Technical Report No. 22

CONVEXITY OF THE RANGE OF CERTAIN INTEGRALS

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ORA Project 024160

submitted for:

UNITED STATES AIR FORCE
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
GRANT NO. AFSOR-69-1662
ARLINGTON, VIRGINIA

administered through:

OFFICE OF RESEARCH ADMINISTRATION ANN ARBOR

May 1971
ADDENDUM III. CONVEXITY OF THE RANGE OF CERTAIN INTEGRALS

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In this appendix we consider any vector function \( f(t) = (f_1, \ldots, f_n) \) whose components are \( L \)-integrable in \([a, b]\), and prove that the set function
\[ K(E) = \int_E f(E) dt \]
has for range a convex closed set when \( E \) describes all measurable subsets of \([a, b]\). This result, which is proved here rather elementarily through a set of lemmas, is actually a particular case of an analogous one concerning nonatomic vector valued measure functions and due to A. Lyapunov [76].

III 1. SOME PRELIMINARY LEMMAS

If \([a, b]\) is any given interval of length \( l = b - a \), and \( \alpha, 0 \leq \alpha \leq 1 \), any number, then the point \( t = a + \alpha(b - a) = a + \alpha l \) divides \([a, b]\) into two parts of measures \( \alpha l \) and \((1 - \alpha)l\). If we divide \([a, b]\) into two equal parts, and we divide each part as above, the corresponding set
\[ D^2_\alpha = [a \leq t < a + \alpha l/2] \cup [a + l/2 \leq t < a + l/2 + \alpha l/2] \]
has still measure \( \alpha l \), and is the union of two disjoint intervals. In general, if we divide \([a, b]\) into \( 2^k \) equal parts, and in each part we take corresponding subintervals, then the set
\[ D^k_\alpha = \bigcup_{i=1}^{2^k} \left[ a + 2^{-k}(i - 1)l, a + 2^{-k}(i - 1 + \alpha)l \right] \quad (III. 1.1) \]
has measure \( \alpha l \), and is the union of \( 2^k \) disjoint intervals. Also, for
\[ 0 \leq \alpha < \alpha' \leq 1 \text{ and the same } k \text{ we have } D^k_\alpha \subseteq D^k_{\alpha'} \text{ and meas } |D^k_{\alpha'} - D^k_\alpha| = (\alpha' - \alpha)l. \]

(III 1.1) Given any vector function \( f(t) = (f_1, \ldots, f_n) \), \( a \leq t \leq b \), whose components are \( L \)-integrable in \([a, b]\), and any \( \varepsilon > 0 \), there is an integer \( K \) such that for all \( k \geq K \) and \( \alpha \), \( 0 \leq \alpha \leq 1 \), we have

\[
\left| \int_{D^k_\alpha} f(t)dt - \alpha \int_{a}^{b} f(t)dt \right| \leq \varepsilon.
\]

In other words, if \( a^\circ_o = (a_1^{\circ_o}, \ldots, a_n^{\circ_o}) \) denotes the integral of \( f(t) \) on \([a, b]\), then the integral on \( D^k_\alpha \), thought of as a function of \( \alpha \), \( 0 \leq \alpha \leq 1 \), is uniformly approximated by the linear function \( a^\circ_o \alpha \), \( 0 \leq \alpha \leq 1 \).

**Proof.** It is not restrictive to assume \( a = 0 \), \( b = 1 \). We know that there is a continuous vector function \( g(t) \), \( 0 \leq t \leq 1 \), such that

\[
\int_{0}^{1} |f(t) - g(t)|dt \leq \varepsilon/4.
\]

