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# EXISTENCE THEOREMS FOR WEAK AND USUAL OPTIMAL SOLUTIONS IN LAGRANGE PROBLEMS WITH UNILATERAL CONSTRAINTS

## I. Closure Theorems

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# EXISTENCE THEOREMS FOR WEAK AND USUAL OPTIMAL SOLUTIONS IN LAGRANGE PROBLEMS WITH UNILATERAL CONSTRAINTS

## I. CLOSURE THEOREMS*

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In the present papers (I, II, and III) we prove existence theorems for weak and usual optimal solutions of nonparametric Lagrange problems with (or without) unilateral constraints.

We shall consider arbitrary pairs $x(t), u(t)$ of vector functions, $u(t)$ measurable with values in $E_m$, $x(t)$ absolutely continuous with values in $E_n$, and we discuss the existence of the absolute minimum of a functional

$$I[x,u] = \int_{t_1}^{t_2} f_0(t,x(t),u(t))dt,$$

with side conditions represented by a differential system

$$dx/dt = f(t,x(t),u(t)), \quad t_1 \leq t \leq t_2,$$

constraints

$$(t,x(t)) \epsilon A, \quad u(t) \epsilon U(t,x(t)), \quad t_1 \leq t \leq t_2,$$

and boundary conditions

$$(t_1,x(t_1),t_2,x(t_2)) \epsilon B,$$

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where A is a given closed subset of the tx-space $E_1 \times E_n$, where B is a given closed subset of the $t_1x_1t_2x_2$-space $E_{2n+2}$, and where U(t,x) denotes a given closed variable subset of the u-space $E_m$, depending on time t and space x. Here A may coincide with the whole space $E_1 x E_m$, and U may be fixed and coincide with the whole space $E_m$.

In the particular situation, where the space U is compact for every (t,x) these problems reduce to Pontryagin problems; in the particular situation where the space U is fixed and coincides with the whole space $E_m$, then these problems have essentially the same generality of usual Lagrange problems. Throughout these papers we shall assume U(t,x) to be any closed subset of $E_m$.

In paper I we prove closure theorems for usual solutions. In II we shall prove existence theorems. These will contain as particular cases the Filippov existence theorem for problems of optimal control (U(t,x) compact), existence theorems for usual Lagrange problems ($U = E_m$), and the Nagumo-Tonelli existence theorem for free problems ($m = n$, $f = u$). In III we shall prove existence theorems for weak (or generalized) solutions introduced as measurable probability distributions of usual solutions (Gamkrelidze chattering states).

In successive papers we shall extend the present results to multidimensional Lagrange problems involving partial differential equations in Sobolev's spaces with unilateral constraints.

We begin with an analysis of the concept of upper semicontinuity of variable subsets in $E_m$. The usual concept of upper semicontinuity is replaced by two others (properties (U) and (Q), §3), which are essentially more general than the uppersemicontinuity, in the sense that closed sets $U(t,x)$, for which uppersemicontinuity property hold, certainly satisfy property (U), and closed and convex sets $Q(t,x)$, for which upper semicontinuity property hold, certainly satisfy property (Q). We then extend (§4) the closure theorem of A. F. Filippov in various ways, so as to include, among other things, the use of pointwise and not necessarily uniform convergence of some components of a sequence of trajectories. In part II we shall prove existence theorem of optimal smooth solutions (§7) by a new analysis of a minimizing sequence, and by using the above extensions of Filippov's closure theorem as a replacement for Tonelli's semicontinuity argument. We shall then deduce (§8) existence theorems for the case where f is linear in u, and for free problems of the calculus of variations ($m = n$, $f = u$). Finally, we shall prove (§9) existence theorems for weak solutions in the general case above, for the case in which f is linear, and for free problems.

## 1. THE PROBLEM

We denote by x a variable n-vector $x = (x^1,\ldots,x^n) \in E_n$, by u a variable m-vector $u = (u^1,\ldots,u^m) \in E_m$, and by $t \in E_1$ the independent variable. We denote by A an arbitrary subset of the (t,x)-space, $A \subset E_1 x E_n$, and, for any $(t,x) \in A$, we denote by $U = U(t,x)$ a variable suspace of the u-space, $U(t,x) \subset E_m$. In the terminology of control problems, u is the control variable and $U(t,x)$ the control space. We denote by $f_i(t,x,u)$, $i = 0,1,\dots,n$, given real functions defined for all $(t,x)\epsilon A$, and all $u\epsilon U(t,x)$, and by $f$ the n-vector function $f = (f_1,\dots,f_n)$. We denote by $B$ a given subset of the $(2n + 2)$-space $(t_1,x_1,t_2,x_2)$. We are interested in the determination of a measurable vector function $u(t)$, $t_1 \leq t \leq t_2$, (control function, or steering function, or strategy), and a corresponding absolute continuous vector function $x(t)$, $t_1 < t \leq t_2$, satisfying almost everywhere the differential system

$$dx/dt = f(t,x(t),u(t)), \qquad t_1 \leq t \leq t_2,$$

satisfying the boundary conditions

$$(t_1,x(t_1),t_2,x(t_2))\epsilon B,$$

satisfying the constraints

$$(t,x(t))\epsilon A, \qquad t_1 \leq t \leq t_2,$$

$$u(t)\epsilon U(t,x(t)), \qquad \text{a.e. in } [t_1,t_2],$$

and for which the integral (cost functional)

$$I[x,u] = \int_{t_1}^{t_2} f_o(t,x(t),u(t))dt$$

has its minimum value (see § 2 for details). We shall assume that $U(t,x)$ is closed for every $(t,x)\epsilon A$.

## 2. THE SPACE OF CONTINUOUS VECTOR FUNCTIONS

Let $X$ be the collection of all continuous n-dim. vector functions $x(t)$ defined on arbitrary finite intervals of the t-axis:

$$x(t) = (x^1,\dots,x^n), \quad a \leq t \leq b, \quad x(t)\epsilon E_n,$$

If $x(t)$, $a \leq t \leq b$, and $y(t)$, $c \leq t \leq d$, are any two elements of X, we shall define a distance $\rho(x,y)$. First, let us extend $x(t)$ and $y(t)$ outside their intervals of definition by constancy and continuity in $(-\infty, +\infty)$, and then let

$$\rho(x,y) = |a-c| + |b-d| + \max |x(t)-y(t)|,$$

where max is taken in $(-\infty, +\infty)$. It is known that X is a complete metric space when equipped with the metric $\rho$. Ascoli's theorem can now be expressed by saying that any sequence of equicontinuous vector functions $x_n$ of X, whose graphs in the tx-space are equibounded, possesses at least one subsequence which is convergent in the $\rho$-metric toward an element x of X.

## 3. ADMISSIBLE PAIRS u(t), x(t)

Let A be a closed subset of the (t,x)-space $E_1 x E_n$. For every $(t,x) \epsilon A$ let $U(t,x)$, or control space, be a subset of the u-space $E_m$. Let M be the set of all $(t,x,u)$ with $(t,x) \epsilon A, u \epsilon U(t,x)$. Let $f(t,x,u) = (f_1, \dots, f_n)$ be a continuous vector function defined on M. We shall denote by $Q(t,x)$ the set of all values in $E_n$ taken by $f(t,x,u)$ when u describes $U(t,x)$, or $Q(t,x) = f(t,x,U(t,x))$. A vector function $u(t) = (u^1, \dots, u^m)$, $t_1 \leq t \leq t_2$ (control function) and a vector function $(x,t) = (x^1, \dots, x^n)$, $t_1 \leq t \leq t_2$ (trajectory) are said to be an <u>admissible</u> pair provided (a) $u(t)$ is measurable in $[t_1,t_2]$; (b) $x(t)$ is absolutely continuous (AC) in $[t_1,t_2]$, (c) $(t,x(t)) \epsilon A$ for every $t \epsilon [t_1,t_2]$; (d) $u(t) \epsilon U(t,x(t))$ a.e. in $[t_1,t_2]$; (e) $dx/dt = f(t,x(t), u(t))$ a.e. in $[t_1,t_2]$. By the expression the <u>vector function x(t), $t_1 \leq t \leq t_2$, is a trajectory</u>, we shall mean below that there exists a vector function

$u(t)$, $t_1 \leq t \leq t_2$, such that the pair $u(t)$, $x(t)$ satisfies (abcde). We say also that $x(t)$ is generates by $u(t)$.

4. UPPER SEMICONTINUITY OF VARIABLE SETS

In view of using sets $U(t,x)$, $Q(t,x)$ which are closed but not necessarily compact, we need a concept of upper semicontinuity which is essentially more general than the usual one. We shall introduce two modifications of the usual definition of upper semicontinuity, and we shall denote them as "property (U)" and "property (Q)" since we shall usually use them for the sets $U(t,x)$ and $Q(t,x)$ above, respectively.

We shall discuss properties (U) and (Q) first in relation to arbitrary variable sets $U(t,x)$, $Q(t,x)$ which are functions of $(t,x)$ in A. Then we shall discuss their relations when $Q(t,x)$ is assumed to be the image of $U(t,x)$ as mentioned in no. 2. Properties proved for $U(t,x)$ under conditions (U) or (Q), will be used for $Q(t,x)$ when this set satisfies conditions (U) or ($\phi$).

(A) The Property (U)

Given any set F in a linear space E we shall denote by cl F, coF, bdF, int F respectively the closure of F, the convex hull of F, the boundary of F, the set of all interior points of F. Thus, cl co F denotes the closure of the convex hull of F. We know that F, cl F, co F, co cl F are all contained in cl co F.

For every $(t,x) \in A$ and $\delta > 0$ let $N_\delta(t,x)$ denote the closed $\delta$-neighborhood of $(t,x)$ in A, that is, the set of all $(t',x') \in A$ at a distance $\leq \delta$ from $(t,x)$.

A variable subset $U(t,x)$, $(t,x)\epsilon A$, is said to be an <u>upper semicontinuous</u> function of $(t,x)$ at the point $(\bar{t},\bar{x})\epsilon A$ provided, given $\epsilon > 0$, there is a number $\delta = \delta(\bar{t},\bar{x},\epsilon) > 0$ such that $(t,x) \epsilon N_\delta(\bar{t},\bar{x})$ implies $U(t,x) \subset [U(\bar{t},\bar{x})]_\epsilon$ where $[U]_\epsilon$ denotes the closed $\epsilon$-neighborhood of $U$ in $E_m$.

