T H E   U N I V E R S I T Y   O F   M I C H I G A N

COLLEGE OF LITERATURE, SCIENCE, AND THE ARTS
Department of Mathematics

Progress Report No. 18

EXISTENCE THEOREMS FOR WEAK AND USUAL OPTIMAL SOLUTIONS IN LAGRANGE PROBLEMS WITH UNILATERAL CONSTRAINTS

II. Existence Theorems

Lamberto Cesari

ORA Project 07100

under contract with:

NATIONAL SCIENCE FOUNDATION
GRANT NO. GP-3920
WASHINGTON, D.C.

administered through:

OFFICE OF RESEARCH ADMINISTRATION   ANN ARBOR

September 1965

EXISTENCE THEOREMS FOR WEAK AND USUAL OPTIMAL SOLUTIONS
IN LAGRANGE PROBLEMS WITH UNILATERAL CONSTRAINTS

II. EXISTENCE THEOREMS*

Lamberto Cesari
Department of Mathematics
The University of Michigan
Ann Arbor, Michigan

In the present paper (II) we prove existence theorems for weak and usual optimal solutions of nonparametric Lagrange problems with (or without) unilateral constraints.

We shall consider arbitrary pairs x(t), u(t) of vector functions, u(t) measurable with values in E_m, x(t) absolutely continuous with values in E_n, and we discuss the existence of the absolute minimum of a functional

$$I[x,u] = \int_{t_1}^{t_2} f_o(t,x(t), u(t))dt,$$

with side conditions represented by a differential system

$$dx/dt = f(t,x(t), u(t)), \qquad t_1 \leq t \leq t_2$$

constraints

$$(t,x(t)) \in A, \qquad u(t) \in U(t, x(t)), \qquad t_1 \leq t \leq t_2,$$

and boundary conditions

*Research partially supported by NSF-grant GP-3920 at The University of Michigan.

$$(t_1, x(t_1), t_2, x(t_2)) \epsilon B,$$

where A is a given subset of the tx-space $E_1 \times E_n$, where B is a given subset of the $t_1x_1t_2x_2$-space E_{2n+2}, and where U(t,x) denotes a given closed variable subset of the u-space E_m, depending on time t and space x. Here A may coincide with the whole space $E_1 \times E_n$, and U may be fixed and coincide with the whole space E_m.

In the particular situation where U(t,x) is compact for every (t,x), these problems reduce to Pontryagin problems; in the particular situation where U is fixed and coincides with the whole space E_m, then these problems have essentially the same generality of usual Lagrange problems. Throughout this paper we shall assume U(t,x) to be any closed subset of E_m.

In the present paper (II) we prove existence theorems for optimal (usual) solutions for the problem above. These contain as particular cases the Filippov existence theorem for problems of optimal control (U(t,x) compact), existence theorems for usual Lagrange problems ($U = E_m$), and the Nagumo-Tonelli existence theorem for free problems ($m = n$, $f = u$). Also, we obtain as corollaries existence theorems for the case in which f is linear in u.

In III we will discuss weak solutions as measurable probability distributions of usual solutions (Gamkrelidze's chattering states), and we will prove corresponding existence theorems in the present very general situation where U(t,x) is any arbitrary closed (not necessarily compact) variable set.

All these existence theorems are proved by a new analysis of a minimizing sequence and by using the extensions of Filippov's clossure theorems proved in paper I, as a replacement for Tonelli's semicontinuity argument.

To simplify references we continue the numeration of sections of paper I. References to paper I will be given by "I" followed by section number.

In successive papers we shall extend the present results to multidimensional Lagrange problems involving partial differential equations in Sobolev's spaces with unilateral constraints.

6. Notations for Lagrange Problems with Unilateral Constraints

Let A be a closed set of the (t,x)-space $E_1 \times E_n$, and, for every $(t,x)\in A$, let U(t,x) be a given subset of E_m. Let $f_i(t,x,u)$, $i = 0,1,\ldots,n$, be continuous functions in the set $M \subset E_1 \times E_n \times E_m$ of all (t,x,u) with $u\in U(t,x)$, $(t,x)\in A$. Let f and $\tilde{f}$ be the n-dim. and (n + 1)-dim. vector functions

$$f = (f_1,\ldots,f_n), \qquad \tilde{f} = (f_o,f_1,\ldots,f_n).$$

As usual we say that $u(t) = (u^1,\ldots u^m)$, $x(t) = (x^1,\ldots,x^n)$, $t_1 < t < t_2$, is an admissible pair provided (a) u(t) is measurable in $[t_1,t_2]$; (b) x(t) is AC in $[t_1,t_2]$, (c) $[t,x(t)]\in A$ for every $t\in[t_1,t_2]$; (d) $u(t) \in U(t,x(t))$ a.e. in $[t_1,t_2]$; (e) $f_i(t,x(t), u(t))$ is L-integrable in $[t_1,t_2]$, $i = 0,1,\ldots,n$, and $dx_i/dt = f_i(t,x(t),u(t))$, $i = 1,\ldots,n$, a.e. in $[t_1,t_2]$. Thus, by introducing the auxiliary variable x^o, the differential equation $dx^o/dt = f_o(t,x(t),u(t))$, the boundary condition $x^o(t_1) = 0$, the vector $\tilde{x} = (x^o,x^1,\ldots,x^n)$, and the set $\bar{A} = A \times E_1 \subset E_{n+2})$, the pair [u(t),x(t)] is admissible if and only if the pair [u(t),x(t)] is admissible according to the definitions of no. 2 for the set $\bar{A}$ of the tx-space $E_1 \times E_{n+1}$, the sets $U(t,x) \subset E_m$, and the vector function $\tilde{f}(t,x,u)$.

