## 9. Existence Theorems for Multidimensional Problems

### 9.1. First Existence Theorems

We begin with the situation described in (7.1) concerning problems with distributed controls. We are interested, for the moment, therefore, with the problem of the minimum of a functional

$$I[x,u] = \int_G f_o(t, (\mathcal{M}x)(t), u(t))dt, \qquad (9.1.1)$$

with state equation

$$(\mathcal{L}x)(t) = f(t, (\mathcal{M}x)(t), u(t)) \quad \text{a.e. in } G, \qquad (9.1.2)$$

and constraints

$$(\mathcal{M}x)(t) \in A(t), \; u(t) \in U(t, (\mathcal{M}x)(t)) \quad \text{a.e. in } G. \qquad (9.1.3)$$

Let G be bounded and open, A, U(t,y), M closed, Y, Z, S, T, $\mathcal{M}$, $\mathcal{L}$, as in (7.1.2). Let $f(t,y,u) = (f_1,\dots,f_r)$, $f_o(t,y,u)$ be given functions defined on M. A pair x, u, $x \in S$, $u \in T$, is said to be admissible if x, u satisfies requirements (a-h) of (7.1). We shall consider classes $\Omega$ of admissible pairs, and we shall seek the minimum of I[x,u] in $\Omega$.

We shall say that a class $\Omega$ of admissible pairs x, u is <u>closed</u> provided $\Omega$ has the following property: if $x \in S$, if $x_k$, $u_k$, $k = 1, 2, \dots$, are admissible pairs and belong to $\Omega$, if $x_k \to x$ weakly in S, and $\underline{\lim}\, I[x_k, u_k] < +\infty$ as $k \to \infty$, then at least one of the elements $u \in T$ guaranteed by lower closure

theorem of (8.3) is such that x, u belongs to $\Omega$ (besides x, u being an admissible pair, and $I[x,u] \leq \underline{\lim}\, I[x_k, u_k]$ as stated in lower closure theorem). Obviously, the class of all admissible pairs is certainly closed in this sense, under the same general assumptions of lower closure theorem of (8.3).

Given a family $\Omega = \{(x,u)\}$ of admissible pairs x, u, $x \in S$, $u \in T$, we shall also consider the corresponding class $\{x\}_\Omega$ of the elements x in $\Omega$, in symbols $\{x\}_\Omega = \{x \in S \mid (x,u) \in \Omega \text{ for some } u \in T\}$. Analogously, we shall consider the class $\{u\}_\Omega = \{u \in T, (x,u) \in \Omega \text{ for some } x \in S\}$.

We shall consider below closed classes $\Omega$ of admissible pairs x, u such that the corresponding set $\{x\}_\Omega$ is weakly sequentially compact in S.

(9.1.i) <u>Existence Theorem</u> (for problems with distributed controls). Let G, A, U(t,y), M be as in (7.1.2) with G bounded and open, A, M closed, let $f(t,y,u) = (f_1,\dots,f_r)$, $f_0(t,y,u)$ be continuous on M, $f_0$ scalar, and let S, Y, V, T, $\mathcal{M}$, $\mathcal{L}$, as in (7.2), with $\mathcal{M}$, $\mathcal{L}$, satisfying property (H). Let us assume that the sets $\tilde{\tilde{Q}}(t,y) = [\tilde{z} = (z^0, z) \mid z^0 \geq f_0(t,y,u), u \in U(t,y)]$ be convex, closed, and satisfy property (Q) in A. Let us assume that $(\psi_t)$ for some scaler L-integrable function $\psi(t) \geq 0$, $t \in G$, we have $f_0(t,y,u) \geq -\psi(t)$ for all $(t,y,u) \in M$. Let $\Omega$ be a nonempty closed class of admissible pairs x, u, that is, $x \in S$, $u \in T$, x, u satisfying (a-h) of (8.2), in particular satisfying the functional equation (9.1.2) and constraints (9.1.3), and let us assume that the corresponding set $\{x\}_\Omega$ is weakly sequentially compact in S. Then, the functional (9.1.1) possesses an absolute minimum in $\Omega$.

*Proof.* For every pair $x(t)$, $u(t)$, $t \in G$, we have $f_o \geq -\psi$, hence

$$I[x,u] = \int_G f_o(t, (\mathcal{M}x)(t), u(t))\, dt \geq -\int_G \psi(t)dt = -L_o,$$

since $\psi$ is L-integrable in G and $L_o$, therefore, is a finite constant. Let i denote the infimum of $I[x,u]$ in $\Omega$, hence $i \geq -L_o$, and also $i < +\infty$ since $\Omega$ is not empty, and thus i is finite. Let $x_k(t)$, $u_k(t)$, $t \in G$, $k = 1, 2, \ldots$, be a minimizing sequence for I in $\Omega$, that is, a sequence of admissible pairs, all in $\Omega$, with $I[x_k, u_k] \to i$ as $k \to \infty$. We can even assume that $i \leq I[x_k, u_k] \leq i + k^{-1} \leq i + 1$ for all $k = 1, 2, \ldots$.

By hypothesis the sequence $x_k$, $k = 1, 2, \ldots$, is weakly sequentially compact. Hence, there is a subsequence, say still [k] for the sake of simplicity, such that $[x_k]$ is weakly convergent in S toward an element $x \in S$. Thus, the sequence $x_k$, $u_k$, $k = 1, 2, \ldots$, has now the two properties requested in the lower closure property: $x_k \to x$ weakly in S, $i = \underline{\lim}\, I[x_k, u_k] < +\infty$. Thus, by lower closure theorem (8.3.i) there are elements $u \in T$ such that x, u is an admissible pair (satisfying (a-h) of (7.2)), and $I[x,u] \leq \underline{\lim}\, I[x_k, u_k] = i$. By closure property of the class $\Omega$, at least one of these elements u is such that the pair x, u belongs to $\Omega$. For this pair we have now $I[x,u] \geq i$, and finally, by comparison, $I[x,u] = i$. This proves the existence theorem.

We consider now the situation described in (7.5) concerning problems with both distributed and boundary controls. Thus, we are interested with the problem of the minimum of a functional

$$I[x,u,v] = \int_G f_o(t, (\mathcal{M}x)(t), u(t))dt + \int_\Gamma g_o(t, (\mathcal{K}x)(t), v(t))d\sigma, \qquad (9.1.4)$$

with state equations

$$(\mathcal{L}x)(t) = f(t, (\mathcal{M}x)(t), u(t)) \quad \text{a.e. in } G, \tag{9.1.5}$$

$$(\mathcal{J}x)(t) = g(t, (\mathcal{K}x)(t), v(t)) \quad \sigma - \text{a.e. in } \Gamma, \tag{9.1.6}$$

and constraints

$$(\mathcal{M}x)(t)\epsilon A(t),\ u(t)\epsilon U(t, (\mathcal{M}x)(t)) \quad \text{a.e. in } G, \tag{9.1.7}$$

$$(Kx)(t)\epsilon B(t),\ v(t)\epsilon V(t, (Kx)(t)) \quad \sigma - \text{a.e. in } \Gamma, \tag{9.1.8}$$

Let G, A(t), A, U(t,y), M, Y, Z, S, T, $\mathcal{M}$, $\mathcal{L}$ be as before. Let $\Gamma$ be a closed subset, say $\Gamma \subset \partial G$, which is the finite union of parts $\Gamma_1,\dots,\Gamma_n$, each $\Gamma_i$ the image of a fixed interval $I \subset E_{\nu-1}$ under a transformation of class K, say $\zeta_i$: $I \to \Gamma_i$. Thus a natural area measure function $\sigma$ is defined on $\Gamma$. Let B(t), B, $V(t,\mathring{y})$, $\mathring{M}$, $\mathring{Y}$, $\mathring{Z}$, $\mathring{T}$, K, I as in (7.5). Let $f(t,y,u) = (f_1,\dots,f_r)$, $f_o(t,y,u)$ be given continuous functions defined on M, and $g(t,\mathring{y},v) = (g_1,\dots,g_{r'})$, $g_o(t,\mathring{y},v)$ continuous functions on $\mathring{M}$. For every $(t,y)\epsilon A$ and for every $(t,\mathring{y})\epsilon B$, let $\tilde{\tilde{Q}}(t,y)$ and $\tilde{\tilde{R}}(t,y)$ be the sets

$$\tilde{\tilde{Q}}(t,y) = [(z^o,z)\epsilon E_{r+1} | z^o \geq f_o(t,y,u),\ z = f(t,y,u), u\epsilon U(t,y)],$$

$$\tilde{\tilde{R}}(t,\mathring{y}) = [(z^o, z)\epsilon E_{r+1} | z^o \geq g_o(t,\mathring{y},v),\ z = g(t,\mathring{y},v), v\epsilon V(t,\mathring{y})].$$

(9.1.ii) <u>Existence Theorem</u> (for problem with distributed and boundary controls). Let G, A, U(t,y), M be as in (7.1.2) with G bounded and open, A, M closed, let let $\Gamma$, B, $\mathring{M}(t,\mathring{y})$, $\mathring{M}$ be as in (7.5) with $\Gamma$, B, $\mathring{M}$ closed, let f, $f_o$ be continuous on M and g, $g_o$ continuous on $\mathring{M}$, and let S, Y, $\mathring{Y}$, $\mathring{Z}$, T, $\mathring{T}$, $\mathcal{M}$, $\mathcal{L}$, $\mathcal{K}$, $\mathcal{J}$ as in (7.5), with $\mathcal{M}$, $\mathcal{L}$, $\mathcal{K}$, $\mathcal{J}$ satisfying axiom (H) of (7.5). Let us assume that the sets $\tilde{\tilde{Q}}(t,y)$ satisfy property (Q) in A, and that the sets $\tilde{\tilde{R}}(t,\mathring{y})$ satisfy property (Q) in $\mathring{M}$. Let us assume that $f_o(t,y,u) \geq -\varphi(t)$ for all $(t,y,u) \in M$ and some $\varphi \geq 0$, $\varphi \in L_1(G)$, and that $g_o(t,\mathring{y},v) \geq -\psi(t)$ for all $(t,\mathring{y},v) \in \mathring{M}$ and some $\psi \geq 0$, $\psi \in L_1(\Gamma)$. Let $\Omega$ be a nonempty closed class of admissible triples x, u, v (that is, $x \in S$, $u \in T$, $v \in \mathring{T}$, x, u, v satisfying ( a-h ) of (7.5), in particular satisfying state equations (9.1.5), (9.1.6), and constraints (9.1.7), (9.1.8), and let us assume that the corresponding sets $\{x\}_\Omega$ is weakly sequentially relatively compact. Then, the functional (9.1.4) possesses an absolute minimum in $\Omega$.