Then \( g(t) \) is uniformly continuous in \([0, 1]\), and hence there is \( \delta > 0 \) such that \( t, t' \in [0, 1], |t - t'| \leq \delta \) implies \( |g(t) - g(t')| \leq \varepsilon/4 \). Let \( K \) be the smallest integer with \( 1/2^k < \delta \). For any \( k \geq K \) let \( g_k(t) \), \( 0 \leq t \leq 1 \), be the step function defined by \( g_k(t) = g(t_{i-1}) \) for all \( t_{i-1} \leq t \leq t_i, i = 1, \ldots, 2^k \), where \( t_i = i/2^k \). Then \( |g(t) - g_k(t)| \leq \varepsilon/4 \) for all \( 0 \leq t \leq 1 \). Thus

\[
\triangle = \int_{0}^{1} |f(t) - g_k(t)|dt \leq \int_{0}^{1} |f - g|dt + \int_{0}^{1} |g - g_k|dt \leq \varepsilon/4 + \varepsilon/4 = \varepsilon/2,
\]

and
\[
\int_{D_k}^k f(t) dt - \alpha f_0^1 f(t) dt \leq \int_{D_k}^k g(t) dt + \int_{D_k}^k g_k^1 g(t) dt - \alpha g_0^1 g_k^1 f(t) dt = s_1 + s_2 + s_3.
\]

Here

\[
s_2 = \sum_{i=1} g_k(t_i)(\alpha/2^k) - \alpha \sum_{i=1} g(t_i)(1/2^k) = 0,
\]

\[
|s_1| \leq \int_{D_k}^k |f - g_k| dt \leq \int_{D_k}^k |f - g| dt \leq \varepsilon/2,
\]

\[
|s_3| \leq \int_{D_k}^k |g_k - f| dt \leq \varepsilon/2,
\]

and finally \( \Delta \leq \varepsilon/2 + 0 + \varepsilon/2 = \varepsilon \). Statement (III 1.1) is thereby proved.

Statement (III 1.1) has a stronger form which of course is less easy to prove.

(III 1.ii) Given any vector function \( f(t) = (f_1, \ldots, f_n) \), \( a \leq t \leq b \), whose components are \( L \)-integrable in \([a, b]\), then for every \( \alpha, 0 \leq \alpha \leq 1 \), there is a measurable subset \( E_\alpha \) of \([a, b]\) such that

\[
\int_{E_\alpha} f(t) dt = \alpha f_a^b f(t) dt, \quad 0 \leq \alpha \leq 1.
\]  

(III. 1.2)

In other words, if \( f_\alpha = (a_1, \ldots, a_n) \) is the integral of \( f \) on \([a, b]\), the integral at the first number of (III 1.3) thought of as a function of \( \alpha \), is a linear function of \( \alpha \), say \( a_\alpha \), \( 0 \leq \alpha \leq 1 \).

This statement is a particular case of the following one which we shall prove below.
(III 1.iii) Given any vector function \( f(t) = (f_1, \ldots, f_n) \), \( a \leq t \leq b \), whose components are L-integrable functions in \([a, b]\), and any measurable subset \( A \) of \([a, b]\), then for every \( \alpha, 0 \leq \alpha \leq 1 \), there is a measurable subset \( B_\alpha \) of \( A \) with

\[
\int_{B_\alpha} f(t) \, dt = \alpha \int_A f(t) \, dt, \quad 0 \leq \alpha \leq 1.
\]  

(III. 1.3)

Proof. The proof of (III 1.iii) is made up of parts. (a). Let us prove (iii) for \( n = 1 \) and \( f \) a nonnegative scalar function. If \( \varphi(t) \) denotes the characteristic function of \( A \), say \( \varphi = 1 \) on \( A \) and \( \varphi = 0 \) otherwise, then

\( f(t) \varphi(t) \) is L-integrable in \([a, b]\), and hence \( F(t) = \int_a^t f(\tau) \varphi(\tau) \, d\tau \), \( a \leq t \leq b \), is a continuous function taking all values from \( F(a) = 0 \) to \( F(b) \).

Thus, there is some \( c, a \leq c \leq b \), with \( F(c) = \alpha F(b) \), and, if \( B_\alpha = [a, c] \cap A \), also

\[
\int_{B_\alpha} f \, dt = \int_a^c f \varphi \, dt = F(c) = \alpha F(b) = \alpha \int_a^b f \varphi \, dt = \alpha \int_A f \, dt.
\]

Thus, (III 1.iii) is proved for \( n = 1 \) and \( f \) scalar nonnegative.

