THE UNIVERSITY OF MICHIGAN
COLLEGE OF LITERATURE, SCIENCE, AND THE ARTS
Department of Mathematics

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EXISTENCE THEOREMS FOR f LINEAR IN u OR IN x

Lamberto Cesari

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5. EXISTENCE THEOREMS FOR f LINEAR IN u OR IN x

5.1. LINEAR PROBLEMS

We briefly consider in this chapter problems of minimum in which the system has some linear character, say $f(t,x,u) = (f_1,\ldots,f_n)$ is either linear in u, or in x, or both,

$$dx/dt = B(t,x)\, u + C(t,x)\ , \tag{5.1.1}$$

$$dx/dt = A(t,u)\, x + C(t,u)\ , \tag{5.1.2}$$

$$dx/dt = A(t)\, x + B(t)\, u + C(t)\ , \tag{5.1.3}$$

while the functional has one of the usual forms

$$I[x,u] = g(t_1,x(t_1),\ t_2,x(t_2))\ , \tag{5.1.4}$$

$$I[x,u] = \int_{t_1}^{t_2} f_o(t,x(t),u(t))dt\ . \tag{5.1.5}$$

In case the functional has the Lagrange form (5.1.5), the particular cases

$$f_o = B_o(t,x)\, u + C_o(t,x) \tag{5.1.6}$$

$$f_o = A_o(t,u)\, x + C_o(t,u) \tag{5.1.7}$$

$$f_o = A_o(t)\, x + B_o(t)\, u + C_o(t) \tag{5.1.8}$$

are of some interest. Above A,B,C denote matrices of the types n x n, n x m, n x 1 respectively, and A_o, B_o, C_o of the types 1 x n, 1 x m, 1 x 1.

Problems as above are all particular cases of those considered in Chapter 3 and 4, and the existence theorems already proved apply.

Nevertheless, for problems with f linear in u (systems (5.1.1) and

(5.1.3), functionals (5.1.4) or (5.1.5), we list below (5.2) a few corollaries. For problems with both f and f_o linear in x, namely of the forms $f = A(t)\,x + C(t,u)$, $f_o = A_o(t)\,x + C_o(t,u)$ (particular cases of (5.1.2), (5.1.3), (5.1.7), (5.1.8)), or alternatively for f as stated and functional (5.1.4), a completely different theorem will be stated and proved in (5.3).

5.2. EXISTENCE THEOREMS FOR f LINEAR IN u

As usual, A denotes a subset of the tx-space E_{1+n}, B a subset of the $t_1x_1t_2x_2$ - space E_{2n+2}, U(t,x) a subset of the u-space E_m, g, f_o are scalars, f an n-vector, and $M, Q, \tilde{\tilde{Q}}$ are the usual sets $M = [(t,x,u) | (t,x) \in A,\ u \in U(t,x)]$, $Q(t,x) = f(t,x,U(t,x)) \subset E_n$ and $\tilde{\tilde{Q}}(t,x) = [\tilde{z} = (z^o, z) | z^o = f_o(t,x,u),\ z = f(t,x,u),\ u \in U(t,x)] \subset E_{n+1}$. Finally, Ω denotes a class of admissible pairs.

(5.2.i) (A corollary of theorem 1). If A is compact, B closed, M compact, $f = B(t,x)\,u + C(t,x)$, B,C matrices with entries continuous on A, $g(t_1,x_1,t_2,x_2)$ continuous on B, Ω closed and not empty, then the functional (5.1.4) has an absolute minimum in Ω.

If A is not compact but closed, then (5.2.i) still holds under conditions (a), (b), or (a), (b), (c), of (3.3), according as A is contained in a slab $[(t,x) | t_o \leq t \leq T,\ x \in E_n]$, t_o, T finite, or not. If A is contained in such a slab, and the entries of the matrices $B = (b_{ij}(t,x))$, $C = (c_i(t,x))$ satisfy relations $|b_{ij}(t,x)|$, $|c_i(t,x)| \leq H + K|x|$ for some constant H,K and all $(t,x) \in A$, then certainly condition (a_o) of (3.3) is satisfied, and so is (a). If A is not contained in such a slab, it is enough to know that the relations above for all $(t,x) \in A$ with $|t| \leq N$ and constants H,K depending on N. This is certainly the case if f is of the form (5.1.3) with matrices A(t),

$B(t)$, $C(t)$ continuous on the real axis. For alternate conditions see (3.3).