Again let $U(t,x)$, $(t,x)\epsilon A$, $U(t,x)\subset E_m$, be a variable subset of $E_m$, which is a function of $(t,x)$ in $A$. For every $\delta > 0$ let $U(t,x,\delta) = \cup U(t',x')$, where the union is taken for all $(t',x') \leq N_\delta(t,x)$. We shall say that $U(t,x)$ has the property (U) at $(\bar{t},\bar{x})$ in A, if

$$U(\bar{t},\bar{x}) = \bigcap_{\delta>0} \text{cl } U(\bar{t},\bar{x},\delta).$$

We shall say that $U(t,x)$ has <u>property (U)</u> in A, if $U(t,x)$ has property (U) at every $(t,x)$ of A.

(i) If $U(t,x)$ has property (U) at $(\bar{t},\bar{x})$, then $U(\bar{t},\bar{x})$ is closed. Indeed,

$$U(\bar{t},\bar{x}) \subset \text{cl } U(\bar{t},\bar{x}) \subset \bigcap_{\delta>0} \text{cl } U(\bar{t},\bar{x},\delta) = U(\bar{t},\bar{x}),$$

and hence the $\subset$ signs can be replaced by $=$ signs.

(ii) If A is closed, and $U(t,x)$ is any variable set which is a function of $(t,x)$ in A and has property (U) in A, then the set of all $(t,x,u)\epsilon A\times E_m$ with $u\epsilon U(t,x)$, $(t,x)\epsilon A$, is closed.

<u>Proof</u>. If $(\bar{t},\bar{x},\bar{u})\epsilon \text{cl } M$ and $\epsilon > 0$, then there are $\infty$-many points $(t,x,u)\epsilon M$ with $|t-\bar{t}| < \epsilon$, $|x-\bar{x}| < \epsilon$, $|u-\bar{u}| < \epsilon$. Thus, $(\bar{t},\bar{x})\epsilon A$ since A is closed, $(t,x)\epsilon N_{2\epsilon}(\bar{t},\bar{x})$, $u \epsilon U(t,x), u\epsilon U(\bar{t},\bar{x},2\epsilon)$, and $\bar{u}\epsilon \bigcap_\epsilon \text{cl } U(\bar{t},\bar{x},2\epsilon) = U(\bar{t},\bar{x})$, $\bar{u}\epsilon U(\bar{t},\bar{x})$, since U has property (U) at $(\bar{t},\bar{x})$. This proves that $(\bar{t},\bar{x},\bar{u})\epsilon M$, that is, M is closed.

Note that the sets $U(\bar{t},\bar{x},\delta)$ are not necessarily closed even if A is closed all sets $U(t,x)$ are closed, and we take for $N_\delta(\bar{t},\bar{x})$ the closed $\delta$-neighborhood if $(\bar{t},\bar{x})$ in A as stated. This can be seen by the following example. Let $A = [0 \le t \le 1, 0 \le x \le 1]$ a subset of $E_2$, and $U(t,x) = [z = (z_1,z_2) | z_2 \ge tz_1, -\infty < z_1 < +\infty]$ for $0 < t \le 1$, and $U(0,x) = [z_2 \ge 0, z_1 = 0]$ for $t = 0$. Then $U(0,x,\delta) = [z = (z_1,z_2) | z_2 \ge \delta z_1$ for $-\infty < z_1 \le 0$, and $z_2 > 0$ for $0 < z_1 < \infty$ ] for any $\delta > 0$. The sets $U(0,x,\delta)$ are not closed. Here $U(t,x)$ does not satisfy property (U) at the points $(0,x)$. Nevertheless, the statement holds

(iii) If A is closed, and $U(t,x)$ satisfies property (U) in A, then the sets $U(t,x,\delta)$, $(t,x)\epsilon A$, $\delta > 0$, are all closed, and hence $U(t,x) = \cap_\delta U(t,x,\delta)$ for every $(t,x)\epsilon A$.

<u>Proof</u>. Let $M_\delta$ denote the set of all points $(t,x,u)$ with $(t,x)\epsilon N_\delta(\bar{t},\bar{x})$, $u\epsilon U(t,x)$. Obviously $N_\delta(\bar{t},\bar{x}) \subset A \subset E_{n+1}$; $M_\delta \subset E_{n+1} x E_m$, and $N_\delta(\bar{t},\bar{x})$ is compact and $M_\delta$ is closed by force of (ii) above. Let $\bar{u}$ be a point of accumulation of $U(t,x,\delta)$, and for any $\eta > 0$ let $V_\eta(\bar{u})$ denote the $\eta$-neighborhood of $\bar{u}$ in $E_m$. Then $M \cap (V_\eta(\bar{u}) x E_{n+1}) \subset N_\delta(\bar{t},\bar{x}) \, x V_\eta(\bar{u})$, hence $M \cap (V_\eta(\bar{u}) x E_{n+1})$ is bounded. Since both $M_\delta$ and $V_\eta(\bar{u}) x E_{n+1}$ are closed sets, the set $M \cap (V_\eta(\bar{u}) x E_{n+1})$ is closed and bounded, and therefore a compact subset of $E_{n+1} \times E_m$. Now the set $U(\bar{t},\bar{x},\delta) \cap V_\eta(\bar{u})$ is the projection of $M \cap (V_\eta(\bar{u}) x E_{n+1})$ on the u-space $E_m$, and therefore $U(\bar{t},\bar{x},\delta) \cap V_\eta(\bar{u})$ is compact. Thus $\bar{u} \epsilon U(\bar{t},\bar{x},\delta) \cap V_\eta(\bar{u})$, and finally $u \epsilon \, U(\bar{t},\bar{x},\delta)$. Thus, $U(\bar{t},\bar{x},\delta)$ is closed, or cl $U(\bar{t},\bar{x},\delta) = U(\bar{t},\bar{x},\delta)$, and $U(\bar{t},\bar{x}) = \cap_\delta$ cl $U(\bar{t},\bar{x},\delta) = \cap_\delta U(\bar{t},\bar{x},\delta)$.

(iv) If A is closed and $U_j(t,x)$, $(t,x)\epsilon A$, $j = 1,\dots,\nu$, $\nu$ finite, are variable subsets of $E_m$ all satisfying property (U) in A, then their union and their intersections $V(t,x) = \cup_j U_1(t,x)$, $W(t,x) = \cap_j U_1(t,x)$, $(t,x)\epsilon A$, are subsets of $E_m$ satisfying property (U) in A. The same holds for their product $V(t,x) = U_1 \times \dots \times U_\nu$.

The proof is straightforward.

Under the hypotheses of (ii) the set M is closed but not necessarily compact as the trivial example $U(t,x) = E_m$, $M = A \times E_m$, shows. The set M is closed but not necessarily compact even if we assume that A is compact, and that every $U(t,x)$ is compact. This is proved by the following example. Let $m = n = 1$, $A = [(t,x)\epsilon E_2,\ 0 \leq t \leq 1,\ 0 \leq x \leq 1]$, $U(0,x) = [u\epsilon E_1 | 0 \leq u \leq 1]$, and, if $t \neq 0$, $U(t,x) = [u\epsilon E_1 | 0 \leq u \leq 1, \text{ and } u = t^{-1}]$. Then M is the set of all $(t,x,u)$ with $0 \leq t \leq 1$, $0 \leq x \leq 1$, and $0 \leq u \leq 1$, or $u = t^{-1}$ if $t \neq 0$. Obviously, M is closed but not compact. Nevertheless, the statement holds:

(v) If A is compact, if the variable set $U(t,x)$ is compact and convex for every $(t,x)\epsilon A$ and possesses property (U) in A, if for every $(t,x)\epsilon A$ there is some $\delta = \delta(t,x) > 0$ such that $U(t,x) \cap U(t',x') \neq \emptyset$ for every $(t',x') \epsilon N_\delta(t,x)$, then M is compact.

Proof. If M is not compact, then there is some sequence of elements $(t_k,x_k,u_k)\epsilon M$, $k = 1,2,\dots$, with $(t_k,x_k)\epsilon A$, $|t_k| + |x_k| + |u_k| \to +\infty$. Since A is compact and hence bounded, we have $|u_k| \to +\infty$. On the other hand, there is some subsequence, say still $(t_k,x_k)$, with $t_k \to \bar{t}$, $x_k \to \bar{x}$, $(\bar{t},\bar{x})\epsilon A$. Given $\epsilon > 0$, we have $u_k \epsilon U(\bar{t},\bar{x},\epsilon)$ for all k sufficiently large, as well as $U(\bar{t},\bar{x}) \cap U(t_k,x_k) \neq \emptyset$. Since $U(\bar{t},\bar{x})$ is compact, there is a solid sphere S containing all of $U(\bar{t},\bar{x})$ in its interior, say $U(\bar{t},\bar{x}) \subset \text{int } S \subset E_m$. On the other hand, if $\bar{u}_k \epsilon U(\bar{t},\bar{x}) \cap U(t_k,x_k)$, we have $\bar{u}_k$ $\epsilon$ int S, and $u_k \epsilon E_m - S$, again for k large. Since both $\bar{u}_k$ and $u_k$ belong to the convex set $U(t_k,x_k)$, the segment $\bar{u}_k u_k$ is contained in $U(t_k,x_k)$. In particular, if $u_k'$ is the point where the segment $\bar{u}_k u_k$ intersects bd S, we have $u_k' \epsilon U(t_k,x_k)$, $u_k' \epsilon U(\bar{t},\bar{x},\epsilon)$, and $u_k' \epsilon$bd S. If u' is any point of accumulation of $[u_k']$, then u'$\epsilon$bd S, and u'$\epsilon$ cl $U(\bar{t},\bar{x},\epsilon)$ for ever $\epsilon > 0$. Hence, $u' \epsilon \cap_\epsilon \text{cl } U(\bar{t},\bar{x},\epsilon) = U(\bar{t},\bar{x})$, a contradiction, since $U(\bar{t},\bar{x}) \subset$ int S. We have proved that M is compact.

(vi) If the set U(t,x) is closed for every (t,x)$\epsilon$A and is an uppersemicontinuous function of (t,x) in A, then U(t,x) has property (U) in A.

<u>Proof</u>. By hypothesis $U(t,x,\delta) \subset [U(t,x)]_\epsilon$, where $U_\epsilon$ is closed. Hence cl $U(t,x,\delta) \subset [U(t,x)]_\epsilon$ for $\delta = \delta(t,x,\epsilon)$ and any $\epsilon > 0$. Since U(t,x) is closed, then $[U(t,x)]_\epsilon \to U(t,x)$ as $\epsilon \to 0+$. Thus $\cap_\delta \text{cl } U(t,x,\delta) \subset U(t,x)$. Since the opposite inclusion is trivial, we have $\cap_\delta \text{cl } U(t,x,\delta) = U(t,x)$. Statement (vi) is thereby proved.