(b) Let us assume that we know how to determine \( B_{1/2} \) for every \( A \) and a given vector function \( f = (f_1, \ldots, f_n) \) whose components are nonnegative L-integrable, and let us prove that we can determine all sets \( B_\alpha \), \( 0 \leq \alpha \leq 1 \), and that we can determine them in such a way that \( \alpha < \alpha' \) implies \( B_\alpha \subset B_{\alpha'} \).

For the sake of simplicity we shall use the notation

\[\mu(E) = \int_E f \, dt, \mu_i(E) = \int_E f_i \, dt, \quad i = 1, \ldots, n.\]  

(III. 1.4)

First, for \( B_{1/2}' = A - B_{1/2} \), we have
\[ \mu(B'_{1/2}) = \mu(A) - \mu(B_{1/2}^I) = \mu(A) - (1/2)\mu(A) = (1/2)\mu(A). \]

Then let us determine sets \( B_{1/4} \subset B_{1/2} \subset B_{3/4} \subset B_{1/2} \) such that

\[ \mu(B_{1/4}) = (1/2)\mu(B_{1/2}), \quad \mu(B_{3/4}') = (1/2)\mu(B_{1/2}') \]

and then, for \( B_{3/4} = B_{1/2} \cup B_{3/4}' \), also

\[ \mu(B_{1/4}) = (1/4)\mu(A), \quad \mu(B_{3/4}') = (3/4)\mu(A), \]

and if \( B_0 = \emptyset, B_1 = A \), we have \( B_0 \subset B_{1/4} \subset B_{1/2} \subset B_{3/4} \subset B_1 \).

By repeating this process we obtain sets \( B_{i/2^r}, i = 0, 1, \ldots, 2^r, r = 1, 2, \ldots, \)
so that \( \mu(B_{i/2^r}) = i/2^r \), and for \( i < j, \lambda = i/2^r, \lambda' = j/2^r \), also \( B_\lambda \subset B_{\lambda'} \).

Now let \( \alpha \) be any number \( 0 < \alpha < 1 \), and let \( [\lambda_s], [\lambda'_s] \) be sequences of numbers
\( \lambda_s = i/2^r, \lambda'_s = j/2^s \), such that \( \lambda_s < \lambda_{s+1} < \alpha < \lambda'_{s+1} < \lambda'_s, \lambda \rightarrow \alpha, \lambda' \rightarrow \alpha \). For

\[ B_{\lambda \alpha} = \bigcup_{\lambda_s} B_{\lambda_s}, \quad B'_{\lambda \alpha} = \bigcap_{\lambda'_s} B_{\lambda'_s} \]

we have \( B_{\lambda \alpha} \subset B'_{\alpha} \) and

\[ \lambda_s \int_A f \, dt = \int_{B_{\lambda \alpha}} f \, dt \leq \int_{B_{\lambda \alpha}} f \, dt \leq \int_{B'_{\lambda \alpha}} f \, dt \leq \int_{B'_{\alpha}} f \, dt = \lambda'_s \int_A f \, dt, \]

where \( \leq \) means that such a relation holds for each component. As \( s \rightarrow \infty \) we obtain

\[ \alpha = \int_{B_{\lambda \alpha}} f \, dt = \int_{B'_{\lambda \alpha}} f \, dt, \quad 0 \leq \alpha \leq 1. \]

