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EXISTENCE THEOREMS FOR WEAK AND USUAL OPTIMAL SOLUTIONS
IN LAGRANGE PROBLEMS WITH UNILATERAL CONSTRAINTS

I. Closure Theorems

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EXISTENCE THEOREMS FOR WEAK AND USUAL OPTIMAL SOLUTIONS 
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I. CLOSURE THEOREMS*

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In the present papers (I, II, and III) we prove existence theorems for 
weak and usual optimal solutions of nonparametric Lagrange problems with 
(or without) unilateral constraints.

We shall consider arbitrary pairs $x(t), u(t)$ of vector functions, $u(t)$ 
measurable with values in $E_m$, $x(t)$ absolutely continuous with values in 
$E_n$, and we discuss the existence of the absolute minimum of a functional

$$I[x,u] = \int_{t_1}^{t_2} F(t,x(t),u(t)) dt,$$

with side conditions represented by a differential system

$$\frac{dx}{dt} = f(t,x(t),u(t)), \quad t_1 \leq t \leq t_2,$$

constraints

$$(t,x(t)) \in A, \quad u(t) \in U(t,x(t)), \quad t_1 \leq t \leq t_2,$$

and boundary conditions

$$(t_1,x(t_1),t_2,x(t_2)) \in B,$$

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where A is a given closed subset of the tx-space $E_1 \times E_n$, where $B$ is a given closed subset of the $t_1x_1t_2x_2$-space $E_{2n+2}$, and where $U(t,x)$ denotes a given closed variable subset of the u-space $E_m$, depending on time $t$ and space $x$. Here $A$ may coincide with the whole space $E_1xE_m$, and $U$ may be fixed and coincide with the whole space $E_m$.

In the particular situation, where the space $U$ is compact for every $(t,x)$ these problems reduce to Pontryagin problems; in the particular situation where the space $U$ is fixed and coincides with the whole space $E_m$, then these problems have essentially the same generality of usual Lagrange problems. Throughout these papers we shall assume $U(t,x)$ to be any closed subset of $E_m$.

In paper I we prove closure theorems for usual solutions. In II we shall prove existence theorems. These will contain as particular cases the Filippov existence theorem for problems of optimal control ($U(t,x)$ compact), existence theorems for usual Lagrange problems ($U = E_m$), and the Nagumo-Tonelli existence theorem for free problems ($m = n, f = u$). In III we shall prove existence theorems for weak (or generalized) solutions introduced as measurable probability distributions of usual solutions (Gambrelidze chattering states).

In successive papers we shall extend the present results to multi-dimensional Lagrange problems involving partial differential equations in Sobolev's spaces with unilateral constraints.
We begin with an analysis of the concept of upper semicontinuity of variable subsets in $E_m$. The usual concept of upper semicontinuity is replaced by two others (properties (U) and (Q), §5), which are essentially more general than the upper semicontinuity, in the sense that closed sets $U(t,x)$, for which upper semicontinuity property hold, certainly satisfy property (U), and closed and convex sets $Q(t,x)$, for which upper semicontinuity property hold, certainly satisfy property (Q). We then extend (§4) the closure theorem of A. F. Filippov in various ways, so as to include, among other things, the use of pointwise and not necessarily uniform convergence of some components of a sequence of trajectories. In part II we shall prove existence theorem of optimal smooth solutions (§7) by a new analysis of a minimizing sequence, and by using the above extension of Filippov's closure theorem as a replacement for Tonelli's semicontinuity argument. We shall then deduce (§8) existence theorems for the case where $f$ is linear in $u$, and for free problems of the calculus of variations ($m = n, f = u$). Finally, we shall prove (§9) existence theorems for weak solutions in the general case above, for the case in which $f$ is linear, and for free problems.

1. THE PROBLEM

We denote by $x$ a variable $n$-vector $x = (x^1, \ldots, x^n) \in E_n$, by $u$ a variable $m$-vector $u = (u^1, \ldots, u^m) \in E_m$, and by $t \in E_1$ the independent variable. We denote by $A$ an arbitrary subset of the $(t,x)$-space, $A \subset E_1 \times E_n$, and, for any $(t,x) \in A$, we denote by $U = U(t,x)$ a variable subset of the $u$-space, $U(t,x) \subset E_m$. In the terminology of control problems, $u$ is the control variable.
and \( U(t,x) \) the control space. We denote by \( f_i(t,x,u), i = 0,1,\ldots,n \), given real functions defined for all \((t,x)\in A\), and all \( u\in U(t,x) \), and by \( f \) the \( n \)-vector function \( f = (f_1,\ldots,f_n) \). We denote by \( B \) a given subset of the 
\((2n + 2)\)-space \((t_1,x_1,t_2,x_2)\). We are interested in the determination of a measurable vector function \( u(t), t_1 \leq t \leq t_2 \), (control function, or steering function, or strategy), and a corresponding absolute continuous vector function \( x(t), t_1 < t < t_2 \), satisfying almost everywhere the differential system

\[
\frac{dx}{dt} = f(t,x(t),u(t)), \quad t_1 \leq t \leq t_2,
\]

satisfying the boundary conditions

\[
(t_1,x(t_1),t_2,x(t_2))\in B,
\]

satisfying the constraints

\[
(t,x(t))\in A, \quad t_1 \leq t \leq t_2,
\]

\[
u(t)\in U(t,x(t)), \quad \text{a.e. in } [t_1,t_2],
\]

and for which the integral (cost functional)

\[
I[x,u] = \int_{t_1}^{t_2} f_0(t,x(t),u(t))dt
\]

has its minimum value (see §2 for details). We shall assume that \( U(t,x) \) is closed for every \((t,x)\in A\).

2. THE SPACE OF CONTINUOUS VECTOR FUNCTIONS

Let \( X \) be the collection of all continuous \( n \)-dim. vector functions \( x(t) \) defined on arbitrary finite intervals of the \( t \)-axis:

\[
x(t) = (x^1,\ldots,x^N), \quad a \leq t \leq b, \quad x(t)\in \mathbb{R}^N,
\]
If \( x(t), a \leq t \leq b, \) and \( y(t), c \leq t \leq d, \) are any two elements of \( X, \) we shall define a distance \( \rho(x,y). \) First, let us extend \( x(t) \) and \( y(t) \) outside their intervals of definition by constancy and continuity in \(( -\infty, +\infty), \) and then let

\[
\rho(x,y) = |a-c| + |b-d| + \max |x(t)-y(t)|,
\]

where \( \max \) is taken in \(( -\infty, +\infty). \) It is known that \( X \) is a complete metric space when equipped with the metric \( \rho. \) Ascoli's theorem can now be expressed by saying that any sequence of equicontinuous vector functions \( x_n \) of \( X, \) whose graphs in the \( tx\)-space are equibounded, possesses at least one subsequence which is convergent in the \( \rho\)-metric toward an element \( x \) of \( X. \)

3. ADMISSIBLE PAIRS \( u(t), x(t) \)

Let \( A \) be a closed subset of the \( (t,x)\)-space \( E_1 \times E_2. \) For every \((t,x)\varepsilon A\) let \( U(t,x), \) or control space, be a subset of the \( u\)-space \( E_m. \) Let \( M \) be the set of all \((t,x,u)\) with \((t,x)\varepsilon A, u\varepsilon U(t,x). \) Let \( f(t,x,u) = (f_1, \ldots, f_n) \) be a continuous vector function defined on \( M. \) We shall denote by \( Q(t,x) \) the set of all values in \( E_n \) taken by \( f(t,x,u) \) when \( u \) describes \( U(t,x), \) or \( Q(t,x) = f(t,x,U(t,x)). \) A vector function \( u(t) = (u^1, \ldots, u^m), t_1 \leq t \leq t_2 \) (control function) and a vector function \( (x,t) = (x^1, \ldots, x^n), t_1 \leq t \leq t_2 \) (trajectory) are said to be an admissible pair provided (a) \( u(t) \) is measurable in \([t_1,t_2]; \)
(b) \( x(t) \) is absolutely continuous (AC) in \([t_1,t_2], \)
(c) \((t,x(t))\varepsilon A \) for every \( t \varepsilon [t_1,t_2]; \)
(d) \( u(t)\varepsilon U(t,x(t)) \) a.e. in \([t_1,t_2]; \)
(e) \( dx/dt = f(t,x(t), u(t)) \) a.e. in \([t_1,t_2]. \) By the expression the vector function \( x(t), t_1 \leq t \leq t_2, \) is a trajectory, we shall mean below that there exists a vector function
u(t), t₁ ≤ t ≤ t₂, such that the pair u(t), x(t) satisfies (a b c d e). We say also that x(t) is generates by u(t).

4. UPPER SEMICONTINUITY OF VARIABLE SETS

In view of using sets U(t,x), Q(t,x) which are closed but not necessarily compact, we need a concept of upper semicontinuity which is essentially more general than the usual one. We shall introduce two modifications of the usual definition of upper semicontinuity, and we shall denote them as "property (U)" and "property (Q)" since we shall usually use them for the sets U(t,x) and Q(t,x) above, respectively.

We shall discuss properties (U) and (Q) first in relation to arbitrary variable sets U(t,x), Q(t,x) which are functions of (t,x) in A. Then we shall discuss their relations when Q(t,x) is assumed to be the image of U(t,x) as mentioned in no. 2. Properties proved for U(t,x) under conditions (U) or (Q), will be used for Q(t,x) when this set satisfies conditions (U) or (Q).

(A) The Property (U)

Given any set F in a linear space E we shall denote by cl F, co F, bd F, int F respectively the closure of F, the convex hull of F, the boundary of F, the set of all interior points of F. Thus, cl co F denotes the closure of the convex hull of F. We know that F, cl F, co F, co cl F are all contained in cl co F.

For every (t,x) ∈ A and δ > 0 let Nδ(t,x) denote the closed δ-neighborhood of (t,x) in A, that is, the set of all (t',x') ∈ A at a distance ≤ δ from (t,x).
A variable subset $U(t,x)$, $(t,x) \in A$, is said to be an upper semicontinuous function of $(t,x)$ at the point $(\bar{t},\bar{x}) \in A$ provided, given $\varepsilon > 0$, there is a number $\delta = \delta(\bar{t},\bar{x},\varepsilon) > 0$ such that $(t,x) \in N_\delta(\bar{t},\bar{x})$ implies $U(t,x) \subset [U(\bar{t},\bar{x})]_\varepsilon$ where $[U]_\varepsilon$ denotes the closed $\varepsilon$-neighborhood of $U$ in $E_m$.