The uppersemicontinuity property implies property (U), but the converse is not true, that is, the uppersemicontinuity property for closed sets is more restrictive than the property (U). This is shown by the following example in which all sets are closed. Take n = 2 and

$$U(t,x) = [(u^1,u^2) \epsilon E_2 | 0 \leq u^1 < +\infty,\ 0 \leq u^2 \leq tu^1]$$

for every $(t,x) \epsilon A = [(t,x) \epsilon E_2 | 0 \leq t \leq 1,\ 0 \leq x \leq 1]$. Then, for $\delta > 0$, we have

$$U(t,x,\delta) = [(u^1,u^2) \epsilon E_2 | 0 \leq u^1 < +\infty,\ 0 \leq u^2 \leq (t+\delta)u^1],$$

hence $U(t,x) = \cap_\delta \text{cl } U(t,x,\delta)$ and U(t,x) has property (U) in A. On the other hand,

$$[U(t,x)]_\epsilon = [(u^1,u^2)\epsilon E_2 | 0 \le u^1 < +\infty, -\epsilon \le u^2 \le tu^1+\epsilon(1+t^2)^{1/2}] \cup N_1,$$

where $N_1 = N_\epsilon(0,0)$ if $t = 0$, and, if $t \neq 0$,

$$N_1 = N_\epsilon(0,0) \cup [(u^1,u^2)\epsilon E_2 | u^1 \le 0,\ u^2 \ge -t^{-1}u^1,\ -tu^1+u^2 \le \epsilon(1+t^2)^{1/2}].$$

Obviously $U(t',x') - [U(t,x)]_\epsilon \neq \emptyset$ for $t' > t$, hence $U(t,x)$ is not an uppersemicontinuous function of $(t,x)$.

(vii) If A is compact, if $U(t,x)$ is compact for every $(t,x)\epsilon A$ and is an upper semicontinuous function of $(t,x)$ in A, then M is compact.

(viii) If A is closed and $U_j(t,x)$, $(t,x)\epsilon A$, $j = 1,\dots,\nu$, $\nu$ finite, are variable subsets of $E_m$ all uppersemicontinuous functions of $(t,x)$ in A, then their union $V(t,x)$ and their intersections $W(t,x)$ are semicontinuous functions of $(t,x)$ in A. The same holds for their product $V(t,x) = U_1 \times\dots\times U_\nu$, as well as for their convex hull $Z(t,x)$, that is, for the set $Z(t,x)$ of all $u = p_1u_1 +\dots+ p_\nu u_\nu$ with $u_j \epsilon U_j(t,x)$, $p_j \ge 0$, $j = 1,\dots,\nu$, $p_1 +\dots+ p_\nu = 1$.

The proof is straightforward.

(B) The Property (Q)

Let $U(t,x)$, $(t,x)\epsilon A$, $U(t,x)\epsilon E_m$, be any variable subset of $E_m$, which is a function of $(t,x)$ in A. By using the same notations as in (A), we shall say that $U(t,x)$ has property (Q) at $(\bar{t},\bar{x})$ in A, if

$$U(t,x) = \bigcap_{\delta>0} \mathrm{cl\ co}\ U(\bar{t},\bar{x},\delta).$$

We shall say that $U(t,x)$ has property (Q) in A if $U(t,x)$ has property (Q) at every $(t,x)$ of A.

(ix) Property (Q) at some $(\bar{t},\bar{x})$ implies property (U) at the same $(\bar{t},\bar{x})$, and

$$U(\bar{t},\bar{x}) = \bigcap_\delta \mathrm{cl\ co}\ U(\bar{t},\bar{x},\delta) = \bigcap_\delta \mathrm{cl}\ U(\bar{t},\bar{x},\delta) = \bigcap_\delta U(\bar{t},\bar{x},\delta).$$

Indeed

$$U(\bar{t},\bar{x}) \subset \bigcap_{\delta>0} \text{cl } U(\bar{t},\bar{x}) \subset \bigcap_{\delta>0} \text{cl } U(\bar{t},\bar{x},\delta) \subset \bigcap_{\delta>0} \text{cl co } U(\bar{t},\bar{x},\delta),$$

where first and last sets coincide by property (Q) at $(\bar{t},\bar{x})$, and hence the inclusion signs $\subset$ can be replaced by = signs.

(xi) If A is closed, and U(t,x) is any variable set which is a function of (t,x) in A and has property (Q) in A, then the set M of all $(t,x,u)\epsilon A\times E_m$ with $u\epsilon U(t,x)$, $(t,x)\epsilon A$, is closed.

Under the hypothesis of (i) the set M is closed but not necessarily compact as the trivial example $U(t,x) = E_m$, $M = A\times E_m$ shows. Nevertheless, the statement holds:

(xii) If A is compact, if the set U(t,x) is compact for every $(t,x)\epsilon A$ and possesses property (Q) in A, then the set M is compact.

<u>Proof</u>. If M is not compact, then there is some sequence, $(t_k,x_k,u_k)\epsilon M$, $k = 1,2,\ldots$, with $(t_k,x_k)\epsilon A$, $|t_k| + |x_k| + |u_k| \to +\infty$ as $k \to \infty$. Since A is compact and hence bounded, we have $|u_k| \to +\infty$. On the other hand, there is some subsequence, say still $(t_k,x_k)$, with $t_k \to \bar{t}$, $\bar{x}_k \to \bar{x}$, $(\bar{t},\bar{x})\epsilon A$. Given $\epsilon > 0$, we have then $u_k\epsilon U(\bar{t},\bar{x},\epsilon)$ for all k sufficiently large. Since $U(\bar{t},\bar{x})$ is compact, there is a solid sphere S containing all of $U(\bar{t},\bar{x})$ in its interior, say $U(\bar{t},\bar{x})\subset \text{int } S \subset E_m$. On the other hand, if $u\epsilon U(\bar{t},\bar{x})$, we have $u\epsilon$ int S, and $u_k\epsilon E_m - S$, again for k large. Since both u and $u_k$ belong to the

convex set cl co $U(t,x,\epsilon)$, we have $u'_k \epsilon$ cl co $U(t,x,\epsilon)$ where $u'_k$ is the point of intersection of the segment $uu_k$ with the boundary bd S of S. If $u'$ is any point of accumulation of $[u'_k]$, then $u' \epsilon$ bd S, and $u' \epsilon$ cl co $U(\bar{t},\bar{x},\epsilon)$ for every $\epsilon > 0$. Hence $u' \epsilon \bigcap_\delta$ cl co $U(\bar{t},\bar{x},\epsilon) = U(\bar{t},\bar{x})$, a contradiction, since $U(\bar{t},\bar{x}) \subset$ int S. We have proved that M is compact.

(xiii) If for every $(t,x) \epsilon A$ the set $U(t,x)$ is closed and convex, and $U(t,x)$ is an uppersemicontinuous function of $(t,x)$ in A, then $U(t,x)$ has property (Q) in A.

<u>Proof</u>. By hypothesis $U(t,x,\delta) \subset [U(t,x)]_\epsilon$, where $U_\epsilon$ is closed and convex as the closed $\epsilon$-neighborhood of a closed convex set. Hence, $\bigcap_\delta$ cl co $U(t,x,\delta) \subset [U(t,x)]_\epsilon$ for every $\epsilon > 0$. Since $U(t,x)$ is closed, then $[U(t,x)]_\epsilon \to U(t,x)$ as $\epsilon \to 0+$. Thus $\bigcap_\delta$ cl co $U(t,x,\delta) \subset U(t,x)$. Since the opposite inclusion relation $\supset$ is trivial, we have $\bigcap_\delta$ cl co $U(t,x,\delta) = U(t,x)$.

(C) Relations Between Properties of $U(t,x)$ and of $Q(t,x)$

Let us now consider sets $Q(t,x) = f(t,x,U(t,x))$, $(t,x) \epsilon A$, $Q(t,x) \subset E_n$, which are the images of sets $U(t,x) \subset E_m$ for every $(t,x) \epsilon A$.

The hypothesis that A is compact, that f is continuous on M, that $U(t,x)$ has property (Q) [or (U)] in A, and that $Q(t,x)$ is convex for every $(t,x) \epsilon A$, does not imply that $Q(t,x)$ has property (Q) [or U] in A. This can be proved by a simple example. Let $m = n = 1$, $A = [-1 \le t \le 1, 0 \le x \le 1]$, let $U(t,x)$ be the fixed interval $U = [u \epsilon E_1 | 0 \le u < +\infty]$, and $f = (u+1)^{-1} - t$. Then $Q(t,x) = [z \epsilon E_1 | -t < z \le 1 - t]$, and, if $-1+\delta < t < 1-\delta$, then

$$\text{cl co } Q(t,x,\delta) = [-t-\delta \le z \le 1-t+\delta].$$

The intersection of all these sets for $\delta > 0$ is the closed set $[z\epsilon E_1 | -t \leq z \leq 1-t]$ which is larger than $Q(t,x)$, and thus Q has not property (Q) in A. Actually, $Q(t,x)$ is not closed, and hence $Q(t,x)$ has neither property (Q), nor property (U).

Even the stronger hypothesis that A is compact, that f is continuous on M, that $U(t,x)$ has property (Q) in A, and that $Q(t,x)$ is compact and convex for every $(t,x)\epsilon A$, does not imply that $Q(t,x)$ has property (Q) in A. This can be proved by the following example. Let $m = 1$, $n = 1$, $A = [(t,x)\epsilon E_2,$ $0 \leq t \leq 1, 0 \leq x \leq 1]$, $U = U(t,x) = [u\epsilon E_1 | 0 \leq u \leq +\infty]$, and $f(t,x,u) = tu \exp(1-tu)$, $(t,x,u)\epsilon A\times U$. For $t = 0$ we have $f \equiv 0$, hence $Q(0,x) = [z=0]$. For $0 < t \leq 1$, we have $Q(t,x) = [0 \leq z \leq 1]$. All sets $Q(t,x)$ are compact and convex, but $Q(t,x)$ does not satisfy property (Q) nor property (U) in A.

(xiv) If A is closed and f continuous on M, if $U(t,x)$ is an upper semi-continuous function of $(t,x)$, then $Q(t,x)$ possesses the same property, and also has property (U). If we know that $Q(t,x)$ is convex, then $Q(t,x)$ has also property (Q).

