t) = B_2[v_2(t) - x(-a,t)] \text{ on } s_3$$

and initial data

$$x(\zeta,o) = x_o(\zeta) \text{ on } s_1.$$

Here a, b, A, B_1, and B_2 are possible constants, x_o is a given function on s_1, and h a given function on s_4. We have two control functions v_1 on s_2 and v_2 on s_3, with $v_1(t) \in V_1$, $v_2(t) \in V_2$, and control spaces $V_1 = [\alpha_1 \leq v_1 \leq \beta_1]$, $V_2 = [\alpha_2 \leq v_2 \leq \beta_2]$, α_1, β_1, α_2, β_2 given positive constants. Here we have $f_o = o$, $g_o = h(\zeta) - x(\zeta,b)$ on s_4, $f = o$, $g = B_1(v_1 - x)$ on s_2, $g = B_2(v_2 - x)$ on s_3.

We may take $S = W_2^2(G)$, $Z = L_1(G)$, $Y = o$, $\dot{Z} = L_2(s_2)$, $\dot{Y} = L_2(s_2)$ for s_2, $\dot{Z} = L_2(s_3)$, $\dot{Y} = L_2(s_3)$ for s_3, $\mathcal{L} = x_t - Ax_{\zeta\zeta}$, $\mathfrak{M}x = o$, $\mathfrak{J}x = \lambda x_\zeta$ on s_2 and s_3,

$\mathcal{K}x = x$ on s_2 and s_3. Then $x_k \to x$ weakly in S implies $\mathcal{L}x_k \to \mathcal{L}x$ weakly in Z, $\mathcal{M}x_k \to \mathcal{M}x$ strongly in Y, $\mathcal{J}x_k \to \mathcal{J}x$ strongly in $\dot{Z}$, $\mathcal{K}x_k \to \mathcal{K}x$ strongly in $\dot{Y}$.

7. Let G be the interval $G = [o < \zeta < u,\ o < \eta < b,\ o < t < T]$, and let us consider the problem of the minimum of

$$I = \int_o^T [h(t) - \tilde{y}(t)]^2 dt$$

with the differential equations

$$x_t = A_1 x_{\zeta\zeta} + B_1 x_\eta + C_1 y_t \text{ in } G,$$

$$y_t = A_2 y_{\zeta\zeta} + B_2 x_{\zeta\zeta} + C_2 y_n \text{ in } G,$$

and where

$$\tilde{y}(t) = a^{-1} \int_o^a y(\zeta, b, t) d\zeta \ .$$

The boundary data are

$$x(\zeta, o, t) = x_o(\zeta, t)$$

$$y(\zeta, o, t) = y_o(\zeta, t)$$

$$\lambda x_\zeta(a, y, t) = \alpha[v(y,t) - x(a,y,t)]$$

$$x_\zeta(o, y, t) = o$$

$$y_\zeta(a, y, t) = \alpha_1 x(a, y, t)$$

$$y_\zeta(o,y,t) = o$$

$$x(\zeta,\eta,o) = x_1(\zeta,\eta)$$

$$y(\zeta,\eta,o) = y_1(\zeta,\eta)$$

Here $A_1,\dots,C_1$ are given bounded measurable functions in G, and h, x_o, y_o, x_1, y_1 also are given functions, and a, b, t, α, α_1 are given positive constants. There is only one (boundary) control v on the face $\zeta = a$ of G, with given control space $V = [\alpha_2 \le v \le \beta_2]$, α_2, β_2 positive constants. We may take $S = (W_2^2(G))^2$, $\mathcal{L}(x,y) = (A_1\, x_{\zeta\zeta},\ A_2\, y_{\zeta\zeta} + B_2\, x_{\zeta\zeta})$, $Z = (L_2(G))^2$, $Y = (L_2(G))^6$, $\mathfrak{M}(x,y) = (x, x_\zeta, x_t, y, y_\zeta, y_t)$, $f = (f_1, f_2)$, $f_1 = x_t - B_1\, x_\eta - C_1\, y_t$, $f_2 = y_t - C_2\, y_\eta$. Thus $x_k \to x$, $y_k \to y$ weakly in S implies $\mathcal{L}(x_k,y_k) \to \mathcal{L}(x,y)$ weakly in Z, and $\mathfrak{M}(x_k,y_k) \to \mathfrak{M}(x,y)$ strongly in Y.

On the faces of G we have either fixed boundary data, or the differential equations

$$\lambda x_\zeta = \alpha(v-x) \quad \text{on } \zeta = a$$

$$x_\zeta = o \quad \text{on } \zeta = o$$

$$y_\zeta = \alpha_1\, x \quad \text{on } \zeta = a$$

$$y_\zeta = o \quad \text{on } \zeta = o$$

on the faces indicated. On these four sides we take, therefore, $\dot{Y} = L_2$, $\dot{Z} = L_2$, $\mathcal{J}x = \lambda x_\zeta$, $= x_\zeta$, $= y_\zeta$, $= y_\zeta$, and $g = \alpha(v-x)$, $= o$, $= \alpha_1\, x$, $= o$ respectively.

On each of these four faces we take $\mathcal{K}x = x$, and thus $x_k \to x$, $y_k \to y$ in S implies $(x_k, y_k) \to (x,y)$, $\mathcal{K}x_k \to \mathcal{K}x$ strongly in $\dot{Y}$ and $\dot{Z}$ respectively.

On the segment $\Gamma = [o \leq t \leq T]$ we have $g_o = (h-\tilde{y})^2$, where $\tilde{y} = \mathcal{K}y$, and h is a given function on Γ. Note that $\mathcal{K}$ is defined by the average process $\mathcal{K}y = a^{-1} \int_o^a y(\zeta,b,t)d\zeta$, and we may take $\dot{Y} = L_2(\Gamma)$, $\dot{Z} = o$, $g = o$, so that K: $S \to \dot{Y}$. As $x_k \to x$, $y_k \to y$ weakly in $S = (W_2^2(G))^2$, then $\mathcal{J}(x_\eta, y_\eta) \to \mathcal{J}(x,y)$ strongly in $\dot{Z}$, $y_k(\zeta,b,t) \to y(\zeta,b,t)$ strongly on the L_2-space on the face $\eta = b$, and finally $\mathcal{K}y_\eta \to \mathcal{K}y$ strongly in the space $\dot{Y} = L_2(\Gamma)$ corresponding to Γ. Hypothesis (H') is thus satisfied with the remark made in (   ).

8. Let G be the interval $G = [o < \zeta < a, o < t < b]$, and let us consider the problem of the minimum of

$$I = \int_o^a\int_o^b \exp(\alpha+\beta x(\zeta,t))d\zeta dt + R\int_o^b v(t)dt,$$

with differential equation

$$x_t + A(\zeta,t)x_\zeta = -\gamma \exp(\alpha+\beta x(\zeta,t)),$$

boundary data

$$x(o,t) = v(t), \quad o \leq t \leq b,$$

and fixed boundary data on the sides $\zeta = a$ and $t = o$. Here we have only one (boundary) control v on the side $\zeta = o$ of G, and α, β, γ, R are given positive constants, and A a given bounded measurable function. Here we have $f_o = \exp(\alpha+\beta x)$, $g_o = Rv$, $f = -\gamma \exp(\alpha+\beta x)$, $\mathcal{M}x = x$, $\mathcal{L}x = x_t + Ax_\zeta$, $S = W_2^1(G)$,