Again let $U(t,x)$, $(t,x) \in A$, $U(t,x) \subset E_m$, be a variable subset of $E_m$, which is a function of $(t,x)$ in $A$. For every $\delta > 0$ let $U(t,x,\delta) = \cup U(t',x')$, where the union is taken for all $(t',x') \leq N_\delta(t,x)$. We shall say that $U(t,x)$ has the property $(U)$ at $(\bar{t},\bar{x})$ in $A$, if

$$U(\bar{t},\bar{x}) = \bigcap_{\delta > 0} \text{cl } U(\bar{t},\bar{x},\delta).$$

We shall say that $U(t,x)$ has property $(U)$ in $A$, if $U(t,x)$ has property $(U)$ at every $(t,x)$ of $A$.

(i) If $U(t,x)$ has property $(U)$ at $(\bar{t},\bar{x})$, then $U(\bar{t},\bar{x})$ is closed.

Indeed,

$$U(\bar{t},\bar{x}) \subset \text{cl } U(\bar{t},\bar{x}) \subset \bigcap_{\delta > 0} \text{cl } U(\bar{t},\bar{x},\delta) = U(\bar{t},\bar{x}),$$

and hence the $\subset$ signs can be replaced by $=$ signs.

(ii) If $A$ is closed, and $U(t,x)$ is any variable set which is a function of $(t,x)$ in $A$ and has property $(U)$ in $A$, then the set of all $(t,x,u) \in AxE_m$ with $u \in U(t,x)$, $(t,x) \in A$, is closed.

Proof. If $(\bar{t},\bar{x},\bar{u}) \in \text{cl } M$ and $\varepsilon > 0$, then there are $\infty$-many points $(t,x,u) \in M$ with $|t-\bar{t}| < \varepsilon$, $|x-\bar{x}| < \varepsilon$, $|u-\bar{u}| < \varepsilon$. Thus, $(\bar{t},\bar{x}) \in A$ since $A$ is closed, $(t,x) \in N_\varepsilon(\bar{t},\bar{x})$, $u \in U(t,x)$, $u \in U(\bar{t},\bar{x},2\varepsilon)$, and $\bar{u} \in \text{cl } U(\bar{t},\bar{x},2\varepsilon) = U(\bar{t},\bar{x})$, $\bar{u} \in U(\bar{t},\bar{x})$, since $U$ has property $(U)$ at $(\bar{t},\bar{x})$. This proves that $(\bar{t},\bar{x},\bar{u}) \in M$, that is, $M$ is closed.
Note that the sets $U(\overline{t}, \overline{x}, \delta)$ are not necessarily closed even if $A$ is closed all sets $U(t, x)$ are closed and we take for $N_\delta(\overline{t}, \overline{x})$ the closed $\delta$-neighborhood if $(\overline{t}, \overline{x})$ in $A$ as stated. This can be seen by the following example. Let $A = \{0 \leq t \leq 1, 0 \leq x \leq 1\}$ a subset of $E_2$, and $U(t, x) = \{z = (z_1, z_2) | z_2 \geq t z_1, -\infty < z_1 < +\infty\}$ for $0 < t \leq 1$, and $U(0, x) = \{z_2 \geq 0, z_1 = 0\}$ for $t = 0$. Then $U(0, x, \delta) = \{z = (z_1, z_2) | z_2 \geq \delta z_1 \text{ for } -\infty < z_1 \leq 0, \text{ and } z_2 > 0 \text{ for } 0 < z_1 < +\infty\}$ for any $\delta > 0$. The sets $U(0, x, \delta)$ are not closed. Here $U(t, x)$ does not satisfy property $(U)$ at the points $(0, x)$. Nevertheless, the statement holds

(iii) If $A$ is closed, and $U(t, x)$ satisfies property $(U)$ in $A$, then the sets $U(t, x, \delta), (t, x) \in A, \delta > 0$, are all closed, and hence $U(t, x) = \bigcap_\delta U(t, x, \delta)$ for every $(t, x) \in A$.

Proof. Let $M_\delta$ denote the set of all points $(t, x, u)$ with $(t, x) \in N_\delta(\overline{t}, \overline{x})$, $u \in U(t, x)$. Obviously $N_\delta(\overline{t}, \overline{x}) \subset A \subset E_{n+1}$; $M_\delta \subset E_n \times E_m$, and $N_\delta(\overline{t}, \overline{x})$ is compact and $M_\delta$ is closed by force of (ii) above. Let $\overline{u}$ be a point of accumulation of $U(t, x, \delta)$, and for any $\eta > 0$ let $V_\eta(\overline{u})$ denote the $\eta$-neighborhood of $\overline{u}$ in $E_m$. Then $M \cap (V_\eta(\overline{u}) \times E_{n+1}) \subset N_\delta(\overline{t}, \overline{x}) \times V_\eta(\overline{u})$, hence $M \cap (V_\eta(\overline{u}) \times E_{n+1})$ is bounded. Since both $M_\delta$ and $V_\eta(\overline{u}) \times E_{n+1}$ are closed sets, the set $M \cap (V_\eta(\overline{u}) \times E_{n+1})$ is closed and bounded, and therefore a compact subset of $E_{n+1} \times E_m$.

Now the set $U(\overline{t}, \overline{x}, \delta) \cap V_\eta(\overline{u})$ is the projection of $M \cap (V_\eta(\overline{u}) \times E_{n+1})$ on the u-space $E_m$, and therefore $U(\overline{t}, \overline{x}, \delta) \cap V_\eta(\overline{u})$ is compact. Thus $\bigcap u \in U(\overline{t}, \overline{x}, 8) \cap V_\eta(\overline{u})$, and finally $u \in U(\overline{t}, \overline{x}, \delta)$. Thus, $U(\overline{t}, \overline{x}, \delta)$ is closed, or $\bigcap U(\overline{t}, \overline{x}, \delta) = U(\overline{t}, \overline{x}, \delta)$, and $U(\overline{t}, \overline{x}, \delta) = \bigcap_\delta \overline{U(\overline{t}, \overline{x}, \delta)} = \overline{U(\overline{t}, \overline{x}, \delta)}$.  

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(iv) If $A$ is closed and $U_j(t,x), (t,x) \in A, j = 1, \ldots, v$, $v$ finite, are
variable subsets of $E_m$ all satisfying property (U) in $A$, then their union
and their intersections $V(t,x) = U_j U_1(t,x), W(t,x) = \cap_j U_j(t,x), (t,x) \in A$,
are subsets of $E_m$ satisfying property (U) in $A$. The same holds for their
product $V(t,x) = U_1 x \ldots x U_v$.

The proof is straightforward.

Under the hypotheses of (ii) the set $M$ is closed but not necessarily com-
 pact as the trivial example $U(t,x) = E_m, M = A \times E_m$, shows. The set $M$ is closed
but not necessarily compact even if we assume that $A$ is compact, and that
every $U(t,x)$ is compact. This is proved by the following example. Let $m =
n = 1, A = [(t,x) \in E_2, 0 \leq t \leq 1, 0 \leq x \leq 1], U(0,x) = [u \in E_1 | 0 \leq u \leq 1], and,
if $t \neq 0, U(t,x) = [u \in E_1 | 0 \leq u \leq 1, and u = t^{-1}]$. Then $M$ is the set of all
$(t,x,u)$ with $0 \leq t \leq 1, 0 \leq x \leq 1, and 0 \leq u \leq 1, or u = t^{-1}$ if $t \neq 0$. Ob-
viously, $M$ is closed but not compact. Nevertheless, the statement holds:

(v) If $A$ is compact, if the variable set $U(t,x)$ is compact and convex
for every $(t,x) \in A$ and possesses property (U) in $A$, if for every $(t,x) \in A$
there is some $\delta = \delta(t,x) > 0$ such that $U(t,x)\cap U(t',x') \neq \emptyset$ for every $(t',x')$
$\in M(t,x)$, then $M$ is compact.

Proof. If $M$ is not compact, then there is some sequence of elements
$(t_k, x_k, u_k) \in M, k = 1, 2, \ldots$, with $(t_k, x_k) \in A, |t_k| + |x_k| + |u_k| + + \infty$. Since
$A$ is compact and hence bounded, we have $|u_k| + + \infty$. On the other hand, there
is some subsequence, say still $(t_k, x_k)$, with $t_k \to \bar{t}, x_k \to \bar{x}, (\bar{t}, \bar{x}) \in A$. Given
$\varepsilon > 0$, we have $u_k \in U(\bar{t}, \bar{x}, \varepsilon)$ for all $k$ sufficiently large, as well as
$U(\bar{t}, \bar{x}) \cap U(t_k, x_k) \neq \emptyset$. Since $U(\bar{t}, \bar{x})$ is compact, there is a solid sphere $S$
containing all of $U(\bar{t}, \bar{x})$ in its interior, say $U(\bar{t}, \bar{x}) \subset \text{int S} \subseteq E_m$. On the
other hand, if \( \bar{u}_k \in \overline{U(t, x)} \cap U(t_k, x_k) \), we have \( \bar{u}_k \in \text{int } S \), and \( u_k \in \mathbb{E}_m - S \), again for \( k \) large. Since both \( \bar{u}_k \) and \( u_k \) belong to the convex set \( U(t_k, x_k) \), the segment \( \overline{u}_k u_k \) is contained in \( U(t_k, x_k) \). In particular, if \( u' \) is the point where the segment \( \overline{u}_k u_k \) intersects \( \partial S \), we have \( u_k' \in U(t_k, x_k) \), \( u_k' \in \overline{U(t, x, \varepsilon)} \), and \( u_k' \in \partial S \). If \( u' \) is any point of accumulation of \( \{u_k'\} \), then \( u' \in \partial S \), and \( u' \in \overline{U(t, x, \varepsilon)} \) for every \( \varepsilon > 0 \). Hence, \( u' \in \partial \overline{U(t, x, \varepsilon)} = U(t, x) \), a contradiction, since \( U(t, x) \subseteq \text{int } S \). We have proved that \( M \) is compact.

(vi) If the set \( U(t, x) \) is closed for every \( (t, x) \in A \) and is an uppersemi-continuous function of \( (t, x) \) in \( A \), then \( U(t, x) \) has property \( (U) \) in \( A \).

**Proof.** By hypothesis \( U(t, x, \delta) \subseteq \overline{U(t, x)} \), where \( U_{\varepsilon} \) is closed. Hence \( \overline{U(t, x, \delta)} \subseteq \overline{U(t, x)} \) for \( \delta = \delta(t, x, \varepsilon) \) and any \( \varepsilon > 0 \). Since \( U(t, x) \) is closed, then \( \overline{U(t, x)} \subseteq U(t, x) \) as \( \varepsilon \to 0^+ \). Thus \( \partial \delta \overline{U(t, x, \delta)} \subseteq \overline{U(t, x)} \).