Proof. The proof is the same as for (9.1.i) with obvious modifications concerning the terms from the boundary. Instead of lower closure theorem (8.3.i), we shall use now (8.3.ii).

The next existence theorem is a modification of (9.1.i) whose assumption $(\psi_t^*)$ of (8.5) will replace $(\psi_t)$, and we shall use the integral $\int_G |(\mathcal{L}x)(t)|dt$, with $|(\mathcal{L}x)(t)|$ Euclidean norm in $E_r$. Note that for p = 1, we have

$$\int_G |(\mathcal{L}x)(t)|dt = \int_G |y(t)|dt = \int_G (\Sigma_j (y^j(t))^2)^{1/2} dt \leq$$

$$\Sigma_j \int_G |y^j(t)|dt \leq s^{1/2}[\Sigma_j (\int_G |y^j(t)|dt)^2]^{1/2} = s^{1/2}||y||_1$$

Analogously, for $p > 1$, we have

$$\int_G |(\mathcal{L}x)(t)|dt \leq \Sigma_j (\text{meas } G)^{1/q} (\int_G |y^j(t)|^p dt)^{1/p} \leq$$

$$(\text{meas } G)^{1/q}\ s^{1/2} (\Sigma_j (\int_G |y^j(t)|^p dt)^{2/p})^{1/2}$$

$$= s^{1/2} (\text{meas } G)^{1/q} ||y||_p$$

For $p = \infty$ we have

$$\int_G |(\mathcal{L}x)(t)|dt \leq \Sigma_j\ (\text{meas } G)\ (\text{Ess Sup}|y^j(t)|)$$

$$= (\text{meas } G) s^{1/2} [\Sigma_j (\text{Ess Sup}|y^j(t)|)^2]^{1/2}$$

$$= s^{1/2} (\text{meas } G) ||y||_\infty$$

Thus, in any case, we have

$$\int_G |(\mathcal{L}x)(t)|dt \leq C_p ||y||_p \tag{10.1.4}$$

for some constant $C_p$ which depends only on G and p, $1 \leq p \leq \infty$.

(9.1.iii) Existence Theorem (for problems with distributed controls).

The same as (9.1.i) with $(\psi_t^*)$ replacing $(\psi_t)$, and $\Omega$ any nonempty closed class of admissible pairs x, u such that $\int_G |(\mathcal{L}x)(t)|dt \leq K$ for some constant K.

<u>Proof</u>. We proceed as in the proof of (8.5.ii) by constructing the closed set $H \subset E_\nu$ of measure zero, the N disjoint open sets $G_\gamma$, $\gamma = 1,\dots,N$, whose union is G-H, and the functions

$$\bar{f}_{o\gamma}(t,y,u) = f_o(t,y,u) + \Sigma_j\ b_{\gamma j}\ f_j(t,y,u) \geq -\psi_\gamma(t)$$

for $t \in G_\gamma$, $(t,y,u) \in \Omega$, $\psi_\gamma \in L(G_\gamma)$, $b_{\gamma j}$ constants. Let b be the largest of all constants $|b_{\gamma j}|$ and let $L_\gamma = \int_{G\gamma} \psi_\gamma(t)dt$. Then, for every pair x, u in $\Omega$ we have

$$I[x,u] = \int_G f_o \, dt = \sum_\gamma \int_{G_\gamma} f_o \, dt = \sum_\gamma \int_{G_\gamma} (\bar{f}_o - \sum_j b_{\gamma j} f_j) \, dt \geq$$

$$\sum_\gamma \int_{G_\gamma} \psi_\gamma(t)dt - rNb \int_G |(\mathcal{L}x)(t)| dt$$

$$= \sum_\gamma L_\gamma - rNb \, K.$$

Thus, if i denotes the infimum of I[x,u] in $\Omega$, certainly i is finite. The proof is now identical to the proof of (9.1.i) where lower closure theorem (8.5.ii) is used instead of (8.5.i).

An analogous form for theorem (9.1.ii) holds, similar to (9.1.iii), but we do not repeat it here.

Another extension of existence theorems (9.1.i) and (9.1.ii) can be obtained by observing that in many cases hypothesis $(H_1)$ of (8.5) holds in the stronger form $(H_1')$, that is, $x_k \to x$ weakly in S implies that $\mathcal{J}x_k \to \mathcal{J}x$ strongly in $\mathring{\mathcal{J}}$ (instead of weakly as in $(H_1)$). In such a situation the existence can be guaranteed by less stringent requirements on the other data. Indeed, it is enough then to demand that the set $\mathring{\tilde{\tilde{R}}}(t,y)$ satisfy property (U) (instead of (Q)).

More generally, we may assume the modified form of hypothesis $(H_1)$:

$(H_g)$ the same as $(H_1)$ where the weak convergences of $\mathcal{L}x_k$ and of $\mathcal{J}x_k$ are are replaced by: $(\mathcal{L}x_k)^i \to (\mathcal{L}x)^i$ weakly in $L_1(G)$, $i = 1,\dots,\rho$; $(\mathcal{L}x_k)^i \to (\mathcal{L}x)^i$

strongly in $L_1(G)$, $i = \rho+1,\dots,r$; $(\mathcal{J}x_k)^i \to (\mathcal{J}x)$ weakly in $L_1(\Gamma)$, $i = 1,\dots,\rho'$; $(\mathcal{J}x_k)^i \to (\mathcal{J}x)^i$ strongly in $L_1(\Gamma)$, $i = \rho'+1,\dots,r'$, where $\rho$, $\rho'$ are fixed numbers $0 \leq \rho \leq r$, $0 \leq \rho \leq r'$.

In this situation (9.1.ii) becomes

(9.1.iv). Existence Theorem (for problems with distributed and boundary controls. The same as (9.1.ii) with $(H_g)$ replacing $(H_1)$, the sets $\tilde{\tilde{Q}}(t,y)$ satisfying property $(Q_\rho)$ in A and the sets $\tilde{\tilde{R}}(t,y)$ satisfying property $(Q_{\rho'})$ in B.

The proof is the same as for (9.1.ii).

## 9.2 Extensions

We shall now separate the m components $u = (u^1,\dots,u^m)$ of the variable u in two classes, say $u' = (u^1,\dots,u^\alpha)$ and $u'' = (u^{\alpha+1},\dots,u^m)$, so that we can write $u = (u',u'')$, and we shall assume that $U(t,y) = U'(t,y) \times U''(t,y)$ for all $(t,y) \in A$ with $U'(t,y) \subset E_\alpha$, $U''(t,y) \subset E_{m-\alpha}$, $0 \leq \alpha \leq m$. Since $\alpha$ and m are arbitrary, u will play the same role that u has played so far, and we extend the previous considerations by allowing u" to play a different role.

Let T be of the form $T = T' \times T''$, where T' is now the space (or set) of all measurable vector functions $u'(t) = (u^1,\dots,u^\alpha)$, $t \in G$, and T" is a Banach space of vector functions $u''(t) = (u^{\alpha+1},\dots,u^m)$, $t \in G$, with norm $||u''||$, say

$$T'' = (L_p(G))^{m-\alpha}, \quad 1 \leq p \leq +\infty,$$

$$||u''|| = \left(\Sigma_{j=\alpha+1}^m ||u^j||_{L_p}^2\right)^{1/2} .$$

We shall assume that $f_o(t,y,u')$, $f(t,y,u') = (f_1,\dots,f_r)$ depend on $t,y,u'$ only, thus, $f_o$, $f$ are defined on the set $M'$ of all $(t,y,u')$ with $(t,y) \in A$, $u' \in U'(t,y)$. In the present situation, instead of the sets $\tilde{\tilde{Q}}(t,y)$ we shall consider the sets $Q'(t,y) = [\tilde{z} = (t^o,t) | z^o \geq f_o(t,y,u'),\ z = f(t,y,u')$, $u' \in U'(t,y)] \subset E_{n+1}$, and the sets $U''(t,y) \subset E_{m-\alpha}$. We shall consider now $\mathcal{M}$ and $\mathcal{L}$ as operators $\mathcal{M}$: $S \times T'' \to Y$, $\mathcal{L}$: $S \times T'' \to Z$ defined on $S \times T''$, that is, $y = \mathcal{M}(x,u'') \in Y$, $v = \mathcal{L}(x,u'') \in Z$ for all $(x,u'') \in S \times T''$. Also, we shall denote by $Z_o$ a space of real-valued function $z_o(t)$, $t \in G$, which are $L_p$-integrable in G for some $p$, $1 \leq p_o \leq +\infty$, or $Z_o \subset Lp_o(G)$, and we shall take in $Z_o$ the norm $||z_o||_L$. We shall then consider an operator $\mathcal{L}_o(x,u'')$, or $\mathcal{L}_o$: $S \times T'' \to Z_o$, mapping each pair $(x,u'') \in S \times T''$ into an element $z_o = \mathcal{L}_o(x,u'') \in Z_o$. Instead of axiom $(H_1)$ of (7.5), we shall now assume: $(H_1^*)$, if $x$, $x_k \in S$, $u''$, $u''_k \in T''$, $k = 1, 2,\dots$, if $x_k \to x$ weakly in S and $u''_k \to u''$ weakly in $T''$, then $y_k \to y$ strongly in Y, $z_k \to$ weakly in Z, $z_{ok} \to {}_o$ weakly in $Z_o$, where $y = \mathcal{M}(x,u'')$, $z = \mathcal{L}(x,u'')$, $z_o = \mathcal{L}_o(x,u'')$, $y_k = \mathcal{M}(x_k, u''_k)$, $z_k = \mathcal{L}(x_k, u''_k)$, $z_{ok} = \mathcal{L}_o(x_k, u''_k)$.