<u>Proof</u>. Each set $Q(t,x)$ is a compact subset of $E_n$ as the continuous image of the compact set $U(t,x)$. Let us prove that M is closed. Let $(\bar{t},\bar{x},\bar{u})$ be a point of accumulation of M. Then there is a sequence $(t_k,x_k,u_k)$ of points of M with $t_k \to \bar{t}$, $x_k \to \bar{x}$, $u_k \to \bar{u}$, and $(t_k,x_k)\epsilon A$, $u_k \epsilon U(t\ ,x\ )$. Then $(\bar{t},\bar{x})\epsilon A$ since A is closed, $(t_k,x_k)\epsilon N_\delta(\bar{t},\bar{x})$ for all k sufficiently large, and $u_k \epsilon U(t_k,x_k) \subset [U(\bar{t},\bar{x})]_\epsilon$. Thus, $u\epsilon[U(\bar{t},\bar{x})]_\epsilon$ for every $\epsilon > 0$, and hence $\bar{u}\epsilon U(\bar{t},\bar{x})$ since this set is compact. We have proved

that $(\bar{t},\bar{x},\bar{u})\epsilon M$, and that M is closed. Let us prove that $Q(t,x)$ is an uppersemicontinuous function of $(t,x)$. Given $(t,x)\epsilon A$ and $\epsilon > 0$, let $\delta = \delta(t,x,\epsilon) > 0$ be the number relative to the definition of uppersemicontinuity of $U(t,x)$, and let $M'$ be the set of all $(t',x',u')$ with $(t',x') \epsilon N_\delta(t,x)$, $u'\epsilon U(t',x')$, and $M''$ be the set of all $(t',x',u')$ with $(t',x')\epsilon N_\delta (t,x)$, $u'\epsilon[U(t,x)]_\epsilon$. Since $U(t,x)$ is compact, also $[U(t,x)]_\epsilon$ is compact. Hence $M'' = N_\delta(t,x) x [U(t,x)]_\epsilon$, and $M' = M \cap M''$. The set $M'$ is compact as the intersection of the closed set M with the compact cylinder $M''$. The function f is continuous on $M'$ and hence bounded and uniformly continuous. Hence, there is some $\eta$, $0 < \eta \leq \min [\delta,\epsilon]$, such that $(t'',x'')\epsilon N_\eta(t',x')$, $|u'-u''| \leq \eta$, $(t',x',u')$, $(t'',x'',u'')\epsilon M'$ implies $|f(t',x',u')-f(t'',x'',u'')| \leq \epsilon$. Also, let $\sigma = \min [\eta, \delta(t,x,\eta)]$. Then, for every $(t',x')\epsilon N_\sigma(t,x)$, we have $U(t',x') \epsilon [U(t,x)]_\eta$, hence, if $u'\epsilon U(t',x')$, there is some $u''\epsilon U(t,x)$ with $|u'-u''| \leq \eta$, and finally $|f(t',x',u')-f(t,x,u'')| \leq \epsilon$. Thus, $Q(t',x') \subset [(Q(t,x)]_\epsilon$ for every $(t',x')\epsilon N_\sigma(t,x)$. This proves that $Q(t,x)$ has the $\epsilon\delta$-property above. The last part of statement (xiv) is now a consequence of statements (vi) and (xiii).

Remark. The statements and examples above show that properties (U) and (Q) are generalizations of the concept of upper semicontinuity for closed, or closed and convex sets, respectively.

(xv) If A is a closed subset of the tx-space $E_1 x E_n$, if $U(t,x)$, $(t,x)\epsilon A$, $U(t,x)\subset E_m$, is a variable subset of $E_m$ satisfying property (U) in A, if M denotes the set of all $(t,x,u)$ with $(t,x)\epsilon A$, $u\epsilon U(t,x)$, if $f_o$ is a continuous scalar function from M into the reals, if $\tilde{U}(t,x)$ denotes the variable subset of $E_{m+1}$ defined by $\tilde{U}(t,x) = [\tilde{u} = (u^o,u)\in E_{m+1} | u^o \geq f_o(t,x,u),\ u\in U(t,x)]$, then $\tilde{U}(t,x)$ satisfies property (U).

Proof. First, let us prove that each set $\tilde{U}(t_o,x_o,\delta)$ is closed. Indeed, if $\tilde{u} = (u^o,u)$ is a point of accumulation of $\tilde{U}(t_o,x_o,\delta)$, then there is a sequence $\tilde{u}_k = (u_k^o,u_k)$ with $u_k^o \to u^o$, $u_k \to u$, $\tilde{u}_k\in\tilde{U}(t_o,x_o,\delta)$. Hence, there is a corresponding sequence of points $(t_k,x_k)\in N_\delta(t_o,x_o)$ with $u_k^o \geq f_o(t_k,x_k,u_k)$, $u_k\in U(t_k,x_k)$. Thus $u_k\in U(t_o,x_o,\delta)$. Since $N_\delta(t_o,x_o)$ is a compact part of the closed set A, there is a subsequence, say still $(t_k,x_k)$, with $t_k \to \bar{t}$, $x_k \to \bar{x}$, $(\bar{t},\bar{x})\in N_\delta(t_o,x_o)\subset A$. Thus $(t_k,x_k,u_k)\in M$, $(t_k,x_k,u_k) \to (\bar{t},\bar{x},u)$, and M is a closed set by force of (ii). By the continuity of $f_o$ we have then $(\bar{t},\bar{x},u)\in M$, $u\in U(\bar{t},\bar{x})$, $u^o \geq f_o(\bar{t},\bar{x},u)$. Thus $\tilde{u} = (u^o,u)\in\tilde{U}(\bar{t},\bar{x})$, and $\tilde{u}\in\tilde{U}(t_o, x_o,\delta)$.

Now let $\tilde{u} = (u^o,u)$ be a point $\tilde{u}\in \bigcap_\delta \text{cl } U(t_o,x_o,\delta)$. Thus, there is a sequence of numbers $\delta_k > 0$, $\delta_k \to 0$, with $\tilde{u}\in \text{cl } \tilde{U}(t_o,x_o,\delta_k)$, and hence $\tilde{u}\in\tilde{U}(t_o,x_o,\delta_k)$ because these last sets are closed. Thus, there is also a sequence of points $(t_k,x_k)\in N_{\delta_k}(t_o,x_o)$ with $\tilde{u}\in\tilde{U}(t_k,x_k)$, or $u^o \geq f_o(t_k,x_k,u)$, $u\in U(t_k,x_k)$. Hence, for every $\eta > 0$, we have $u\in U(t_o,x_o,\eta)$ for every k sufficiently large (so that $\delta_k \leq \eta$), and, by property (U) of $U(t,x)$ at $(t_o,x_o)$, also $u\in \bigcap_\eta \text{cl } U(t_o,x_o,\eta) = U(t_o,x_o)$. Thus, $u\in U(t_o,x_o)$, $(t_o,x_o,u)\in M$, and by $u^o \geq f_o(t_k,x_k,u)$ and the continuity of $f_o$, also $u^o \geq f_o(t_o,x_o,u)$. We have proved that $\tilde{u} = (u^o,u)\in\tilde{U}(t_o,x_o)$, hence

$$\bigcap_\delta \text{cl } U(t_o,x_o,\delta)\subset U(t_o,x_o).$$

Since the opposite inclusion relation is trivial, equality sign holds, and $\tilde{U}(t,x)$ has property (U) at $(t_o,x_o)$, and, thus, everywhere in A. Statement

(xv) is thereby proved.

The set $\tilde{U}(t,x)$ of statement (xv) has not necessarily property (Q) even if we assume that $U(t,x)$ has property Q and $f_o(t,x,u)$ is convex in u for every $(t,x)\epsilon A$. This can be seen by a simple example. Let $A = [-1 \leq t \leq 1,\ 0 \leq x \leq 1]$ and let $U = U(t,x)$ be the fixed set $U(t,x) = E_1$, that is, $U = [-\infty < u^1 < +\infty]$. Then, each set $U(t,x)$ is closed and convex, and obviously $U(t,x)$ possesses property (Q), and M is the cylinder of all $(t,x,u)$ with $(t,x)\epsilon A$, $u\epsilon E_1$. Finally, let $f_o(t,x,u) = tu^1$, so that $f_o$ is continuous in M and, for every $(t,x)\epsilon A$, $f_o = tu^1$ is linear in $u^1$, hence certainly convex in $u^1$. Now we have

$$\tilde{U}(t,x) = [(u^o,u^1)\epsilon E_2 | -\infty < u^1 < +\infty,\ tu^1 \leq u^2 < +\infty]$$

$$\tilde{U}(0,x,\delta) = [(u^o,u^1)\epsilon E_2 | -\infty < u^1 < +\infty,\ -\delta|u^1| \leq u^2 < +\infty .$$

Consequently, co $\tilde{U}(0,x,\delta) = E_2$, and hence

$$\cap_\delta \text{ cl co } \tilde{U}(0,x,\delta) = E_2,$$

while

$$\tilde{U}(0,x) = [(u^1,u^2)\epsilon E_2 | -\infty < u^1 < +\infty,\ u^2 \geq 0.$$

This shows that $\tilde{U}(t,x)$ does not have property (Q) at the points (0,x) of A.

A scalar function $f_o(t,x,u)$, $(t,x,u)\epsilon M$, is said to be convex in u at $(t_o,x_o)\epsilon A$ if

$$f_o(t_o,x_o,u_o) \leq \sum_{i=1}^{N} \lambda_i f_o(t_o,x_o,u_i),$$

whenever

$$u_o = \sum_{i=1}^{N} \lambda_i u_i,$$

where $u_i \in U(t_o,x_o)$, $\lambda_i \geq 0$, $i = 1,\ldots, N$, $\lambda_1 + \ldots + \lambda_N = 1$.

A scalar function $f_o(t,x,u)$, $(t,x,u) + M$, is said to be quasi normally convex in u at $(t_o,x_o,u_o)\epsilon M$ provided, given $\epsilon > 0$, there are a number $\delta = \delta(t_o,x_o,u_o,\epsilon) > 0$, and a linear scalar function $z(u) = z + b\cdot u$, $b = (b_1,\ldots,b_m)$, $r$, $b_1,\ldots,b_m$ real, such that

(a) $f_o(t,x,u) \geq z(u)$ for all $(t,x)\epsilon N_\delta(t_o,x_o), u\epsilon U(t,x)$,

(b) $f_o(t,x,u) \leq z(u)+\epsilon$ for all $(t,x)\epsilon N_\delta(t_o,x_o), u\epsilon U(t,x)$, $|u-u_o| \leq \delta$.

The scalar function $f_o(t,x,u)$ is said to be normally convex in u at $(t_o,x_o,u_o)$ if, given $\epsilon > 0$, there are numbers $\delta = \delta(t_o,x_o,u_o,\epsilon) > 0$, $\nu = \nu(t_o,x_o,u_o,\epsilon) > 0$, and a linear scalar function $z(u) = r + b\cdot u$ as above such that (b) holds and

(a') $f_o(t,x,u) \geq z(u) + \nu|u-u_o|$ for all $(t,x)\epsilon N_\delta(t_o,x_o), u\epsilon U(t,x)$.