This proves (b).
(c) Assume that (iii) has been proved for some vector function $f = (f_1, \ldots, f_n)$ whose components are nonnegative $L$-integrable. Let $E$, $F$ be any two measurable subsets of $A \subset [a, b]$. Then for every $\alpha$, $0 \leq \alpha \leq 1$, there is some subset $C(\alpha)$ of $E \cup B$ with $C(0) = E$, $C(1) = F$, such that

$$\int_{C(\alpha)} f \, dt = (1 - \alpha) \int_{E} f \, dt + \alpha \int_{F} f \, dt, \quad 0 \leq \alpha \leq 1,$$

$$\left| \int_{C(\alpha)} f \, dt - \int_{C(\alpha')} f \, dt \right| \leq |\alpha - \alpha'| \left( \int_{E-F} f_1 \, dt + \int_{F-E} f_1 \, dt \right) \quad (III.1.5)$$

$$0 \leq \alpha, \alpha' \leq 1, \ i = 1, \ldots, n.$$

Indeed, let us apply (iii) to the sets $E-F$ and $F-E$, and the number $\alpha$. Let $B_\alpha \subset E-F$, $B'_\alpha \subset F-E$ be the corresponding sets and take

$$C(\alpha) = (E \cap F) \cup (E-F-B_\alpha) \cup B'_\alpha.$$

Then

$$\mu(C(\alpha)) = \mu(E \cap F) + \mu(E - F) - \mu(B_\alpha) + \mu(B'_\alpha)$$

$$= \mu(E \cap F) + \mu(E - F) - \alpha \mu(E - F) + \alpha \mu(F - E)$$

$$= \mu(E \cap F) + (1 - \alpha)\mu(E - F) + \alpha \mu(F - E)$$

$$= (1 - \alpha)\mu[(E \cap F) \cup (E - F)] + \alpha \mu[(E \cap F) \cup (F - E)]$$

$$= (1 - \alpha)\mu(E) + \alpha \mu(F).$$

In addition, for each component $f_i$ and $0 \leq \alpha, \alpha' \leq 1$, we have
\[|\mu_i(C(\alpha)) - \mu_i(C(\alpha'))| = |(-\alpha + \alpha')\mu(E - F) + (\alpha - \alpha')\mu(F - E)| \leq |\alpha - \alpha'| (\mu(E - F) + \mu(F - E)), \ i = 1, \ldots, n.\]

Thus (c) is proved.

(d) Statement (iii) has been proved for \(n = 1\) and \(f\) scalar nonnegative. Assume that (iii) has been proved for \(n = 1\) and vectors \(f\) with nonnegative components, and let us prove it for \(n\). Let \(\tilde{f}\) be the \((n - 1)\) -vector
\[\tilde{f} = (f_1', \ldots, f_{n-2}', \tilde{f}_{n-1})\] with \(\tilde{f}_{n-1} = f_{n-1} + f_n\), and let \(\tilde{\mu}, \mu_1, \mu_i\) be the set functions defined by (III.1.4) with \(f\) replaced by \(\tilde{f}, f_1', \tilde{f}_i\). First by (iii) with \(\alpha = 1/2\) applied to \(\tilde{f}\) there is a subdivision of \(A\) into two parts \(E, F\), with \(E \cap F = \emptyset, E \cup F = A\), and
\[\tilde{\mu}(E) = \tilde{\mu}(F) = (1/2)\tilde{\mu}(A).\] (III 1.6)

Also, by force of (c), there are sets \(C(\alpha) \subset E \cup F = A, 0 \leq \alpha \leq 1,\) with
\(C(0) = E, C(1) = F,\) and
\[\tilde{\mu}(C(\alpha)) = (1 - \alpha)\tilde{\mu}(E) + \alpha \tilde{\mu}(F) = (1 - \alpha)(1/2)\tilde{\mu}(A) + \alpha(1/2)\tilde{\mu}(A)\]
\[= (1/2)\tilde{\mu}(A), \quad 0 \leq \alpha \leq 1.\] (III 1.7)