In the present situation we say that a pair $x(t)$, $u(t)$, $t \in G$, is admissible provided $x \in S$, $u = (u',u'')$, $u \in T'$, $u'' \in T''$, $y = \mathcal{M}(x,u'') \in Y$, $= \mathcal{L}(x,u'') \in Z$, $(t,y(t)) \in A$ a.e. in G, $u'(t) \in U'(t,y(t))$, $u''(t) \in U''(t,y(t))$, a.e. in G, $(t) = f(t,y(t))$ a.e. in G, $f_o(t,y(t), u'(t))$ is L-integrable in G, and $z_o = \mathcal{L}(x,u'') \in Z_o$.

We deal here with the problem of the minimum of a functional

$$I[x,u',u''] = \int_G [f_o(t,(\mathcal{M}(x,u''))(t), u'(t)) + (\mathcal{L}_o(x,u''))(t)]dt \tag{9.2.1}$$

in a class $\Omega$ of admissible pairs $x(t)$, $u(t)$, $t \in G$, $u = (u',u'')$, and the constraints and functional relation are now of the forms

$$(\mathcal{L}(x,u''))(t) = f(t,(\mathcal{M}(x,u''))(t), u'(t)) \qquad \text{a. . in } G, \qquad (9.2.2)$$

$$(t,(\mathcal{M}(x,u''))(t)) \in A, \ u'(t) \in U'(t,(\mathcal{M}(x,u''))(t)) \qquad \text{a.c. in } G. \qquad (9.2.3)$$

We shall say now that the functional (9.2.1) possesses the property of lower closure at the elements $x \in S$, $u'' \in T''$, provided the following occurs: for every sequence $x_k$, $u_k$, $k = 1, 2, \ldots$, $u_k = (u_k', u_k'')$ of admissible pairs, $x_k \in S$, $u_k' \in T'$, $u_k'' \in T''$, $x_k \to x$ weakly in $S$, $u_k'' \to u''$ weakly in $T''$, $\underline{\lim}\, I[x_k, u_k', u_k'']$ $< +\infty$, there is an element $u' \in T$, such that if $u = (u', u'')$, then $x$, $u$ is an admissible pairs, and $I[x, u', u''] \leq \underline{\lim}\, I[x_k, u_k', u_k'']$.

With this definition the lower closure theorem (9.3.i) holds only with formal changes.

(9.2.i) (a lower closure theorem). Let $G$, $A$, $U'(t,y)$, $U''(t,y)$, $M'$, and $Y$, $Z$, $Z_o$, $S$, $T'$, $T''$, $\mathcal{M}$, $\mathcal{L}$, $\mathcal{L}_o$, $f(t,y,u') = (f_1,\ldots,f_r)$, $f_o(t,y,u')$ as in (8.1.2), or detailed above, with $G$ bounded and open, $A$, $M$ closed, and $f$, $f_o$, continuous on $M'$. Let us assume that condition $(H_1^*)$ holds, that the sets $\tilde{\tilde{Q}}'(t,y) \subset E_{r+1}$ are convex, closed, and satisfy property (Q) in A, and that property $(\psi_t)$ holds. Then the functional (9.2.1) has the property of lower closure at all $x,u''$, $x \in S$, $u'' \in T''$.

The proof is similar to the one for (9.3.i) and is left as an exercise for the reader.

We shall consider below classes $\Omega$ of admissible pairs x, u, $u = (u', u'')$, $x \in S$, $u' \in T'$, $u'' \in T''$. We shall say that such a class $\Omega$ is closed provided: if $x \in S$, $u'' \in T''$, if $x_k$, $u_k$ with $u_k = (u'_k, u''_k)$, k = 1, 2,..., are admissible pairs and belong to $\Omega$, if $x_k \to x$ weakly in S and $u''_k$ weakly in $T''$, and $\underline{\lim}$ $I[x_k, u'_k, u''_k] < +\infty$ as $k \to \infty$, than at least one of the elements $u' \in T'$ guaranteed by lower closure theorem (10.2.i) is such that the pair x, u with $u = (u', u'')$ belongs to $\Omega$.

Finally, given any class $\Omega$ of admissible pairs x, u with $u = (u', u'')$, we shall consider the sets $\{x\}_\Omega = \{x | x \in S, (x, u) \in \Omega$ for some $u = (u', u'') \in T' \times T''\}$, and the sets $\{u''\}_\Omega = \{u'' | u'' \in T'', (x, u', u'') \in \Omega$ for some $x \in S, u' \in T'\}$.

We are now in a position to state an existence theorem for the present situation.

( 9.2.ii) (an existence theorem). Let G, A, $U'(t,y)$, $U''(t,y)$, M' and Y, Z, $Z_o$, S, T', T'', $\mathcal{M}$, $\mathcal{L}$, $\mathcal{L}_o$, $f_o(t,y,u')$ as in (8.1.2), or detailed above, with G bounded and open, A, M closed, and f, $f_o$ continuous on M'. Let us assume that condition ($H_1^*$) holds, that the sets $\tilde{\tilde{Q}}'(t,y) \subset E_{r+1}$ are closed, convex, and satisfy property (Q) in A, and that property ($\psi_t$) holds. Let $\Omega$ be a nonempty closed class of admissible pairs x, u with $u = (u', u'')$, $x \in S$, $u' \in T$, $u'' \in T''$, and let us assume that the corresponding sets $\{x\}_\Omega$, $\{u''\}_\Omega$ are weakly sequentially compact (in S and T'' respectively). Then the functional (9.2.1) possesses an absolute minimum in $\Omega$. If condition ($\psi_t^*$) holds instead of ($\psi$), then this is still true under the additional hypothesis that $\int |\mathcal{L}(x,u'')(t)| dt \leq K$, for all pairs x,u in $\Omega$, $u = (u', u'')$, and some constant K.

The proof is similar to the one for (9.3.i) and is left as an exercise for the reader. We do not exclude in all theorems above that either $U' = E_\alpha$, or $U'' = E_{m-\alpha}$, or both. Also. in the search for the minimum of $I[x,u]$ in $\Omega$ it is always possible to restrict $\Omega$ to the nonempty subclass $\Omega_o \subset \Omega$ of all pairs x, u in $\Omega$ with $I[x,u] \leq N$ for some constant N.

## 9.3. Lagrange Problems of Optimization With a Total Differential System of Partial Differential Equations

This is the problem which was sketched briefly in (7.2),no. 12. Let G be an open bounded subset of the t-space $E_\nu$, $t = (t^1,\dots,t^\nu)$, of class $K_1$ (see App. C ), that is with "smooth" boundary $\partial G$ in the sense of Morrey, and take for S the Sobolev space $S = (W_p^\ell(G))^n$ so that every element $x \in S$ is an n-vector function $x(t) = (x^1,\dots,x^n)$, $t \in G$, whose components $x^i \in L_p(G)$, $1 \leq p \leq +\infty$, possess first order generalized partial derivatives $\nabla x = [\partial x^i/\partial t^s,\ i = 1,\dots,n,\ s = 1,\dots,\nu]$ a.e. in G, and all $x^i$, $\partial x^i/\partial t^s \in L_p(G)$, $i = 1,\dots,n$, $s = 1,\dots,\nu$. Let $Y = (L_p(G))^\nu$, $Z = (L_p(G))^{n\nu}$, let $\mathcal{M}$: $S \to Y$ be the identity operator, mapping $x \in S$ into $y = x$ as an element of Y, and let $\mathcal{L}$: $S \to V$ be defined by $\mathcal{L}x = \nabla x$. We have taken $s = n$, $r = n\nu$. Now property (H) is trivial, since $x_k \to x$ weakly in S implies that $x_k \to x$ strongly in Y, and $\nabla x_k \to \nabla x$ weakly in V, that is, $\mathcal{M}x_k \to \mathcal{M}x$ strongly in Y, and $\mathcal{L}x_k \to \mathcal{L}x$ weakly in Z. Let T denote the space (set) of all vector functions $u(t) = (u^1,\dots,u^m)$, $t \in G$, whose coordinates $u^j$ are measurable in G, $j = 1,\dots,m$. We are now in a position to study the problem of the minimum of a multiple integral

$$I[x,u] = \int_G f_o(t, x(t), u(t))dt \qquad (9.3.1)$$

with side conditions expressed by a total differential (or Dieudonné-Rashevski) system

$$\partial x^i/\partial t^s = f_{is}(t, x(t), u(t)), \text{ a.e. in } G, \; i = 1,\dots,n, \; s = 1,\dots,\nu, \tag{9.3.2}$$

constraints of the form

$$x(t) \in A(t), u(t) \in U(t, x(t)), \text{ a.e. in } G, \tag{9.3.3}$$

and possible boundary conditions concerning the values of the functions $x^i$ on the boundary $\partial G$ of G. If f denotes the $n\nu$-vector function $f(t,x,u) = (f_{is}$, $i = 1,\dots,n$, $s = 1,\dots,\nu)$, then the differential system can be written in the form $(\nabla x)(t) = f$, or

$$dx/dt = f(t, x(t), u(t)), \text{ a.e. in } G.$$

In this situation, closed classes $\Omega$ such that $\{x\}_\Omega$ is weakly sequentially compact in S can be obtained without difficulties. If we denote by (B) the set of required boundary conditions concerning the values of the functions $x^i(t)$, $i = 1,\dots,n$, on the boundary $\partial G$ of G (see App. D ), we shall only need that the following assumption, or property (P), is satisfied: if $(x,u)$, $(x_k, u_k)$, $k = 1, 2,\dots$, are admissible pairs, in the given class $\Omega$, if $x_k \to x$ weakly in S as $k \to \infty$, and all $x_k$ satisfy boundary conditions (B), then x also satisfies boundary conditions (B).

In most cases $\partial G$ is divided up into regular finitely many parts (see App. C ) and some $x^i$ are assigned on certain parts $\Gamma_j$. These conditions certainly satisfy property (P) in all cases we shall consider below.