The scalar function $f_o(t,x,u)$ is said to be quasi normally convex in u, or normally convex in u, if it has these properties at every $(t_o,x_o,u_o)\epsilon M$.

For the case where $U = U(t,x)$ is the fixed set $U = E_m$, the following statement gives a useful characterization of the functions $f_o$ which are normally convex in u.

(xvi) If A is closed, and $f_o(t,x,u)$ is continuous on $M = AxE_m$, then $f_o$ is normally convex in u if and only if $f_o$ is convex in u at every $(t_o,x_o)\epsilon A$, and for no points $(t_o,x_o)\epsilon A$, $u_o,u_1\epsilon E_m$, $u_1 \neq 0$, the relation holds $f_o(t_o,x_o,u_o) = 2^{-1}[f_o(t_o,x_o,u_o+\lambda u_1) + f_o(t_o,x_o,u_o-\lambda u_1)$ for all $\lambda \geq 0$.

This statement was proved in [9a] and [10]. In particular, if for every $(t,x)\epsilon A$, $f_o(t,x,u)$ is convex in u and $f_o(t,x,u)/|u| \to +\infty$ as $|u| \to +\infty$, then certainly $f_o(t,x,u)$ is normally convex in u.

(xvii) If A is closed subset of the tx-space $E_1 \times E_n$, if $U(t,x)$, $(t,x)\epsilon A$, $U(t,x)\subset E_m$, is a variable subset of $E_m$ satisfying property (Q) in A, if M denotes the set of all $(t,x,u)$ with $(t,x)\epsilon A$, $u\epsilon U(t,x)$, if $f_o$ is a continuous function from M into the reals, which is convex in u for every $(t,x)\epsilon A$, if either ($\alpha$) the sets $U(t,x)$ are all contained in a fixed solid sphere S of $E_m$, or ($\beta$) the function $f_o(t,x,u)$ is quasi normally convex in u at every $(t_o,x_o,u_o)$ of M, then the set $\tilde{U}(t,x)$ of statement (xv) has property (Q) in A.

Proof. Let $\tilde{u} = (u^o,u)$ be a point $\tilde{u} = \cap_\delta$ cl co $\tilde{U}(t_o,x_o,\delta)$. Then there is a sequence $[\delta_k]$ of numbers $\delta_k > 0$, $\delta_k \to 0$, with $\tilde{u}\epsilon$cl co $\tilde{U}(t_o,x_o,\delta_k)$. Hence, there is a sequence of pairs of points $\tilde{u}_{k_1}$, $\tilde{u}_{k_2}\epsilon E_{m+1}$ and of points $\tilde{v}_k$ of the segment $(\tilde{u}_{k_1}\tilde{u}_{k_2})\epsilon E_{m+1}$, such that

$$\tilde{v}_k \to \tilde{u}, \qquad \tilde{u}_{k_1},\tilde{u}_{k_2}\epsilon\tilde{U}(t_o,x_o,\delta_k),$$

$$\tilde{v}_k = \alpha_k\tilde{u}_{k_1} + (1-\alpha_k)\tilde{u}_{k_2}, \qquad 0 \le \alpha_k \le 1, \qquad k = 1,2,\dots .$$

We shall use the notation $\tilde{v}_k = (v_k^o,v_k)$, $\tilde{u} = (u^o,u)$, $\tilde{u}_{kj} = (u_{kj}^o,u_{kj})$, $j = 1,2$. Then we have

$$v_k^o \to u^o,\ v_k \to u,\ u_{k_1},u_{k_2}\ \epsilon U(t_o,x_o,\delta_k)$$

$$v_k^o = \alpha_k u_{k_1}^o + (1-\alpha_k)u_{k_2}^o, \quad v_k = \alpha_k u_{k_1} + (1-\alpha_k)u_{k_2}.$$

Consequently, there are points such that

$$(t_{k_1},x_{k_1}),\ (t_{k_2},x_{k_2})\epsilon N_{\delta k_1}(t_o,x_o)\subset A,$$

$$u_{k_1}\epsilon U(t_{k_1},x_{k_1}),u_{k_2}\epsilon U(t_{k_2},x_{k_2}).$$

The sequence $[\alpha_k]$ is bounded, hence there is a convergent subsequence, say still $\alpha_k$, so that $\alpha_k \to \alpha$ for some $0 \leq \alpha \leq 1$.

For every $\eta > 0$ and k sufficiently large (so that $\delta_k \leq \eta$), we have $u_{k_1}, u_{k_2} \epsilon U(t_o,x_o,\eta)$, hence

$$u_{k_1}, u_{k_2} \epsilon \,\text{cl co}\, U(t_o,x_o,\eta).$$

As a consequence

$$v_k = \alpha_k u_{k_1} + (1-\alpha_k) u_{k_2} \epsilon \,\text{cl co}\, U(t_o,x_o,\eta)$$

for all k sufficiently large. As $k \to \infty$, we obtain $u \epsilon \,\text{cl co}\, U(t_o,x_o,\eta)$. By the property (U), finally

$$u \epsilon \textstyle\bigcap_\eta \text{cl co}\, U(t_o,x_o,\eta) = U(t_o,x_o). \qquad (1)$$

Assume first that condition ($\alpha$) holds. Then both sequences $[u_{k_1}]$, $[u_{k_2}]$ are bounded, and hence there is a subsequence, say still $[u_{k_1}]$, $[u_{k_2}]$, for which both $u_{k_1}$ and $u_{k_2}$ are convergent in $E_m$, say $u_{k_1} \to u_1$, $u_{k_2} \to u_2$, $u_1, u_2 \epsilon E_m$. For such a subsequence, we have

$$v_k^o = \alpha_k u_{k_1}^o + (1-\alpha_k) u_{k_2}^o \geq \alpha_k f_o(t_{k_1}, x_{k_1}, u_{k_1}) + (1-\alpha_k) f_o(t_{k_2}, x_{k_2}, u_{k_2}),$$

$$v_k = \alpha_k u_{k_1} + (1-\alpha_k) u_{k_2},$$

$$(t_{k_1}, x_{k_1}, u_{k_1}),\ (t_{k_2}, x_{k_2}, u_{k_2}) \epsilon M,$$

where M is closed. By taking limits as $k \to \infty$, we have

$$u^o \geq \alpha f_o(t_o,x_o,u_1) + (1-\alpha) f_o(t_o,x_o,u_2)$$

$$u = \alpha u_1 + (1-\alpha) u_2,$$

$$(t_o,x_o,u_1),(t_o,x_o,u_2) \epsilon M.$$

By the convexity of $f_o$ in u at $(t_o,x_o)$ we have now

$$u^o \geq f_o(t_o,x_o,\alpha u_1 + (1-\alpha)u_2) = f_o(t_o,x_o,u).$$

This proves that $\tilde{u} = (u^o,u)\epsilon\tilde{U}(t_o,x_o)$, hence

$$\cap_\delta \text{ cl co } \tilde{U}(t_o,x_o) \subset \tilde{U}(t_o,x_o). \tag{2}$$

Since the opposite inclusion is trivial, = sign holds in this relation, and $\tilde{U}(t,x)$ has property (Q) at $(t_o,x_o)$. Since $(t_o,x_o)\epsilon A$ is arbitrary, $\tilde{U}(t,x)$ has property (Q) in A.

Assume now that condition $(\beta)$ holds. As stated by relation (1) above, $u\epsilon U(t_o,x_o)$, hence $(t_o,x_o,u)\epsilon M$. By the quasi normal convexity of $f_o$ in u at $(t_o,x_o,u)$ we deduce the existence of a number $\delta > 0$ and of a linear scalar function $z(u) = r + b \cdot v$ such that (a) $f_o(t,x,v) = z(v)$ for all $(t,x)\epsilon N_\delta(t_o,x_o)$, $v\epsilon U(t,x)$ and (b) $f_o(t,x,w) \leq z(v) + \epsilon$ for all $(t,x)\epsilon N_\delta(t_o,x_o)$, $v\epsilon U(t,x)$, $|u-v| \leq \delta$. By combining (a) and (b) we have then (c) $z(u) \leq f_o(t_o,x_o,u) = z(u) + \epsilon$.

Now we have $v_k = \alpha_k u_{k1} + (1-\alpha_k)u_{ko}$ for some $0 \leq \alpha_k \leq 1$, and $v_k \to u$, $(t_{kj},x_{kj}) \to (t_o,x_o)$, j = 1,2. Thus, for k sufficiently large, $(t_{kj},x_{kj})\epsilon N_\delta (t_o,x_o)$, j = 1,2, and, by property (a),

$$\begin{aligned} v_k^o &\geq \alpha_k f_o(t_{k_1},x_{k_1},u_{k_1}) + (1-\alpha_k)f_o(t_{k_2},x_{k_2},u_{k_2}) \\ &\geq \alpha_k z(u_{k_1}) + (1-\alpha_k)z(u_{k_2}) = \\ &= z(\alpha_k u_{k_1} + (1-\alpha_k)u_{k_2}) = z(v_k). \end{aligned}$$

As $k\to+\infty$, we have then $u^o \geq z(u)$, and finally by (c) above, $u^o \geq f_o(t_o,x_o,u) -\epsilon$, where $\epsilon > 0$ is arbitrary. We conclude that $u^o \geq f_o(t_o,x_o,u)$, with $u\epsilon U(t_o,x_o)$. Thus $\tilde{u} = (u^o,v)\epsilon\tilde{U}(t_o,x_o)$, and again we have proved inclusion (2). The same reasoning above yields that $\tilde{U}(t,x)$ has property (Q) in A.

(xviii) If A is a closed subset of the tx-space $E_1 \times E_n$, if $U(t,x)$, $(t,x)\epsilon A$, is a variable subset of $E_m$ satisfying property (U) in A, if M denotes the set of all $(t,x,u)$ with $(t,x)\epsilon A$, $u\epsilon U(t,x)$, if $\tilde{f} = (f_o,f)$ is a continuous function from M into the $\tilde{z}$-space $E_{n+1}$, $z = (z^o,z)$, if $Q(t,x)\subset E_n$, $\tilde{\tilde{Q}}(t,x)\subset E_{n+1}$ are the sets

$$Q(t,x) = f(t,x,U(t,x)) = [z\epsilon E_n | z = f(t,x,u), u\epsilon U(t,x)],$$

$$\tilde{\tilde{Q}}(t,x) = [\tilde{z} = (z^o,z)\epsilon E_{n+1} | z^o \geq f_o(t,x,u), z = f(t,x,u), u\epsilon U(t,x)],$$

and (a) for every $(t,x)\epsilon A$, $Q(t,x)$ is a convex subset of $E_n$; (b) $Q(t,x)$ has propriety (Q) in A; (c) for every $(t,x)\epsilon A$, $z = f(t,x,u)$ is a 1-1 map from $U(t,x)$ onto $Q(t,x)$ with a continuous inverse $u = f^{-1}(t,x,z)$, $z\epsilon Q(t,x)$; (d) the real valued function $F_o(t,x,z) = f_o(t,x,f^{-1}(t,x,z))$, $(t,x)\epsilon A$, $z\epsilon Q(t,x)$, is continuous in the set M' of all $(t,x,z)$ with $(t,x)\epsilon A$, $z\epsilon Q(t,x)$, and $F_o(t,x,z)$ is convex in z and also quasi normally convex, then the set $\tilde{\tilde{Q}}(t,x)$ is convex and has property (Q) in A.

