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EXISTENCE IN THE LARGE OF PERIODIC SOLUTIONS OF HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

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EXISTENCE IN THE LARGE OF PERIODIC SOLUTIONS OF HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

by

Lamberto Cesari *

The problem of existence of solutions $\phi(x,y)$ periodic in x and in y of period T for an hyperbolic partial differential system of the form

$$u_{xy} = f(x,y,u,u_x,u_y), \tag{1}$$

where $u = (u_1,\dots,u_n)$, and $f = (f_1,\dots,f_n)$ is periodic in x and y of period T, presents a number of difficulties when no damping of any sort is assumed. In this paper we analyze this difficult problem in the line of our previous work on ordinary and partial differential equations. We conclude with criteria of existence for solutions to the problem above. These criteria can then be used for the analogous problem for the equation

$$u_{xx} - u_{yy} = g(x,y,u,u_x,u_y). \tag{2}$$

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1. THE MODIFIED PROBLEM

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We shall first associate to (1), the analogous weaker problem, or modified problem:

Given two periodic functions $u_o(x)$, $v_o(y)$ of class C^1 in $(-\infty,+\infty)$ and of period T,

$$u_o(x+T) = u_o(x), \qquad v_o(x+T) = v_o(x),$$

determine a function $\Phi(x,y)$ continuous with its partial derivatives Φ_x,Φ_y,Φ_{xy}, two functions $m(y)$ and $n(x)$ both continuous, and a constant μ, such that

$$\Phi(x+T,y) = \Phi(x,y) = \Phi(x,y+T), \qquad m(x+T) = m(x), \qquad n(y+T) = n(y),$$

$$\int_0^T m(y)dy = 0, \qquad \int_0^T n(x)dx = 0, \tag{3}$$

and

$$\Phi_{xy} = f(x,y,\Phi,\Phi_x,\phi_y) - m(y) - n(x) - \mu. \tag{4}$$

For this modified problem we shall prove theorems of existence, uniqueness, and continuous dependence on the boundary values and parameters. In (4) we assume

$$f(x+T,y,z,p,q) = f(x,y,z,p,q) = f(x,y+T,z,p,q).$$

Then the function ϕ is a periodic solution of the original problem (1) if and only if we can determine $u_o(x)$, $v_o(y)$ in such a way that

$$\mu = 0, \qquad m(y) \equiv 0, \qquad n(x) \equiv 0.$$

Criteria for this occurrence are given in Section 2.

2. THEOREM I (existence theorem for the modified problem)

If $T > 0$, and $N,N_1,N_2,L,M,b_1,b_2,M_1,M_2,M_3 \geq 0$ are constants, if A and R are the sets

$A = [0 \leq x \leq T, \quad 0 \leq y \leq T], \qquad R = [0 \leq x \leq T,\ 0 \leq y \leq T, \quad |z| \leq M_1,$

$|p| \leq M_2, \quad |q| \leq M_3],$

if

$$M_1 \geq N + (N_1+N_2)T/2 + 3L\,T^2, \qquad M_2 \geq N_1 + 3L\,T, \qquad M_3 \geq N_2 + 3L\,T, \tag{5}$$

if

$$u_o(x), \qquad 0 \leq x \leq T, \qquad v_o(y), \qquad 0 \leq y \leq T,$$

are vector functions which are continuous with $u_o'(x)$, $v_o'(y)$, if

$$f(x,y,z,p,q), \qquad (x,y,z,p,q)\epsilon R,$$

is continuous in R, and

$$u_o(T) = u_o(o), \qquad v_o(T) = v_o(o) = 0, \qquad u_o'(T) = u_o'(o), \qquad v_o'(T) = v_o'(o), \tag{6}$$

$$|u_o(o)| \leq N, \qquad |u_o(x_1)-u_o(x_2)| \leq N_1|x_1-x_2|, \qquad |v_o(y_1)-v_o(y_2)| \leq N_2|y_1-y_2|, \tag{7}$$

$$f(T,y,z,p,q) = f(o,y,z,p,q), \qquad f(x,T,z,p,q) = f(x,o,z,p,q), \tag{8}$$

$$|f(x,y,z,p,q)| \leq L, \quad |f(x,y,z,p_1,q_1)-f(x,y,z,p_2,q_2)| \leq b_1|p_1-p_2|+b_2|q_1-q_2| \tag{9}$$

then for

$$2T\,b_1 < 1, \qquad 2T\,b_2 < 1, \tag{10}$$

there exist a vector function $\phi(x,y)$, $(x,y)\epsilon A$, continuous in A together with $\phi_x, \phi_y, \phi_{xy}$, continuous vector functions $m(y), o \leq y \leq T, n(x), o \leq x \leq T$, and a constant μ, such that

$$\Phi(x,o) = \Phi(x,T) = u_o(x), \qquad \Phi_y(x,o) = \Phi_y(x,T), \tag{11}$$

$$\Phi(o,y) = \Phi(T,y) = u_o(o) + v(y), \qquad \phi_x(o,y) = \phi_x(T,y), \tag{12}$$

$$m(o) = m(T), \qquad n(o) = n(T), \tag{13}$$

$$\phi_{xy}(x,y) = f(x,y,\phi(x,y), \phi_x(x,y), \phi_y(x,y)) - m(y) - n(x) - \mu, \tag{14}$$

$$\int_o^T m(\eta)d\eta \;=\; 0 \qquad \int_o^T n(\xi)d\xi \;=\; 0, \tag{15}$$

$$\mu \;=\; T^{-2} \int_o^T \int_o^T f(\xi,\eta,\phi(\xi,\eta),\ \phi_x(\xi,\eta),\ \phi_y(\xi,\eta))d\xi\ d\eta, \tag{16}$$

$$m(y) \;=\; T^{-1} \int_o^T f(\xi,y,\phi(\xi,y),\phi_x(\xi,y),\ \phi_y(\xi,y))d\xi \;-\; \mu, \tag{17}$$

$$n(x) \;=\; T^{-1} \int_o^T f(x,\eta,\phi(x,\eta),\phi_x(x,\eta),\phi_y(x,\eta))d\eta \;-\; \mu, \tag{18}$$

for all $o \leq x \leq T$, $o \leq y \leq T$. Thus, by extending all functions $\phi(x,y)$, $m(y)$, $n(x)$, $f(x,y,z,p,q)$ for all $-\infty < x,y < +\infty$, $|z|\leq M_1$, $|p|\leq M_2$, $|q|\leq M_3$, by means of the periodicity of period T in x and y, Equation (14) is satisfied in the whole xy-plane.

3. PROOF OF THEOREM I

First let us note that relations (11), (12), (14), (15), imply (16), (17), (18). Indeed, by integration of (14) on A, we deduce (16). Then, by integration of (14) again on $o \leq x \leq T$, or on $o \leq y \leq T$, we deduce (17) and (18) respectively. Note that (8), (11), (12), (17), (18) imply (13), and that (14), (16), (17), (18) imply (15).

Let us first prove that every vector function $\phi(x,y)$, $(x,y)\epsilon A$, satisfying

$$\begin{aligned} &\phi(x,o) \;=\; \phi(x,T) \;=\; u_o(x), \qquad \phi(o,y) \;=\; \phi(T,y) \;=\; u_o(o) + v_o(y), \\ &|\phi(x_1,y_1) - \phi(x_1,y_2) = \phi(x_2,y_1) + \phi(x_2,y_2)| \leq 6L|x_1-x_2|\,|y_1-y_2|, \end{aligned} \tag{19}$$

also satisfies the relations

$$\begin{aligned} &|\phi(x,y)|\leq M_1, \qquad |\phi(x_1,y) - \phi(x_2,y)|\leq M_2|x_1-x_2|, \\ &|o(x,y_1) - o(x,y_2)\leq M_3|y_1-y_2|. \end{aligned} \tag{20}$$

Indeed, we have, for $o \leq x \leq T$, $o \leq y \leq T$,

$$|\phi(x_1,y) - \phi(x_2,y) - \phi(x_1,o) + \phi(x_2,o)| \leq 6L|x_1-x_2|y,$$

where

$$|\phi(x_1,o) - \phi(x_2,o)| \;=\; |u_o(x_1)-u_o(x_2)| \;\leq N_1|x_1-x_2|,$$

and hence

$$|\phi(x_1,y) - \phi(x_2,y)| \leq (N_1+6Ly)|x_1-x_2|$$

Analogously, we have

$$|\phi(x_1,y) = \phi(x_2,y)| \leq (N_1+6L(T-y))|x_1-x_2|.$$

Since, either $o \leq y \leq T/2$, or $o \leq T-y \leq T/2$, we have

$$|\phi(x_1,y) = \phi(x_2,y)| \leq (N_1+3LT)|x_1-x_2| \leq M_2|x_1-x_2|.$$

Analogously, we prove that

$$|\phi(x,y_1) = \phi(x,y_2)| \leq (N_2+3LT)|y_1-y_2| \leq M_3|y_1-y_2|.$$

Finally, for $0 \leq x,y \leq T/2$,

$$|\phi(x,y)| \leq |\phi(o,o)| + |\phi(o,y) - \phi(o,o)| + |\phi(x,y) - \phi(o,y)| \leq N + N_2\, y + (N_1+6Ly)x$$

$$\leq N + (N_1+N_2)T/2 + 3LT^2 \leq M_1.$$

Analogous reasoning holds for (x,y) in the remaining quaadrants of A. Thus $|\phi(x,y)| \leq M_1$, (x,y) A. We have proved that relations (19) imply (20). Also, the vector functions $\phi(x,y)$ satisfying (20) are all Lipschitzian in A, and hence have partial derivatives ϕ_x, ϕ_y a.e. in A satisfying $|\phi_x| \leq M_2$, $|\phi_y| \leq M_3$ a.e. in A.