(5.2.ii) (A corollary of Theorem 2). If A is compact, B closed, M compact, $f = B(t,x)\, u + C(t,x)$, B,C matrices with entries continuous on A, $f_0(t,x,u)$ continuous on M and convex in u, Ω closed and not empty, then the functional (5.1.5) has an absolute minimum in Ω.

If A is not compact but closed, then (5.2.ii) still holds under conditions (a), (b), or (a'), (b), (c), of (3.4), and the same remarks hold as before. For alternate conditions see again (3.4). If f_0 is linear in u, say of the form (5.1.6) or (5.1.8), then certainly f_0 is convex in u.

(5.2.iii) (A Corollary of Theorem 3). Let A be compact, B closed, M closed, $f = B(t,x)\, u + C(t,x)$, B,C continuous matrices with entries continuous on A, $f_0(t,x,u)$ continuous on M, convex in u, satisfying (ψ) $f_0(t,x,u) \geq \psi(t)$, where ψ is a locally L-integrable given function. Let us assume that the sets $\tilde{\tilde{Q}}(t,x)$ satisfy condition (Q) at all $(t,x) \in A$ with exception perhaps of a set of points in A whose t coordinate lie in a set of measure zero on the t-axis. Let Ω be a closed nonempty family of admissible pairs satisfying a relation $\int_{t_1}^{t_2} |x'(t)|^p \, dt \leq L$ for some $p > 1$ and $L \geq 0$. Then the functional (5.1.5) has an absolute minimum in Ω.

Condition (ψ) can be replaced by the much weaker condition (ψ^*) of (2.10). Condition (Q) and condition (ψ^*) and (ψ) are certainly satisfied under conditions (α) and (X) of (2.12). (The last situation is an extension of the so-called "normal convexity" case for free problems.)

If A is not compact, but A is closed, then theorem (5.2.iii) (and variants above) still holds provided condition (b), or conditions (b), (a') of (4.3) hold, according as A is contained in a slab $[t_o \leq t \leq T, x \in E_n]$, t_o, T finite, or not. For alternate conditions see (4.3).

(5.2.iv) (A Corollary of Theorem 3). Let A be compact, B closed, M closed, $f = B(t,x)\,u + C(t,x)$, B,C continuous matrices with entries continuous on A, $f_o(t,x,u)$ continuous on M, and convex in u. Let us assume that the sets $\tilde{\tilde{Q}}(t,x)$ satisfy condition (Q) at all $(t,x) \in A$ with exception perhaps of a set of points in A whose t coordinate lie in a set of measure zero on the t-axis. Let us assume that the following growth condition is satisfied: (γ) given $\epsilon > 0$ there is a locally integrable function $\psi_\varepsilon(t)$ such that $|B(t,x)\,u + C(t,x)| \leq \psi_\varepsilon(t) + \varepsilon\, f_o(t,x,u)$ for all $(t,x,u) \in M$. If Ω is a closed nonempty family of admissible pairs, then the functional (5.1.5) has an absolute minimum in Ω.

Condition (Q) is certainly satisfied under conditions (α) and (X) of (2.12), this situation being an extension of the so called "normal convexity" case for free problems.

If a growth condition hold as (Φ): there is a scalar continuous function $\Phi(\xi)$, $0 \leq \xi < +\infty$, such that $\Phi(\xi)/\xi \to +\infty$ as $\xi \to \infty$, and $f_o(t,x,u) \geq \Phi(|u|)$ for all $(t,x,u) \in M$, then certainly condition (γ) holds, and all sets $\tilde{\tilde{Q}}(t,x)$ have property (Q).

If A is not compact, but A is closed, then theorem (5.2.iv) (and variants) still holds provided conditions (a), (b), or conditions (a'), (b), (c') of (4.3) hold, according as A is contained in a slab $[t_o \leq t \leq T, x \in E_n]$, t_o, T finite, or not. For alternate conditions, in particular conditions (aa'), see (4.3).

Analogous corollaires of theorem 4 and others are left as exercises for the reader.