Since the opposite inclusion is trivial, we have \( \partial \delta \overline{U(t, x, \delta)} = U(t, x) \).

Statement (vi) is thereby proved.

The uppersemi-continuity property implies property \( (U) \), but the converse is not true, that is, the uppersemi-continuity property for closed sets is more restrictive than the property \( (U) \). This is shown by the following example in which all sets are closed. Take \( n = 2 \) and

\[
U(t, x) = \{(u^1, u^2) \in \mathbb{E}_2 | 0 \leq u^1 < +\infty, 0 \leq u^2 \leq tu^1\}
\]

for every \( (t, x) \in A = [(t, x) \in \mathbb{E}_2 | 0 \leq t \leq 1, 0 \leq x \leq 1] \). Then, for \( \delta > 0 \), we have

\[
U(t, x, \delta) = \{(u^1, u^2) \in \mathbb{E}_2 | 0 \leq u^1 < +\infty, 0 \leq u^2 \leq (t+\delta)u^1\},
\]

hence \( U(t, x) = \partial \delta \overline{U(t, x, \delta)} \) and \( U(t, x) \) has property \( (U) \) in \( A \). On the other hand,
\[ [U(t,x)]_\varepsilon = \{(u^1,u^2) \in E_2 | 0 \leq u^1 < +\infty, -\varepsilon \leq u^2 \leq tu^1 + \varepsilon(1+t^2)^{1/2} \} \cup N_1, \]

where \( N_1 = N_\varepsilon (0,0) \) if \( t = 0 \), and, if \( t \neq 0 \),

\[ N_1 = N_\varepsilon (0,0) \cup \{(u^1,u^2) \in E_2 | u^1 \leq 0, u^2 \geq -t^{-1}u^1, -tu^1 + u^2 \leq \varepsilon(1+t^2)^{1/2} \}. \]

Obviously \( U(t',x') - [U(t,x)]_\varepsilon \neq \emptyset \) for \( t' > t \), hence \( U(t,x) \) is not an upper semicontinuous function of \( (t,x) \).

(vii) If \( A \) is compact, if \( U(t,x) \) is compact for every \( (t,x) \in A \) and is an upper semicontinuous function of \( (t,x) \) in \( A \), then \( M \) is compact.

(viii) If \( A \) is closed and \( U_j(t,x) \), \( (t,x) \in A \), \( j = 1, \ldots, v \), \( v \) finite, are variable subsets of \( E_m \) all upper semicontinuous functions of \( (t,x) \) in \( A \), then their union \( V(t,x) \) and their intersections \( W(t,x) \) are semicontinuous functions of \( (t,x) \) in \( A \). The same holds for their product \( V(t,x) = U_1 \times \cdots \times U_v \), as well as for their convex hull \( Z(t,x) \), that is, for the set \( Z(t,x) \) of all \( u = p_{1}u_{1} + \cdots + p_{v}u_{v} \) with \( u_j \in U_j(t,x) \), \( p_{j} \geq 0 \), \( j = 1, \ldots, v \), \( p_{1} + \cdots + p_{v} = 1 \).

The proof is straightforward.

(B) The Property (Q)

Let \( U(t,x) \), \( (t,x) \in A \), \( U(t,x) \in E_m \), be any variable subset of \( E_m \), which is a function of \( (t,x) \) in \( A \). By using the same notations as in (A), we shall say that \( U(t,x) \) has property (Q) at \((\bar{t},\bar{x}) \) in \( A \), if

\[ U(t,x) = \bigcap_{\delta > 0} \text{cl} \text{co} U(t,x,\delta). \]

We shall say that \( U(t,x) \) has property (Q) in \( A \) if \( U(t,x) \) has property (Q) at every \( (t,x) \) of \( A \).

(ix) Property (Q) at some \((\bar{t},\bar{x}) \) implies property (U) at the same \((\bar{t},\bar{x}) \), and

\[ U(\bar{t},\bar{x}) = \bigcap_{\delta > 0} \text{cl} \text{co} U(\bar{t},\bar{x},\delta) = \bigcap_{\delta > 0} \text{cl} U(\bar{t},\bar{x},\delta) = \bigcap_{\delta > 0} U(\bar{t},\bar{x},\delta). \]
Indeed
\[ \bigcap_{\varepsilon > 0} \text{cl } U(t, x) = \bigcap_{\varepsilon > 0} \text{cl } \bigcap_{\varepsilon > 0} \text{cl } U(t, x, \varepsilon), \]
where first and last sets coincide by property (Q) at \((t, x)\), and hence the inclusion signs \(\subset\) can be replaced by \(=\) signs.

(xi) If \(A\) is closed, and \(U(t, x)\) is any variable set which is a function of \((t, x)\) in \(A\) and has property (Q) in \(A\), then the set \(M\) of all \((t, x, u) \in A \times \mathbb{R}_m\) with \(u \in U(t, x)\), \((t, x) \in A\), is closed.

Under the hypothesis of (i) the set \(M\) is closed but not necessarily compact as the trivial example \(U(t, x) = \mathbb{R}_m\), \(M = A \times \mathbb{R}_m\) shows. Nevertheless, the statement holds:

(xii) If \(A\) is compact, if the set \(U(t, x)\) is compact for every \((t, x) \in A\) and possesses property (Q) in \(A\), then the set \(M\) is compact.

**Proof.** If \(M\) is not compact, then there is some sequence, \((t_k, x_k, u_k) \in M\), \(k = 1, 2, \ldots\), with \((t_k, x_k) \in A\), \(|t_k| + |x_k| + |u_k| \to + \infty\) as \(k \to \infty\). Since \(A\) is compact and hence bounded, we have \(|u_k| \to + \infty\). On the other hand, there is some subsequence, say still \((t_k, x_k)\), with \(t_k \to t, x_k \to x, (t, x) \in A\). Given \(\varepsilon > 0\), we have then \(u_k \in U(t, x, \varepsilon)\) for all \(k\) sufficiently large. Since \(U(t, x)\) is compact, there is a solid sphere \(S\) containing all of \(U(t, x)\) in its interior, say \(U(t, x) \subset \text{int } S \subset \mathbb{R}_m\). On the other hand, if \(u \in U(t, x)\), we have \(u \in \text{int } S\), and \(u_k \in S - S\), again for \(k\) large. Since both \(u\) and \(u_k\) belong to the
convex set \( \text{cl co } U(t,x,\varepsilon) \), we have \( u_k' \in \text{cl co } U(t,x,\varepsilon) \) where \( u_k' \) is the point of intersection of the segment \( uu_k \) with the boundary \( \text{bd } S \) of \( S \). If \( u' \) is any point of accumulation of \( \{u_k'\} \), then \( u' \in \text{bd } S \), and \( u' \in \text{cl co } U(t,x,\varepsilon) \) for every \( \varepsilon > 0 \). Hence \( u' \in \text{cl co } U(t,x,\varepsilon) = U(t,x) \), a contradiction, since \( U(t,x) \subset \text{int } S \). We have proved that \( M \) is compact.

(xiii) If for every \((t,x) \in A\) the set \( U(t,x) \) is closed and convex, and \( U(t,x) \) is an uppersemicontinuous function of \((t,x)\) in \( A \), then \( U(t,x) \) has property (Q) in \( A \).

**Proof.** By hypothesis \( U(t,x,\varepsilon) \subset [U(t,x)]_{\varepsilon} \), where \( U_{\varepsilon} \) is closed and convex as the closed \( \varepsilon \)-neighborhood of a closed convex set. Hence, \( \cap_{\varepsilon} \text{cl co } U(t,x,\varepsilon) \subset \cap_{\varepsilon} [U(t,x)]_{\varepsilon} \) for every \( \varepsilon > 0 \). Since \( U(t,x) \) is closed, then \( \cap_{\varepsilon} [U(t,x)]_{\varepsilon} \to U(t,x) \) as \( \varepsilon \to 0^+ \). Thus \( \cap_{\varepsilon} \text{cl co } U(t,x,\varepsilon) \subset U(t,x) \). Since the opposite inclusion relation \( \supset \) is trivial, we have \( \cap_{\varepsilon} \text{cl co } U(t,x,\varepsilon) = U(t,x) \).

(C) Relations Between Properties of \( U(t,x) \) and of \( Q(t,x) \)

Let us now consider sets \( Q(t,x) = f(t,x,U(t,x)) \), \((t,x) \in A \), \( Q(t,x) \subset E_n \), which are the images of sets \( U(t,x) \subset E_m \) for every \((t,x) \in A \).

The hypothesis that \( A \) is compact, that \( f \) is continuous on \( M \), that \( U(t,x) \) has property (Q) [or (U)] in \( A \), and that \( Q(t,x) \) is convex for every \((t,x) \in A \), does not imply that \( Q(t,x) \) has property (Q) [or U] in \( A \). This can be proved by a simple example. Let \( m = n = 1 \), \( A = [-1 \leq t \leq 1, 0 \leq x \leq 1] \), let \( U(t,x) \) be the fixed interval \( U = [u \in E_1 \mid 0 \leq u < +\infty] \), and \( f = (u + 1)^{-1} - t \). Then \( Q(t,x) = [z \in E_1 \mid -t < z \leq 1 - t] \), and, if \(-1+\delta < t < 1-\delta \), then

\[
\text{cl co } Q(t,x,\varepsilon) = [-t-\delta \leq z \leq 1-t+\delta] .
\]
The intersection of all these sets for $\delta > 0$ is the closed set $[z \in B_1]$ 
$-t \leq z \leq 1-t$ which is larger than $Q(t,x)$, and thus $Q$ has not property $(Q)$ 
in $A$. Actually, $Q(t,x)$ is not closed, and hence $Q(t,x)$ has neither 
property $(Q)$, nor property $(U)$.

Even the stronger hypothesis that $A$ is compact, that $f$ is continuous on 
$M$, that $U(t,x)$ has property $(Q)$ in $A$, and that $Q(t,x)$ is compact and convex 
for every $(t,x) \in A$, does not imply that $Q(t,x)$ has property $(Q)$ in $A$. This 
can be proved by the following example. Let $m = 1$, $n = 1$, $A = \{(t,x) \in B_2, 
0 \leq t \leq 1, 0 \leq x \leq 1\}$, $U = U(t,x) = \{ueB_1 | 0 \leq u \leq +\infty\}$, and $f(t,x,u) = 
tu \exp (1-tu), (t,x,u) \in AxU$. For $t = 0$ we have $f \equiv 0$, hence $Q(0,x) = [z=0]$. 
For $0 < t \leq 1$, we have $Q(t,x) = [0 \leq z \leq 1]$. All sets $Q(t,x)$ are compact 
and convex, but $Q(t,x)$ does not satisfy property $(Q)$ nor property $(U)$ in $A$.