The vector function $f(x,y,z,p,q)$ is continuous in R. Hence, there are continuous monotone functions $\omega_1(\alpha)$, $\omega_2(\beta)$, $\omega_3(\gamma)$ in $[o, +\infty)$ such that $\omega_1(o) = \omega_2(o) = \omega_3(o) = 0$, and

$$|f(x_1,y,z,p,q) - f(x_2,y,z,p,q)| \leq \omega_1(|x_1-x_2|),$$

$$|f(x,y_1,z,p,q) - f(x,y_2,z,p,q)| \leq \omega_2(|y_1-y_2|), \qquad (21)$$

$$|f(x,y,z_1,p,q) - f(x,y,z_2,p,q)| \leq \omega_3(|z_1-z_2|),$$

for all

$$o \leq x,x_1,x_2,y,y_1,y_2 \leq T, \qquad |z|,|z_1|,|z_2| \leq M_1, \qquad |p| \leq M_2, \qquad |q| \leq M_3.$$

The vector functions $u_o'(x)$, $v_o'(y)$ are continuous in $[o,T]$. Hence, there are continuous monotone functions $\omega_4(\alpha)$, $\omega_5(\beta)$, $o \leq \alpha,\beta < +\infty$, with $\omega_4(o) = \omega_5(o) = o$, such that

$$|u_o'(x_1) - u_o'(x_2)| \leq \omega_4(|x_1-x_2|), \qquad |v_o'(y_1) - v_o'(y_2)| \leq \omega_5(|y_1-y_2|). \qquad (22)$$

Let

$$\eta_1(\beta) = (1-2T\ b_2)^{-1}[\omega_5(\beta)+2T\ \omega_2(\beta)+2T\ \omega_3(M_3\beta)+12LT\ b_1\beta], \tag{23}$$

$$\eta_2(\alpha) = (1-2T\ b_1)^{-1}[\omega_4(\alpha)+2T\ \omega_1(\alpha)+2T\ \omega_3(M_2\alpha)+12LT\ b_2\alpha]. \tag{24}$$

Both $\eta_1(\beta)$ and $\eta_2(\alpha)$, $o \leq \alpha,\beta < +\infty$, are continuous monotone functions with $\eta_1(o) = \eta_2(o) = 0$.

Let E be the linear space of all vector functions $\phi(x,y)$, $(x,y)\epsilon A$, continuous in A together with their partial derivatives ϕ_x, ϕ_y with norm $\|\phi\| = \max|\phi| + \max|\phi_x| + \max|\phi_y|$ where max is taken in A.

Let K be the subset of E made up of all vector functions $o(x,y)\epsilon E$ satisfying relations (19) and in addition

$$\phi_x(o,y) = \phi_x(T,y), \quad \phi_y(x,o) = \phi_y(x,T), \quad |\phi_x(x_1,y) - \phi_x(x_2,y)| \leq \eta_2(|x_1-x_2|), \quad |\phi_x(x,y_1) - \phi_x(x,y_2)| \leq 6L|y_1-y_2|, \quad |\phi_y(x_1,y) - \phi_y(x_2,y)| \leq 6L|y_1-y_2|, \quad |\phi_y(x,y_1) - \phi_y(x,y_2)| \leq \eta_1)|y_1-y_2|). \tag{25}$$

Then the vector functions $\phi\epsilon K$ satisfy relations (20) and then $|\phi| \leq M_1$, $|\phi_x| \leq M_2$, $|\phi_y| \leq M_3$ everywhere in A. As a consequence the vector function

$$F(x,y) = f(x,y,\ \phi(x,y),\ \phi_x(x,y),\ \phi_y(x,y)),\ (x,y)\epsilon A, \tag{26}$$

is defined everywhere in A and is continuous in A.

For $\phi\epsilon K$ the vector functions m(y) and n(x) defined by (17) and (18) are continuous in [o,T]. With μ defined by (16), the vector function

$$\psi(x,y) = u_o(x) + v_o(y) + \int_o^x \int_o^y [F(\xi,\eta) - m(\eta) - n(\xi) - \mu]\ d\xi\ d\eta \tag{27}$$

is continuous in A together with its partial derivatives

$$\psi_x(x,y) = u_o'(x) + \int_o^y [F(x,\eta) - m(\eta) - n(x) - \mu]d\eta, \tag{28}$$

$$\psi_y(x,y) = v_o'(y) + \int_o^x [F(\xi,y) - m(y) - n(\xi) - \mu]d\xi. \tag{29}$$

Thus, relations (16), (17), (18), (26), and (27) define a map $\mathcal{C} : \psi = \mathcal{C}\phi$, or $\mathcal{T}: K \to E$. Let us prove that actually $\mathcal{T}: K \to K$.

Since $|f| \leq L$, by (16), (17), (18) we deduce

$$|\mu| \leq L, \qquad |m(y)|,\ |n(x)| \leq 2L,$$

On the other hand, by (6), (8), (25), (26), (27), (28), (29), we have with the usual convections

$$F(o,y) = F(T,y), \qquad F(x,o) = F(T,o), \qquad m(o) = m(T), \qquad n(o) = n(T),$$

$$\int_o^T m(\eta)d\eta = 0, \qquad \int_o^T n(\xi)d\xi = 0. \tag{30}$$

$$\psi(x,o) = \psi(x,T) = u_o(x), \qquad \psi_y(x,o) = \psi_y(x,T), \qquad \psi(o,y) = \psi(T,y)$$

$$= u_o(o) + v_o(y), \qquad \psi_x(o,y) = \psi_x(T,y), \tag{31}$$

$$|\psi(x_1,y_1) - \psi(x_1,y_2) - \psi(x_2,y_1) + \psi(x_2,y_2)|$$

$$= |\int_{x_1}^{x_2}\int_{y_1}^{y_2} [F(\xi,\eta) - m(\eta) - n(\xi) - \mu]d\xi\, d\eta| \leq 6L|x_1-x_2|\,|y_1-y_2|. \tag{32}$$

In other words $\psi = \mathcal{T}\phi$ for $\phi \epsilon K$ satisfies reactions (19) and, hence, relations (20) as proved in No. 3. Also, we have

$$|\psi_x(x,y_1) - \psi_x(x,y_2)| = |\int_{y_1}^{y_2} [F(x,\eta) - m(\eta) - u(x) - \mu]d\eta,$$

$$|\psi_y(x_1,y) - \psi_y(x_2,y)| = |\int_{x_1}^{x_2} [F(\xi,y) - m(y) - n(\xi) - \mu]d\xi,$$

and hence

$$|\psi_x(x,y_1) - \psi_x(x,y_2)| \leq 6L|y_1-y_2|, \qquad |\psi_y(x_1,y) - \psi_y(x_2,y)| \leq 6L|x_1-x_2|. \tag{33}$$

We have further, from (17),

$$|m(y_1) - m(y_2)| = |T^{-1}\int_0^T [f(\xi,y_1,\phi(\xi,y_1),\phi_x(\xi,y_1),\phi_y(\xi,y_1))$$

$$- f(\xi,y_2,\phi(\xi,y_2),\phi_x(\xi,y_2), \phi_y(\xi,y_2))]d\xi| \leq T^{-1}\int_0^T [\omega_2(|y_1-y_2|) + \omega_3(|\phi(\xi,y_1)-\phi(\xi,y_2)|)$$

$$+ b_1|\phi_x(\xi,y_1) - \phi_x(\xi,y_2)| + b_2|\phi_y(\xi,y_1) - \phi_y(\xi,y_2)|]d\xi$$

$$\leq \omega_2(|y_1-y_2|) + \omega_3(M_3|y_1-y_2|) + 6L\, b_1|y_1-y_2| + b_2\, \eta_1(|y_1-y_2|),$$

and analogously

$$|n(x_1) - n(x_2)| \leq \omega_1(|x_1-x_2|) + \omega_3(M_2|x_1-x_2|) + b_1\, \eta_2(|x_1-x_2|) + 6L\, b_2|x_1-x_2|.$$

We have now, from (29)

$$|\psi_y(x,y_1) - \psi_y(x,y_2)| = |v_o'(y_1)-v_o'(y_2) + \int_0^x [f(\xi,y_1,\phi(\xi,y_1),\phi_x(\xi,y_1),\phi_y(\xi,y_1))$$

$$- m(y_1) - f(\xi,y_2,\phi(\xi,y_2),\phi_x(\xi,y_2),\phi_y(\xi,y_2)) + m(y_2)]d\xi| \leq \omega_5(|y_1-y_2|)$$

$$+ \int_0^T [\omega_2(|y_1-y_2|) + \omega_3(|\phi(\xi,y_1)-\phi(\xi,y_2)|) + b_1|\phi_x(\xi,y_1)-\phi_x(\xi,y_2)| + b_2|\phi_y(\xi,y_1)$$

$$- \phi_y(\xi,y_2)| + |m(y_1)-m(y_2)|]d\xi \leq \omega_5(|y_1-y_2|) + 2T\, \omega_2(|y_1-y_2|) + 2T\, \omega_3(M_3|y_1-y_2|)$$

$$+ 12LT\, b_1|y_1-y_2| + 2T\, b_2\, \eta_1(|y_1-y_2|) = (1-2T\, b_2)\eta_1(|y_1-y_2|) + 2T\, b_2\, \eta_1(|y_1-y_2|)$$

$$= \eta_1(|y_1-y_2|). \qquad (34)$$

Analogously, we have

$$|\psi_x(x_1,y) - \psi_x(x_2,y)| \leq \eta_2(|x_1-x_2|). \qquad (35)$$

Relations (31), (32), (33), (34) show that $\psi = \mathcal{T}\phi$ for $\phi \in K$ satisfies all relations (19) and (20). Thus $\psi \in K$, and $\mathcal{T}: K \to K$.