Let us prove that \(\mu_{n-1}(C(\alpha))\) is a continuous function of \(\alpha\) in \([0, 1]\).
Indeed, \(\mu_{n-1}(C(\alpha))\) is a scalar, namely the integral of \(f_{n-1} \geq 0\) over \(C(\alpha),\) and
\[ |\mu_{n-1}(C(\alpha)) - \mu_{n-1}(C(\alpha'))| = |\mu_{n-1}[C(\alpha) - C(\alpha')] - \mu_{n-1}[C(\alpha') - C(\alpha)]| \]

\[ \leq |\mu_{n-1}[C(\alpha) - C(\alpha')] + \mu_{n-1}[C(\alpha') - C(\alpha)]| \]

\[ \leq |\tilde{\mu}_{n-1}[C(\alpha) - C(\alpha')] + \tilde{\mu}_{n-1}[C(\alpha') - C(\alpha)]| \]

\[ \leq |\alpha - \alpha'| \left( \tilde{\mu}_{n-1}(E - F) + \tilde{\mu}_{n-1}(F - E) \right) \leq 2|\alpha - \alpha'| \int_A (f_{n-1} + f_n) \, dt. \]

This proves that \( \mu_{n-1}(C(\alpha)) \) is a continuous function of \( \alpha \) for \( 0 \leq \alpha \leq 1 \). On the other hand, \( \mu_{n-1}(C(0)) = \mu_{n-1}(E), \mu_{n-1}(C(1)) = \mu_{n-1}(F) \). Since \( E \) and \( F \) are complementary in \( A \) then \( \mu_{n-1}(E) \leq (1/2)\mu_{n-1}(A) \) according as

\[ \mu_{n-1}(F) \geq (1/2)\mu_{n-1}(A). \]

Thus, as \( \alpha \) describes \( [0, 1] \), \( \mu(C(\alpha)) \) describes an interval which contains \( (1/2)\mu_{n-1}(A) \). We conclude that there is some \( \alpha \), \( 0 \leq \alpha \leq 1 \), such that \( \mu_{n-1}(C(\alpha)) = (1/2)\mu_{n-1}(A) \). For this particular value of \( \alpha \), we have from (III 1.7)

\[ \int_{C(\alpha)} f_i \, dt = (1/2) \int_A f_i \, dt, \quad i = 1, \ldots, n-2, \]

\[ \int_{C(\alpha)} (f_{n-1} + f_n) \, dt = (1/2) \int_A (f_{n-1} + f_n) \, dt, \]

\[ \int_{C(\alpha)} f_{n-1} \, dt = (1/2) \int_A f_{n-1} \, dt, \]

and hence, by difference, also

\[ \int_{C(\alpha)} f \, dt = (1/2) \int_A f \, dt, \]

or

\[ \int_{C(\alpha)} f \, dt = (1/2) \int_A f \, dt. \] (III 1.8)
We have proved that for the n-vector \( f = (f_1, \ldots, f_n) \) we can determine a subset \( B_{1/2} = C(\alpha) \subset A \) satisfying (III 1.6), where \( A \) is any measurable subset of \([a, b]\). Thus, by (b), we can determine analogous sets \( B_\alpha \) for all \( \alpha \), \( 0 \leq \alpha \leq 1 \), and (III 1.iii) is proved for vector valued functions with non-negative components.

(e) We have now to prove (III 1.iii) for vector functions \( f = (f_1, \ldots, f_n) \) with L-integrable components of arbitrary signs. For every \( i = 1, \ldots, n \), and \( j = 1, 2 \), we consider the sets \( A_{ij} \) where \( f_i \geq 0 \) and \( A_{ij} \) where \( f_i < 0 \). We divide \([a, b]\) into \( 2^n \) disjoint measurable sets

\[
A_r = A_{i_1 j_1} \cap A_{i_2 j_2} \cap \cdots \cap A_{i_n j_n},
\]

where \( r \) denotes any one of the \( 2^n \) systems \((j_1, j_2, \ldots, j_n)\) of indices 1 and 2. On each set \( A_r \) the components \( f_i \) have constant signs, and there are, therefore, sets \( B_{r\alpha} \subset A_r \) with \( \int_{B_{r\alpha}} f \, dt = \alpha \int_A f \, dt \), \( 0 \leq \alpha \leq 1 \). The sets \( B_{r\alpha} = \bigcup_{r\alpha} B_{r\alpha} \) then satisfy the requirements of (III 1.iii). Statement (III 1.iii) is thereby proved.