Actually, property (P) will be subsumed in the property of closure of the class $\Omega$. Indeed, for any admissible pair x, u, the element $x \in S$ satisfies the boundary conditions independently of the associated control function u. Hence, if we request that $\Omega$ is made up of pairs for which x satisfies boundary condition (B) and we request that $\Omega$ is closed according to the definition of no. 9.1, then certainly the given boundary conditions (B) must satisfy property (P).

Case 1. M compact, that is, A compact, M closed, all sets $U(t,x)$ compact and uniformly bounded. In this situation the variables t, x, u for $(t,x,u) \in M$, and the continuous functions $f(t,x,u)$, $f_o(t,x,u)$ are bounded, say $|t|$, $|x|$, $|u|$, $|f(t,x,u)|$, $|f_o(t,x,u)| \leq N$ for some constant N and all $(t,x,u) \in M$. The condition $(\psi_t)$ is certainly satisfied since $f_o \geq -N$. For every admissible pair x, u we certainly have $|x^i(t)| \leq N$, $|\partial x^i/\partial t^s| = |f_{is}(t,x,u)| \leq N$. Hence the functions $x^i(t)$ are equibounded, equilipschitzian, equicontinuous, and the functions $\partial x^i/\partial t^s$ are equiabsolutely continuous. The set $\{x\}_\Omega$ is therefore weakly sequentially compact in the topology of $S = (W_p(G))^\nu$ for any choice of the number $p$, $1 \leq p \leq +\infty$. The sets $\tilde{\tilde{Q}}(t,x) \subset E_{n\nu+1}$ need now to be convex, and in this situation they certainly satisfy property (Q) in A.

(9.3.i) Let G be bounded, open, and of class K, let A, $U(t,x)$, M be as defined with M compact. Let $f_o(t,x,u)$, $f(t,x,u) = (f_{is})$ be continuous on M, and let us assume that the sets $\tilde{\tilde{Q}}(t,x) = [(z^o,z) | z^o \geq f_o(t,x,u),\ z = f(t,x,u),\ u \in U(t,x)]$ $\subset E_{n\nu+1}$ be convex for every $(t,x) \in A$. Let (B) be a given system of boundary conditions concerning the values of x on $\partial G$, and let $\Omega$ be a nonempty closed class of pairs $x(t) = (x^1,\dots,x^n)$, $u(t) = (u^1,\dots,u^m)$, $t \in G$, each x satisfying boundary conditions (B), with $x^i \in W^1_p(G)$, $1 \leq p \leq +\infty$, $i = 1,\dots,n$, $u^j$ measurable in G, $j = 1,\dots m$, x, u satisfying (9.3.2), (9.3.3). Then the integral (9.3.1) has an absolute minimum in $\Omega$.

It may be required that the elements $x \in S$ of the pairs x, u in $\Omega$ satisfy relations of the form

$$||x^i||_p \leq N_i \qquad \text{for } i \in \{\beta\}$$

$$||\partial x^i/\partial t^s||_p \leq N_{is} \qquad \text{for } i \in \{\beta\}_s \qquad (9.3.4)$$

for given constants N, $N_{is}$ and all i of given systems $\{\beta\}$, $\{\beta\}_s$, $s = 1,\dots,\nu$, of indices 1, 2,...,n, (which may be empty). These conditions are preserved by weak convergence in S, and hence can be used in defining closed classes $\Omega$ of pairs x, u.

Case 2. A, M closed, all $p_i > 1$. In this situation condition $(\psi_t)$, condition $(\psi_t^*)$ must be required explicitly, and so property (Q) of the sets $\tilde{\tilde{Q}}(t,x)$. The weak compactness of the sets $\{x\}_\Omega$ can now be guaranteed by means of relations (9.3.4). Existence theorem (9.1.i) now yields.

(9.3.ii) Let G be bounded, open, and of class $K_1$, let A, U(t,x), M be as defined, with A and M closed. Let $f_o(t,x,u)$, $f(t,x,u) = (f_{is})$ be continuous on M, let us assume that $(\psi_t)$: $f_o(t,x,u) \geq -\psi(t)$ for all $(t,x,u) \in M$ and some

L-integrable function $\psi(t) \geq 0$, $t \in G$, and let us assume that the sets $\tilde{\tilde{Q}}(t,x) = [(z^o z) | z^o \geq f_o(t,x,u),\ z = f(t,x,u),\ u \in U(t,x)] \subset E_{n\nu+1}$ are closed, convex, and satisfy property (Q) in A. Let (B) be a given system of boundary conditions concerning the values of x on $\partial G$, and let $\Omega$ be a nonempty closed class of pairs $x(t) = (x^1,\ldots,x^n)$, $u(t) = (u^1,\ldots u^m)$, $t \in G$, each x satisfying boundary conditions (B), with $x^i \in W_p^1(G)$, $1 < p \leq +\infty$, $i = 1,\ldots,n$, $u^j$ measurable in G, $j = 1,\ldots,m$, x, u satisfying (9.3.2), (9.3.3), and given inequalities

$$||x^i||_p \leq N_i \qquad \text{for } i \in \{\beta\},$$

$$||\partial x^i/\partial t^s||_p \leq N_{is} \qquad \text{for } i \in \{\beta\}_s, \tag{9.3.5}$$

for given constants $N_i$, $N_{is}$ and all i of given systems $\{\beta\}$, $\{\beta\}_s$, $s = 1,\ldots,\nu$, of indices $1,\ldots,n$ (which may be empty). Assume that $(x,u) \in \Omega$, $I[x,u] \leq L_o$ implies

$$||x^i||_p \leq L_i,$$

$$||\partial x^i/\partial t^s||_p \leq L_{is}, \tag{9.3.6}$$

for some constants $L_i$, $L_{is}$ (which may depend on L, $N_i$, $N_{is}$, G, (B), $\Omega$) and for all $i = 1,\ldots,n$, which are not in $\{\beta\}$, $\{\beta\}_s$ respectively. Then the integral (9.3.1) has an absolute minimum in $\Omega$.

In (9.3.ii) condition $(\psi_t)$ can be replaced by the much weaker assumption $(\psi_t^*)$: for every $\bar{t} \in \mathrm{cl}G$ there is a neighborhood $N(\bar{t})$ of $\bar{t}$ in $E_\nu$, real constants

$b_{is}$, $i = 1,\dots,n$, $s = 1,\dots,\nu$, and some L-integrable function $\psi(t)$, $t \in N(\bar{t})$, such that

$$\bar{f}_o(t,x,u) = f_o(t,x,u) + \Sigma_{i=1}^{n} \Sigma_{j=1} b_{is} f_{is}(t,x,u) \geq \psi(t)$$

for all $(t,x,u) \in M$, $t \in N(\bar{t}) \cap G$.

Indeed, in the present situation, the additional condition $\int_G |(\mathcal{L}x)(t)| dt \leq K$ requested in (10.1.ii), reduces $\int_G |\partial x^i/\partial t^s| dt \leq L'_{is}$, $i = 1,\dots,n$, $s = 1,\dots,\nu$, for constants $L'_{is}$, and these relations are certainly satisfied as a consequence of relations ( 9.3.5), ( 9.3.6) and ( 9.1.4).

Case 3. A, M closed, all $p_i = 1$. In this situation the weak compactness of the sets $\{x\}_\Omega$ is not guaranteed by relations ( 9.3.5), ( 9.3.6). Nevertheless a growth condition, as the condition $(\varepsilon)$ below, guarantees that for the pairs x, u with $I[x,u] \leq L_o$, the derivatives $\partial x^i/\partial t^s$, $i = 1,\dots n$, $s = 1,\dots,\nu$, are equi-absolutely integrable in G. As a consequence $\int_G |\partial x^i/\partial t^s| dt \leq L_{is}$ for suitable constants $L_{is}$ and all $i = 1,\dots,n$, $s = 1,\dots,\nu$. If we know that also $\int_G |x^i| dt \leq L_i$, $i = 1,\dots,n$, then the set $\{x\}_\Omega$ is certainly weakly compact in the topology of $S = (W_1^1(G))^n$. Thus, existence theorem (9.1.i) yields

( 9.3.iii) Let G be bounded, open, and of class $K_1$, let A, U(t,x), M be defined, with A and M closed. Let $f_o(t,x,u)$, $f(t,x,u) = (f_{is})$ be continuous on M, and let us assume that the sets $\tilde{\tilde{Q}}(t,x) = [(z^o,z) | z^o \geq f_o(t,x,u),\ z = f(t,x,u),\ u \in U(t,x)] \subset E_{n\nu+1}$ are closed, convex, and satisfy property (Q) in A. Let us assume that the following growth condition holds $(\varepsilon)$: for every $\varepsilon > 0$ there is some L-integrable function $\psi_\varepsilon(t) \geq 0$, $t \in G$, which may depend upon $\varepsilon$, such that $|f(t,x,u)| \leq \psi_\varepsilon(t) + \varepsilon f_o(t,x,u)$ for all $(t,x,u) \in M$. Let (B) be a given system of boundary conditions concerning the values of x on $\partial G$, and let $\Omega$ be a nonempty closed class of pairs $x(t) = (x^1,\dots,x^n)$, $u(t) = (u^1,\dots,u^m)$, $t \in G$, each x satisfying boundary conditions (B), with $x^i \in W_1^1(G)$, $i = 1,\dots,n$, $u^j$ measurable in G, $j = 1,\dots,m$, u satisfying (10.3.2), (10.3.3), and given inequalities

$$||x^i|| \leq N_i \quad \text{for } i \in \{\beta\}$$

for given constants $N_i$ and all i of a given system $\{\beta\}$ of indices $1,\dots,n$ (which may be empty). Assume that $(x,u) \in \Omega$, $I[x,u] \leq L_o$ implies

$$||x^i|| \leq L_i$$

for some constants $L_i$ (which may depend on $L_o$, $N_i$, G, (B), $\Omega$) and for all $i = 1,\dots,n$, which are not in $\{\beta\}$. Then the integral (9.3.i) has an absolute minimum in $\Omega$.

Note that the usual condition $(\psi_t)$: $f_o \geq -\psi$ in G, is certainly satisfied here since assumption $(\varepsilon)$ for $\varepsilon = 1$ certainly implies $0 \leq \psi_1(t) + f_o$, or $f_o \geq - \psi_1$.