The transformation $\mathcal{T}: K \to K$, $K \subset E$, is continuous in K with respect to the norm $\|\phi\|$ of E. Indeed, for two vector functions $\phi_j \in K$, $j = 1,2$, we have $\psi_j = \mathcal{T}\phi_i$, $F_j = F_{\phi_j}$, $m_j = m_{\phi_j}(y)$, $n_j = n_{\phi_j}(x)$, $\mu_j = \mu_{\phi_j}$, $j = 1,2$, and from (17),

$$|\mu_1-\mu_2| = |T^{-2}\int_0^T\int_0^T f(\xi,\eta,\phi_1(\xi,\eta),\phi_{1x}(\xi,\eta),\phi_{1y}(\xi,\eta))$$

$$- f(\xi,\eta,\phi_2(\xi,\eta),\phi_{2x}(\xi,\eta),\phi_{2y}(\xi,\eta))]d\xi\, d\eta| \leq [\omega_3\|\phi_1-\phi_2\|+b_1\|\phi_1-\phi_2\|+b_2\|\phi_1-\phi_2\|].$$

Then from (18) we have

$$|m_1(y)-m_2(y)| = |T^{-1}\int_0^T [f(x,y,\phi_1(x,y),\phi_{1x}(x,y),\phi_{1y}(x,y))$$

$$- f(x,y,\phi_2(x,y),\phi_{2x}(x,y),\phi_{2y}(x,y))]dx + \mu_1 - \mu_2|$$

$$\leq 2[\omega_3(\|\phi_1-\phi_2\|)+b_1\|\phi_1-\phi_2\|+b_2\|\phi_1-\phi_2\|]$$

and analogously from (19),

$$|n_1(x)-n_2(x)| \leq 2[\omega_3(\|\phi_1-\phi_2\|)+b_1\|\phi_1-\phi_2\|+b_2\|\phi_1-\phi_2\|].$$

We reduce from (27)

$$|\psi_1(x,y)-\psi_2(x,y)| = |\int_0^x \int_0^y [F_1(\xi,\eta)-m_1(\eta)-n_1(\xi)-\mu_1-F_2(\xi,\eta)+m_2(\eta)+n_2(\xi)$$

$$+ \mu_2]\, d\xi\, d\eta| \leq 6T^2[\omega_3(\|\phi_1-\phi_2\|)+(b_1+b_2)\|\phi_1-\phi_2\|]$$

Analogously, from (28) and (29),

$$|\psi_{1x}(x,y)-\psi_{2x}(x,y)| \leq 6T[\omega_3(\|\phi_1-\phi_2\|)+(b_1+b_2)\|\phi_1-\phi_2\|],$$

$$|\psi_{1y}(x,y)-\psi_{2y}(x,y)| \leq 6T[\omega_3(\|\phi_1-\phi_2\|)+(b_1+b_2)\|\phi_1-\phi_2\|].$$

Thus $\|\psi_1-\psi_2\| \to 0$ as $\|\phi_1-\phi_2\| \to 0$ uniformly in K. Finally, the set K is obviously convex, closed and compact with respect to the norm $\|\phi\|$ of E. By Schauder's fixed point theorem we conclude that there is an element $\phi(x,y)\epsilon K$ such that $\phi = \mathcal{C}\,\phi$, or

$$\phi(x,y) = u_o(x)+v_o(y)+\int_0^x \int_0^y [f(\xi,\eta,\phi(\xi,\eta),\phi_x(\xi,\eta),\phi_y(\xi,\eta))-m(\eta)-n(\xi)-\mu]d\xi\, d\eta,$$

$$m(y) = T^{-1}\int_0^T f(\xi,y,\phi(\xi,y),\phi_x(\xi,y),\phi_y(\xi,y))d\xi - \mu,$$

$$n(x) = T^{-1}\int_0^T f(x,\eta,\phi(x,\eta),\phi_x(x,\eta),\phi_y(x,\eta))d\eta - \mu,$$

$$\mu = T^{-2}\int_0^T \int_0^T f(\xi,\eta,\phi(\xi,\eta),\phi_x(\xi,\eta),\phi_y(\xi,\eta))d\xi\, d\eta,$$

for all $o \leq x,y \leq T$. Obviously ϕ_x, ϕ_y, ϕ_{xy} exist everywhere in A, are continuous in A, and, everywhere in A, we have

$$\phi_{xy} = f(x,y,\phi,\phi_x,\phi_y) - m(y) - n(x) - \mu .$$

Theorem I is thereby proved.

4. REMARK 1

If f is Lipschitzian with respect to all variables x,y,z,p,q in R, and if $u_o'(x)$, $v_o'(y)$ also are Lipschitzian, then m(y), n(x), as well as ϕ, ϕ_x, ϕ_y, ϕ_{xy} are Lipschitzian. Indeed, if $\omega_1(\alpha) = k_1\alpha$, $\omega_2(\beta) = k_2\alpha$, $\omega_3(\gamma) = b_o\gamma$, $\omega_4(\alpha) = k_4\alpha$, $\omega_5(\beta) = k_5\beta$, then

$$\eta_1(\beta) = (1-2T\ b_2)^{-1}(k_5+2T\ k_2+2T\ b_o\ M_3+12LT\ b_1)\beta = k_6\beta,$$

$$\eta_2(\alpha) = (1-2T\ b_1)^{-1}(k_4+2T\ k_1+2T\ b_o\ M_2+12LT\ b_2)\alpha = k_7\alpha,$$

and then

$$|m(y_1)-m(y\beta)| \leq (k_2+b_oM_3+6L\ b_1+b_2k_6)\beta = k_8\beta ,$$

$$|n(x_1)-n(x_2)| \leq (k_1+b_oM_2+b_1k_7+6L\ b_2)\alpha = k_9\alpha .$$

Formulas (33), (34), (35) show that ϕ_x, ϕ_y are also uniformly Lipschitzian and so is $\phi_{xy} = f-m-n-\mu$.

5. REMARK 2

The conditions of Theorem I do not assure uniqueness, as the following example shows. Take T = 1, $u_o(x) = 0$, $v_o(y) = 0$, $f = |z|^{1/2}\sin 2\pi\ x \sin 2\pi y$, for $o \leq x,y \leq 1$, and all y,z,p,q. Then the equation

$$u_{xy} = |u|^{1/2} \sin 2\pi x \sin 2\pi y \qquad (36)$$

besides the trivial solution $\phi_1(x,y) = 0$, has also the solution $\phi_2(x,y) = (16\pi^4)^{-1} \sin^4\pi x \sin^4\pi y$, $o \leq x,y \leq 1$, and both satisfy the boundary conditions. We have here $m_1(y) = m_2(y) = 0$, $n_1(x) = n_2(x) = 0$. Note that we may take $N = N_1 = N_2 = 0$, $L = 1$, $M_1 = 1$, $M_2 = M_3 = 2$, $b_1 = b_2 = 0$. All conditions of Theorem I are satisfied.

6. THE LIPSCHITZIAN CASE

We shall assume now that $\omega_3(\gamma) = b_o|\gamma|$, so that f is now Lipschitzian in z,y,q with constants b_o, b_1, b_2. In this situation, for given boundary values $u_o(x)$, $v_o(x)$ and different functions ϕ_1, $\phi_2 \epsilon K$ we have

$|\mu_1-\mu_2| \leq (b_o+b_1+b_2)\|\phi_1-\phi_2\|$, $|m_1(y)-m_2(y)|$, $|n_1(x)-n_2(x)| \leq 2(b_o+b_1+b_2)\|\phi_1-\phi_2\|$,

$|\psi_1(x,y)-\psi_2(x,y)| \leq 6T^2(b_o+b_1+b_2)\|\phi_1-\phi_2\|$, $|\psi_{1x}(x,y)-\psi_{2x}(x,y)|$, $|\psi_{1y}(x,y)-\psi_{2y}(x,y)|$

$\leq 6T(b_o+b_1+b_2)\|\phi_1-\phi_2\|$.

Thus

$$\|\psi_1-\psi_2\| = \|\mathcal{T}\phi_1-\mathcal{T}\phi_2\| \leq 6T(T+2)(b_o+b_1+b_2)\|\phi_1-\phi_2\|.$$

If we assume now that

$$6T(T+2)\ (b_o+b_1+b_2) < 1, \qquad (37)$$

then $\mathcal{T}$: $K \to K$ is a contraction into. This remark yields:

7. THEOREM II (uniqueness)

Under the same hypotheses of Theorem I, if $\omega_3(\gamma) = b_o|\gamma|$, and (37) holds, then $\mathcal{T}$: $K \to K$ is a contraction and problem (11-18) has one and only one solution.