(xiv) If $A$ is closed and $f$ continuous on $M$, if $U(t,x)$ is an upper semi-
continuous function of $(t,x)$, then $Q(t,x)$ possesses the same property,
and also has property $(U)$. If we know that $Q(t,x)$ is convex, then $Q(t,x)$ 
has also property $(Q)$.

Proof. Each set $Q(t,x)$ is a compact subset of $E_n$ as the continuous 
image of the compact set $U(t,x)$. Let us prove that $M$ is closed. Let 
$(t,\bar{x},\bar{u})$ be a point of accumulation of $M$. Then there is a sequence 
$(t_k, x_k, u_k)$ of points of $M$ with $t_k \rightarrow t, x_k \rightarrow \bar{x}, u_k \rightarrow \bar{u}$, and $(t_k, x_k) \in A$, 
$u_k \in U(t, x)$. Then $(t, x) \in A$ since $A$ is closed, $(t_k, x_k) \in \bar{B}_\varepsilon(t, x)$ for all $k$ sufficiently large, and $u_k \in U(t_k, x_k) = [U(t, x)]_\varepsilon$. Thus, $u \in [U(t, x)]_\varepsilon$ for 
every $\varepsilon > 0$, and hence $\bar{u} \in U(t, x)$ since this set is compact. We have proved
that \((T,\mathcal{X},\mathcal{P})\) is \(M\), and that \(M\) is closed. Let us prove that \(Q(t,x)\) is an upper semicontinuous function of \((t,x)\). Given \((t,x)\) is \(A\) and \(\epsilon > 0\), let 
\(\delta = \delta(t,x,\epsilon) > 0\) be the number relative to the definition of upper semicontinuity of \(U(t,x)\), and let \(M'\) be the set of all \((t',x',u')\) with \((t',x')\) \(\in\) \(N_\delta(t,x)\), \(u' \in U(t',x')\), and \(M''\) be the set of all \((t',x',u')\) with \((t',x')\) \(\in\) \(N_\delta(t,x)\), \(u' \in U(t',x')\). Since \(U(t,x)\) is compact, also \([U(t,x)]_\epsilon\) is compact. Hence \(M'' = N_\delta(t,x)\times[U(t,x)]_\epsilon\), and \(M' = M''M''\). The set \(M'\) is compact as the intersection of the closed set \(M\) with the compact cylinder \(M''\). The function \(f\) is continuous on \(M'\) and hence bounded and uniformly continuous.

Hence, there is some \(\eta, 0 < \eta \leq \min \{\delta, \epsilon\}\), such that \((t'',x'')\) \(\in\) \(N_\eta(t',x')\), 
\(|u' - u''| \leq \eta\), \((t',x',u')\), \((t'',x'',u'')\) \(\in\) \(M'\) implies \(|f(t',x',u') - f(t'',x'',u'')| \leq \epsilon\). Also, let \(\sigma = \min \{\eta, \delta(t,x,\eta)\}\). Then, for every \((t',x')\) \(\in\) \(N_\eta(t,x)\), we have \(U(t',x') \subseteq [U(t,x)]_\eta\), hence, if \(u' \in U(t',x')\), there is some \(u'' \in U(t,x)\) with 
\(|u' - u''| \leq \eta\), and finally \(|f(t',x',u') - f(t,x,u'')| \leq \epsilon\). Thus, \(Q(t',x')\) \(\subseteq\) \([Q(t,x)]_\epsilon\) for every \((t',x')\) \(\in\) \(N_\eta(t,x)\). This proves that \(Q(t,x)\) has the \(\epsilon\)\(\delta\)-property above. The last part of statement (xiv) is now a consequence of statements (vi) and (xiii).

Remark. The statements and examples above show that properties (U) and (Q) are generalizations of the concept of upper semicontinuity for closed, or closed and convex sets, respectively.

(xv) If \(A\) is a closed subset of the \(tx\)-space \(B_1x_{\mathcal{M}}\), if \(U(t,x)\), \((t,x)\) \(\in\) \(A\), \(U(t,x)\) \(\subseteq\) \(E_0\), is a variable subset of \(E_0\) satisfying property (U) in \(A\), if \(M\) denotes the set of all \((t,x,u)\) with \((t,x)\) \(\in\) \(A\), \(u \in U(t,x)\), if \(f_0\) is a continuous scalar function from \(M\) into the reals, if \(\tilde{U}(t,x)\) denotes the variable sub-
set of $E_{m+1}$ defined by $\tilde{U}(t,x) = \{ \tilde{u} = (u^0,u) \in E_{m+1} | u^0 \geq f_o(t,x,u), u \in U(t,x) \}$, then $\tilde{U}(t,x)$ satisfies property (U).

**Proof.** First, let us prove that each set $\tilde{U}(t_0,x_0,\delta)$ is closed. Indeed, if $\tilde{u} = (u^0,u)$ is a point of accumulation of $\tilde{U}(t_0,x_0,\delta)$, then there is a sequence $\tilde{u}_k = (u_k^0,u_k)$ with $u_k^0 \to u^0$, $u_k \to u$, $\tilde{u}_k \in \tilde{U}(t_0,x_0,\delta)$. Hence, there is a corresponding sequence of points $(t_k,x_k) \in N_0(t_0,x_0)$ with $u_k^0 \geq f_o(t_k,x_k,u_k)$, $u_k \in U(t_k,x_k)$. Thus $u_k \in U(t_0,x_0,\delta)$. Since $N_0(t_0,x_0)$ is a compact part of the closed set $A$, there is a subsequence, say still $(t_k,x_k)$, with $t_k \to \tilde{t}$, $x_k \to \tilde{x}$, $(\tilde{t},\tilde{x}) \in N_0(t_0,x_0) \subset A$. Thus $(t_k,x_k,u_k) \in M$, $(t_k,x_k,u_k) \to (\tilde{t},\tilde{x},u)$, and $M$ is a closed set by force of (ii). By the continuity of $f_o$ we have then $(\tilde{t},\tilde{x},u) \in M$, $u^0 \geq f_o(\tilde{t},\tilde{x},u)$. Thus $\tilde{u} = (u^0,u) \in \tilde{U}(\tilde{t},\tilde{x})$, and $\tilde{u} \in \tilde{U}(t_0,x_0,\delta)$.

Now let $\tilde{u} = (u^0,u)$ be a point $\tilde{u} \in \bar{N}_0 \cap U(t_0,x_0,\delta)$. Thus, there is a sequence of numbers $\delta_k > 0$, $\delta_k \to 0$, with $\tilde{u} \in \bar{U}(t_0,x_0,\delta_k)$, and hence $\tilde{u} \in \tilde{U}(t_0,x_0,\delta_k)$ because these last sets are closed. Thus, there is also a sequence of points $(t_k,x_k) \in N_0(t_0,x_0)$ with $u_k \in \tilde{U}(t_k,x_k)$, or $u^0 \geq f_o(t_k,x_k,u)$, $u \in U(t_k,x_k)$. Hence, for every $\eta > 0$, we have $u \in U(t_0,x_0,\eta)$ for every $k$ sufficiently large (so that $\delta_k \leq \eta$), and, by property (U) of $U(t,x)$ at $(t_0,x_0)$, also $u \in \bar{U}(t_0,x_0,\eta) = U(t_0,x_0)$. Thus, $u \in U(t_0,x_0)$, $(t_0,x_0,u) \in M$, and by $u^0 \geq f_o(t_k,x_k,u)$ and the continuity of $f_o$, also $u^0 \geq f_o(t_0,x_0,u)$. We have proved that $\tilde{u} = (u^0,u) \in \tilde{U}(t_0,x_0)$, hence

$$\bar{N}_0 \cap U(t_0,x_0,\delta) \subset U(t_0,x_0).$$

Since the opposite inclusion relation is trivial, equality sign holds, and $\tilde{U}(t,x)$ has property (U) at $(t_0,x_0)$, and, thus, everywhere in $A$. Statement
(xv) is thereby proved.

The set $\overline{U}(t,x)$ of statement (xv) has not necessarily property (Q) even if we assume that $U(t,x)$ has property Q and $f_0(t,x,u)$ is convex in $u$ for every $(t,x)\in A$. This can be seen by a simple example. Let $A = [-1 \leq t \leq 1, 0 \leq x \leq 1]$ and let $U = U(t,x)$ be the fixed set $U(t,x) = E_1$, that is, $U = [-\infty < u^1 < +\infty]$. Then, each set $U(t,x)$ is closed and convex, and obviously $U(t,x)$ possesses property (Q), and $M$ is the cylinder of all $(t,x,u)$ with $(t,x)\in A$, $u \in E_1$. Finally, let $f_0(t,x,u) = tu^1$, so that $f_0$ is continuous in $M$ and, for every $(t,x)\in A$, $f_0 = tu^1$ is linear in $u^1$, hence certainly convex in $u^1$. Now we have

$$\overline{U}(t,x) = [(u^0, u^1) \in E_2 | - \infty < u^1 < +\infty, tu^1 \leq u^2 < +\infty]$$

$$\overline{U}(0,x,\delta) = [(u^0, u^1) \in E_2 | - \infty < u^1 < +\infty, -\delta |u^1| \leq u^2 < +\infty].$$

Consequently, $\overline{U}(0,x,\delta) = E_2$, and hence

$$\text{co} \text{ cl} \text{ co} \overline{U}(0,x,\delta) = E_2,$$

while

$$\overline{U}(0,x) = [(u^1, u^2) \in E_2 | - \infty < u^1 < +\infty, u^2 \geq 0].$$

This shows that $\overline{U}(t,x)$ does not have property (Q) at the points $(0,x)$ of $A$.

A scalar function $f_0(t,x,u)$, $(t,x,u)\in M$, is said to be convex in $u$ at

$(t_0,x_0)\in A$ if

$$f_0(t_0,x_0,u_0) \leq \sum_{i=1}^{N} \lambda_i f_0(t_0,x_0,u_i),$$

whenever

$$u_0 = \sum_{i=1}^{N} \lambda_i u_i,$$
where $u_i \in U(t_0, x_0)$, $\lambda_i \geq 0$, $i = 1, \ldots, N$, $\lambda_1 + \ldots + \lambda_N = 1$.