We may consider here the McShane-type growth condition, namely (M): for every compact subset $A_o$ of A and $\varepsilon > 0$ there is a number $\bar{u} \geq 0$ such that $(t,x) \in A_o$, $u \in U(t,x)$, $|u| \geq \bar{u}$ implies $1 \leq \varepsilon f_o(t,x,u)$, $|f(t,x,u)| \leq \varepsilon f_o(t,x,u)$.

Under this condition (M), both condition ($\varepsilon$) above is satisfied, and the sets $\tilde{\tilde{Q}}(t,x)$, if convex, are closed and satisfy property (Q) in A (see App. A ).

Finally, we may consider the Tonelli-Nagumo-type growth condition, namely, ($\Phi$): for every compact subset $A_o$ of A there are constants C, D $\geq$ 0 and a scalar continuous function $\Phi(\zeta)$, $0 \leq \zeta < +\infty$, such that $\Phi(\zeta)/\zeta \to +\infty$ as $\zeta \to +\infty$, and $f_o(t,x,u) \geq \Phi(|u|)$, $|f(t,x,u)| \leq C + D|u|$ for all $(t,x) \in A_o$, $u \in U(t,x)$. Under this condition ($\Phi$), certainly condition (M) holds, and hence condition ($\varepsilon$) and property (Q) of the sets $\tilde{\tilde{Q}}(t,x)$, if convex.

## 9.4. Free Problems for Multiple Integrals

This is the problem which was sketched briefly in (7.2), no. 11.

Let G be an open bounded subset of the t-space $E_\nu$, $t = (t^1,\dots t^\nu)$, of class $K_1$ (see App. C ), that is, with "smooth" boundary $\partial G$ in the sense of Morrey, and take for S the Sobolev space $S = (W_p^1(G))^n$, for some p, $1 \leq p \leq +\infty$ so that every element $x \in S$ is an n-vector function $x(t) = (x^1,\dots x^n)$, $t \in G$, whose components $x^i \in L_p(G)$ possess first order generalized partial derivatives $\nabla x = [\partial x^i/\partial t^s$, $i = 1,\dots,n$, $s = 1,\dots,\nu]$ a.e. in G, and all $x^i$, $\partial x^i/\partial t^s$ $\in L_p(G)$, $i = 1,\dots,n$, $s = 1,\dots,\nu$. Let $Y = (L_p(G))^n$, $Z = (L_p(G))^{n\nu}$, let $\mathcal{M}$: $S \to Y$ be the identity operator, mapping $x \in S$ into $y = x$, as an element of Y, and let $\mathcal{L}$: $S \to Z$ be defined by $\mathcal{L}x = \nabla x$. We have taken $s = n$, $r = n\nu$. Property (H) is now trivial as pointed out in (9.3). Let $m = n\nu = r$ and let T denote the space (set) of all vector functions $u(t) = (u_{is}$, $i = 1,\dots,n$, $s = 1,\dots,\nu)$, whose components $u_{is}$ are measurable in G, and take $U = E_{n\nu}$. Then the set M has the form $M = A \times E_{n\nu}$, and we take for $f(t,x,u) = (f_{is},\ i = 1,\dots,n,\ s = 1,\dots,\nu)$ the vector defined on M by taking $f_{is} = u_{is}$, or $f = u$. The equation $\mathcal{L}x = f$ reduces now to $\nabla x = u$. In other words $u(t) = (\nabla x)(t)$ is defined a.e. in G by the element x of S. We are now in a position to study the minimum of a multiple integral.

$$I[x] = \int_G f_o(t,\ x(t),\ (\nabla x)(t))dt \qquad (9.4.1)$$

with the only constraint $(t,\ x(t)) \in A \subset E_{\nu+n}$, and possible boundary conditions concerning the values of the functions $x^i$ on the boundary $\partial G$ of G.

The integral (9.4.1) is said to be lower semicontinuous at an element $x \in S$, provided $x_k \in S$, $k = 1, 2, \dots$, $x_k \to x$ weakly in S as $k \to \infty$, $\underline{\lim}\ I[x_k] < +\infty$ implies $I[x] \leq \underline{\lim}\ I[x_k]$.

In the present situation, lower closure theorems (8.3.i), (8.3.ii) yield the following lower semicontinuity statement.

(9.4.i) (a semicontinuity theorem). Let $G \subset E_\nu$ be bounded and open, A a closed set in $E_{\nu+n}$ whose projection on $E_\nu$ is cl G, let M be the closed set $M = A \times E_\nu$, and $f_o(t,x,u)$ be a real continuous function on M which is convex in u for every $(t,x) \in A$. For every $(t,x) \in A$ let $\tilde{\tilde{Q}}(t,x)$ denote the closed convex subset of $E_{n\nu+1}$ defined by $\tilde{\tilde{Q}}(t,x) = [(z^o,u) z^o \geq f_o(t,x,u),\ u \in E_{n\nu}]$. Let us assume that these sets $\tilde{\tilde{Q}}(t,x)$ satisfy property (Q) in A and that $(\psi_t)$ $f_o(t,x,u) \geq -\psi(t)$ for all $(t,x) \in A$, $u \in E_{n\nu}$, $t \in Q$, and some L-integrable function $\psi(t) \geq 0$, $t \in G$. Then the functional (9.4.1) is lower semicontinous at every element $x \in S$.

We may consider here the following Tonelli-Nagumo condition $(\Phi_o)$: for every compact subset $A_o$ of $A$ there is a scalar continuous function $\Phi(\zeta)$, $0 \leq \zeta < +\infty$, with $\Phi(\zeta)/\zeta \to +\infty$ as $\zeta \to +\infty$, and $f_o(t,x,u) \geq \Phi(|u|)$ for all $(t,x) \in A_o$ and $u \in E_{n\nu}$. Under this condition $(\Phi_o)$ certainly the sets $\tilde{\tilde{Q}}(t,x)$ satisfy property (Q) in A (see App. A ), and condition $(\psi_t)$ holds.

Condition $(\psi)$ in (10.4.i) can be replaced by the following much weaker condition $(\psi_t^*)$: for every $\bar{t} \in \mathrm{cl}G$ there is a neighborhood $N(\bar{t})$ of $\bar{t}$ in $E_\nu$, real constants $b_{is}$, $i = 1,\dots,n$, $s = 1,\dots,\nu$, and an L-integrable function $\psi(t)$, $t \in N(\bar{t})$ such that

$$\bar{f}_o(t,x,u) = f_o(t,x,u) + \sum_{i=1}^{n} \sum_{s=1}^{\nu} b_{is} u_{is} \geq \psi(t)$$

for all $t \in N(\bar{t}) \cap G$, $(t,x) \in A$, $u \in E_{n\nu}$.

We shall now consider boundary conditions (B) concerning the values of x on the boundary $\partial G$ of G as in (9.3), and we shall assume that theses conditions satisfy property (P) as in (9.3). We shall also consider corresponding nonempty closed classes $\Omega$ of (admissible) elements $x \in S$. In the present situation, which is a particular case of (9.3), case 1 of (9.3) is vacuous. The existence theorems (9.3.ii) (case 2), and (9.3.iii) (case 3) yields now as corollaries the following statements.

(9.4.ii) (an existence theorem). Let $G \subset E_\nu$ be bounded, open, and of class $K_1$, let A be a closed subset of $E_{\nu+n}$ whose projection on $E_\nu$ is cl G, and let M be the closed set $M = A \times E_{n\nu}$. Let $f_o(t,x,u)$ be a scalar continuous function on M which is convex in u for every $(t,x) \in A$, let us assume that $(\psi)$:

$f_o(t,x,u) \geq -\psi(t)$ for all $(t,x,u) \in M$ and some L-integrable function $\psi(t) \geq 0$, $t \in G$, and let us assume that the closed convex sets $\tilde{\tilde{Q}}(t,x) = [(z^o,u) | z^o \geq f_o(t,x,u),\ u \in E_{n\nu}] \subset E_{n\nu+1}$ satisfy property (Q) in A. Let (B) be a given system of boundary conditions concerning the values of x on $\partial G$, and let $\Omega$ be a nonempty closed class of functions $x(t) = (x^1,\dots,x^n)$, $t \in G$, satisfying boundary condition (B), with $x^i \in W^1_p(G)$, $1 < p < +\infty$, $i = 1,\dots n$, $x(t) \in A(t)$ a.e. in G, and $f_o(t, x(t), (\nabla x)(t))$ L-integrable in G, and x satisfying given inequalities

$$||x^i||_p \leq N_i \qquad \text{for } i \in \{\beta\},$$

$$||\partial x^i/\partial t^s||_p \leq N_{is} \quad \text{for } i \in \{\beta\}_s, \tag{10.4.5}$$

for given constants $N_i$, $N_{is}$, and all i of given systems $\{\beta\}$, $\{\beta\}_s$, $s = 1,\dots,\nu$, of indices $1,\dots,n$ (which may be empty). Assume that $x \in \Omega$, $I[x] \leq L_o$ implies

$$||x^i||_p \leq L_i, \quad ||\partial x^i/\partial t^s||_p \leq L_{is} \tag{10.4.6}$$

for some constants $L_i$, $L_{is}$ (which may depend on $L_o$, $N_i$, $N_{is}$, G, (B), $\Omega$) and for all $i = 1,\dots,n$, which are not in $\{\beta\}$, $\{\beta\}_s$ respectively. Then the integral (9.4.1) has an absolute minimum in $\Omega$.