(III 1.iv) Given any vector function \( f(t) = (f_1, \ldots, f_n) \), \( a \leq t \leq b \), whose components are L-integrable in \([a, b]\), and any two fixed measurable sets \( E, F \subset [a, b] \), then for every \( \alpha \), \( 0 \leq \alpha \leq 1 \), there is some set \( C(\alpha) \subset E \cup F \), with \( C(0) = E \), \( C(1) = F \), and

\[
\int_{C(\alpha)} f \, dt = (1 - \alpha) \int_E f \, dt + \alpha \int_F f \, dt.
\]

This statement is a consequence of parts (b) and (c) of the proof of (III 1.iii).
III 2. THE MAIN STATEMENTS

(III 2.1) Given any vector function \( f(t) = (f_1, \ldots, f_n) \), \( a \leq t \leq b \), whose components are L-integrable, and any measurable subset \( A \) of \( [a, b] \), then

\[
\mu(E) = \int_E f(t) dt \quad \text{(III 2.1)}
\]

describes a convex set \( H \) as \( E \) describes all possible measurable subsets \( E \) of \( A \) (in other words, the range of \( \mu(E) \) is convex).

**Proof.** If \( \mu_1, \mu_2 \in H \), then there are measurable sets \( E_1, E_2 \) in \( A \) such that \( \mu_i = \mu(E_i) = \int_{E_i} f \, dt \), \( i = 1, 2 \). Among all measurable subsets of \( A \) there certainly are the sets \( C(\alpha), \ 0 \leq \alpha \leq 1 \), defined in (III 1.iv). Then

\[
\mu(C(\alpha)) = (1 - \alpha) \int_E f \, dt + \alpha \int_F f \, dt = (1 - \alpha)\mu_1 + \alpha\mu_2,
\]

that is, all points of the segment \( (1 - \alpha)\mu_1 + \alpha\mu_2 \), \( 0 \leq \alpha \leq 1 \), belong to \( H \), and \( H \) is proved to be convex.

(III 2.1i) Given any two vector functions \( f(t) = (f_1, \ldots, f_n) \), \( g(t) = (g_1, \ldots, g_n) \), \( a \leq t \leq b \), whose components are L-integrable, let \( E \) denote any measurable subset of \( [a, b] \) and \( h_E(t) \), \( a \leq t \leq b \), the function \( h_E(t) = f(t) \) for \( t \in E \), \( h_E(t) = g(t) \) for \( t \in F = [a, b] - E \). Then

\[
\mu(E) = \int_a^b h_E(t) dt \quad \text{(III 2.2)}
\]

describes a convex subset \( H \) of the space \( E_n \) as \( E \) describes all measurable subsets of \( [a, b] \).
Proof. For every $E$ as above and $F = [a, b] - E$, we have
\[
\mu(E) = \int_a^b h_E dt = \int_E f dt + \int_{F_E} g dt = \int_E (f - g) dt + \int_{a}^{b} g dt.
\]
If $\mu_0$ is the fixed value of the last integral, and we apply (III 2.i) to the function $f-g$, we see that the set $H$ of (III 2.ii) is simply a translation of the convex set $H$ of (III 2.1) relative to $f-g$.

(III 2.iii) The set $H$ of statement (III 2.1) is closed.

Proof of (III 2.iii) for $n = 1$. We have $\mu(E) = \int_E f dt$ where $f$ is a scalar. If $f \geq 0$ the statement is trivial since the values taken by $\mu(E)$ fill the closed segment $[0, \mu(A)]$. Otherwise, let $A^+, A^-$ be the subsets of all $t \in A$ where $f \geq 0$, or $f \leq 0$, and then $A^+ \cap A^- = \emptyset$, $A^+ \cup A^- = A$. For every set $E \subset A$, let $E^+ = E \cap A^+$, $E^- = E \cap A^-$, and then $\mu(E) = \mu(E^+) + \mu(E^-)$. Then $\mu(E)$ takes on its maximum value $\mu^+ = \mu(A^+) \geq 0$ for $E = A^+$, and its minimum value $\mu^- = \mu(A^-) \leq 0$ for $E = A^-$, and the values taken by $\mu(E)$ fill the segment $[\mu^-_E, \mu^+]$. We shall prove (III 2.iii) for $n > 1$ below.