5.3. EXISTENCE THEOREMS FOR f LINEAR IN x

Let us consider first problems with system of the form (5.1.2), or

$$dx/dt = A(t,u)\, x + C(t,u)\ .$$

Existence theorems 1 to 4 of Chapters 1 and 2 naturally apply. However, unlike the case of f linear in u, these theorems do not yield statements which are in any way simpler than the original ones, and therefore, we do not restate them.

Of some interest may be the remark that, if the entries of the matrices $A = (a_{ij}(t,u))$ and $C = (c_i(t,u))$ are bounded, say $|a_{ij}| \leq M_o$, $|c_i| \leq M_o$, than the condition (a) of (4.3) is satisfied:

$$x^1 f_1 + \cdots + x^n f_n = \Sigma_{ij}\, a_{ij}\, x^i x^j + \Sigma_i\, c_i x^i \leq M_o\, \Sigma_{ij} |x^i x^j| + M_o\, \Sigma_i |x^i|$$

$$\leq M_o (|x^1| + \cdots + |x^n|)^2 + M_o\, \Sigma_i\, ((x^i)^2 + 1)$$

$$\leq (M_o n^2 + M_o)(\Sigma_i (x^i)^2 + 1)\ .$$

An analogous remark holds for condition (a').

Let us consider now problems where the system has the more particular form

$$dx/dt = A(t)\, x + C(t,u) \qquad (5.3.1)$$

with compact control space U, A closed of the form $A = [t_o,T] \times E_n$, or $A = E_1 \times E_n$, and functional of the (Mayer) form (5.1.4). (Below we shall consider the analogous case with functional of the form (5.1.5) and $f_o = A_o(t)\, x + C_o(t,u)$).

(5.3.i) Existence Theorem (for Mayer problems and linear systems (5.3.1)). Let $A = [t_o, T] \times E_n$, t_o, T finite, U a fixed compact set of the u-space E_m, $f = A(t)\,x + C(t,u)$, A, C matrices with entries continuous in $[t_o, T]$ and $[t_o, T] \times U$ respectively. Let B be a closed subset of the $t_1 x_1 t_2 x_2$-space and $g(t_1, x_1, t_2, x_2)$ a scalar function continuous on B. Let P be a compact subset of A, and let r be the class, which we suppose not empty, of all admissible pairs x, u whose trajectory x possesses at least one point $(t^*, x(t^*)) \in P$. Then the functional (5.1.4) has an absolute minimum in Ω.

If $A = E_1 \times E_n$ then (5.3.i) still holds under the additional condition (d) of (3.3) (or alternates). Note that conditions (a) and (a') hold here automatically as consequences of the hypotheses (U compact, A, C continuous).

Proof. Let $A(t) = [a_{ij}(t)]$ $(i,j = 1,\ldots,n)$, $C(t,u) = [c_i(t,u)]$, $i = 1,\ldots,n$. Let $X(t,t^*)$ denote the fundamental solution of the homogeneous system $dx/dt = A(t)\,x(t)$ which satisfies $X(t^*,t^*) = I$, the unit matrix, in other words the n columns of X are independent solutions of the homogeneous system with initial values at t^* given by $(1,0,\ldots,0),\ldots,(0,\ldots,0,1)$ respectively. Then we know from differential equation theory that any solution $x(t)$ of the canonical system (5.3.1) satisfies the relation

$$x(t) = X(t,t^*)[x(t^*) + \int_{t^*}^{t} X^{-1}(\tau,t^*)C(\tau,u(\tau))d\tau] .$$

Since $(t^*, x(t^*) \in P$, P compact, and the vector functions and matrices $C(t,u)$, $X(t,t^*)$, $X^{-1}(t,t^*)$ are uniformly bounded for $t, t^* \in [t_o, T]$, and $u \in U$ since U is compact, we conclude that $x(t)$, $t_1 \leq t \leq t_2$, with $[t_1,t_2] \subset [t_o, T]$, admits of a uniform bound. Then, there is some constant N such that $|x(t_1)| \leq N$, $|x(t_2)|$

$\leq N$ for all admissible pairs $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$. Also, $A(t)\,x(t) + C(t, u(t))$, $t_1 \leq t \leq t_2$, admits of a uniform bound, say still N, and then $|dx/dt| \leq N$, and the trajectories $x(t)$, $t_1 < t < t_2$, are uniformly Lipschitzian with constant N.