(9.4.ii) (an existence theorem). Let $G \subset E_\nu$ be bounded, open, and of class $K_1$, let A be a closed subset of $E_{\nu+n}$ whose projection on $E_\nu$ is clG, and let M be the closed set $M = A \times E_{n\nu}$. Let $f_o(t,x,u)$ be a scalar continuous function on M which is convex in u for every $(t,x) \in A$, let us assume that the closed convex sets $\tilde{\tilde{Q}}(t,x) = [(z^o,u) | z^o \geq f_o(t,x.u),\ u \in E_{n\nu}] \subset E_{n\nu+1}$ satisfy property (Q) in A. Let us assume that the following growth condition holds ($\varepsilon$): for every $\varepsilon > 0$ there is some L-integrable function $\psi_\varepsilon(t) \geq 0$, $t \in G$, which may depend on $\varepsilon$, such that $|u| \leq \psi_\varepsilon(t) + \varepsilon f_o(t,x,u) \in M$. Let (B) be a given system of boundary conditions concerning the values of x on $\partial G$, and let $\Omega$ be a nonempty closed class of functions $x(t) = (x^1,\ldots,x^n)$, $t \in G$, satisfying boundary conditions (B), with $x^i \in W_1^1(G)$, $i = 1,\ldots,n$, $x(t) \in A(t)$ a.e. in G, $f_o(t, x(t), \nabla(t))$ L-integrable in G, each $x \in \Omega$ satisfying given inequalities

$$||x^i||_1 \leq N_i \quad \text{for } i \in \{\beta\},$$

for given constants $N_i$ and all i of a given system $\{\beta\}$ of indices $1,\ldots,n$ (which may be empty). Assume that $x \in \Omega$, $I[x] \leq L_o$ implies

$$||x^i||_1 \leq L_i$$

for some constant $L_i$ (which may depend on $L_o$, $N_i$, G, (B), $\Omega$) and for all $i = 1,\ldots,n$, which are not in $\{\beta\}$. Then the integral (9.4.1) has an absolute minimum in $\Omega$.

The reader may note that if we assume that a growth condition $(\phi_o)$ is satisfied, then certainly condition ($\varepsilon$) holds, and the closed convex sets $\tilde{\tilde{Q}}(t,x)$ certainly satisfy condition (Q) in A.

## 9.5 Abstract Free Problems

This is the problem we have sketched briefly in (7.2), no. 13.

Let $G \subset E_\nu$ be open and bounded, let S be an arbitrary Banach space of elements x, let Y, V, $\mathcal{M}$, $\mathcal{L}$, s, r, A be defined as in (7.1.2), but now we take $m = r$, $U(t,y) = E_r$ for all $(t,y) \in A$, hence $M = A \times E_e$, and we define $f(t,y,u) = (f_1,\dots,f_r)$ by taking $f_i = u^i$, $i = 1,\dots,r$, or $f = u$, while $f_o(t,y,u)$ is a scalar function on M. We take for T the class of all vector functions $u(t) = (u^1,\dots,u^r)$, $t \in G$, with $u^i$ measurable in G. Then the equation $\mathcal{L}x = f$ reduces to $\mathcal{L}x = u$, that is, $u \in T$ is uniquely determined by $x \in S$. The problem (7.1.1-3) reduces now to the problem of the minimum of the integral

$$I[x] = \int_G f_o(t,(\mathcal{M}x)(t),(\mathcal{L}x)(t))dt. \tag{9.5.1}$$

The concept of lower closure reduces now to the concept of lower semicontinuity of (9.5.1) with respect to weak convergence in S. The lower closure theorem (8.3.i) reduces to the statement.

(9.5.i) (a lower semicontinuity statement). Let G, S, Y, V, $\mathcal{M}$, $\mathcal{L}$, A, $M = A \times E_r$, as above, with G open and bounded, A closed, $f_o(t,x,u)$ continuous on M, and convex in u for every $(t,x) \in A$. Let us assume that the closed convex sets $\tilde{\tilde{Q}}(t,x) = [(z^o,u) \mid z^o \geq f_o(t,x,u),\ u \in E_r]$ satisfy property (Q) in A. Let us assume that ($\psi$): $f_o(t,x,u) \geq -\psi(t)$ for all $(t,x) \in A$, $u \in E_r$, and some L-integrable function $\psi$ in A. Then the functional (9.5.1) is lower semicontinuous at every element $x \in S$.

Statement (9.5.i) holds even under the weaker assumption ($\psi_t^*$) instead of ($\psi$). If the usual Tonelli-Nagumo condition ($\Phi_o$): for every compact subset $A_o$ of A there is a scalar continuous function $\Phi(\zeta)$, $0 \leq \zeta < +\infty$, with $\Phi(\zeta)/\zeta \to +\infty$ as $\zeta \to +\infty$ and $f_o(t,x,u) \geq \Phi(|u|)$ for all $(t,x) \in A_o$, $u \in E_r$, then certainly the convex closed sets $\tilde{\tilde{Q}}(t,x)$ satisfy property (Q) in A and property ($\psi$) holds.

The existence theorem (9.1.i) now yields:

(9.5.ii) (an existence theorem). Under the same hypotheses of (9.5.i), let $\Omega$ be a nonempty closed class of elements $x \in S$ for which $f_o(t,(\mathcal{M}x)(t),(\mathcal{L}x)(t))$ is L-integrable in G, and which is weakly sequentially compact in S. Then the functional (9.5.1) possesses an absolute minimum in $\Omega$.

Under the hypothesis ($\Phi_o$) above, not only the sets $\tilde{\tilde{Q}}(t,x)$ satisfy property (Q), and condition ($\psi$) is satisfied, but also it happens that $x \in \Omega$, $I[x] \leq L_o$ implies that the functions $(\mathcal{L}x)(t)$, $t \in G$, are equiabsolutely integrable in G, and in most situations the weak sequential compactness of the corresponding subset of $\Omega$ can be deduced.

As mentioned in (7.4) the space S can be obtained by completion and graph norm. In this situation we consider first a Banach space X with norm $||x||$, we take a linear subspace $X_o$ in X in which linear operators $\mathcal{L}$ and $\mathcal{M}$ are defined, $\mathcal{M}$: $X_o \to Y$, $\mathcal{L}$: $X_o \to V$, and then S is the completion of $X_o$ with the graph norm $|||x||| = (||x||^2 + ||\mathcal{M}x||^2 + ||\mathcal{L}x||^2)^{1/2}$. In this situation, we need only require, instead of (H), the simpler requirement ($H_o$) of (7.4).

## 9.6 Lagrange Problems of Optimization With a "Normal" System of Partial Difference Equations

Let G be an open bounded subset of the t-space $E_\nu$, $t = (t^1,\dots,t^\nu)$, of some class $K_\ell$, $\ell \geq 1$, (see App. C), that is, with boundary $\partial G$ smooth of order $\ell$ in the sense of Morrey, and take for S a Sobolev space $S = \Pi_{i=1}^n W_p^{\ell_i}(G)$ for some numbers p, $1 \leq p \leq +\infty$, and integers $\ell_i$, $1 \leq \ell_i \leq \ell$. Thus every element $x \in S$ is an n-vector function $x(t) = (x^1,\dots,x^n)$, $t \in G$, whose components $x^i \in L_p(G)$, possess generalized partial derivatives $D^\alpha x^i$ of all orders $0 \leq |\alpha| \leq \ell_i$, and all $D^\alpha x^i \in L_p(G)$, $0 \leq |\alpha| \leq \ell_i$. Let $s_i$ denote the number of multi-indices $\alpha$ with $0 \leq |\alpha| \leq \rho_i - 1$, let $s = s_1+\dots+s_n$, and let $\mathcal{M}$ denote the operator $\mathcal{M}x = \nabla' x = [D^\alpha x^i,\ 0 \leq |\alpha| \leq \ell_i - 1,\ i = 1,\dots,n]$ for $x \in S$, so that $\mathcal{M}: S \to Y$ where $Y = \Pi_{i=1}^n (L_p(G))^{s_i}$. Let $\sigma_i$ denote the number of multi-indices $\alpha$ with $|\alpha| = \ell_i$, let $\sigma = \sigma_1+\dots+\sigma_n$, and let $\nabla'' x$ denote the $\sigma$-vector $[D^\alpha x^i,\ |\alpha| = \ell_i,\ i = 1,\dots,n]$. We shall now define the operator $\mathcal{L}$. For every $i = 1,\dots,n$, let $\{\alpha\}_i$ denote a given collection of $r_i$ multi-indices $\alpha$ with $|\alpha| = \ell_i$, so that $0 \leq r_i \leq \sigma_i$, and let $r = r_1+\dots+r_n$. We take for $\mathcal{L}$ the operator defined by $\mathcal{L}x = [D^\alpha x^i,\ \alpha \in \{\alpha\}_i,\ i = 1,\dots n]$ for $x \in S$, so that $\mathcal{L}: S \to Z$ where $Z = \Pi_{i=1}^n (L_p(G))^{r_i}$. Thus, $\mathcal{M}x$ is an s-vector $y = y(t) = (y^1,\dots,y^s)$, $t \in G$, and $\mathcal{L}x$ is an r-vector $z = z(t) = (z^1,\dots,v^r)$, $t \in G$. Again, as in (9.3), property (H) is trivial. Indeed, if $x_k \to x$ weakly in S, then $\nabla' x_k \to \nabla' x$ strongly in Y, and $\nabla'' x_k \to \nabla'' x$ weakly in $Z = \Pi_{i=1}^n (L_p(G))^{r_i}$; in particular, $\mathcal{M}x_k \to \mathcal{M}x$ strongly in Y, and $\mathcal{L}x_k \to \mathcal{L}x$ weakly in Z. Let $A \subset E_{\nu+r}$ be any set whose projection on $E_\nu$ is clG, and for any $(t,y) \in A$ let $U(t,y)$ be a given subset of the u-space $E_m$, $u = (u^1,\dots,u^m)$. Now M will be the set of all $(t,y,u) \in E_{\nu+n+m}$ with $(t,y) \in A$, $u \in U(t,y)$, and we denote by $f(t,y,u) = (f_1,\dots,f_r)$, $f_0(t,y,u)$ given functions defined on M. The general problem (7.1.1-3) reduces now to the problem of the minimum of the multiple integral

$$I[x,u] = \int_G f_0(t,(\nabla x')(t),u(t))dt, \tag{9.6.1}$$

with side conditions represented by the r partial differential equations.

$$D^\alpha x^i = f_{i\alpha}(t,(\nabla x')(t),u(t)), \qquad \alpha \in \{\alpha\}_i, \; i = 1,\dots,n, \tag{9.6.2}$$

and by the usual constraints,

$$(t,(\nabla x')(t)) \in A, \quad u(t) \in U(t,(\nabla x')(t)) \quad \text{a.e. in } G. \tag{9.6.3}$$

If we denote the differential operator $\mathcal{L}$ by D, then system (10.6.2) can be written in the simple form

$$(Dx)(t) = f(t,(\nabla x')(t),u(t)) \quad \text{a.e. in } G$$

Such a system is often denoted a "normal" system of partial differential equations.