(III 2.iv) If $H$ is the convex set of (III 2.1) and $\Pi: p \cdot x - c = 0$ any supporting plane for $H$ with $\Pi \cap \text{cl} \ H \neq 0$, then $\Pi \cap H \neq 0$, that is, there is some $\xi \in H$ with $p \cdot \xi - c = 0$, and some measurable set $E_0 \subset A$ with $\xi = \int_{E_0} f dt$.

Proof. We may assume $p \cdot x - c \geq 0$ for all $x \in H$, and hence also for all $x \in \text{cl} \ H$. Thus, $(\text{cl} \ H) \cap \Pi \neq 0$ implies $c = \text{Inf} (p \cdot x)$ where Inf is taken for all $x \in H$, that is, $c = \text{Inf} p \cdot \int_E f dt = \text{Inf} \int_E (p \cdot f) dt$, where Inf is taken for all measurable subsets $E$ of $A$. On the other hand $\nu(E) = \int_E (p \cdot f) dt$ is our usual function $\mu$ relative to the scalar function
\[ g(t) = p \cdot f(t), \; t \in A. \] By (III 2.iii), \( \nu(E) \) takes on its maximum and minimum values. Thus, there is some measurable set \( E_0 \subset A \) with \( C = \nu(E_0) = \int_{E_0} (p \cdot f) dt = p \cdot \int_{E_0} f dt \), that is, \( p \cdot \xi - c = 0 \) for \( \xi = \int_{E_0} f dt \).

Proof of (III 2.iii) for \( n > 1 \). We have proved (III 2.iii) for \( n = 1 \). Let us assume that (III 2.iii) has been proved for \( 1, 2, \ldots, n - 1 \), and let us prove it for \( n \). Let \( f(t) = (f_1, \ldots, f_n), \; t \in A \), \( A \) measurable, and let \( H \) be the range of the function \( \mu(E) = \int_E f dt \) as \( E \) describes all measurable subset \( E \) of \( A \). By (III 2.1) we know that \( H \) is convex, and we have to prove that \( H \) is closed. Suppose that this is not true, so that \( \text{cl} H \cap H \neq \emptyset \), and let \( \zeta \) be a point \( \zeta \in \text{cl} H - H \). Then \( \zeta \in \text{bd} H \), and by (Vol. I, App. C2) there is a supporting hyperplane \( \Pi: \; p \cdot x - c = 0 \) through \( \zeta \), thus \( p \cdot x - c \geq 0 \) for all \( x \in H \), and \( p \cdot \xi = c \).

By (III 2.iv) there is a point \( \xi' \in H \) on \( \Pi \), that is, \( p \cdot \xi' - c = 0 \), and since \( \xi' \in H \), there is a measurable subset \( E_0 \) of \( A \) with \( \xi' = \int_{E_0} f dt \). Thus

\[ c = p \cdot \xi' = p \cdot \int_{E_0} f dt = \int_{E_0} (p \cdot f) dt. \]

Then for every measurable set \( E \subset A \) we have

\[ \nu(E) = p \cdot \mu(E) - c = p \cdot \int_E f dt - p \cdot \int_{E_0} (p \cdot f) dt = \int_{E_0} p \cdot f dt + \int_{E_0} (-p \cdot f) dt. \]

If \( g(t), \; t \in A \) denotes the scalar \( g(t) = p \cdot f(t) \) for \( t \in A \), then

\[ g(t) = - p \cdot f(t) \]

for \( t \in E_0 \), then

\[ \nu(E) = \int_{(E_0 - E)}^{(E_0 - E)} g(t) dt \geq 0 \]

for every measurable subset \( E \subset A \). This implies that \( g(t) \geq 0 \) almost everywhere.
where in A.