The boundary values are represented by the pair of functions $w = (u_o(x), v_o(y))$ of class C^1 and satisfying (6) and (7). Therefore, they form a subset $\mathcal{B}$ of the linear space of all w of class C^1 satisfying (6) only, and we take in this linear space the norm

$$\|w\| = \max|u_o(x)| + \max|u_o'(x)| + \max|v_o(y)| + \max|v_o'(y)|. \qquad (38)$$

The solution of the problem (11-18) is actually the system $W = [\phi(x,y), m(y), n(x), \mu]$. These quadruples also can be thought of as inbedded in a linear space on which we take the norm

$$\|W\| = \max|\phi| + \max|\phi_x| + \max|\phi_y| + \max|m| + \max|n| + |\mu|. \qquad (39)$$

We shall prove that the solution, or system W, is a continuous function $\mathcal{F}$ of the boundary values, or system w, and we write

$$W = \mathcal{F} w, \qquad w \in \mathcal{B}.$$

We shall need the numbers

$$\Delta = (1-6T\ b_1)(1-6T\ b_2) - 36T^2 b_1 b_2, \qquad k = 6\ T^2\ b_o + 72\ T^3\ b_o(b_1+b_2) \qquad (40)$$

8. THEOREM III (continuous dependence upon the boundary values)

Under the conditions of Theorem II, if in addition $\Delta > 0$, and $0 < k < 1$, then the solution $W = (\phi, m, n, \mu)$ of problem (11-18) is a continuous function $\mathcal{F}$ of the boundary values $w = (u_o, v_o) \in \mathcal{B}$ in the topology determined by the norms (38) and (39).

9. PROOF OF THEOREM III

Let $w_1 = [u_{o1}(x), v_{o1}(y)]$, $w_2 = [u_{o2}(x), v_{o2}(y)]$ be a pair of boundary values as in Theorems I and II, and let $W_1 = [\phi_1, m_1, n_1, \mu_1]$, $W_2 = [\phi_2, m_2, n_2, \mu_2]$ be the corresponding solutions. Let

$$\epsilon = \|w_1 - w_2\| = \max|u_{o1}(x) - u_{o2}(x)| + \max|u_{o1}'(x) - u_{o2}'(x)| + \max|v_{o1}(y) - v_{o2}(y)|$$

$$+ \max|v_{o1}'(y) - v_{o2}'(y)|,$$

$$\alpha = \max|\phi_1(x,y) - \phi_2(x,y)|, \qquad \beta = \max|\phi_{1x}(x,y) - \phi_{2x}(x,y)|,$$

$$\gamma = \max|\phi_{1y}(x,y) - \phi_{2y}(x,y)|, \qquad \delta = \max|m_1(y) - m_2(y)|,$$

$$\delta' = \max|n_1(x) - n_2(x)|, \qquad \delta'' = |\mu_1 - \mu_2|.$$

We shall denote by F_1 and F_2 the functions F relative to ϕ_1 and ϕ_2. Then we have

$$\delta'' = |\mu_1 - \mu_2| = \left|T^{-2}\int_0^T\int_0^T [F_1(x,y) - F_2(x,y)]dx\,dy\right| \le b_o\alpha + b_1\beta + b_2\gamma,$$

$$|m_1(y) - m_2(y)| = \left|T^{-1}\int_0^T [F_1(x,y) - F_2(x,y)]dx - \mu_1 + \mu_2\right| \le b_o\alpha + b_1\beta + b_2\gamma + \delta'',$$

$$|n_1(x) - n_2(x)| = \left|T^{-1}\int_0^T [F_1(x,y) - F_2(x,y)]dy - \mu_1 + \mu_2\right| \le b_o\alpha + b_1\beta + b_2\gamma + \delta'',$$

Hence

$$\delta'' \le b_o\alpha + b_1\beta + b_2\gamma, \qquad \delta, \delta' \le b_o\alpha + b_1\beta + b_2\gamma + \delta''. \tag{41}$$

Analogously,

$|\phi_1(x,y)-\phi_2(x,y)| =$

$$|u_{o1}(x)+v_{o1}(y)-u_{o2}(x)-v_{o2}(y) + \int_o^x\int_o^y [F_1(x,y)-m_1(y)-n_1(x)-\mu_1-F_2(x,y)+m_2(y)$$

$+n_2(x)+\mu_2]dx\,dy| \leq \epsilon + T^2(b_o\alpha+b_1\beta+b_2\gamma+\delta+\delta'+\delta'')$, $|\phi_{1x}(x,y)-\phi_{2x}(x,y)|$

$\leq \epsilon + T(b_o\alpha+b_1\beta+b_2\gamma+\delta+\delta'+\delta'')$, $|\phi_{1y}(x,y)-\phi_{2y}(x,y)| \leq \epsilon + T(b_o\alpha+b_1\beta+b_2\gamma+\delta+\delta'+\delta'')$,

and hence

$$\alpha \leq \epsilon + T^2(b_o\alpha+b_1\beta+b_2\gamma+\delta P\delta'+\delta''), \qquad \beta,\gamma \leq \epsilon + T(b_o\alpha+b_1\beta+b_2\gamma+\delta+\delta'+\delta''). \qquad (42)$$

Relations (41), (42) yield

$$\delta'' \leq (b_o\alpha+b_1\beta+b_2\gamma), \quad \delta,\delta' \leq 2(b_o\alpha+b_1\beta+b_2\gamma), \quad \alpha \leq \epsilon + 6T^2(b_o\alpha+b_1\beta+b_2\gamma),$$
$$\beta,\gamma \leq \epsilon + 6T(b_o\alpha+b_1\beta+b_2\gamma). \qquad (43)$$

The last relation can be written in the form

$$\beta = 6T\,b_o\xi_1\alpha + 6Tb_1\xi_1\beta + 6Tb_2\xi_1\gamma + \epsilon, \qquad \gamma = 6T\,b_o\xi_2\alpha + 6Tb_1\xi_2\beta + 6Tb_2\xi_2\gamma + \epsilon$$

where $o \leq \xi_1, \xi_2 \leq 1$ are convenient numbers, and then

$$(1-6Tb_1\xi_1)\beta - 6Tb_2\xi_1\gamma = 6Tb_o\xi_1\alpha + \epsilon\xi_1, - 6Tb_2\xi_2\beta + (1-6Tb_2\xi_2)\gamma = 6Tb_o\xi_2\alpha+\epsilon\xi_2$$

If Δ' is the determinant of this system we have $\Delta' \geq \Delta > o$, $o \quad 6Tb_j < 1$, $j = 1,2$, and

$$\beta = \Delta'^{-1}\{(1-6Tb_2\xi_2)(6Tb_o\xi_1\alpha+\epsilon\xi_1)+(6Tb_2\xi_1)(6Tb_o\xi_2\alpha+\epsilon\xi_2)\} \leq 2\Delta^{-1}(\epsilon+6Tb_o\alpha).$$

Analogously, we have

$$\gamma \leq 2\Delta^{-1}(\epsilon+6Tb_o\alpha).$$

Finally, by (43)

$$\alpha \leq \epsilon + 36T^3\,M_o\Delta^{-1}(b_1+b_2)\alpha + 6T^2M_o\alpha.$$

Since the number k defined in (40) is $o < k < 1$, we have

$$\alpha \leq (1-k)^{-1}(1+12\Delta^{-1}T^2(b_1+b_2)\epsilon.$$

This proves that $\alpha,\beta,\gamma,\delta,\delta',\delta'' \to 0$ as $\epsilon \to 0$ uniformly in K. Theorem III is thereby proved.

10. A METHOD OF SUCCESSIVE APPROXIMATIONS

Under the hypotheses of Theorem III, the usual method of successive approximations defined by $\phi_{k+1} = T\phi_k, k = 0,1...$, converges toward the solution ϕ of problem (11-18), where ϕ_0 is an arbitrary element of K. It may be convenient to use as first approximation

$$\Phi_0 = u_0(x) + v_0(y).$$

Then

$$F_0(x,y) = f(x,y,u_0(x)+v_0(y), \quad u_0'(x), v_0'(y)), \ \mu_0 = T^{-2}\int_0^T\int_0^T F_0(\xi,\eta)d\xi\, d\eta,$$

$$m_0(y) = T^{-1}\int_0^T F_0(\xi,y)d\xi, \ u_0(x) = T^{-1}\int_0^T F_0(x,\eta)d\eta,$$

$$\phi_1(x,y) = u_0(x)+v_0(y) + \int_0^x\int_0^y [F_0(\xi,\eta)-m_0(\eta)-n_0(\xi)-\mu_0]d\xi\, d\eta,$$

and successively,

$$F_k(x,y) = f(x,y,\phi_k(x,y),\phi_{kx}(x,y)+\phi_{ky}(x,y)),$$

$$\mu = T^{-2}\int_0^T\int_0^T F_k(\xi,\eta)d\xi\, d\eta, \quad m_k(y) = T^{-1}\int_0^T F_k(\xi,y)d\xi,$$

$$n_k(x) = T^{-1}\int_0^T F_k(x,\eta)d\eta, \quad \phi_{k+1}(x,y) = u_0(x)+v_0(y) \tag{45}$$

$$+\int_0^T\int_0^T [F_k(\xi,\eta)-m_k(\eta)-n_k(\xi)-\mu_k]d\xi\, d\eta, \qquad k = 1,2,\ldots .$$

Then we have $\phi_k \to \phi$, $\phi_{kx} \to \phi_x$, $\phi_{ky} \to \phi_y$, $m_k \to m$, $n_k \to n$, $\mu_k \to \mu$ uniformly for $o \le x \le T$, $o \le y \le T$, and consequently we have also $\phi_{kxy} \to \phi_{xy}$ as $k \to \infty$ uniformly.