Leaving details for the reader to supply, we state here the existence theorems which follow immediately from (9.1.i) and (9.1.ii) in the present situation in the three cases corresponding essentially to those described in detail in (9.3):

(9.6.i) Let G be bounded, open, and of class $K_\ell$, let A, U(t,y), M be as above with M compact. Let $f_o(t,y,u)$, $f(t,y,u) = (f_1,\dots,f_r)$ be continuous on M, and let us assume that the sets $\tilde{\tilde{Q}}(t,y) = [(z^o,z) | z^o \geq f_o(t,y,u),\ z = f(t,y,u),\ u \in U(t,y)] \subset E_{r+1}$ be convex for every $(t,y) \in A$. Let (B) be a given system of boundary conditions concerning the values of the functions $x^i$ and $D^\alpha x^i$, $0 \leq |\alpha| \leq \ell_{i-1}$, on $\partial G$, and let $\Omega$ be a nonempty closed class of pairs $x(t) = (x^1,\dots,x^n)$, $u(t) = (u^1,\dots,u^m)$, $t \in G$, each x satisfying boundary conditions (B), with $x^i \in W_p^{\ell_i}(G)$, $1 < p \leq +\infty$, $i = 1,\dots,n$, $u^j$ measurable in G, $j = 1,\dots,m$, x, u satisfying (9.6.2), (9.6.3).

Also, let us assume that $(x,u) \in \Omega$, $I[x] \leq L_o$ implies

$$||D^\beta x^i||_p \leq N_{i\beta} \quad \text{for all } \beta \in \{\beta_i\},\ i = 1,\dots,n,$$

for some constants $N_{i\beta}$ and all multi-indices $\beta$, $0 \leq |\beta| \leq \ell_i$, of given collections $\{\beta\}_i$, and each $\{\beta\}_i$ contains at least all such multi-indices $\beta$ which are not in $\{\alpha\}_i$, $i = 1,\dots n$. Then, the integral (9.6.i) has an absolute minimum in $\Omega$.

(9.6.ii) Let G be bounded, open, and of class $K_\rho$, let A, U(t,y), M be as above with A and M closed. Let $f_o(t,y,u)$, $f(t,y,u) = (f_1,\dots,f_r)$ be continuous on M, and let us assume that the sets $\tilde{\tilde{Q}}(t,y) = [(z^o,z) | z^o \geq f_o(t,y,u),$ $z = f(t,y,u),\ u \in U(t,y)] \subset E_{r+1}$ be closed, convex, and satisfy property (Q) in A. Let us assume that $(\psi)$: $f_o(t,y,u) \geq -\psi(t)$ for all $(t,y,u) \in M$ and some L-integrable function $\psi(t) \geq 0$, $t \in G$. Let (B) be a given system of boundary conditions concerning the values of the functions $x^i$ and $D^\alpha x^i$, $0 \leq |\alpha| \leq \ell_{i-1}$, on $\partial G$, and let $\Omega$ be a nonempty closed class of pairs $x(t) = (x^1,\dots,x^n)$, $u(t) = (u^1,\dots,u^m)$, $t \in G$, each $x$ satisfying boundary conditions (B), with $x^i \in W_p^{\ell_i}(G)$, $1 \leq \ell_i \leq \ell$, $1 < p \leq +\infty$, $i = 1,\dots,n$, $u^j$ measurable in $G$, $j = 1,\dots,m$, satisfying (9.6.2), (9.6.3). Also, let us assume that $(x,u) \in \Omega$, $I[x] \leq L_o$ implies $\|D^\beta x^i\|_p \leq N_{i\beta}$ for some constants $N_{i\beta}$ and all multi-indices $\beta$ with $0 \leq |\beta| \leq \ell_i$. Then the integral (9.6.1) has an absolute minimum in $\Omega$.

Note that the constants $N_{i\beta}$ may depend on $L_o$, besides $G$, $\Omega$, (B). Some of the constants $N_{i\beta}$ instead may be fixed and used to define the class $\Omega$, and the remaining constants $N_{i\beta}$ may then depend on $L_o$ and the given constants $N_{i\beta}$. In (9.6.ii) condition $(\psi_t^*)$ may well replace $(\psi)$ without any other change. Under the usual condition $(\Phi)$: for every compact part $A_o$ of $A$ there are constants $C, D \geq 0$ and a scalar continuous function $\Phi(\zeta)$, $0 \leq \zeta < +\infty$, such that $\Phi(\zeta)/\zeta \to +\infty$ as $\zeta \to +\infty$, $f_o(t,y,u) \geq \Phi(|u|)$, $|f(t,y,u)| \leq C + D|u|$, then certainly condition $(\psi)$ holds, and all sets $\tilde{\tilde{Q}}(t,x)$, if convex, are closed and satisfy property (Q) in $A$. The same is true under the weaker condition (M): for every compact part $A_o$ of $A$ and $\varepsilon > 0$ there is a number $\bar{u} \geq 0$ such that $(t,y)$ $U(t,y)$, $|u| \geq \bar{u}$ implies $|f(t,y,u)| \leq \varepsilon f_o(t,y,u)$.

(9.6.iii) Let $G$ be bounded, open, and of class $K_\ell$, let $A$, $U(t,y)$, $M$ be as above with $A$ and $M$ closed. Let $f_o(t,y,u)$, $f(t,y,u) = (f_1,\dots,f_r)$ be continuous on $M$, and let us assume that the sets $\tilde{\tilde{Q}}(t,y) = [(z^o, z) | z^o \geq f_o(t,y,u)$,

$z = f(t,y,u),\ u \in U(t,y)] \subset E_{r+1}$ be closed, convex, and satisfy property (Q) in A. Let (B) be a given system of boundary conditions concerning the values of the functions $x^i$ and $D^\alpha x^i$, $0 \leq |\alpha| \leq \ell_i - 1$, on $\partial G$, and let $\Omega$ be a nonempty closed class of pairs $x(t) = (x^1,\dots,x^n)$, $u(t) = (u^1,\dots,u^m)$, $t \in G$, each x satisfying boundary conditions (B) with $x^i \in W_1^{\ell_i}(G)$, $1 \leq \ell_i \leq \ell$, $i = 1,\dots,n$, $u^j$ measurable in G, $j = 1,\dots,m$, satisfying (9.6.2), (9.6.3). Let us assume that the following growth condition holds ($\varepsilon$): for every $\epsilon \geq 0$ there is an L-integrable function $\psi_\varepsilon(t) \geq 0$, $t \in G$, which may depend on $\varepsilon$, such that $|f(t,y,u)| \leq \psi_\varepsilon(t) + \varepsilon f_o(t,y,u)$ for all $(t,y,u) \in M$. Let us assume that (x,u) $\in \Omega$, $I[x,u] \leq L_o$ implies 1. all derivatives $D^\alpha x^i$ with $|\alpha| = \ell_i$, $i = 1,\dots,n$, $\alpha \notin \{\alpha\}_i$, are equiabsolutely integrable in G; 2. $||D^\beta x^i||_1 \leq N_{i\beta}$ for some constants $N_{i\beta}$ and all $\beta$ with $0 \leq |\beta| \leq \ell_i - 1$, and all $\beta$ with $|\beta| = \ell_i$, $\beta \notin \{\alpha\}_i$, $i = 1,\dots,n$. Then the integral (9.6.1) has an absolute minimum in $\Omega$.

In (9 .5.iii) condition ($\psi_t^*$) may replace ($\psi$) without any other change.

Note that for the multi-indices $\alpha \in \{\alpha\}_i$, $|\alpha| = \ell_i$, the derivatives $D^\alpha x^i$ are certainly equiabsolutely integrable as a consequence of assumption ($\varepsilon$) and of $I[x,u] \leq L_o$. For the same $\alpha$ then necessarily we have $||D^\alpha x^i||_1 \leq N_{i\alpha}$ for some constant $N_{i\alpha}$. As above, some constants $N_{i\beta}$ may be given, and others may depend on $L_o$ and on the given constants $N_{i\beta}$. The same remarks hold, as at the end of (9.6.ii), concerning conditions ($\Phi$) and (M).

## 9.7. Lagrange Problems With Linear Partial Differential Equations and Graph Norm

Let G as in (9.6), let $X = \Pi_{i=1}^{n} W_p^{\ell i}(G)$ as in (9.6), let $X_o$ be the linear subspace of all $x(t) = (x^1,\dots,x^n)$, $t \in G$, where each $x^i$ coincides in G with a function of class $C^\infty$ in $E_\nu$, and let $\mathcal{M}$: $X_o \to Y$ be the operator defined as in (9.6) by $\mathcal{M}x = \nabla x' = [D^\alpha x^i,\ 0 \leq |\alpha| \leq \ell_i - 1,\ i = 1,\dots,n]$, hence $\mathcal{M}x$ is a function $y(t) = (y^1,\dots,y^s) \in Y$, $t \in G$, $Y = \Pi_{i=1}^{n} [L_p(G)]^{s_i}$, $s = s_1+\dots+s_n$. Let $r_i \geq 0$, $i = 1,\dots,n$, be arbitrary integers, let $r = r_1+\dots+r_n$, let $V = \Pi_{i=1}^{n} [L_p(G)]^{r_i}$, and let $\mathcal{L}$: $X_o \to Z$ be any linear differential operator with integrable bounded coefficient in G, say $(\mathcal{L}x)^r = \Sigma_{j=1}^{k_i} \Sigma_{j=1}^{n} \Sigma_{|\alpha|=j} A_{isj\alpha}(t)$ $D^\alpha x^s$, of arbitrary orders $k_i$ which can be larger than $\ell_i$. Let S be the completion of $X_o$ by means of the norm $|||x||| = (||x||^2 + ||\mathcal{M}x||^2 + ||\mathcal{L}x||^2)^{1/2}$, when $||x||$, $||\mathcal{M}x||$, $||\mathcal{L}x||$ are the norms in X, Y, Z respectively. Note that if $k_i < \ell_i$ for all i, then S actually coincides with X and the norm $|||x|||$ is equivalent to the norm $||x||$ in X. If $k_i > \ell_i$ for at least one i, then S may be distinct from X and the norm $|||x|||$ is equivalent to the norm $(||x||^2 + ||\mathcal{L}x||^2)^{1/2}$. In any case, property $(H_o)$ obviously holds (see App. B). We are now in a position to consider the problem of the minimum of the multiple integral

$$I[x,u] = \int_G f_o(t,(\nabla x')(t),u(t))dt,$$

with side conditions represented by the differential system

$$(\mathcal{L}x)(t) = f(t,(\nabla x')(t),u(t)), \text{ a.e. in } G,$$

with $f = (f_1,\dots,f_r)$, and possible constraints of the form

$$(t,(\nabla x')(t)) \in A, \quad u(t) \in U(t,(\nabla x')(t)) \quad \text{a.e. in } G,$$

and a possible system (B) of boundary conditions concerning the values of the functions $D^\alpha x^i$, $0 \leq |\alpha| \leq \ell_i - 1$, on the boundary $\partial G$ of G.