Let us prove that the set B_o of all $(t_1,x(t_1),t_2,x(t_2))$, which are terminal points of trajectories, is closed.

Let $(t_1,x_1,t_2,x_2) \in \text{cl}\, B_o$. Then there is a sequence of admissible pairs $[x_k(t),\ u_k(t),\ t_{1k} \leq t \leq t_{2k}]$, $k = 1,2,\ldots$, and points t_k^* such that $t_{1k} \to t_1$, $x_k(t_{1k}) \to x_1$, $t_{2k} \to t_2$, $x_k(t_{2k}) \to x_2$, $t_o \leq t_{1k} \leq t_k^* \leq t_{2k} \leq T$, $(t_k^*,\ x_k(t_k^*)) \in P$, $u_k(t) \in U$ a.e. in $[t_{1k},t_{2k}]$. Since the trajectories $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, are equibounded, equicontinuous, and actually equilipschizian (of constant N and exponent 1), there is a subsequence, say still $[x_k]$, which converges in the metric ρ toward a continuous vector function $x(t)$, $t_1 \leq t \leq t_2$, which is necessarily Lipschitzian (of constant N) and hence AC in $[t_1,t_2]$. Also, P is compact, hence, we can extract the subsequence so that we have also $(t_u^*,x_k(t_k^*)) \to (t^*,x(t^*))$ for some t^*, $t_1 \leq t^* \leq t_2$. It remains to prove that there is a measurable function $u(t)$, $t_1 \leq t \leq t_2$, with values $u(t) \in U$, such that $dx/dt = A(t)\,x(t) + C(t,u(t))$ a.e. in $[t_1,t_2]$, and thus $x(t)$, $u(t)$ is an admissible pair (with $(t_1,x_1,t_2,x_2) \in B_o$ instead of $(t_1,x_1,t_2,x_2) \in B$)).

We shall denote by $[t_1,t_2]$ the interval of definition of $x(t)$, which is from now on a fixed interval. Let Ω_o be the class of all B-measurable functions $u(t)$, $t_1 \leq t \leq t_2$, with $[t_1,t_2]$ fixed as above, and values $u(t) \in U$. For each $u \in \Omega_o$, let $y(t)$, $t_1 \leq t \leq t_2$, be the AC n-vector function defined by the initial value differential problem

$$dy/dt = A(t)\, y(t) + C(t,u(t)) \text{ a.e. in } [t_1,t_2],\ y(t_1) = x(t_1) .$$

Then

$$y(t_2) = X(t_2,t_1)[y(t_1) + \int_{t_1}^{t_2} X^{-1}(t,t_1)\, C(t,u(t))dt] ,$$

where as usual $X(t,t_1)$ denotes the fundamental solution of the homogeneous system $dy/dt = A(t)\, y(t)$ with $X(t_1,t_1) = I$, the unit matrix. Let Ω, denote the class of all n-vector functions $v(t) = C(t,u(t))$ with $u \in \Omega_o$. Let Ω_2 denote the subset of E_n of all n-vectors z of the form

$$z = \int_{t_1}^{t_2} X(t,t_1)\, C(t,u(t))\, dt,\ u \in \Omega_o .$$

Finally, let Ω_3 be the set of all n-vectors $y(t_2)$ defined as above for all $u \in \Omega_o$. Obviously, we have a mapping $u \to v \to z \to y(t_2)$, or $\Omega_o \to \Omega_1 \to \Omega_2 \to \Omega_3$. The class Ω_o has the property that, if $u_1,u_2 \in \Omega_o$ and H is any B-measurable subset of $[t_1,t_2]$, then the function $u(t) = u_1(t)$ if $t \in H$, $u(t) = u_2(t)$ if $t \in [t_1,t_2]-H$, is also an element of Ω_o. Obviously, Ω_1 possesses the same property. By a version of a Lyapunsv's theorem [see App. E], the set Ω_2 is convex and closed. (Then Ω_2 is compact since Ω_2 is certainly bounded). Finally, Ω_3 is the image of Ω_2 by means of the linear transformation $y(t_2) = X(t_2,t_1)\, [x(t_1) + z]$, and hence Ω_3 is also convex and closed (compact).