Proof. Indeed, under the specific hypotheses above, the set $\tilde{\tilde{Q}}(t,x)$ can be represented as

$$\tilde{\tilde{Q}}(t,x) = [\tilde{z} = (z^o,z)\epsilon E_{n+1} | z^o \geq F_o(t,x,z), z\epsilon Q(t,x)],$$

and thus $\tilde{\tilde{Q}}$ is generated from $Q(t,x)$ exactly as $\tilde{U}$ is generated from $U(t,x)$. By statement (xvii) above we conclude that $\tilde{\tilde{Q}}(t,x)$ has property (Q) in A.

Remark. The condition that f is a homeomorphism between U and Q is certainly verified in all free problems, where $m = n$, $f = u$, that is, $f_i = u^i$, $i = 1,2,\ldots,n$ (see no. 9 below). In this situation then we have $F_o(t,x,u) = f_o(t,x,u)$, and the convexity of $f_o$ in u implies the convexity of

$F_o$ in u. We shall need this remark, and the more general statement (xviii) in part II.

## 5. CLOSURE THEOREMS

We shall use here the notations of no. 2 and 3. In particular, a trajectory x(t) is defined as in no. 2.

Closure Theorem I. (A first generalization of Filippov's theorem). Let A be a closed subset of $E_1 \times E_n$, let U(t,x) be a closed subset of $E_m$ for every $(t,x)\epsilon A$, let $f(t,x,u) = (f_1,\dots,f_n)$ be a continuous vector function on M into $E_n$, and let $Q(t,x) = f(t,x,U(t,x))$ be a closed convex subset of $E_n$ for every $(t,x)\epsilon A$. Assume that U(t,x) has property (U) in A, and that Q(t,x) has property (Q) in A. Let $x_k(t)$, $t_{1k} \le t \le t_{2k}$, $k = 1,2,\dots$, be a sequence of trajectories, which is convergent in the metrix $\rho$ toward an absolutely continuous function x(t), $t_1 \le t \le t_2$. Then x(t) is a trajectory.

Remark. If we assume that U(t,x) is compact for every $(t,x)\epsilon A$, and that U(t,x) is an uppersemicontinuous function of (t,x) in A, then by statement (x), the set Q(t,x) has the same property, U(t,x) has property (U), Q(t,x) has property (Q), and closure theorem I reduces to one of A. F. Filippov [2] (not explicitely stated in [2] but contained in the proof of his existence theorem for the Pontryagin problem with U(t,x) always compact).

Proof of Closure Theorem I. The vector functions

$$\varphi(t) = x'(t), \quad t_1 \le t \le t_2,$$
$$\varphi_k(t) = x_k'(t) = f(t,x_k(t),u_k(t)), \quad t_{1k} \le t \le t_{2k}, \quad k = 1,2,\dots, \tag{1}$$

are defined almost everywhere and are L-integrable. We have to prove that

$(t,x(t))\epsilon A$ for every $t_1 \leq t \leq t_2$, and that there is an admissible control function $u(t)$, $t_1 \leq t \leq t_2$, such that

$$\varphi(t) = x'(t) = f(t,x(t),u'(t)), \qquad u(t)\epsilon U(t,x(t)), \tag{2}$$

for almost all $t\epsilon[t_1 t_2]$.

First, $\rho(x_k,x) \to 0$ as $k \to 0$; hence, $t_{1k} \to t_1$, $t_{2k} \to t_2$. If $t\epsilon(t_1,t_2)$, or $t_1 < t < t_2$, then $t_{1k} < t < t_{2k}$ for all k sufficiently large and $(t,x_k(t))\epsilon A$. Since $x_k(t) \to x(t)$ as $k \to \infty$ and A is closed, we conclude that $(t,x(t))\epsilon A$ for every $t_1 < t < t_2$. Since $x(t)$ is continuous, and hence continuous at $t_1$ and $t_2$, we conclude that $(t,x(t))\epsilon A$ for every $t_1 \leq t \leq t_2$.

For almost all $t\epsilon[t_1,t_2]$ the derivative $x'(t)$ exists and is finite. Let $t_o$ be such a point with $t_1 < t_o < t_2$. Then there is a $\sigma > 0$ with $t_1 < t_o - \sigma < t_o + \sigma < t_2$, and, for some $k_o$ and all $k \geq k_o$, also $t_{1k} < t_o - \sigma < t_o + \sigma < t_{2k}$. Let $x_o = x(t_o)$.

We have $x_k(t) \to x(t)$ uniformly in $[t_o - \sigma, t_o + \sigma]$ and all functions $x(t)$, $x_k(t)$ are continuous in the same interval. Thus, they are equicontinuous in $[t_o - \sigma,, t_o + \sigma]$. Given $\epsilon > 0$, there is a $\delta > 0$ such that $t,t'\epsilon[t_o - \sigma, t_o + \sigma]$, $|t-t'| \leq \delta$, $k \geq k_o$, implies

$$|x(t) - x(t')| \leq \epsilon/2 \quad |x_k(t) - x_k(t')| \leq \epsilon/2.$$

We can assume $0 < \delta < \sigma$, $\delta \leq \epsilon$. For any h, $0 < h \leq \delta$, let us consider the averages

$$\begin{aligned} m_h &= h^{-1}\int_0^h \varphi(t_o+s)ds = h^{-1}[x(t_o+h)-x(t_o)], \\ m_{hk} &= h^{-1}\int_0^h \varphi_k(t_o+s)ds = h^{-1}[x_k(t_o+h)-x_k(t_o)]. \end{aligned} \tag{3}$$

Given $\eta > 0$ arbitrary, we can fix h, $0 < h \leq \delta < \sigma$, so small that

$$|m_h - \varphi(t_o)| \leq \eta. \qquad (4)$$

Having so fixed h, let us take $k_1 \geq k_o$ so large that

$$|m_{hk} - m_h| \leq \eta, \quad |x_k(t_o)-x(t_o)| \leq \epsilon/2 \qquad (5)$$

for all $k \geq k_1$. This is possible since $x_k(t) \to x(t)$ as $k \to \infty$ both at $t = t_o$ and $t = t_o + h$. Finally, for $0 \leq s \leq h$,

$$|x_k(t_o+s)-x(t_o)| \leq |x_k(t_o+s)-x_k(t_o)|+|x_k(t_o)-x(t_o)|$$

$$\leq \epsilon/2 + \epsilon/2 = \epsilon,$$

$$|(t_o+s)-t_o| \leq h \leq \delta \leq \epsilon$$

$$f(t_o+s,x_k(t_o+s),u_k(t_o+s))\epsilon Q(t_o+s,x_k(t_o+s)).$$

Hence, by the definition of $Q(t_o,x_o,2\epsilon)$, also

$$\varphi_k(t_o+s) = f(t_o+s,x_k(t_o+s), u_k(t_o+s))\epsilon Q(t_o,x_o,2\epsilon).$$

The second integral relation (3) shows that we have also

$$m_{hk} \in \text{cl co } Q(t_o,x_o,2\epsilon),$$

since the latter is a closed convex set. Finally, by relations (4) and (5), we deduce

$$|\varphi(t_o)-m_{hk}| \leq |\varphi(t_o)-m_h|+|m_h-m_{hk}| \leq 2\eta,$$

and hence

$$\varphi(t_o)\epsilon \ [\text{cl co } Q(t_o,x_o,2\epsilon)]_{2\eta}.$$

Here $\eta > 0$ is an arbitrary number, and the set in brackets is closed.

Hence,

$$\varphi(t_o)\epsilon \text{ cl co } Q(t_o,x_o,2\epsilon),$$

and this relation holds for every $\epsilon > 0$. By property (Q) we have

$$\varphi(t_o) \in \cap_\epsilon \text{ cl co } Q(t_o,x_o,2\epsilon) = Q(t_o,x_o),$$

where $x_o = x(t_o)$, and $Q(t_o,x_o) = f(t_o,x_o,U(t_o,x_o))$. This relation implies that there are points $\bar{u} = \bar{u}(t_o) \in U(t_o,x_o)$ such that

$$\varphi(t_o) = f(t_o,x(t_o),\bar{u}(t_o)). \qquad (6)$$

This holds for almost all $t_o \in [t_1,t_2]$, that is, for all t of a measurable set $I \subset [t_1,t_2]$ with meas $I = t_2-t_1$. If we take $I_o = [t_1,t_2]-I$, then means $I_o = 0$. Hence, there is at least one function $\bar{u}(t)$, defined almost everywhere in $[t_1,t_2]$, for which relation (6) holds a.e. in $[t_1,t_2]$. We have to prove that there is at least one such function which is measurable. For every $t \in I$, let $P(t)$ denote the set

$$P(t) = [u | u \in U(t,x(t)), \varphi(t) = f(t,x(t),u] \subset U(t,x(t)) \subset E_m.$$

We have proved that $P(t)$ is not empty.

For every integer $\lambda = 1,2,\ldots$, there is a closed subset $C_\lambda$ of I, $C_\lambda \subset I \subset [t_1,t_2]$, with meas $C_\lambda > \max [0,t_2-t_1-1/\lambda]$, such that $\varphi(t)$ is continuous on $\lambda$. Let $W_\lambda$ be the set

$$W_\lambda = [(t,u) | t \in C_\lambda, u \in P(t)] \subset E_1 \times E_m.$$

Let us prove that the set $W_\lambda$ is closed. Indeed, if $(\bar{t},\bar{u})$ is a point of accumulation of $W_\lambda$, then there is a sequence $(t_s,u_s), s = 1,2,\ldots$, with $(t_s,u_s) \in W_\lambda$, $t_s \to E$, $u_s \to \bar{u}$. Then $t_s \in C_\lambda$ and $\bar{t} \in C_\lambda$ since $C_\lambda$ is closed. Also $x(t_s) \to x(\bar{t})$, $\varphi(t_s) \to \varphi(\bar{t})$, and since $(t_s, x(t_s)) \in A$, $\varphi(t_s) = f(t_s,x(t_s), u(t_s))$, $(t_s,x(t_s), u(t_s)) \in M$, we have also $(\bar{t},x(\bar{t})) \in A$, $(t,x(\bar{t}),\bar{u}) \in M$, because

A and M are closed, and $\varphi(\bar{t}) = f(\bar{t},x(\bar{t}),\bar{u})$ because f is continuous. Thus, $\bar{u}\epsilon P(\bar{t})$, and $(\bar{t},\bar{x})\epsilon W_\lambda$.