We leave to the reader to deduce from (9.1.i), (9.1.ii) existence theorems for the present situation.

For instance, if we take $\nu = 2$, $n = 1$, $\ell_i = 1$, $m = 1$, $\xi\eta$ coordinates of $E_2$, and we take for $\mathcal{L}$ the Laplacian, we may consider the problem of the minimum of the double integral

$$I[x,u] = \int_G f_o(\xi,\eta,x(\xi,\eta),u(\xi,\eta))d\xi d\eta,$$

with partial differential equation

$$x_{\xi\xi} + x_{\eta\eta} = f(\xi,\eta,x(\xi,\eta),u(\xi,\eta)) \quad \text{a.e. in } G,$$

constraints of the form

$$(\xi,\eta,x(\xi,\eta)) \in A, \; u(\xi,\eta) \in U(\xi,\eta,x(\xi,\eta))$$

and possible boundary conditions concerning the values of $x(\xi,\eta)$ on the boundary, $\partial G$ of G. Here $X_o$ is the linear space of $X = W_p^1(G)$ which coincide with some function of class $C^\infty$ in G, and S is the completion of $X_o$ with respect to the norm

$$|||x||| = (||x||_p^2 + ||x_\xi||^2 + ||x_\eta||^2 + ||x_{\xi\xi} + x_{\eta\eta}||^2)^{1/2}.$$

The minimum above is sought in classes $\Omega$ of pairs x, u with $x \in S$, u measurable in G.

We leave the reader to formulate particular existence theorems for the present problem as corollaries of (9.1.i), (9.1.ii).

## 9.8. A Problem of Optimization With Equations of Evolution

Let $\nu$ be replaced by $\nu+1$ and t replaced by $(t,\tau)$, or $(t,\tau^1,\dots,\tau^\nu)$. Let $G_o$ be a fixed open bounded subset of the $\tau$-space $E_\nu$. Let S be the Banach space of all n-vector functions $x(t,\tau) = (x^1,\dots,x^n)$, $(t,\tau) \in (o,t_2) \times G_o$, each component $x^i \in L_p(G)$ with generalized partial derivatives $\partial x^i/\partial t = x^i_t$ and $D^\alpha_\tau x^i$ for all $0 \le |\alpha| \le h$, $h \ge 1$, all in Lp(G). Thus, for almost all $t \in [o,T]$ the function $x(t,\tau)$ of $\tau$ above belongs to $W^h_p(G_o)$. Here p is any given number $1 \le p \le +\infty$, and S is a Banach space with the norm

$$||x|| = (||x_t||^2_p + \Sigma_{|\alpha|\le h} ||D^\alpha_\tau x||^2_p)^{1/2}$$

where $|| \ ||_p$ denotes $L_p$-norm in $G = G_o \times [o,t_2]$.

Let s denote the total number of partial derivatives $\nabla' x = [D^\alpha_\tau x^i$, $i = 1, \dots, n$, $0 \le |\alpha| \le h-1]$. For the sake of simplicity we assume here that $A_o$ is a fixed subset of the y-space $E_n$, that U is a fixed set of the u-space $E_m$, $y = (y^1,\dots,y^s)$, $u = (u^1,\dots,u^m)$, and thus we take $A = A_o \times c\ell\, G_o \times [o,t_2]$, $M = A \times U$. We take for $f(t,\tau,y,u) = (f_1,\dots,f_n)$, $f_o(t,\tau,y,u)$ given functions on M. Let $Y = (L_p(G))^r$, $Z = (L_p(G))^n$, $r = n$, so that Y is the space of all r-vector valued functions $y(t) = (y^1,\dots,y^s)$, $t \in G$, with $y^i \in L_p(G)$, and Z the space of all n-vector valued functions $z(t) = (z^1,\dots,z^n)$, $t \in G$, with $z^j \in L_p(G)$. Let T be the set of all m-vector valued measurable functions $u(t) = (u^1,\dots,u^m)$, $t \in G$.

Let $\mathcal{M}$ be the operator $\mathcal{M}$: $S \to Y$ defined by $\mathcal{M}x = \nabla' x$. For every $t \in [o,t_2]$, let $\mathcal{L}_\tau$ denote a given n-vector valued linear differential operator of order h in the fixed set $G_o$ of the $\tau$-space $E_\nu$, say

$$(\mathcal{L}_\tau x)(t,\tau) = \Sigma_{|\alpha|\leq h} A_\alpha(t,\tau) D_\tau^\alpha x(t,\tau),$$

where each $A_\alpha(t,\tau)$ is a bounded measurable n-vector function of $t$, $\tau$ in $G_o \times [o,t_2]$. Let $\mathcal{L}$ be the operator $\mathcal{L}$: $S \to V$ defined by $\mathcal{L}x = x_t - \mathcal{L}_\tau x$. With these definitions of X, Y, V, $\mathcal{M}$, $\mathcal{L}$, hypothesis (H) is certainly satisfied.

Now the functional equation $\mathcal{L}x = f$ reduces to the evolution-equation

$$x_t - \mathcal{L}_\tau x = f(t,\tau,\nabla' x(t,\tau),u(t,\tau)) \quad \text{a.e. in } G, \tag{9.8.1}$$

or

$$x_t = \Sigma_{|\alpha|\leq h} A_\alpha(t,\tau) D_\tau^\alpha x + f(t,\tau,\nabla' x(t,\tau),u(t,\tau)),$$

with constraints

$$(\nabla' x)(t,\tau) \in A_o, \quad u(t,\tau) \in U \quad \text{a.e. in } G, \tag{9.8.2}$$

and the functional to be minimized is

$$I[x,u] = \int_o^{t_2} \int_G f_o(t,\tau,\nabla' x(t,\tau),u(t,\tau))\,dt\,d\tau. \tag{9.8.3}$$

A pair x, u is said to be admissible provided $x \in S$, $u \in T$, $y = \mathcal{M}x = \nabla' x \in Y$, $v = x = x_t - {}_\tau x \in V$, relations (9.8.1) and (9.8.2) are satisfied a.e. in G, and $f_o(t,\tau,\nabla' x(t,\tau),u(t,\tau))$ is L-integrable in G. We shall consider classes $\Omega$ of admissible pairs x, u, satisfying suitable boundary conditions. If $\partial G_o$ is sufficiently smooth, say $G_o$ is of class $K_h$ (see App. C), we may consider here usual boundary conditions, say (B), concerning the values of the s functions $D^\alpha x^i$, $i = 1,\dots,n$, $0 \le |\alpha| \le h-1$, on the surface $S_o \times [o,t_2]$ of the cylinder $\partial G_o \times [o,t_2]$, beside initial conditions concerning the values of $x(t,\tau)$ for $t = 0$ and $t = t_2$. We shall only assume that the boundary conditions (B) satisfy the usual closure property: (P) If $x_k$, $u_k$, $k = 1, 2,\dots$, are pairs in $\Omega$ (hence $x_k \in S$), all $x_k$ satisfy boundary conditions (B), and $x_k \to x$ weakly in S, then x satisfies boundary conditions (B). This requirement will not be explicitly requested. It will suffice to require, as usual, the closure property of the class $\Omega$.

(9.8.i) (an existence statement). Let $G = G_o \times (o,t_2)$, $A = A_o \times c\ell G \times [o,t_2]$, $U = M = A \times U$, as above, with $G_o$ bounded, open, and of class $K_h$ in $E_\nu$, let $f_o(t,y,u)$, $f(t,y,u) = (f_1,\dots,f_n)$ be continuous on M, $f_o$ scalar, and let S, Y, V, T, $\mathcal{M}, \mathcal{L} = \partial/\partial t - \mathcal{L}_\tau$ as above (satisfying property H). Let us assume that the sets $\tilde{\tilde{Q}}(t,y) = [\tilde{z} = (z^o,z) \mid z^o \ge f_o(t,\tau,y,u),\ z = f(t,\tau,y,u)\ u \in U] \subset E_{n+1}$ be convex, closed, and satisfy property (Q) in A (with exception perhaps of points $(t,\tau,y)$ with $(t,\tau)$ in a set of measure zero in G). Let us assume that for some scalar L-integrable function $(t,\tau) \ge 0$, $(t,\tau) \in G$, we have

$f_o(t,\tau,y,u) \geq -\psi(t,\tau)$ for all $(t,\tau,y,u) \in M$. Let $\Omega$ be a nonempty closed class of admissible pairs x, u (that is, $x \in S$, $u \in T$, x, u satisfying (9.8.1), (9.8.2), with $f_o(t,\tau,\nabla x',u)$ L-integrable in G), and let us assume that $\|x_t\|_p \leq M_o$, $\|D_\tau^\alpha x\|_p \leq M_\alpha$ for all elements $x \in \{x\}_\Omega$, all $\alpha$ with $|\alpha| \leq h$, and some constants $p > 1$, $M_o$, $M_\alpha$. Then, the functional (9.8.3) possesses as absolute minimum in $\Omega$.

We leave to the reader to formulate the analogous statements with $p = 1$ and suitable growth conditions.