11. SMOOTHNESS OF THE SOLUTION

Two theorems can now be stated concerning the smoothness of the solution (ϕ,m,n,μ) of the problem (11-18).

(α) Under the conditions of theorems I, II, III, if $f(x,y,z,p,q)$ is of class C^1 in R and $u_0(x)$, $v_0(y)$ of class C^2, then $m(y)$, $n(x)$ are of class C^1

and $\phi(x,y)$ of class C^2.

This statement was essentially proved in Section 12 of [2a]. A more precise statement is as follows:

(β) Under the conditions of Theorems I, II, III, if $f(x,y,z,p,q)$ is of class C^1 with Lipschitzian first order partial derivatives, if $u_o(x)$, $v_o(y)$ are of class C^2 with Lipschitzian second order derivatives, then $m(y)$, $n(x)$ are of class C^1 with Lipschitzian first derivatives and $\phi(x,y)$ of class C^2 also with Lipschitzian second order partial derivatives.

The proof is the same as for (α). An analogous statement holds:

(γ) Under the conditions of Theorems I, II, III, if $f(x,y,z,p,q)$ is of class C^{1+r} in R with Lipschitzian partial derivatives of the order 1+r, if $u_o(x)$, $v_o(y)$ are of class C^{2+r} with lipschitzian derivatives of the order 2+r, then $m(y)$, $n(x)$ are of class C^{1+r} with Lipschitzian derivatives of order 1+r, and $\phi(x,y)$ is of class C^{2+r} with Lipschitzian partial derivatives of the order 2+r.

2. CRITERIA FOR THE EXISTENCE OF PERIODIC SOLUTIONS IN THE LARGE OF THE ORIGINAL PROBLEM

12. A DIFFERENTIAL EQUATION CONTAINING A SMALL PARAMETER

Let us consider the differential equation

$$u_{xy} = f(x,y,u,u_x,u_y), \qquad f = \epsilon[\psi(x,y)+Cu+\psi_1(y)u_x+\psi_2(x)u_y] + \epsilon^2 g(x,y,u,u_x,u_y), \tag{46}$$

where ϵ is a small parameter, and ψ, ψ_1, ψ_2, g are periodic functions of period T in x and y. The Fourier series of ψ, ψ_1, ψ_2 will be denoted by

$$\psi(x,y) \sim \sum_{m,n=0}^{\infty} (a_{mn} \cos m\omega x \cos n\omega y + b_{mn} \cos m\omega x \sin n\omega y + c_{mn} \sin m\omega x \cos n\omega y$$
$$+ d_{mn} \sin m\omega x \sin n\omega y),$$

$$\psi_1(y) \sim e_o + \sum_1^{\infty} (e_n \cos n\omega y + f_n \sin n\omega y),$$

$$\psi_2(x) \sim g_o + \sum_1^{\infty} (g_n \cos n\omega w + \lambda_n \sin n\omega x).$$

If $u_o(x)$, $v_o(y)$ denote arbitrary boundary values, with $v_o(o) = 0$, and u_o, v_o both of class C^1 and u_o', v_o' Lipschitzian with constants k_1, k_2 respectively, then it is convenient for us to denote their Fourier series as follows:

$$u_o(x) \sim \alpha_o + \sum_1^\infty (\alpha_n \cos n\omega w - \alpha_n + \beta_n \sin n\omega x),$$

$$v_o(y) \sim \sum_1^\infty (\gamma_n \cos n\omega y - \gamma_n + \delta_n \sin n\omega y), \tag{47}$$

where both series $\Sigma\alpha_m$, $\Sigma\gamma_n$ are absolutely convergent. With this notation we have

$$u_o(o) = \alpha_o, \qquad v_o(o) = 0, \qquad T^{-1}\int_o^T u_o(x)dx = \alpha_o - \sum_1^\infty \alpha_n,$$

$$T^{-1}\int_o^T v_o(y)dy = -\sum_1^\infty \gamma_n.$$

If we apply <u>formally</u> the method of successive approximations of No. 10 to Equation (46) with initial values $u_o(x)$, $v_o(y)$, we obtain—at the first approximation and preserving only the terms in ϵ-a quadruple $[\phi_o, \epsilon m_o, \epsilon n_o, \epsilon\mu_o]$, with ϕ_o, m_o, n_o, μ_o given by

$$\phi_o(x,y) = u_o(x) + v_o(y) + \epsilon \int_o^x\int_o^y [F_o(\xi,\eta) - m_o(\xi) - n_o(\eta) - \mu_o]d\xi\, d\eta,$$

$$m_o(y) = T^{-1}\int_o^T F_o(\xi,y)d\xi - \mu, \qquad n_o(x) = T^{-1}\int_o^T F_o(x,\eta)d\eta - \mu, \tag{48}$$

$$\mu_o = T^{-2}\int_o^T\int_o^T F_o(\xi,\eta)d\xi\, d\eta, \qquad F_o(x,y) = \psi(x,y) - Cu_o(x) - Cv_o(y) - \psi_1(y)u_o'(x)$$

$$- \psi_2(x)v_o'(y).$$

If we write

$$m_o(y) \sim \sum_1^\infty (B_n \cos n\omega y + C_n \sin n\omega y), \qquad n_o(x) \sim \sum_1^\infty (D_n \cos n\omega x + E_n \sin n\omega x),$$

$$\mu_o = A_o,$$

we obtain

$$\mu_o = A_o = a_{oo} + C(\alpha_o - \sum_1^\infty \alpha_s - \sum_1^\infty \gamma_s),$$

$$m_o(y) = a_{oo} + \mathscr{H}_1(y) + C(\alpha_o - \sum_1^\infty \alpha_s) + Cv_o(y) + g_o v_o'(y) - \mu, \qquad (49)$$

$$n_o(x) = a_{oo} + \mathscr{H}_2(x) + Cu_o(x) + e_o u_o'(x) + C(- \sum_1^\infty \gamma_s) - \mu,$$

$$\mathscr{H}_1(y) = T^{-1} \int_0^T \psi(\xi,y)d\xi - a_{oo} \sim \sum_1^\infty (a_{on} \cos n\omega y + b_{on} \sin n\omega y),$$

$$\mathscr{H}_2(x) = T^{-1} \int_0^T \psi(x,\gamma)d\gamma - a_{oo} \sim \sum_1^\infty (a_{on} \cos n\omega x + c_{on} \sin n\omega x). \qquad (50)$$

In terms of the Fourier constants relations (49) become

$$A_o = a_{oo} + C(\alpha_o - \sum_1^\infty \alpha_s - \sum_1^\infty \gamma_s), \qquad B_n = C\gamma_n + n\omega g_o \delta_n + a_{on},$$

$$C_n = C\delta_n - n\omega g_o \gamma_n + b_{on}, \qquad D_n = C\alpha_n + n\omega g_o \gamma_n + b_{on}, \qquad (51)$$

$$E_n = C\beta_n - n\omega e_o \alpha_n + c_{no}, \qquad n = 1,2,\dots \; .$$

We shall denote by $u(x)$, $v(y)$ the same functions $u_o(x)$, $v_o(y)$ up to a constant so as to make them with mean value zero, or

$$u(x) = u_o(x) - \alpha_o + \sum_1^\infty \alpha_s \sim \sum_1^\infty (\alpha_s \cos s\omega x + \beta_s \sin s\omega x),$$

$$v(y) = v_o(y) + \sum_1^\infty \gamma_s \sim \sum_1^\infty (\gamma_s \cos s\omega y + \delta_s \sin s\omega y). \qquad (52)$$

If we require $\mu_o = 0$, $m_o(y) \equiv 0$, $n_o(x) \equiv 0$, then relations (48) reduce to

$$Cv(y) + g_o v'(y) = - \mathscr{K}_1(y), \qquad Cu(x) + e_o u'(x) = - \mathscr{K}_2(x).$$

For $e_o = 0$, we have

$$u(x) = - C^{-1} \quad {}_2(x); \tag{53}$$

for $e_o \neq 0$, we have

$$u(x) = \exp(-e_o{}^{-1}Cx) \, [K + e_o{}^{-1} \int_0^x \exp(e_o{}^{-1}C\xi) \, \mathscr{K}_2(\xi)d\xi],$$

$$K = - e_o{}^{-2}C(1-\exp(-e_o{}^{-1}CT))^{-1} \int_0^T \exp(e_o{}^{-1}Cx)dx \int_0^x \exp(e_o{}^{-1}C\xi) \, \mathscr{K}_2(\xi)d\xi, \tag{54}$$

where the constant K is determined in such a way that $\int_0^T u(x)dx = 0$.