For every integer $\ell$ let $W_{\lambda\ell}$, $P_\ell(t)$, be the sets

$$W_{\lambda\ell} = [(t,u)|(t,u)\epsilon W_\lambda, |u| \leq \ell] \subset W_\lambda \subset E_1 \times E_m,$$

$$P_\ell(t) = [u|u\epsilon P(t), |u| \leq \ell] \subset P(t) \subset U(t,x(t)) \subset E_m,$$

$$C_{\lambda\ell} = [t|(t,u)\epsilon W_{\lambda\ell} \text{ for some } u] \subset C_\lambda \subset I \subset [t_1,t_2].$$

Obviously, $W_{\lambda\ell}$ is compact, and so is $C_{\lambda\ell}$ as its projection on the t-axis. Also, $U_\ell C_{\lambda\ell} = C_\lambda$, and $W_{\lambda\ell}$ is the set of all (t,u) with $t\epsilon C_{\lambda\ell}$, $u\epsilon P_\ell(t)$. Thus, for $t\epsilon C_{\lambda\ell}$, P(t) is a compact subset of U(t,x(t)).

For $t\epsilon C_{\lambda\ell}$, the set $P_\ell(t)$ is the nonempty compact subset of all $u = (u^1,\ldots,u^m)\epsilon U(t,x(t))$ with $f(t,x(t),u) = \varphi(t)$. As in Filippov's argument let P, be the subset of P with $u^1$ minimum, let $P_2$ be the subset of $P_1$ with $u^2$ minimum,..., let $P_m$ be the subset of $P_{m-1}$ with $u^m$ minimum. Then $P_m$ is a single point $u = u(t)\epsilon U(t,x(t))$ with $u(t) = (u^1,\ldots,u^m)$, $t\epsilon C_{\lambda\ell}$, $|u(t)| \leq \ell$, and $f(t,x(t),u(t)) = \varphi(t)$. Let us prove that u(t), $t\epsilon C_{\lambda\ell}$ is measurable. We shall prove this by induction on the coordinates. Let us assume that $u^1(t),\ldots,u^{s-1}(t)$ have been proved to be measurable on $C_{\lambda\ell}$ and let us prove that $u^s(t)$ is measurable. For s = 1 nothing is assumed, and the argument below proves that $u^1(t)$ is measurable. For every integer j there are closed subsets $C_{\lambda\ell j}$ of $C_{\lambda\ell}$ with $C_{\lambda\ell j} \subset C_{\lambda\ell}$, $C_{\lambda\ell j} \subset C_{\lambda\ell,j+1}$, meas $C_{\lambda\ell j} >$ [0, meas $C_{\lambda\ell}$-1/j], such that $u^1(t),\ldots,u^{s-1}(t)$ are continuous on $C_{\lambda\ell j}$. The function $\varphi(t)$ is already continuous on $C_\lambda$ and hence $\varphi(t)$ is continuous on every set $C_{\lambda\ell}$ and $C_{\lambda\ell j}$. Let us prove that $u^s(t)$ is measurable on $C_{\lambda\ell j}$.

We have only to prove that, for every real a, the set of all $t \in C_{\lambda \ell j}$ with $\varphi^s(t) \leq a$ is closed. Suppose that this is not the case. Then there is a sequence of points $t_k \in C_{\lambda \ell j}$ with $u^s(t_k) \leq a$, $t_k \to \bar{t} \in C_{\lambda \ell j}$, $u^s(\bar{t}) > a$. Then $\varphi(t_k) \to \varphi(\bar{t})$, $u^\alpha(t_k) \to u^\alpha(\bar{t})$ as $k \to \infty$, $\alpha = 1,\ldots,s-1$. Since $|u^\beta(t_k)| \leq \ell$ for all k and $\beta = s,s+1,\ldots,m$, we can select a subsequence, say still $[t_k]$ such that $u^\beta(t_k) \to \tilde{u}^\beta$ as $k \to \infty$, $\beta = s,s+1,\ldots,m$, for some real numbers $\tilde{u}^\beta$. Then $t_k \to \bar{t}$, $x(t_k) \to x(\bar{t})$, $u(t_k) \to \tilde{u}$, where

$$\tilde{u} = (u^1(\bar{t}),\ldots,u^{s-1}(\bar{t}),\tilde{u}^s,\ldots,\tilde{u}^m).$$

Then, given any number $\eta > 0$, we have

$$u(t_k) \in U(t_k,x(t_k)) \subset \text{cl } U(\bar{t},x(t),\eta)$$

for all k sufficiently large, and, as $k \to \infty$, also

$$\tilde{u} \in \text{cl } U(\bar{t},x(\bar{t}),\eta).$$

By the property (U) we have

$$\tilde{u} \in \cap_\eta \text{ cl } U(\bar{t},x(\bar{t}),\eta) = U(\bar{t},x(\bar{t}))$$

On the other hand $\varphi(t_k) = f(t_k,x(t_k),u(t_k))$, $u^s(t_k) \leq a$, yield as $k \to \infty$,

$$\varphi(\bar{t}) = f(\bar{t},x(\bar{t}),\tilde{u}), \quad \tilde{u}^s \leq a, \tag{6}$$

while $\bar{t} \in C_{\lambda \ell}$ implies

$$\varphi(\bar{t}) = f(\bar{t},x(\bar{t}),u(\bar{t})), \quad u^s(\bar{t}) > a. \tag{7}$$

Relations (6) and (7) are contradictory because of the property of minimum with which $u^s(\bar{t})$ has been chosen. Thus $u^s(t)$ is measurable on $C_{\lambda \ell j}$ for every j, and then $u^s(t)$ is also measurable on $C_{\lambda \ell}$. By induction argument, all components $u^1(t),\ldots,u^m(t)$ of $u(t)$ are measurable on $C_{\lambda \ell}$, hence $u(t)$ is measurable on $C_{\lambda\ell}$. Since $U_\ell C_{\lambda\ell} = C_\lambda$, meas $C_\lambda >$ meas $I - 1/\lambda$, we conclude that $u(t)$ is measurable on every set $C_\lambda$ and hence on I, with meas $I = t_2-t_1$. Thus, $u(t)$ is defined a.e. on $(t_1,t_2)$, $u(t)\in U(t,x(t)$ and $f(t,x(t),u(t)) = \varphi(t)$ a.e. on $[t_1,t_2]$. Closure Theorem I is thereby proved.

Let us denote by $y = (x^1,\ldots,x^s)$ the s-vector made up of certain components, say $x^1,\ldots,x^s$, $0 \leq s \leq n$, of $x = (x^1,\ldots,x^n)$, and by z the complementary (n-s)-vector $z = (x^{s+1},\ldots,x^n)$ of x, so that $x = (y,z)$. Let us assume that $f(t,y,u)$ depends only on the coordinates $x^1,\ldots,x^s$ of x. If $x(t)$, $t_1 \leq t \leq t_2$, is any vector function, we shall denote by $x(t) = [y(t),z(t)]$ the corresponding decomposition of $x(t)$ in its coordinates $y(t) = (x^1,\ldots,x^s)$ and $z(t) = (x^{s+1},\ldots,x^n)$.

We shall denote by $A_o$ a closed subset of points $(t,x^1,\ldots,x^s)$, that is, a closed subset of the ty-space $E_1xE_s$, and let $A = A_o \times E_{n-s}$. Thus, A is a closed subset of the tx-space $E_1xE_n$.

<u>Closure Theorem II</u>. (A further generalization of Filippov's Theorem). Let $A_o$ be a closed subset of the ty-space $E_1xE_s$, and then $A = A_o \times E_{n-s}$ is a closed subset of the tx-space $E_1xE_n$. Let $U(t,y)$ denote a closed subset of $E_m$ for every $(t,y)\in A_o$, let $M_o$ be the set of all $(t,y,u)\in E_{1+s+m}$ with $(t,y)\in A_o$, $u\in U(t,y)$, and let $f(t,y,u) = (f_1,\ldots,f_n)$ be continuous vector function from M into $E_n$. Let $Q(t,y) = f(t,y,U(t,y))$ be a closed convex subset of $E_n$ for every $(t,y)\in A_o$. Assume that $U(t,y)$ has property (U) in $A_o$ and that $Q(t,y)$ has property (Q) in $A_o$.

Let $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k = 1,2,\ldots$, be a sequence of trajectories, $x_k(t) = (y_k(t), z_k(t))$, for which we assume that the s-vector $y_k(t)$ converges in the $\rho$-metric toward an AC vector function $y(t)$, $t_1 \leq t \leq t_2$, and that the (n-k)-vector $z_k(t)$ converges (pointwise) for almost all $t_1 < t < t_2$, toward a vector $z(t)$ which admits of a decomposition $z(t) = Z(t) + S(t)$ where $Z(t)$ is an AC vector function in $[t_1,t_2]$, and $S'(t) = 0$ a.e. in $[t_1,t_2]$ (that is, $S(t)$ is a singular function). Then, the AC vector $X(t) = [y(t),Z(t)]$, $t_1 \leq t \leq t_2$, is a trajectory.

Remark. For $s = n$, this theorem reduces to closure theorem I.

Proof of Closure Theorem II. The vector functions

$$\varphi(t) = X'(t) = (y'(t),Z'(t)), \quad t_1 \leq t \leq t_2,$$

$$\varphi(t) = x_k'(t) = (y_k'(t),z_k'(t)) = f(t,y_k(t),u_k(t)), \quad t_{1k} \leq t \leq t_{2k} \qquad (8)$$
$$k = 1,2,\ldots,$$

are defined almost everywhere and are L-integrable. We have to prove that $[t,y(t),Z(t)]\epsilon A$ for every $t_1 \leq t \leq t_2$, and that there is an admissible control function $u(t)$, $t_1 \leq t \leq t_2$, such that

$$\varphi(t) = X'(t) = (y'(t),Z'(t)) = f(t,y(t),u(t)),$$
$$u(t)\epsilon U(t,y(t)),$$

for almost all $t \,\epsilon[t_1,t_2]$.