Let us extend each vector $u_k(t)$, $t_{1k} \leq t \leq t_{2k}$, in the whole interval $[t_o,T]$ by taking $u_k(t) = w$, a fixed arbitrary point of U. Then let us restrict $u_k(t)$ to the fixed interval $[t_1,t_2]$. Now $u_k(t)$ is an element of Ω_o, and its image in Ω_3 is given by

$$y_k(t_2) = X(t_2,t_1)[y(t_2) + \int_{t_1}^{t_2} X^{-1}(t,t_1)\,C(t,u_k(t))dt] ,$$

while

$$x_k(t_{2k}) = X(t_{2k},t_{1k})[x_k(t_{1k}) + \int_{t_{1k}}^{t_{2k}} X^{-1}(t,t_{1k})\,C(t,u_k(t))dt] .$$

Here $t_{1k} \to t_1$, $t_{2k} \to t_2$, $x(t_{1k}) \to x(t_1) = y(t_1)$ as $k \to \infty$, as well as $X(t_{2k},t_{1k}) \to X(t_2,t_1)$, and $X^{-1}(t,t_{1k}) \to X^{-1}(t,t_1)$ uniformly in $[t_o,T]$. Since $x_k(t_{2k}) \to x(t_2) = x_2$, we conclude that $y_k(t_2)$ has the same limit as $k \to \infty$, or $y_k(t_2) \to x(t_2) = x_2$. Thus, $x(t_2)$ belongs to the closure of Ω_3, and hence $x(t_2)$ belongs to Ω_3 since Ω_3 is closed. In other words, there is some $u \in \Omega_o$ which generates x. Since $g(t_1,x_1,t_2,x_2)$ is continuous on B, and hence on $B \cap B_o$, and this set is not empty and compact, we conclude that the functional $I[x,u] = g(t_1,x(t_1), t_2,x(t_2))$ takes on both minimum and maximum in Ω. Thereby (5.3.i) is proved.

We can prove now, as a corollary of (5.3.i), an analogous statement for Lagrange problems.

(5.3.ii) Existence Theorem (For Lagrange Problems and f and f_o Linear in x). Let $A = [t_o,T] \times E_n$, t_o, T finite, U a fixed compact set of the u-space E_m, $f = A(t)\,x + C(t,u)$, $f_o = A_o(t)\,x + C_o(t,u)$, A, A_o, C, C_o matrices with entries continuous in $[t_o,T]$ and $[t_o,T] \times U$ respectively. Let B be a closed subset of the $t_1x_1t_2x_2$-space. Let P be a compact subset of A and let Ω be the class, which we suppose not empty, of all admissible pairs x,u whose trajectory x possesses at least one point $(t^*,x(t^*)) \in P$. Then the functional (5.1.5) has an absolute minimum in Ω.

If $A = E_1 \times E_n$ then (5.3.i) still holds under the additional condition (c') of (3.4) (or alternate) (which here imply $A_o(t) = 0$). As before, conditions (c) and (c') hold here as a consequence of the hypotheses.

Proof. If we introduce an auxiliary variable x^o satisfying the differential equation and initial condition

$$dx^o/dt = A_o(t)x(t) + C_o(t,u(t)),\ x^o(t_1) = 0 ,$$

then we can write the canonic system together with this equation in the form

$$d\tilde{x}/dt = \tilde{A}(t)\tilde{x}(t) + \tilde{C}(t,u(t)) ,$$

where $\tilde{x} = (x^o, x^1, \ldots, x^n) = (x^o, x)$, $\tilde{A} = (A_o, A)$, $\tilde{C} = (C_o, C)$, and then $I[x,u] = x^o(t_2)$. We can now apply theorem (5.3.i), where A is replaced by $A_1 = [t_o, T] \times E_{n+1}$, B by $B_1 = B \times [x_1^o = 0] \times E_1 \subset E_{2n+4}$, P by $P_1 = P \times [x' = 0]$, g by $\tilde{g}(t_1, \tilde{x}(t_1), t_2, \tilde{x}(t_2)) = x^o(t_2)$. This proves theorem (5.3.ii).