Analogous relations hold for $v(y)$. This determines all the coefficients $\alpha_n, \beta_n, \gamma_n, \delta_n$, $n = 1,2,\ldots$. Actually, relations (53), (54) are equivalent to those we obtain from (51) by taking $B_n = C_n = D_n = E_n = 0$ and solving with respect to $\alpha_n, \beta_n, \gamma_n, \delta_n$:

$$\alpha_n = (C^2+n^2\omega^2 e_o{}^2)^{-1} (-Ca_{no}+n\omega e_o c_{no}), \qquad \beta_n = (C^2+n^2\omega^2 e_o{}^2)^{-1} (-n\omega e_o a_{no}-Cc_{no}),$$

$$\gamma_n = (C^2+n^2\omega^2 g_o{}^2)^{-1} (-Ca_{on}+n\omega g_o b_{on}), \qquad \delta_n = (C^2+n^2\omega^2 g_o{}^2)^{-1} (-n\omega g_o-Cb_{on}),$$

$n = 1,2,\ldots$.

The coefficients $\alpha_n, \beta_n, \gamma_n, \delta_n$, $n = 1,2,\ldots$, being so determined, then equation $\mu_o = A_o = 0$ yields

$$\alpha_o = - C^{-1}a_{oo} + \sum_1^\infty \alpha_s + \sum_1^\infty \gamma_s,$$

provided the series $\Sigma\alpha_s$, $\Sigma\gamma_s$ converge. This will be the case under the hypotheses of the criterion I below. We shall denote the corresponding functions $u_o(x)$, $v_o(y)$ so determined, say $U(x)$, $V(y)$, or

$$U(x) = u_o(x) + \alpha_o - \sum_1^\infty \alpha_s \sim \alpha_o + \sum_1^\infty (\alpha_s \cos s\omega x - \alpha_s + \beta_s \sin s\omega x),$$

$$V(y) = v_o(y) - \sum_1^\infty \beta_s \sim \sum_1^\infty (\gamma_s \cos s\omega y - \gamma_s + \delta_s \sin s\omega y). \tag{55}$$

Under the conditions of Criterion I we shall require that these functions are interior points of the set defined by relations (6) of Theorem I.

13. CRITERION I

If the function f given in (46) for all $|\epsilon| \leq \epsilon_o$ satisfies all conditions of Theorems I, II, III with given constants T, N, N_1, N_2, L, M, M_1, M_2, M_3, b_o, b_1, b_2, and in addition f is Lipschitzian with respect to x and y in R, if $C \neq 0$, and the functions $\mathscr{H}_1(x)$, $\mathscr{H}_2(y)$, U(x), V(y) defined in (50) and (55) are of class C^1 with Lipschitzian first derivative, and

$$|U(o)| \leq N_o < N, \quad |U(x_1)-U(x_2)| \leq N_{10}|x_1-x_2|, \quad N_{10} < N_1,$$

$$|V(y_1)-V(y_2)| \leq N_{20}|y_1-y_2|, \quad N_{20} < N_2,$$

then there is some $\bar{\epsilon}_o$, $0 < \bar{\epsilon}_o \leq \epsilon_o$, such that Equation (46) for all $|\epsilon| \leq \bar{\epsilon}_o$, possesses at least one periodic solution $\phi(x,y)$ of period T in x and y, which is Lipschitzian in E_2 together with ϕ_x, ϕ_y, ϕ_{xy}:

$$\phi_{xy} = f(x,y,\phi,\phi_x,\phi_y), \qquad \phi(x+T,y) = \phi(x,y) = \phi(x,y+T).$$

The periodic functions $u_o(x) = \phi(x,o) = \phi(x,T)$, $v_o(y) = \phi(o,y) - \phi(o,o) = \phi(T,y) - \phi(T,T)$, satisfy moreover relations (6) of Theorem I.

14. PROOF OF CRITERION I

Let us denote by k_{4o}, k_{5o} the Lipschitzian constants of U(x) and V(y) respectively, and let k_4, k_5 be arbitrary numbers $k_4 > k_{4o}$, $k_5 > k_{5o}$. Let us denote as usual by k_1, k_2 the Lipschitzian constants of f with respect to x and y respectively in R.

Let S be the set of all pairs $w = [u_o(x),v_o(y)]$ of functions $u_o(x)$, $v_o(y)$ periodic of period T, of class C^1, with derivatives $u_o'(x)$, $v_o'(y)$ Lipschitzian of constants k_4, k_5, and satisfying relations (6) of Theorem I, that is

$$|u_o(o)| \leq N, \qquad |u_o(x_1)-u_o(x_2)| \leq N_1|x_1-x_2|, \qquad v_o(o) = 0,$$

$$|v_o(y_1)-v_o(y_2)| \leq N_2|y_1-y_2|.$$

Then $w_o = [U,V] \epsilon S$. We shall consider S imbedded in the Banach space of all pairs of periodic functions of class C^1 with norm

$$\|w\| = \max|u(x)| + \max|u'(x)| + \max|v(y)| + \max|v'(y)|. \qquad (56)$$

For every $w = [u_o(x),v_o(y)] \epsilon S$ we shall determine the solution $W = [\phi,\epsilon m,\epsilon n,\epsilon\mu]$

of the modified problem relative to (46). Since this solution can be determined by the method of successive approximations of No. 10, we see that W can be written in the form

$$\Phi(x,y,\epsilon) = \phi_o(x,y) + \epsilon\tilde{\phi}(x,y,\epsilon), \qquad m(y,\epsilon) = m_o(y) + \tilde{m}(y,\epsilon),$$

$$n(x,\epsilon) = n_o(x) + \tilde{n}(x,\epsilon), \qquad \mu(\epsilon) = \mu_o + \tilde{\mu}(\epsilon),$$

where $\tilde{\phi}$, $\tilde{m}$, $\tilde{n}$, $\tilde{\mu} = O(1)$ uniformly as $\epsilon \to 0$, and ϕ_o, m_o, n_o, μ_o are given by formulas (48).

Using the functions $u(x)$, $v(y)$ as in No. 8, equations $\mu = 0$, $m(y) \equiv 0$, $n(x) \equiv 0$ reduce to

$$\begin{aligned} Cv(y) + g_o v'(y) &= -\mathscr{H}_1(y) - \tilde{m}(y,\epsilon), \\ Cu(x) + e_o u'(x) &= -\mathscr{H}_2(x) - \tilde{n}(x,\epsilon), \\ a_{oo} + C(\alpha_o - \sum_1^\infty \alpha_s - \sum_1^\infty \gamma_s) - \tilde{\mu}(\epsilon) &= 0. \end{aligned} \tag{57}$$

For $e_o = 0$ we have

$$u(x) = -C^{-1}(\mathscr{H}_2(x)+\tilde{n}(x,\epsilon)); \tag{58}$$

for $e_o \neq 0$ we have

$$\begin{aligned} u(x) &= \exp(-e_o^{-1}Cx)[K+e_o^{-1}\int_0^x \exp(e_o^{-1}C\xi)(\mathscr{H}_2(\xi)+\tilde{n}(\xi,\epsilon))d\xi, \\ K &= -e_o^{-2}C(1-\exp(-e_o^{-1}CT))^{-1}\int_0^T \exp(e_o^{-1}Cx)dx\int_0^x \exp(e_o^{-1}C\xi)(\mathscr{H}_2(\xi) \\ &\quad +\tilde{n}(\xi,\epsilon))d\xi, \end{aligned} \tag{59}$$

and hence

$$\begin{aligned} u'(x) &= -C^{-1}(d/dx)(\mathscr{H}_2(x)+\tilde{n}(x,\epsilon)) \quad \text{if } e_o = 0, \\ u'(x) &= -e_o^{-1}(Cu(x)+\mathscr{H}_2(x)+\tilde{n}(x,\epsilon)) \quad \text{if } e_o \neq 0. \end{aligned} \tag{60}$$

Analogous formulas hold for $v(y)$.

This determines $u(x)$, $v(y)$ and hence all coefficients α_n, β_n, γ_n, δ_n, $n = 1,2,\ldots$. By Remark 1 we know that m, n are Lipschitzian functions, and so are $\tilde{m}$, $\tilde{n}$ as well as $\mathcal{H}_1$, $\mathcal{H}_2$. Thus u(x), u(y) are periodic functions of mean value zero, of class C^1, with Lipschitzian first derivative. Thus, the series $\sum \alpha_n$, $\sum \gamma_n$ are absolutely convergent, and (47), (57) yield

$$\begin{aligned} \alpha_o &= - C^{-1}(a_{oo}-\tilde{\mu}(\epsilon) + \sum_1^\infty \alpha_s + \sum_1^\infty \gamma_s), \\ &= - C^{-1}(a_{oo}-\tilde{\mu}(\epsilon)) - u(o) - v(o)), \end{aligned} \qquad (61)$$

$$u_o(x) = u(x) + \alpha_o - \sum_1^\infty \alpha_s, \qquad v_o(y) = v(y) - \sum_1^\infty \gamma_s. \qquad (62)$$

Note that these functions, when we take $\tilde{m} = \tilde{n} = 0$, reduce to U(x) and V(y) respectively, and thus the convergence of the series $\sum \alpha_n$, $\sum \gamma_n$ of No. 8 is proved above.