A scalar function $f_o(t, x, u)$, $(t, x, u) + M$, is said to be quasi normally convex in $u$ at $(t_0, x_0, u_0) \in M$ provided, given $\epsilon > 0$, there are a number $\delta = \delta(t_0, x_0, u_0, \epsilon) > 0$, and a linear scalar function $z(u) = z + b \cdot u$, $b = (b_1, \ldots, b_m)$, $r, b_1, \ldots, b_m$ real, such that

(a) $f_o(t, x, u) \geq z(u)$ for all $(t, x) \in N_{\delta}(t_0, x_0), u \in U(t, x),$

(b) $f_o(t, x, u) \leq z(u) + \epsilon$ for all $(t, x) \in N_{\delta}(t_0, x_0), u \in U(t, x), |u - u_0| \leq \delta.$

The scalar function $f_o(t, x, u)$ is said to be normally convex in $u$ at $(t_0, x_0, u_0)$ if, given $\epsilon > 0$, there are numbers $\delta = \delta(t_0, x_0, u_0, \epsilon) > 0$, $\nu = \nu(t_0, x_0, u_0, \epsilon) > 0$, and a linear scalar function $z(u) = r + b \cdot u$ as above such that (b) holds and

(a') $f_o(t, x, u) \geq z(u) + \nu|u - u_0|$ for all $(t, x) \in N_{\delta}(t_0, x_0), u \in U(t, x).$

The scalar function $f_o(t, x, u)$ is said to be quasi normally convex in $u$, or normally convex in $u$, if it has these properties at every $(t_0, x_0, u_0) \in M$.

For the case where $U = U(t, x)$ is the fixed set $U = E_m$, the following statement gives a useful characterization of the functions $f_o$ which are normally convex in $u$.

(xvi) If $A$ is closed, and $f_o(t, x, u)$ is continuous on $M = AxE_m$, then $f_o$ is normally convex in $u$ if and only if $f_o$ is convex in $u$ at every $(t_0, x_0) \in A$, and for no points $(t_0, x_0) \in A$, $u_0, u_1 \in E_m$, $u_1 \neq 0$, the relation holds $f_o(t_0, x_0, u_0) = 2^{-1}[f_o(t_0, x_0, u_0 + \lambda u_1) + f_o(t_0, x_0, u_0 - \lambda u_1)]$ for all $\lambda \geq 0$.

This statement was proved in [9a] and [10]. In particular, if for every $(t, x) \in A$, $f_o(t, x, u)$ is convex in $u$ and $f_o(t, x, u)/|u| \rightarrow +\infty$ as $|u| \rightarrow +\infty$, then certainly $f_o(t, x, u)$ is normally convex in $u$. 

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(xvii) If A is closed subset of the tx-space $E_1 \times E_n$, if $U(t,x)$, $(t,x) \in A$, $U(t,x) \in E_m$, is a variable subset of $E_m$ satisfying property (Q) in A, if $M$ denotes the set of all $(t,x,u)$ with $(t,x) \in A$, $u \in U(t,x)$, if $f_o$ is a continuous function from M into the reals, which is convex in u for every $(t,x) \in A$, if either (α) the sets $U(t,x)$ are all contained in a fixed solid sphere $S$ of $E_m$, or (β) the function $f_o(t,x,u)$ is quasi normally convex in u at every $(t_0,x_0,u_0)$ of $M$, then the set $\tilde{U}(t,x)$ of statement (xv) has property (Q) in A.

**Proof.** Let $\tilde{u} = (u^o,u)$ be a point $\tilde{u} = \cap \, \text{cl co } \tilde{U}(t_0,x_0,\delta)$. Then there is a sequence $[\delta_k]$ of numbers $\delta_k > 0$, $\delta_k \rightarrow 0$, with $\tilde{u} \in \text{cl co } \tilde{U}(t_0,x_0,\delta_k)$. Hence, there is a sequence of pairs of points $\tilde{u}^i_k \in E_{m+1}$ and of points $\tilde{v}_k$ of the segment $(\tilde{u}^i_k \tilde{u}^j_k) \in E_{m+1}$, such that

$$\tilde{v}_k \rightarrow \tilde{u}, \quad \tilde{u}^i_k , \tilde{u}^j_k \in U(t_0,x_0,\delta_k),$$

$$\tilde{v}_k = \alpha_k \tilde{u}^i_k + (1-\alpha_k)\tilde{u}^j_k, \quad 0 \leq \alpha_k \leq 1, \quad k = 1,2, \ldots.$$ 

We shall use the notation $\tilde{v}_k = (v_k^o,v_k)$, $\tilde{u} = (u^o,u)$, $\tilde{u}^i_j = (u^i_k,u^j_k)$, $j = 1,2$. Then we have

$$v_k^o + u^o, \quad v_k + u, \quad u_k^i \in u_k^j \in E(t_0,x_0,\delta_k)$$

$$v_k^o = \alpha_k u_k^i + (1-\alpha_k)u_k^j, \quad v_k = \alpha_k u_k^i + (1-\alpha_k)u_k^j.$$ 

Consequently, there are points such that

$$(t_{k_1},x_{k_1}), \quad (t_{k_2},x_{k_2}) \in N_{\delta_k}(t_0,x_0) \subset A,$$

$$u_{k_1} \in U(t_{k_1},x_{k_1}), \quad u_{k_2} \in U(t_{k_2},x_{k_2}).$$
The sequence \([\alpha_k]\) is bounded, hence there is a convergent subsequence, say still \(\alpha_k\), so that \(\alpha_k \to \alpha\) for some \(0 \leq \alpha \leq 1\).

For every \(\eta > 0\) and \(k\) sufficiently large (so that \(\delta_k \leq \eta\)), we have
\[u_{k_1}, u_{k_2} \in U(t_0, x_0, \eta),\]

hence
\[u_{k_1}, u_{k_2} \in \text{cl} \co U(t_0, x_0, \eta).\]

As a consequence
\[v_k = \alpha_k u_{k_1} + (1-\alpha_k) u_{k_2} \in \text{cl} \co U(t_0, x_0, \eta)\]

for all \(k\) sufficiently large. As \(k \to \infty\), we obtain \(u \in \text{cl} \co U(t_0, x_0, \eta)\).

By the property \((U)\), finally
\[\{u\} \cap \text{cl} \co U(t_0, x_0, \eta) = U(t_0, x_0).\]  \(\text{(1)}\)

Assume first that condition \((\alpha)\) holds. Then both sequences \([u_{k_1}]\), \([u_{k_2}]\) are bounded, and hence there is a subsequence, say still \([u_{k_1}], [u_{k_2}],\)

for which both \(u_{k_1}\) and \(u_{k_2}\) are convergent in \(E_m\), say \(u_{k_1} \to u_1\), \(u_{k_2} \to u_2\),

\(u_1, u_2 \in E_m\). For such a subsequence, we have
\[v_k = \alpha_k u_{k_1} + (1-\alpha_k) u_{k_2} \geq \alpha_k f_o(t_{k_1}, x_{k_1}, u_{k_1}) + (1-\alpha_k) f_o(t_{k_2}, x_{k_2}, u_{k_2}),\]

\[v_k = \alpha_k u_{k_1} + (1-\alpha_k) u_{k_2},\]

\((t_{k_1}, x_{k_1}, u_{k_1}), (t_{k_2}, x_{k_2}, u_{k_2}) \in M,\]

where \(M\) is closed. By taking limits as \(k \to \infty\), we have
\[u_o \geq \alpha f_o(t_0, x_0, u_1) + (1-\alpha) f_o(t_0, x_0, u_2)\]

\[u = \alpha u_1 + (1-\alpha) u_2,\]

\((t_0, x_0, u_1), (t_0, x_0, u_2) \in M.\]

By the convexity of \(f_o\) in \(u\) at \((t_0, x_0)\) we have now
\[ u^0 \geq f_0(t_0, x_0, u) + (1-\alpha)u_2 = f_0(t_0, x_0, u) \]

This proves that \( \tilde{U} = (u^0, u) \in U(t_0, x_0) \), hence

\[ \mathfrak{U}_0 \subset U(t_0, x_0) \subset \tilde{U}(t_0, x_0). \]  

(2)

Since the opposite inclusion is trivial, = sign holds in this relation, and \( \tilde{U}(t, x) \) has property (Q) at \( (t_0, x_0) \). Since \( (t_0, x_0) \in A \) is arbitrary, \( \tilde{U}(t, x) \) has property (Q) in A.

Assume now that condition (\( \beta \)) holds. As stated by relation (1) above, \( u \in U(t_0, x_0) \), hence \( (t_0, x_0, u) \in \mathfrak{U} \). By the quasi normal convexity of \( f_0 \) in u at \( (t_0, x_0, u) \) we deduce the existence of a number \( \delta > 0 \) and of a linear scalar function \( z(u) = r + b \cdot v \) such that (a) \( f_0(t, x, v) = z(v) \) for all \( (t, x) \in \mathfrak{U}_0(t_0, x_0) \), \( v \in U(t, x) \) and (b) \( f_0(t, x, v) \leq z(v) + \varepsilon \) for all \( (t, x) \in \mathfrak{U}_0(t_0, x_0) \), \( v \in U(t, x) \), \( |u - v| \leq \delta \). By combining (a) and (b) we have then (c) \( z(u) \leq f_0(t_0, x_0, u) = z(u) + \varepsilon \).