First, $\rho(y_k,y) \to 0$ as $k \to 0$; hence $t_{1k} \to t_1, t_{2k} \to t_2$. If $t\epsilon(t_1,t_2)$, or $t_1 < t < t_2$, then $t_{1k} < t < t_{2k}$ for all k sufficiently large, and $(t,y_k(t)\epsilon A_0$. Since $y_k(t) \to y(t)$ as $k \to \infty$ and $A_0$ is closed, we conclude that $(t,y(t))\epsilon A_0$ for every $t_1 < t \leq t_2$, and finally $(t,y(t),Z(t))\epsilon A_0 \times E_{n-s}$, or $(t,x(t))\in A$, $t_1 \leq t \leq t_2$.

For almost all $t\in[t_1,t_2]$ the derivative $X'(t) = [y'(t),Z'(t)]$ exists and is finite, $S'(t)$ exists and $S'(t) = 0$, and $z_k(t) \to z(t)$. Let $t_o$ be such a point with $t_1 < t_o < t_2$. Then there is a $\sigma > 0$ with $t_1 < t_o - \sigma < t_o + \sigma < t_2$, and, for some $k_o$ and all $k > k_o$, also $t_{1k} < t_o - \sigma < t_o + \sigma < t_{2k}$. Let $x_o = x(t_o) = (y_o,Z_o)$, or $y_o = y(t_o)$, $Z_o = Z(t_o)$. Let $z_o = z(t_o)$, $S_o = S(t_o)$. We have $S'(t_o) = 0$, hence $z'(t_o)$ exists and $z'(t_o) = Z'(t_o)$. Also, we have $Z_k(t_o) \to z(t_o)$.

We have $y_k(t) \to y(t)$ uniformly in $[t_o - \sigma, t_o + \sigma]$, and all functions $y(t)$, $y_k(t)$ are continuous in the same interval. Thus, they are equicontinuous in $[t_o - \sigma, t_o + \sigma]$. Given $\epsilon > 0$, there is a $\delta > 0$ such that $t,t' \in[t_o - \sigma, t_o + \sigma]$, $|t - t'| \leq \delta$, $k \geq k_o$, implies

$$|y(t) - y(t')| \leq \epsilon/2, \qquad |y_k(t) - y_k(t')| \leq \epsilon/2.$$

We can assume $0 < \delta < \sigma$, $\delta \leq \epsilon$. For any $h, 0 < h \leq \delta$, let us consider the averages

$$m_h = h^{-1}\int_0^h \varphi(t_o + s)ds = h^{-1}[X(t_o + h) - X(t))],$$

$$m_{hk} = h^{-1}\int_0^h \varphi_k(t_o + s)ds = h^{-1}[x_k(t_o + h) - x_k)], \qquad (10)$$

where $X = (y,Z)$, $x_k = (y_k,z_k)$.

Given $\eta > 0$ arbitrary, we can fix $h$, $0 < h \leq \delta < \sigma$, so small that

$$|m_h - \varphi(t_o)| \leq \eta,$$

$$|S(t_o + h) - S(t_o)| < \eta h/4,$$

This is possible since $h^{-1}\int_0^h \varphi(t_o + s)ds \to \varphi(t_o)$ and $[S(t_o + h) - S(t_o)]h^{-1}$ $\to 0$ as $h \to 0+$. Also, we can choose h, in such a way that $z_k(t_o + h)$ $\to z(t_o)$ as $k \to +\infty$. This is possible since $z_k(t) \to z(t)$ for almost all $t_1 < t < t_2$.

Having so fixed h, let us take $k_1 \geq k_o$ so large that

$$|y_k(t_o) - y(t_o)|, \; |y_k(t_o + h) - y(t_o+h)| \leq \min [\eta h/4, \epsilon/2],$$

$$|z_k(t_o) - z(t_o)|, \; |z_k(t_o + h) - z(t_o+h)| \leq \eta h/8.$$

This is possible since $y_k(t) \to y(t)$, $z_k(t) \to z(t)$ both at $t = t_o$ and $t = t_o + h$. Then we have

$$|h^{-1}[y_k(t_o + h) - y_k(t_o)] - h^{-1}[y(t_o + h) - y(t_o)]|$$

$$\leq |h^{-1}[y_k(t_o + h) - y(t_o + h)]| + |h^{-1}[y_k(t_o) - y(t_o)]|$$

$$\leq h^{-1}(\eta h/4) + h^{-1}(\eta h/8) < \eta/2.$$

Analogously, since $z = Z + S$, we have

$$|h^{-1}[z_k(t_o + h) - z_k(t_o)] - h^{-1}[Z(t_o + h) - Z(t_o)]|$$

$$= |h^{-1}[z_k(t_o + h) - z_k(t_o)] - h^{-1}[z(t_o + h) - z(t_o)] + h^{-1}[S(t_o + h) - S(t_o)]|$$

$$\leq |h^{-1}[z_k(t_o + h) - z(t_o + h)]| + |h^{-1}[z_k(t_o) - z(t_o)]| + |h^{-1}[S(t_o + h) - S(t_o)]|$$

$$\leq h^{-1}(\eta h/8) + h^{-1}(\eta h/8) + h^{-1}(\eta h/4) = \eta/2.$$

Finally, we have

$$|m_{hk} - m_h| = |h^{-1}[x_k(t_o + h) - x_k(t_o)] - h^{-1}[X(t_o + h) - X(t_o)]|$$

$$= |h^{-1}[y_k(t_o + h) - y_k(t_o)] - h^{-1}[y(t_o + h) - y(t_o)]| +$$

$$+ \left|h^{-1}[z_k(t_o + h) - z_k(t_o)] - h^{-1}[Z(t_o + h) - Z(t_o)]\right|$$

$$\leq \eta/2 + \eta/2 = \eta.$$

We conclude that for the chosen value of h, $0 < h \leq \delta < \sigma$, and every $k \geq k_1$ we have

$$|m_h - \varphi(t_o)| \leq \eta, \quad |m_{hk} - m_h| \leq \eta, \quad |y_k(t_o) - y(t_o)| \leq \epsilon/2. \tag{11}$$

For $0 \leq s \leq h$ we have now

$$|y_k(t_o + s) - y(t_o)| \leq |y_k(t_o + s) - y_k(t_o)| + |y_k(t_o) - y(t_o)| \leq \epsilon/2 + \epsilon/2 = \epsilon,$$

$$|(t_o + s) - t_o| \leq h \leq \delta \leq \epsilon,$$

$$f(t_o + s, y_k(t_o + s), u_k(t_o + s)) \epsilon Q(t_o + a, y_k(t_o + s)).$$

Hence, by definition of $Q(t_o, y_o, 2\epsilon)$, also

$$\varphi_k(t_o + s) = f(t_o + s, y_k(t_o + s), u_k(t_o + s)) \epsilon Q(t_o, y_o, 2\epsilon).$$

The second integral relation (10) shows that we have also

$$m_{hk} \in \text{cl co } Q(t_o, y_o, 2\epsilon)$$

since the latter is a closed convex set. Finally, by relations (11), we deduce

$$|\varphi(t_o) - m_{hk}| \leq |\varphi(t_o) - m_h| + |m_h - m_{hk}| \leq 2\eta,$$

and hence

$$\varphi(t_o) \in [\text{cl co } Q(t_o, y_o, 2\epsilon)]_{2\eta}.$$

Here $\eta > 0$ is an arbitrary number, and the set in brackets is closed. Hence

$$\varphi(t_o) \in \text{cl co } Q(t_o, y_o, 2\epsilon),$$

and this relation holds for every $\epsilon > 0$. By property (Q) we have

$$\varphi(t_o)\epsilon \cap_\epsilon \text{ cl co } Q(t_o,y_o,2\epsilon) \ = \ Q(t_o,y_o),$$

where $y_o = y(t_o)$, and $Q(t_o,y_o) = f(t_o,y_o,U(t_o,y_o))$. This relation implies that there are points $\bar{u} = \bar{u}(t_o)\epsilon U(t_o,y_o)$ such that

$$\varphi(t_o) = f(t_o,y(t_o),\bar{u}(t_o)).$$

This holds for almost all $t_o \ \epsilon[t_1,t_2]$. Hence, there is at least one function $\bar{u}(t)$, defined a.e. in $[t_1,t_2]$ for which relation (9) holds a.e. in $[t_1,t_2]$. We have to prove that there is at least one such function which is measurable. The proof is exactly as the one for closure theorem I, where we write $y,y_k$ instead of $x,x_k$, and will not be repeated here. Closure theorem II is thereby proved.

<ins>Content</ins>. 

REFERENCES


1. L. Cesari (a) Semicontinuità e convessità nel calcolo delle variazioni, Annali Scuola Normale Sup. Pisa, 14, 1964, 339-423; (b) Un teorema di esistenza in problemi di controlli ottimi, Ibid., 1965, 35-78; (c) An existence theorem in problems of optimal control, J. SIAM Control, to appear; (d) Existence Theorems in problems of Lagrange and optimal control, J. SIAM Control, to appear.

2. A. F. Filippov, On certain questions in the theory of optimal control, Vestnik Moskov. Univ. Ser. Mat. Mech. Astr. 2, 1959, 25-32 (Russian). English translation in J. SIAM Control, (A) 1, 1962, 76-34.

3. R. V. Gamkrelidze, On sliding optimal regimes. Dokl. Akad. Nauk SSSR 143, 1962, 1243-1245 (Russian). English translation in Soviet Math. Doklady 3, 1962, 390-395.

4. L. Markus and E. B. Lee, Optimal control for nonlinear processes, Arch. Rational Mech. Anal. 8, 1961, 36-58.

5. M. Nagumo, Über die gleichmässige Summierbarkeit und ihre Anwendung auf ein Variation problem. Japanese Journ. Math. 6, 1929, 173-182.

6. L. S. Pontryagin, Optimal control processes. Uspekhi Mat. Nauk 14, 1(85), 1959, 3-20. English translation in Automation Express 2, no. 1, 1959, 26-30.

7. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mai shchenko, The mathematical theory of optimal processes, Gosudarst. Moscow 1961. English translations: Interscience 1962; Pergamon Press 1964.

8. E. Roxin, The existence of optimal controls, Mech. Math. J. 9(1962), 109-119.

9. L. Tonelli, (a) Sugli integrali del calcolo delle variazioni in forma ordinaria, Annali Scuola Normale Sup. Pisa (2) 3, 1934, 401-450. Opere scelte, Cremonese, Roma 1962, 3, 192-254; (b) Un teorema, di calcolo delle variaoioni, Rend. Accad. Lincei, 15, 1932, 417-423. Opere scelte, 3, 84-91; (c) Fondamenti di calcolo delle variazioni, Zanichelli, Bologna, 1921-23, 2 vols.

10. L. Turner, The direct method in the calculus of variations, A Ph.D. thesis at Purdue University, Lafayette, Indiana, August 1957.