Actually, for every $w = [u\ (x), v\ (y)] \epsilon S$, we can first determine m, n, μ as in theorems I, II, III, and the method of successive approximations of No. 10, then we determine $\tilde{m}$, $\tilde{n}$, and finally the second members of formula, (58), (59), (60), (62) and analogous ones determine new functions, say $\bar{w} = [\bar{u}_o(x), \bar{v}_o(y)]$. Thus, we have a map $\mathcal{F}$,

$$\bar{w} = \mathcal{F} w, \quad w \epsilon S,$$

whose fixed elements $w = \mathcal{F} w$, if any, have the property that $\mu = 0$, $m(y) \equiv 0$, $n(x) \equiv 0$

We have already chosen the uniform topology of class C^1 on w and $\bar{w}$ by means of (55). Let us choose the uniform topology of class C^o on m, n as in Theorem III, as well as on $\tilde{m}$, $\tilde{n}$. We know already from Theorem III that m, n are continuous functions of w, and so are $\tilde{m}$, $\tilde{n}$. The second members of (58), (59), (60), (62) defines continuous functions of $\tilde{m}$, $\tilde{n}$. Thus $\mathcal{F}$ is a continuous function of w for $w \epsilon S$ in the topology defined by (55).

By Theorems I, II, III we know that $m(y,\epsilon)$, $n(x,\epsilon)$ are Lipschitzian functions. The same property holds for $\tilde{m}(y,\epsilon)$, $\tilde{n}(x,\epsilon)$, but these functions— as well as their Lipschitzian constants—have a uniform bound of the form $M\epsilon$ for some $M > 0$ and all $|\epsilon| \leq \epsilon$. Then, by choosing convenient constants k, k_1, we have

$|\tilde{m}(y,\epsilon)|$, $|\tilde{n}(x,\epsilon)| \leq k\epsilon$, $|\tilde{\mu}| \leq k\epsilon$, $|\tilde{m}(y_1,\epsilon)-\tilde{m}(y_2,\epsilon)| \leq k\epsilon|y_1-y_2|$,

$|\tilde{n}(x_1,\epsilon)-\tilde{n}(x_2,\epsilon)| \leq k\epsilon|x_1-x_2|$, $|\overline{u}_o-U| \leq k_1\epsilon$, $|\overline{v}_o-V| \leq k_1\epsilon$, $|\overline{u}_o'-U'| \leq k_1\epsilon$,

$|\overline{v}_o'-V'| \leq k_1\epsilon$, $|\overline{u}_o(x_1)-U)x_1)-\overline{u}_o(x_2)+U(x_2)| \leq k_1\epsilon$, $|\overline{v}_o(y_1)-V(y_1)-\overline{v}(y_2)+V(y_2)| \leq k_1\epsilon$,

$|\overline{u}_o'(x_1)-U'(x_1)-\overline{u}_o'(x_2)+U'(x_2)| \leq k_1\epsilon$, $|\overline{v}_o'(y_1)-V'(y_1)-\overline{v}'(y_2)+V'(y_2)| \leq k_1\epsilon$.

If k_4, k_5 are the Lipschitzian constants of U', V', and

$$\overline{\epsilon}_o = \min[\epsilon_o, k_1^{-1}(N-N_o),\ k_1^{-1}(N_1-N_{10}),\ k_1^{-1}(N_2-N_{20})$$

then for $|\epsilon| \leq \overline{\epsilon}_o$ we have

$$|\overline{u}_o(o)| \leq U(o) + k_1\epsilon \leq N_o + k_1\epsilon \leq N, \quad |\overline{u}_o(x_1)-u_o(x_2)| \leq (N_{10}+k_1\epsilon)|x_1-x_2|$$

$$\leq N_1|x_1-x_2|, \quad |\overline{u}_o'(x_1)-u_o'(x_2)| \leq (k_4+k_1\epsilon)|x_1-x_2|, \quad |\overline{v}_o(y_1)-v_o(y_2)| \leq$$

$$\leq (N_{20}+k_1\epsilon)|y_1-y_2| \leq N_2|y_1-y_2|, \quad |\overline{v}_o'(y_1)-v_o'(y_2)| \leq (k_5+k_1\epsilon)|y_1-y_2|.$$

This shows that, for $|\epsilon| \leq \overline{\epsilon}$, $\mathcal{T}$ maps S into itself, $\mathcal{T}$: $S \to S$, and S is a convex closed compact subset of a Banach space. By Schauder's fixed point theorem $\mathcal{T}$ possesses at least one fixed element $w = \mathcal{T}w \in S$, $w = [u_o(x), v_o(y)]$, with $u_o(x)$, $v_o(y)$ satisfying relations (6) of Theorem I. Criterion I is thereby proved.

15. EXAMPLE

The equation

$$u_{xy} = \epsilon(1+u) + \epsilon^2 g(x,y,u,u_x,u_y)$$

where g is periodic of period 2π in x and y, has a periodic solution $\phi(x,y)$ of the same period

$$\phi(x,y) = 1 + O(\epsilon)$$

The analogous equation

$$u_{xy} = \epsilon(\sin x - \cos y - \sin y + \cos y + u + u_x - u_y) + \epsilon^2 g(x,y,u,u_x,u_y)$$

with g as above has a periodic solution

$$\phi = \cos x - \cos y + O(\epsilon).$$

16. ANOTHER EQUATION CONTAINING A SMALL PARAMETER

Let us consider the differential equation

$$u_{xy} = f(x,y,u,u_x,u_y), \tag{63}$$

$$f = \psi(x,y) + Cu + \psi_1(y)u_x + \psi_2(x)u_y + \epsilon g(x,y,u,u_x,u_y),$$

where ϵ is a small parameter, and ψ, ψ_1, ψ_2 are as in No. 12. We assume here that, for $\epsilon = 0$, Equation (63) possesses a known periodic solution of period T in x and y,

$$\phi_0(x,y) = u_0(x) + v_0(y),$$

where $u_0(x)$, $v_0(y)$ have Fourier series (47), and hence

$$\psi(x,y) = - Cu_0(x) - Cv_0(y) - \psi_1(y)u_0'(x) - \psi_2(x)v_0'(y).$$

Under the hypotheses below, we shall prove that for $|\epsilon| \neq 0$ sufficiently small, (63) possesses a solution $\phi(x,y) = \phi_0(x,y) + O(\epsilon)$ which is periodic of period T in x and y.

17. CRITERION II

If the function f defined in (63) for all $|\epsilon| \leq \epsilon_0$ satisfies all conditions of Theorems I, II, III with given constants $T, N, N_1, N_2, L, M, M_1, M_2, M_3, b_0, b_1, b_2$ and in addition f is Lipschitzian with respect to s and y in R, if $C \neq 0$, if (63) possesses for $\epsilon = 0$ a solution $\phi_0(x,y) = U(x)+V(y)$ with U, V periodic of period T, if the functions $U(x)$, $V(y)$, $\mathcal{H}_1(x)$, $\mathcal{H}_2(y)$ are of class C^1 with Lipschitzian first derivative, and

$$|U(o)| \leq N_0 < N, \quad |U(x_1)-U(x_2)| \leq N_{10}|x_1-x_2|, \quad N_{10} < N_1,$$

$$|V(y_1)-V(y_2)| \leq N_{20}|y_1-y_2|, \quad N_{20} < N_2,$$

then there is some $\bar{\epsilon}_0$, $0 < \bar{\epsilon}_0 \leq \epsilon_0$, such that Equation (63) for all $|\epsilon| \leq \bar{\epsilon}$, possesses at least one periodic solution $\phi(x,y)$ of period T in x and y, which is Lipschitzian in E_2 together with ϕ_x, ϕ_y, ϕ_{xy}:

$$\phi_{xy} = f(x,y,\phi,\phi_x,\phi_y), \qquad \phi(x+T,y) = \phi(x,y) = \phi(x,y+T).$$

The periodic functions $u_0(x) = \phi(x,o) = \phi(x,T)$, $v_0(y) = \phi(o,y)-\phi(o,o) = \phi(T,y)-\phi(T,T)$, satisfy moreover relations (6) of Theorem I.

18. PROOF OF CRITERION II

As in No. 12 let us apply formally the method of successive approximations of No. 10 to Equation (63) with arbitrary initial values $u_0(x)$, $v_0(y)$. Then the quadruple $[\phi, m, n, \mu]$, solution of the modified problem for Equation (63), is given by

$$\phi(x,y,\epsilon) = \phi_0(x,y) + \tilde{\phi}(x,y,\epsilon), \quad m(y,\epsilon) = m_0(y) + \tilde{m}(y,\epsilon),$$

$$n(x,\epsilon) = n_0(x) + \tilde{n}(x,\epsilon), \quad \mu(\epsilon) = \mu_0 + \tilde{\mu}(\epsilon),$$

where $\tilde{\phi}, \tilde{m}, \tilde{n}, \tilde{\mu} = O(1)$ uniformly as $\epsilon \to 0$, and ϕ_0, m_0, n_0, μ_0 are given by formulas (48). In addition we know that for $u_0(x) \equiv U(x)$, $v_0(y) \equiv V(y)$, we have $\mu_0 = 0$, $m_0(y) \equiv 0$, $n_0(x) \equiv 0$. We can now repeat with obvious variants the argument of the proof of Criterion I.

19. EXAMPLES

The equation

$$u_{xy} = -1 + u + \psi_1(y)u_x + \psi_2(x)u_y + \epsilon g(x,y,u,u_x,u_y)$$

for $\epsilon = 0$ has the obvious solution $u = 1$. Since $C \neq 0$ the same equation has a periodic solution ϕ of period T in x and y for all $|\epsilon|$ sufficiently small.