Now we have \( v_k = \alpha_k u_{k1} + (1-\alpha_k)u_{k0} \) for some \( 0 \leq \alpha_k \leq 1 \), and \( v_k + u, (t_{k,j}, x_{k,j}) \to (t_0, x_0), j = 1, 2 \). Thus, for \( k \) sufficiently large, \( (t_{k,j}, x_{k,j}) \in \mathfrak{U}_0 \) (\( t_0, x_0 \), \( j = 1, 2 \)), and, by property (a),

\[ v_k \geq \alpha_k f_0(t_{k1}, x_{k1}, u_{k1}) + (1-\alpha_k) f_0(t_{k2}, x_{k2}, u_{k2}) \]

\[ \geq \alpha_k z(u_{k1}) + (1-\alpha_k) z(u_{k2}) = z(\alpha_k u_{k1} + (1-\alpha_k) u_{k2}) = z(v_k). \]

As \( k \to \infty \), we have then \( u^0 \geq z(u) \), and finally by (c) above, \( u^0 \geq f_0(t_0, x_0, u) \) \(-\varepsilon\), where \( \varepsilon > 0 \) is arbitrary. We conclude that \( u^0 \geq f_0(t_0, x_0, u) \), with \( u \in U(t_0, x_0) \). Thus \( U = (u^0, v) \in \tilde{U}(t_0, x_0) \), and again we have proved inclusion (2). The same reasoning above yields that \( \tilde{U}(t, x) \) has property (Q) in A.
(xviii) If $A$ is a closed subset of the $tx$-space $E_1 \times E_n$, if $U(t,x)$, $(t,x) \in A$, is a variable subset of $E_m$ satisfying property $(U)$ in $A$, if $M$ denotes the set of all $(t,x,u)$ with $(t,x) \in A$, $u \in U(t,x)$, if $\tilde{f} = (f_0, \tilde{f})$ is a continuous function from $M$ into the $z$-space $E_{n+1}$, $z = (z^0, z)$, if $Q(t,x) \subseteq E_n$, $Q(t,x) \subseteq E_{n+1}$ are the sets

$$Q(t,x) = f(t,x, U(t,x)) = \{z \in E_n : z = f(t,x,u), u \in U(t,x)\},$$

$$\tilde{Q}(t,x) = \{z^0, z \in E_{n+1} : z^0 \geq f_0(t,x,u), z = f(t,x,u), u \in U(t,x)\},$$

and (a) for every $(t,x) \in A$, $Q(t,x)$ is a convex subset of $E_n$; (b) $Q(t,x)$ has property $(Q)$ in $A$; (c) for every $(t,x) \in A$, $z = f(t,x,u)$ is a 1-1 map from $U(t,x)$ onto $Q(t,x)$ with a continuous inverse $u = f^{-1}(t,x,z)$, $z \in Q(t,x)$; (d) the real valued function $F_0(t,x,z) = f_0(t,x, f^{-1}(t,x,z))$, $(t,x) \in A$, $z \in Q(t,x)$, is continuous in the set $M'$ of all $(t,x,z)$ with $(t,x) \in A$, $z \in Q(t,x)$, and $F_0(t,x,z)$ is convex in $z$ and also quasi normally convex, then the set $\tilde{Q}(t,x)$ is convex and has property $(Q)$ in $A$.

**Proof.** Indeed, under the specific hypotheses above, the set $\tilde{Q}(t,x)$ can be represented as

$$\tilde{Q}(t,x) = \{z^0, z \in E_{n+1} : z^0 \geq F_0(t,x,z), z \in Q(t,x)\},$$

and thus $\tilde{Q}$ is generated from $Q(t,x)$ exactly as $\tilde{U}$ is generated from $U(t,x)$.

By statement (xvii) above we conclude that $Q(t,x)$ has property $(Q)$ in $A$.

**Remark.** The condition that $f$ is a homeomorphism between $U$ and $Q$ is certainly verified in all free problems, where $m = n$, $f = u$, that is, $f_i = u^i$, $i = 1, 2, \ldots, n$ (see no. 9 below). In this situation then we have $F_0(t,x,u) = f_0(t,x,u)$, and the convexity of $f_0$ in $u$ implies the convexity of
in $u$. We shall need this remark, and the more general statement (xviii) in part II.

5. CLOSURE THEOREMS

We shall use here the notations of no. 2 and 3. In particular, a trajectory $x(t)$ is defined as in no. 2.

Closure Theorem I. (A first generalization of Filippov's theorem).

Let $A$ be a closed subset of $E_1 \times E_n$, let $U(t,x)$ be a closed subset of $E_m$ for every $(t,x) \in A$, let $f(t,x,u) = (f_1, \ldots, f_n)$ be a continuous vector function on $M$ into $E_n$, and let $Q(t,x) = f(t,x,U(t,x))$ be a closed convex subset of $E_n$ for every $(t,x) \in A$. Assume that $U(t,x)$ has property (U) in $A$, and that $Q(t,x)$ has property (Q) in $A$. Let $x_k(t)$, $t_1 \leq t \leq t_2$, $k = 1, 2, \ldots$, be a sequence of trajectories, which is convergent in the metric $\rho$ toward an absolutely continuous function $x(t)$, $t_1 \leq t \leq t_2$. Then $x(t)$ is a trajectory.

Remark. If we assume that $U(t,x)$ is compact for every $(t,x) \in A$, and that $U(t,x)$ is an uppersemicontinuous function of $(t,x)$ in $A$, then by statement (x), the set $Q(t,x)$ has the same property, $U(t,x)$ has property (U), $Q(t,x)$ has property (Q), and closure theorem I reduces to one of A. F. Filippov [2] (not explicitly stated in [2] but contained in the proof of his existence theorem for the Pontryagin problem with $U(t,x)$ always compact).

Proof of Closure Theorem I. The vector functions

$$
q(t) = x'(t), \quad t_1 \leq t \leq t_2, \\
q_k(t) = x'_k(t) = f(t, x_k(t), u_k(t)), \quad t_{1k} \leq t \leq t_{2k}, \quad k = 1, 2, \ldots, (1)
$$

are defined almost everywhere and are $L$-integrable. We have to prove that
(t,x(t))∈A for every $t_1 \leq t \leq t_2$, and that there is an admissible control function $u(t), t_1 \leq t \leq t_2$, such that

$$\varphi(t) = x'(t) = f(t,x(t),u(t)), \quad u(t)\in U(t,x(t)), \quad (2)$$

for almost all $t \in [t_1,t_2]$.

First, $\rho(x_k,x) \to 0$ as $k \to 0$; hence, $t_{1k} + t_1, t_{2k} + t_2$. If $t \in (t_1,t_2)$, or $t_1 < t < t_2$, then $t_{1k} < t < t_{2k}$ for all $k$ sufficiently large and $(t,x_k(t))\in A$. Since $x_k(t) + x(t)$ as $k \to \infty$ and $A$ is closed, we conclude that $(t,x(t))\in A$ for every $t_1 < t < t_2$. Since $x(t)$ is continuous, and hence continuous at $t_1$ and $t_2$, we conclude that $(t,x(t))\in A$ for every $t_1 \leq t \leq t_2$.

For almost all $t \in [t_1,t_2]$ the derivative $x'(t)$ exists and is finite. Let $t_0$ be such a point with $t_1 < t_0 < t_2$. Then there is a $\sigma > 0$ with $t_1 < t_0 - \sigma < t_0 + \sigma < t_2$, and, for some $k_0$ and all $k \geq k_0$, also $t_{1k} < t_0 - \sigma < t_0 + \sigma < t_{2k}$. Let $x_0 = x(t_0)$.

We have $x_k(t) + x(t)$ uniformly in $[t_0 - \sigma, t_0 + \sigma]$ and all functions $x(t), x_k(t)$ are continuous in the same interval. Thus, they are equicontinuous in $[t_0 - \sigma, t_0 + \sigma]$. Given $\epsilon > 0$, there is a $\delta > 0$ such that $t,t'\in [t_0 - \sigma, t_0 + \sigma], |t-t'| \leq \delta, k \geq k_0$, implies

$$|x(t) - x(t')| \leq \epsilon/2, \quad |x_k(t) - x_k(t')| \leq \epsilon/2.$$

We can assume $0 < \delta < \sigma, \delta \leq \epsilon$. For any $h, 0 < h \leq \delta$, let us consider the averages

$$m_h = h^{-1} \int_0^h \varphi(t_0+s)ds = h^{-1}[x(t_0+h)-x(t_0)],$$

$$m_{hk} = h^{-1} \int_0^h \varphi_k(t_0+s)ds = h^{-1}[x_k(t_0+h)-x_k(t_0)]. \quad (3)$$
Given $\eta > 0$ arbitrary, we can fix $h$, $0 < h \leq \delta < \sigma$, so small that

$$|mh - \varphi(t_0)| \leq \eta.$$  \hfill (4)

Having so fixed $h$, let us take $k_1 \geq k_0$ so large that

$$|mh_k - mh| \leq \eta, \quad |x_k(t_0) - x(t_0)| \leq \varepsilon/2$$  \hfill (5)

for all $k \geq k_1$. This is possible since $x_k(t) \to x(t)$ as $k \to \infty$ both at $t = t_0$ and $t = t_0 + h$. Finally, for $0 \leq s \leq h$,

$$|x_k(t_0 + s) - x(t_0)| \leq |x_k(t_0 + s) - x_k(t_0)| + |x_k(t_0) - x(t_0)|$$

$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

$$|(t_0 + s) - t_0| \leq h \leq \delta \leq \varepsilon$$

$$f(t_0 + s, x_k(t_0 + s), u_k(t_0 + s)) \epsilon Q(t_0 + s, x_k(t_0 + s)).$$

Hence, by the definition of $Q(t_0, x_0, 2\varepsilon)$, also

$$\varphi_k(t_0 + s) = f(t_0 + s, x_k(t_0 + s), u_k(t_0 + s)) \epsilon Q(t_0, x_0, 2\varepsilon).$$

The second integral relation (3) shows that we have also

$$m_{hk} \epsilon \text{cl conv } Q(t_0, x_0, 2\varepsilon),$$

since the latter is a closed convex set. Finally, by relations (4) and (5), we deduce

$$|\varphi(t_0) - m_{hk}| \leq |\varphi(t_0) - mh| + |mh - m_{hk}| \leq 2\eta,$$

and hence

$$\varphi(t_0) \epsilon [\text{cl conv } Q(t_0, x_0, 2\varepsilon)] 2\eta].$$

Here $\eta > 0$ is an arbitrary number, and the set in brackets is closed.

Hence,

$$\varphi(t_0) \epsilon \text{cl conv } Q(t_0, x_0, 2\varepsilon),$$
and this relation holds for every $\varepsilon > 0$. By property (Q) we have
\[ \phi(t_0) \in \cap_{\varepsilon} \text{cl co } Q(t_0, x_0, 2\varepsilon) = Q(t_0, x_0), \]
where $x_0 = x(t_0)$, and $Q(t_0, x_0) = f(t_0, x_0, U(t_0, x_0))$. This relation implies that there are points $\bar{U} = \overline{U(t_0)} \in U(t_0, x_0)$ such that
\[ \phi(t_0) = f(t_0, x(t_0), \bar{U}(t_0)). \] (6)
This holds for almost all $t_0 \in [t_1, t_2]$, that is, for all $t$ of a measurable set $I \subset [t_1, t_2]$ with measure $I = t_2 - t_1$. If we take $I_0 = [t_1, t_2] - I$, then means $I_0 = 0$. Hence, there is at least one function $\bar{U}(t)$, defined almost everywhere in $[t_1, t_2]$, for which relation (6) holds a.e. in $[t_1, t_2]$. We have to prove that there is at least one such function which is measurable.