Let $A_o [A_\delta, \delta > 0]$, be the set of all $t \in A$ with $g(t) \leq 0 [g(t) \leq \delta]$. Then all sets $A, A_\delta$ are measurable, $A_o \subset A_\delta \subset A_o$, $A_\delta - A_o \rightarrow 0$, meas

$(A_\delta - A_o) \rightarrow 0$ as $\delta \rightarrow 0 + 0$. Let $\mu', \mu''$ be the functions $\mu'(E) = \mu(E - A_o) = \int_A f \mu o \partial, \mu''(E) = \mu(E \cap A_o) = \int_A f \mu o \partial$, both defined for all measurable subsets $E-A_\partial$ of $E \cap A_o$ of $E$. Let $H'$, $H''$ be the ranges of $\mu'$ and $\mu''$. Since $p \cdot \mu''(E) = \int_A (p \cdot f) dt = 0$, we see that the range $H''$ of $\mu''$ is contained in the hyperplane $\Pi$: $p \cdot x = 0$. By a change of coordinates we could, therefore, represent $H''$ by means of an $(n-1)$-vector function, that is, as the range of the values $\int_A f(h) dt, h = (h', \ldots, h^{n-1})$, for an L-integrable $(n-1)$-vector function $g$. By the induction argument, $H''$ is therefore a convex closed subset.

Let us prove that $(*) \nu(E_s) \rightarrow 0$ implies $\mu'(E_s) \rightarrow 0$ for any sequence $[E_s]$ of measurable subsets of A. Let $E' = E_s - A_\delta$, and let us assume that this statement is not true. Then there is some $m > 0$ and a sequence, say still $[E_s]$, with $|\mu'(E_s)| \leq m > 0$, and hence also $|\mu(E')| = |\mu(E_s - A_\delta)| = |\mu'(E_s)| \leq m > 0$. Since $f$ is L-integrable in $A$, there is some $\sigma > 0$ such that $\int_A |f| dt < m/2$ on every measurable subset $E$ of $A$ of measure $\leq \sigma$. This implies that meas $E_s \geq \sigma$ for every $s$, since otherwise $|\mu(E')| = \int_A f dt < \int_A f dt < m/2$, a contradiction. Finally, if we take $\delta > 0$ so that meas $(A_\delta - A_o) < \sigma/2$, we see that the set $E'' = E_s - A_\delta = E_s - (A_\delta - A_o)$ has measure $\geq \sigma/2$, and hence $p \cdot f \geq \delta$ everywhere in $E''$, and $\nu(E_s) = \nu(E') \geq \nu(E'') \geq \delta \sigma/2$, while $\nu(E_s) \rightarrow 0$, a contradiction. We have proved $(*)$.

Since $\xi \in \text{bd } H$, there is a sequence $[\xi_s]$ with $\xi_s \in H, \xi_s \rightarrow \xi$, and hence
a sequence of sets \( B_s \subset A \) with \( \mu(B_s) = \xi_s, s = 1, 2, \ldots \). Then \( \nu(B_s) = p \cdot \mu(B_s) - c = p \cdot \xi_s - c + p \cdot \xi - c = 0 \). By force of (*) we have then \( \mu'(B_s) \to 0 \) as \( s \to \infty \), or \( \mu(B_s - A_o) \to 0 \). On the other hand \( \mu''(B_s) = \mu(B_s \cap B_o) = \mu(B_s) - \mu(B_s - A_o) \to \xi \) as \( s \to \infty \). This proves that \( \xi \in \text{cl } H'' \).

Since \( H'' \) is closed we have \( \xi \in H'' \), that is, there is some set \( E_1 \subset A \) with \( \xi = \mu''(E_1) = \mu(E_1 \cap A_o) \). This proves that there is some measurable subset \( E = E_1 \cap A_o \subset A_o \subset A \) with \( \mu(E) = \xi \), that is, \( \xi \in H \). We have proved that \( H \) is closed, and thus, by induction argument, (III 2.iii) is proved for every \( n \).