Analogously, the equation

$$u_{xy} = -\cos x + \sin x + \cos y - \sin y + u + u_x + u_y + \epsilon g(x,y,u,u_x,u_y)$$

has, for $\epsilon = 0$, the obvious solution $u = \cos x - \cos y$. Since $C \neq 0$, the same equation has a periodic solution of period 2π in x and y for every $|\epsilon| \neq 0$ sufficiently small.

20. APPLICATION TO THE WAVE EQUATION

Let us consider the differential equation

$$u_{tt} - u_{\xi\xi} = f(t,\xi,u,u_t,u_\xi), \tag{64}$$

where f is periodic in t and ξ of period T. Then the transformation

$$t = x + y, \quad \xi = x - y, \quad x = 2^{-1}(t+\xi), \quad y = 2^{-1}(t-\xi), \tag{65}$$

changes (64) into

$$u_{xy} = F(x,y,u,u_x,u_y),$$

where

$$F = f(x+y, x-y, u, 2^{-1}u_x+2^{-1}u_y, 2^{-1}u_x-2^{-1}u_y) \tag{66}$$

and F is periodic of period T in x and y. The theorems I, II, III and the criteria should now be applied to (66). Other transformations beside (65) can be used.

As an example, let us consider the equation

$$u_{tt} - u_{\xi\xi} = \epsilon[\lambda(t,\xi)+C_0u+\lambda_1(t,\xi)u_t+\lambda_2(t,\xi)u_\xi] + \epsilon^2 g(t,\xi,u,u_t,u_\xi), \tag{67}$$

$$\begin{aligned} \lambda(t,\xi) &= A_0+B_1 \cos 2T + C_1 \sin 2t + B_2 \cos 2\xi + C_2 \sin 2\xi \\ &+ D_1 \cos(t+\xi) + E_1 \sin(t+\xi) + D_2 \cos(t+\xi) + E_2 \sin(t-\xi), \\ \lambda_1(t,\xi) &= A + B\cos(t+\xi) + C\sin(t+\xi) + D\cos(T-\xi) + E\sin(t-\xi), \\ \lambda_2(t,\xi) &= A' - B\cos(t+\xi) - C\sin(t+\xi) + D\cos(t-\xi) + E\sin(t-\xi), \end{aligned} \tag{68}$$

where A_0, $B, \ldots, E$ are constants, $C_0 \neq 0$, and g is of period u in t and ξ. By the transformation

$$t = 2^{-1}(x+y), \quad \xi = 2^{-1}(x-y), \quad x = t+\xi, \quad y = t-\xi, \tag{69}$$

Equation (67) is changed into

$$u_{xy} = \epsilon[\psi(x,y)+4^{-1}C_0u+\psi_1(y)u_x+\psi_2(x)u_y] + \epsilon^2 g(2^{-1}x+2^{-1}y, 2^{-1}x-2^{-1}y, u, u_x+u_y, u_x-u_y), \tag{70}$$

where the second member has period 2π in x and y, and with

$$\begin{aligned} \psi(x,y) &= a_{00} + a_{10}\cos x + c_{01}\sin x + a_{01}\cos y + b_{01}\sin y + a_{11}\cos x\cos y \\ &+ b_{11}\cos x\sin y + c_{11}\sin x\cos y + d_{11}\sin x\sin y, \\ \psi_1(y) &= e_0 + e_1\cos y + f_1\sin y, \qquad \psi_2(x) = g_0 + g_1\cos x + h_1\sin y, \end{aligned} \tag{71}$$

$4a_{00} = A_0$, $4a_{10} = D_1$, $4c_{01} = E_1$, $4a_{01} = D_2$, $4b_{01} = E_2$,

$4a_{11} = B_1+B_2$, $4b_{11} = C_1-C_2$, $4c_{11} = C_1+C_2$, $4d_{11} = -B_1+B_2$, $4e_0 = A+A'$,

$2e_1 = D$, $2f_1 = E$, $4g_0 = A-A'$, $2g_1 = B$, $2h_1 = C$.

By Criterion I we conclude that, if $C_0 \neq 0$ and $|\epsilon|$ sufficiently small then Equation (67) has at least a periodic solution $\phi(x,y,\epsilon)$ of period 2π in x and y, and then Equation (64) has at least a solution

$$u(t,\xi,\epsilon) = \phi(t+\xi,t-\xi,\epsilon),$$

also of period 2π in t and ξ.

21. ANOTHER EXAMPLE

Let us consider the differential equation

$$u_{tt} - u_{\xi\xi} = \lambda(t,\xi) + C_0 u + \lambda_1(t,\xi)u_t + \lambda_2(t,\xi)u_\xi + \epsilon g(t,\xi,u,u_t,u_\xi), \quad (72)$$

where λ, λ_1, λ_2 are given by (68) and again $C_0 \neq 0$. By the same transformation (69) Equation (72) is changed into

$$u_{xy} = \psi(x,y) + 4^{-1}C_0 u + \psi_1(y)u_x + \psi_2(x)u_y + \epsilon g(2^{-1}x+2^{-1}y, 2^{-1}x-2^{-1}y, u, u_x+u_y, u_x-u_y), \quad (73)$$

where ψ, ψ_1, ψ_2 are given by formulas (71). It is immediately seen that (73) for $\epsilon = 0$ has a solution of the form

$$u(x,y) = U(x) + V(y), \qquad U(x) = \alpha_0 + \alpha_1 \cos x + \beta_1 \sin x,$$

$$V(y) = \gamma_1 \cos y + \delta_1 \sin y$$

if and only if

$$B_1 = \Delta_1 C_0(-ED_1+DE_1) + \Delta_1(A+A')(DD_1+EF_1) + \Delta_2 C_0(CD_2+BE_2) + \Delta_2(A-A')(BD_2-CE_2),$$

$$B_2 = \Delta_1 C_0(ED_1+DE_1) + \Delta_1(A+A')(DD_1-EE_1) + \Delta_2 C_0(-CD_2+BE_2) + \Delta_2(A-A')(BD_2+CE_2),$$

$$C_1 = \Delta_1 C_0(-DD_1-EE_1) + \Delta_1(A+A')(-ED_1+DE_1) + \Delta_2 C_0(-BD_2+CE_2) + \Delta_2(A-A')(CD_2+BE_2),$$

$$C_2 = \Delta_1 C_0(-DD_1+EE_1) + \Delta_1(A+A')(ED_1+DE_1) + \Delta_2 C_0(BD_2+CE_2) + \Delta_2(A-A')(CD_2-BE_2),$$

$$\Delta_1 = (C_0^2+(A+A')^2)^{-1}, \qquad \Delta_2 = (C_0^2+(A-A')^2)^{-1}. \quad (74)$$

In this situation, then

$$\alpha_1 = \Delta_1(-C_0D_1+(A+A')E_1), \qquad \beta_1 = \Delta_1(-C_0E_1-(A+A')D_1),$$

$$\gamma_1 = \Delta_2(-C_0D_2+(A-A')E_2), \qquad \delta_1 = \Delta_2(-C_0E_2-(A-A')D_2), \qquad \alpha_0 = -C_0^{-1}A_0.$$

By Criterion II we conclude that, if $C_0 \neq 0$, $|\epsilon|$ sufficiently small, and relations (74) hold, then Equation (73) has a periodic solution $\phi(x,y)$ of period 2π in x and y, and (72) has a solution

$$u(t,\xi,\epsilon) = \phi(t+\xi,t-\xi,\epsilon)$$

also of period 2π in t and ξ.

For instance, for the equation

$$u_{tt} - u_{\xi\xi} = D_1 \cos(t+\xi) + E_1 \sin(t+\xi) + D_2 \cos(t+\xi)+E_2 \sin(t-\xi) + u + u_t$$
$$+ \epsilon g(t,\xi,u,u_t,u_\xi),$$

where D_1, E_1, D_2, E_2 are arbitrary constants, and g periodic of period π in t and ξ, we see that relations (74) are all satisfied with

$$B_1 = C_1 = B_2 = C_2 = 0, \quad B = C = D = E = 0, \quad A_0 = 0, \quad C_0 = 1, \quad A = 1,$$

$$A' = 0, \quad \Delta_1 = \Delta_2 = 2^{-1}.$$

The corresponding Equation (73) is

$$4u_{xy} = D_1 \cos x + E_1 \sin x + D_2 \cos y + E_2 \sin y + u + u_x + u_y + \epsilon g.$$

For $\epsilon = 0$ this equation has the periodic solution $\phi_0(x,y)$ of period 2π in x, y given by

$$-2\phi_0(x,y) = (D_1-E_1)\cos x + (D_1+E_1) \sin x + (D_2-E_2)\cos y + (D_2+E_2) \sin y,$$

and hence (75) for $\epsilon = 0$ has the periodic solution

$$u_0(t,\xi) = -2^{-1}(D_1-E_1)\cos(t+\xi)-2^{-1}(D_1+E_1)\sin(t+\xi)-2^{-1}(D_2+E_2)\cos(t-\xi)$$
$$-2^{-1}(D_2+E_2)\sin(t-\xi).$$

Thus, for all $|\epsilon|$ sufficiently small Equation (75) has a periodic solution of period 2π in t and ξ of the form $u(t,\xi) = u_0(t,\xi)+O(\epsilon)$.

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