For every $t \in I$, let $P(t)$ denote the set
\[ P(t) = \{ u | \text{ue}U(t, x(t)), \phi(t) = f(t, x(t), u) \subset U(t, x(t)) \subset E_m \}. \]
We have proved that $P(t)$ is not empty.

For every integer $\lambda = 1, 2, \ldots$, there is a closed subset $C_\lambda$ of $I$, $C_\lambda \subset \subset [t_1, t_2]$, with measure $C_\lambda > \max [0, t_2 - t_1 - 1/\lambda]$, such that $\phi(t)$ is continuous on $\lambda$. Let $W_\lambda$ be the set
\[ W_\lambda = \{ (t, u) | t \in C_\lambda, u \in P(t) \} \subset E_1 \times E_m. \]

Let us prove that the set $W_\lambda$ is closed. Indeed, if $(\bar{t}, \bar{u})$ is a point of accumulation of $W_\lambda$, then there is a sequence $(t_s, u_s), s = 1, 2, \ldots$, with $(t_s, u_s) \in W_\lambda$, $t_s \to \bar{t}$, and $u_s \to \bar{u}$. Then $t_s \in C_\lambda$ and $u_s \in C_\lambda$ since $C_\lambda$ is closed. Also $x(t_s) \to \bar{x}(\bar{t})$, $\phi(t_s) \to \phi(\bar{t})$, and since $(t_s, x(t_s)) \in A$, $\phi(t_s) = f(t_s, x(t_s), u(t_s))$, $(t_s, x(t_s), u(t_s)) \in M$, we have also $(\bar{t}, \bar{x}(\bar{t})) \in A$, $(t, x(\bar{t}), \bar{u}) \in M$, because
A and M are closed, and $\varphi(t) = f(t, x(t), u)$ because $f$ is continuous. Thus, $\bar{u} \in P(t)$, and $(t, x) \in W_{\lambda}^n$.

For every integer $\ell$, let $W_{\lambda, \ell}$, $P_{\lambda}(t)$, be the sets

$$W_{\lambda, \ell} = \{ (t, u) | (t, u) \in W_{\lambda}^n, |u| \leq \ell \} \subseteq W_{\lambda}^n \subseteq E_1 \times E_m,$$

$$P_{\lambda}(t) = \{ u | u \in P(t), |u| \leq \ell \} \subseteq P(t) \subseteq U(t, x(t)) \subseteq E_m,$$

$$C_{\lambda, \ell} = \{ t | (t, u) \in W_{\lambda, \ell} \text{ for some } u \} \subseteq C_\lambda \subseteq \mathbb{I} \subseteq [t_1, t_2].$$

Obviously, $W_{\lambda, \ell}$ is compact, and so is $C_{\lambda, \ell}$ as its projection on the $t$-axis.

Also, $U_{\ell}C_{\lambda, \ell} = C_\lambda$, and $W_{\lambda, \ell}$ is the set of all $(t, u)$ with $t \in C_{\lambda, \ell}$, $u \in P_{\lambda}(t)$. Thus, for $t \in C_{\lambda, \ell}$, $P(t)$ is a compact subset of $U(t, x(t))$.

For $t \in C_{\lambda, \ell}$, the set $P_{\lambda}(t)$ is the nonempty compact subset of all $u = (u^1, \ldots, u^m) \in U(t, x(t))$ with $f(t, x(t), u) = \varphi(t)$. As in Filippov's argument let $P$, be the subset of $P$ with $u^1$ minimum, let $P_2$ be the subset of $P_1$ with $u^2$ minimum, $\ldots$, let $P_m$ be the subset of $P_{m-1}$ with $u^m$ minimum. Then $P_m$ is a single point $u = u(t) \in U(t, x(t))$ with $u(t) = (u^1, \ldots, u^m)$, $t \in C_{\lambda, \ell}$, $|u(t)| \leq \ell$, and $f(t, x(t), u(t)) = \varphi(t)$. Let us prove that $u(t), t \in C_{\lambda, \ell}$ is measurable. We shall prove this by induction on the coordinates. Let us assume that $u^1(t), \ldots, u^{s-1}(t)$ have been proved to be measurable on $C_{\lambda, \ell}$ and let us prove that $u^s(t)$ is measurable. For $s = 1$ nothing is assumed, and the argument below proves that $u^1(t)$ is measurable. For every integer $j$ there are closed subsets $C_{\lambda, \ell, j}$ of $C_{\lambda, \ell}$ with $C_{\lambda, \ell} \subseteq C_{\lambda, \ell, j} \subseteq C_{\lambda, \ell, j+1}$, meas: $C_{\lambda, \ell, j} > [0, \text{ meas: } C_{\lambda, \ell-1/j}]$, such that $u^1(t), \ldots, u^{s-1}(t)$ are continuous on $C_{\lambda, \ell, j}$.

The function $\varphi(t)$ is already continuous on $C_\lambda$ and hence $\varphi(t)$ is continuous on every set $C_{\lambda, \ell}$ and $C_{\lambda, \ell, j}$. Let us prove that $u^s(t)$ is measurable on $C_{\lambda, \ell, j}$. 

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We have only to prove that, for every real \( a \), the set of all \( t \in C_{\lambda,j} \) with \( \varphi^s(t) \leq a \) is closed. Suppose that this is not the case. Then there is a sequence of points \( t_k \in C_{\lambda,j} \) with \( u^s(t_k) \leq a \), \( t_k + \tau \in C_{\lambda,j} \), \( u^s(\tau) > a \).

Then \( \varphi(t_k) + \varphi(\tau), u^\alpha(t_k) + u^\alpha(\tau) \) as \( k \to \infty \), \( \alpha = 1, \ldots, s-1 \). Since 
\[ |u^\beta(t_k)| \leq \epsilon \] for all \( k \) and \( \beta = s, s+1, \ldots, m \), we can select a subsequence, say still \( t_k \) such that \( u^\beta(t_k) + u^\beta \) as \( k \to \infty \), \( \beta = s, s+1, \ldots, m \), for some real numbers \( \tilde{u}^\beta \). Then \( t_k + \tau, x(t_k) + x(\tau), u(t_k) \to \tilde{u} \), where 
\[ \tilde{u} = (u^1(\tau), \ldots, u^{s-1}(\tau), \tilde{u}^s, \ldots, \tilde{u}^m). \]

Then, given any number \( \eta > 0 \), we have 
\[ u(t_k) \in U(t_k, x(t_k)) \subseteq \text{cl } U(\tau, x(t), \eta) \]
for all \( k \) sufficiently large, and, as \( k \to \infty \), also 
\[ \tilde{u} \in \text{cl } U(\tau, x(\tau), \eta). \]

By the property (U) we have 
\[ \tilde{u} \in \bigcap_{\eta} \text{cl } U(\tau, x(\tau), \eta) = U(\tau, x(\tau)) \]
On the other hand \( \varphi(t_k) = f(t_k, x(t_k), u(t_k)) \), \( u^s(t_k) \leq a \), yield as \( k \to \infty \), 
\[ \varphi(\tau) = f(\tau, x(\tau), \tilde{u}), \; \tilde{u}^s \leq a, \] (6)
while \( \tau \in C_{\lambda,j} \) implies 
\[ \varphi(\tau) = f(\tau, x(\tau), u(\tau)), \; u^s(\tau) > a. \] (7)

Relations (6) and (7) are contradictory because of the property of minimum with which \( u^s(\tau) \) has been chosen. Thus \( u^s(t) \) is measurable on \( C_{\lambda,j} \) for every \( j \), and then \( u^s(t) \) is also measurable on \( C_{\lambda,j} \). By induction argument, all components \( u^j(t), \ldots, u^m(t) \) of \( u(t) \) are measurable on \( C_{\lambda,j} \),
hence $u(t)$ is measurable on $C_\Lambda$. Since $U_\Lambda C_\Lambda = C_\Lambda$, meas $C_\Lambda >$ meas $I - 1/\Lambda$, we conclude that $u(t)$ is measurable on every set $C_\Lambda$ and hence on $I$, with meas $I = t_2 - t_1$. Thus, $u(t)$ is defined a.e. on $(t_1, t_2)$, $u(t) \epsilon U(t, x(t))$ and $f(t, x(t), u(t)) = \varphi(t)$ a.e. on $[t_1, t_2]$. Closure Theorem I is thereby proved.

Let us denote by $y = (x^1, \ldots, x^8)$ the $s$-vector made up of certain components, say $x^1, \ldots, x^8$, $0 \leq s \leq n$, of $x = (x^1, \ldots, x^n)$, and by $z$ the complementary $(n-s)$-vector $z = (x^{s+1}, \ldots, x^n)$ of $x$, so that $x = (y, z)$. Let us assume that $f(t, y, u)$ depends only on the coordinates $x^1, \ldots, x^s$ of $x$. If $x(t), t_1 \leq t \leq t_2$, is any vector function, we shall denote by $x(t) = [y(t), z(t)]$ the corresponding decomposition of $x(t)$ in its coordinates $y(t) = (x^1, \ldots, x^s)$ and $z(t) = (x^{s+1}, \ldots, x^n)$.

We shall denote by $A_0$ a closed subset of points $(t, x^1, \ldots, x^8)$, that is, a closed subset of the ty-space $E_1 \times E_8$, and let $A = A_0 \times E_{n-s}$. Thus, $A$ is a closed subset of the tx-space $E_1 \times E_n$.

**Closure Theorem II.** (A further generalization of Filippov's Theorem). Let $A_0$ be a closed subset of the ty-space $E_1 \times E_8$, and then $A = A_0 \times E_{n-s}$ is a closed subset of the tx-space $E_1 \times E_n$. Let $U(t, y)$ denote a closed subset of $E_m$ for every $(t, y) \epsilon A_0$, let $M_0$ be the set of all $(t, y, u) \epsilon E_1 + \epsilon E_m$ with $(t, y) \epsilon A_0$, $u \epsilon U(t, y)$, and let $f(t, y, u) = (f_1, \ldots, f_n)$ be continuous vector function from $M$ into $E_n$. Let $Q(t, y) = f(t, y, U(t, y))$ be a closed convex subset of $E_n$ for every $(t, y) \epsilon A_0$. Assume that $U(t, y)$ has property (U) in $A_0$ and that $Q(t, y)$ has property (Q) in $A_0$.
Let \( x_k(t), t_{1k} \leq t \leq t_{2k}, k = 1, 2, \ldots \), be a sequence of trajectories, \( x_k(t) = (y_k(t), z_k(t)) \), for which we assume that the \( s \)-vector \( y_k(t) \) converges in the \( \rho \)-metric toward an AC vector function \( y(t), t_1 \leq t \leq t_2 \), and that the \((n-k)\)-vector \( z_k(t) \) converges (pointwise) for almost all \( t_1 < t < t_2 \), toward a vector \( z(t) \) which admits of a decomposition \( z(t) = Z(t) + S(t) \) where \( Z(t) \) is an AC vector function in \( [t_1, t_2] \), and \( S'(t) = 0 \) a.e. in \( [t_1, t_2] \) (that is, \( S(t) \) is a singular function). Then, the AC vector \( X(t) = [y(t), Z(t)] \), \( t_1 \leq t \leq t_2 \), is a trajectory.

**Remark.** For \( s = n \), this theorem reduces to closure theorem I.

**Proof of Closure Theorem II.** The vector functions

\[
\varphi(t) = X'(t) = (y'(t), Z'(t)), \quad t_1 \leq t \leq t_2,
\]

\[
\varphi(t) = x'_k(t) = (y'_k(t), z'_k(t)) = f(t, y_k(t), u_k(t)), \quad t_{1k} \leq t \leq t_{2k}, \quad k = 1, 2, \ldots,
\]

are defined almost everywhere and are \( L \)-integrable. We have to prove that \( [t, y(t), Z(t)] \in A \) for every \( t_1 \leq t \leq t_2 \), and that there is an admissible control function \( u(t), t_1 \leq t \leq t_2 \), such that

\[
\varphi(t) = X'(t) = (y'(t), Z'(t)) = f(t, y(t), u(t)),
\]

\[
u(t) \in U(t, y(t)),
\]

for almost all \( t \in [t_1, t_2] \).

First, \( \rho(y_k, y) \to 0 \) as \( k \to 0 \); hence \( t_{1k} \to t_1, t_{2k} \to t_2 \). If \( t \in (t_1, t_2) \), or \( t_1 < t < t_2 \), then \( t_{1k} < t < t_{2k} \) for all \( k \) sufficiently large, and \( (t, y_k(t)) \in A_0 \). Since \( y_k(t) \to y(t) \) as \( k \to \infty \) and \( A_0 \) is closed, we conclude that \( (t, y(t)) \in A_0 \) for every \( t_1 < t < t_2 \), and finally \( (t, y(t), Z(t)) \in A_0 \times \mathbb{E}_{n-s} \).
or \((t,x(t))\in A, t_1 \leq t \leq t_2\).

For almost all \(t\in [t_1,t_2]\) the derivative \(X'(t) = [y'(t),Z'(t)]\) exists and is finite, \(S'(t)\) exists and \(S'(t) = 0\), and \(z_k(t) \to z(t)\). Let \(t_0\) be such a point with \(t_1 < t_0 < t_2\). Then there is a \(\sigma > 0\) with \(t_1 < t_0 - \sigma < t_0 + \sigma < t_2\), and, for some \(k_0\) and all \(k > k_0\), also \(t_{1k} < t_0 - \sigma < t_0 + \sigma < t_{2k}\). Let \(x_0 = x(t_0) = (y_0,Z_0)\), or \(y_0 = y(t_0), Z_0 = Z(t_0)\). Let \(z_0 = z(t_0), S_0 = S(t_0)\). We have \(S'(t_0) = 0\), hence \(z'(t_0)\) exists and \(z'(t_0) = Z'(t_0)\).

Also, we have \(Z_k(t_0) \to z(t_0)\).

We have \(y_k(t) \to y(t)\) uniformly in \([t_0 - \sigma, t_0 + \sigma]\), and all functions \(y(t), y_k(t)\) are continuous in the same interval. Thus, they are equicontinuous in \([t_0 - \sigma, t_0 + \sigma]\). Given \(\varepsilon > 0\), there is a \(\delta > 0\) such that \(t,t' \in [t_0 - \sigma, t_0 + \sigma], |t - t'| \leq \delta, k \geq k_0\), implies

\[
|y(t) - y(t')| \leq \varepsilon/2, \quad |y_k(t) - y_k(t')| \leq \varepsilon/2.
\]

We can assume \(0 < \delta < \sigma, \delta \leq \varepsilon\). For any \(h, 0 < h \leq \delta\), let us consider the averages

\[
m_h = h^{-1} \int_0^h \varphi(t_0 + s)ds = h^{-1}[X(t_0 + h) - X(t)],
\]

\[
m_{hk} = h^{-1} \int_0^h \varphi_k(t_0 + s)ds = h^{-1}[x_k(t_0 + h) - x_k],
\]

where \(X = (y,Z), x_k = (y_k,z_k)\).

Given \(\eta > 0\) arbitrary, we can fix \(h, 0 < h \leq \delta < \sigma\), so small that

\[
|m_h - \varphi(t_0)| \leq \eta,
\]

\[
|S(t_0 + h) - S(t_0)| < \eta h/4,
\]

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This is possible since $h^{-1} \int_0^h \varphi(t_0 + s) ds + \varphi(t_0)$ and $[S(t_0 + h) - S(t_0)]h^{-1} \to 0$ as $h \to 0^+$. Also, we can choose $h$, in such a way that $z_k(t_0 + h) \to z(t_0)$ as $k \to +\infty$. This is possible since $z_k(t) \to z(t)$ for almost all $t_1 < t < t_2$.

Having so fixed $h$, let us take $k_1 \geq k_0$ so large that

$$|y_k(t_0) - y(t_0)|, |y_k(t_0 + h) - y(t_0 + h)| \leq \min \{\eta h/4, \epsilon/2\},$$

$$|z_k(t_0) - z(t_0)|, |z_k(t_0 + h) - z(t_0 + h)| \leq \eta h/8.$$  

This is possible since $y_k(t) \to y(t)$, $z_k(t) \to z(t)$ both at $t = t_0$ and $t = t_0 + h$. Then we have

$$|h^{-1}[y_k(t_0 + h) - y_k(t_0)] - h^{-1}[y(t_0 + h) - y(t_0)]|$$

$$\leq |h^{-1}[y_k(t_0 + h) - y(t_0 + h)]| + |h^{-1}[y_k(t_0) - y(t_0)]|$$

$$\leq h^{-1}(\eta h/4) + h^{-1}(\eta h/8) < \eta/2.$$  

Analogously, since $z = Z + S$, we have

$$|h^{-1}[z_k(t_0 + h) - z_k(t_0)] - h^{-1}[Z(t_0 + h) - Z(t_0)]|$$

$$= |h^{-1}[z_k(t_0 + h) - z_k(t_0)] - h^{-1}[z(t_0 + h) - z(t_0)] + h^{-1}[S(t_0 + h) - S(t_0)]|$$

$$\leq |h^{-1}[z_k(t_0 + h) - z(t_0 + h)]| + |h^{-1}[z_k(t_0) - z(t_0)]|$$

$$+ |h^{-1}[S(t_0 + h) - S(t_0)]|$$

$$\leq h^{-1}(\eta h/8) + h^{-1}(\eta h/8) + h^{-1}(\eta h/4) = \eta/2.$$  

Finally, we have

$$|m_{hk} - m_h| = |h^{-1}[x_k(t_0 + h) - x_k(t_0)] - h^{-1}[x(t_0 + h) - x(t_0)]|$$

$$= |h^{-1}[y_k(t_0 + h) - y_k(t_0)] - h^{-1}[y(t_0 + h) - y(t_0)]| +$$
\[ + \left| h^{-1}[z_k(t_o + h) - z_k(t_o)] - h^{-1}[Z(t_o + h) - Z(t_o)] \right| \leq \eta/2 + \eta/2 = \eta. \]

We conclude that for the chosen value of \( h \), \( 0 < h \leq \delta < \sigma \), and every \( k \geq k_1 \) we have

\[ |m_h - \varphi(t_o)| \leq \eta, \quad |m_{hk} - m_h| \leq \eta, \quad |y_k(t_o) - y(t_o)| \leq \epsilon/2. \tag{11} \]

For \( 0 \leq s \leq h \) we have now

\[ |y_k(t_o + s) - y(t_o)| \leq |y_k(t_o + s) - y_k(t_o)| + |y_k(t_o) - y(t_o)| \leq \epsilon/2 + \epsilon/2 = \epsilon, \]

\[ |(t_o + s) - t_o| \leq h \leq \delta \leq \epsilon, \]

\[ f(t_o + s, y_k(t_o + s), w_k(t_o + s)) \in Q(t_o + s, y_k(t_o + s)). \]

Hence, by definition of \( Q(t_o, y_o, 2\epsilon) \), also

\[ \varphi_k(t_o + s) = f(t_o + s, y_k(t_o + s), w_k(t_o + s)) \in Q(t_o, y_o, 2\epsilon). \]

The second integral relation (10) shows that we have also

\[ m_{hk} \in cl \, co \, Q(t_o, y_o, 2\epsilon) \]

since the latter is a closed convex set. Finally, by relations (11), we deduce

\[ |\varphi(t_o) - m_{hk}| \leq |\varphi(t_o) - m_h| + |m_h - m_{hk}| \leq 2\eta, \]

and hence

\[ \varphi(t_o) \in [cl \, co \, Q(t_o, y_o, 2\epsilon)]_{2\eta}. \]

Here \( \eta > 0 \) is an arbitrary number, and the set in brackets is closed.

Hence

\[ \varphi(t_o) \in cl \, co \, Q(t_o, y_o, 2\epsilon), \]

and this relation holds for every \( \epsilon > 0 \). By property (Q) we have
\[ \varphi(t_0) \in \text{cl co } Q(t_0, y_0, 2\varepsilon) = Q(t_0, y_0), \]

where \( y_0 = y(t_0) \), and \( Q(t_0, y_0) = f(t_0, y_0, U(t_0, y_0)) \). This relation implies that there are points \( \overline{u} = \overline{u}(t_0) \in U(t_0, y_0) \) such that

\[ \varphi(t_0) = f(t_0, y(t_0), \overline{u}(t_0)). \]

This holds for almost all \( t_0 \in [t_1, t_2] \). Hence, there is at least one function \( \overline{u}(t) \), defined a.e. in \([t_1, t_2]\) for which relation (9) holds a.e. in \([t_1, t_2]\).

We have to prove that there is at least one such function which is measurable. The proof is exactly as the one for closure theorem I, where we write \( y, y_k \) instead of \( x, x_k \), and will not be repeated here. Closure theorem II is thereby proved.

**Content.** 1. The problem. p. 5.—2. The space of continuous vector functions. p. 4.—3. Admissible pairs \( x(t), u(t) \). p. 5.—4. Uppersemicontinuity of variable sets. p. 6.—5. Closure theorems. p. 23.
REFERENCES


REFERENCES (Concluded)