If [u(t),x(t)] is admissible, then u(t) is said to be an admissible control function, $x(t)$ a trajectory, and

$$x^0(t_2) = I[x,u] = \int_{t_1}^{t_2} f_0(t,x(t),u(t))\,dt \qquad (1)$$

the cost functional.

A class Ω of admissible pairs $x(t),u(t)$ is said to be complete if for every sequence $x_k(t)$, $u_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k = 1,2,\ldots$, of admissible pairs all in Ω, with the sequence $[x_k(t)]$ converging in the metric ρ toward a vector function $x(t)$ which is known to be a trajectory generated by some admissible control function $u(t)$, then $[x(t), u(t)]$ belongs to Ω.

Such a class Ω is often defined in terms of boundary conditions. For instance, if B is a given closed set of points (t_1,x_1,t_2,x_2) of the $(2n+2)$-dim. Euclidean space E_{2n+2}, we may define Ω as the class of all admissible pairs $x(t),u(t)$ satisfying

$$(t_1,x(t_1),t_2,x(t_2))\in B. \qquad (2)$$

Then Ω is a complete class in the sense mentioned above, since B is, by hypothesis, a closed set.

We shall denote by B_1 the projection of B on the (t,x_1)-space E_{n+1}, that is, B_1 is the set of all points $(t_1,x_1)\in E_{n+1}$ for $(t_1,x_1,t_2,x_2)\in B$. Analogously, we denote by B_2 the projection of B on the (t_2,x_2)-space E_{n+1}. Obviously, $B \subset B_1 \times B_2$, and $B_1 \times B_2$ may be larger than B.

It is often requested that each trajectory $x(t)$ of a class Ω as above possesses at least one point $(t^*,x(t^*))$ on a given compact subset P of A. Such a condition is certainly satisfied if B is compact, or at least if B is closed and B_1, or B_2, is compact.

For the analysis of problems of Lagrange with unilateral constraints certain variable sets have to be taken into consideration, namely, the set $U(t,x)$ above and the sets

$$Q(t,x) = [z \mid z = f(t,x,u),\ u\in U(t,x)] = f[t,x,U(t,x)] \subset E_n,$$

$$\tilde{Q}(t,x) = [\tilde{z} \mid \tilde{z} = \tilde{f}(t,x,u),\ u\in U(t,x)] = \tilde{f}[t,x,U(t,x)]$$

$$= [\tilde{z} = (z^o,z) \mid z^o = f_o(t,x,u),\ z = f(t,x,u),\ u\in U(t,x)] \subset E_{n+1},$$

$$\tilde{\tilde{Q}}(t,x) = [\tilde{z} = (z^o,z) \mid z^o \geq f_o(t,x,u),\ z = f(t,x,u),\ u\in U(t,x)] \subset E_{n+1}.$$

The sets Q and $\tilde{Q}$ are well known and have been considered by a number of authors (for instance, A. F. Filippov, [2]). The set $\tilde{\tilde{Q}}(t,x)$ is being considered here and in [1c] for the first time. By considering this set, instead of Q or $\tilde{Q}$, we prove here theorems I and II (no. 7) which include a number of existence theorems for both problems of optimal control and the calculus of variations.

7. An Existence Theorem for Lagrange Problems with Unilateral Constraints

Existence Theorem I. Let A be a compact subset of the tx-space $E_1 \times E_n$, and for every $(t,x)\in A$ let $U(t,x)$ be a closed subset of the u-space E_m. Let $\tilde{f}(t,x,u) = (f_o,f_1,\dots f_n) = (f_o,f)$ be a continuous vector function on the set M of all (t,x,u) with $(t,x)\in A$, $u\in U(t,x)$. Assume that, for every $(t,x)\in A$ the set

$$\tilde{\tilde{Q}}(t,x) = [\tilde{z} = (z^0,z) \mid z^0 \geq f_0(t,x,u),\ z = f(t,x,u),\ u \in U(t,x)] \subset E_{n+1}$$

is convex. Assume that $U(t,x)$ satisfies property (U) in A, and $\tilde{\tilde{Q}}(t,x)$ satisfies property (Q) in A. Assume that there is a continuous scalar function $\Phi(\zeta)$, $0 \leq \zeta < +\infty$, with $\Phi(\zeta)/\zeta \to +\infty$ as $\zeta \to +\infty$, such that $f_0(t,x,u) \geq \Phi(|u|)$ for all $(t,x,u) \in M$, and that there are constants $C, D \geq 0$ such that $|f(t,x,u)| \leq C + D|u|$ for all $(t,x,u) \in M$. Then the cost functional $I[x,u] = \int_{t_1}^{t_2} f_0(t,x,u)\,dt$ has an absolute minimum in any nonempty complete class Ω of admissible pairs $x(t)$, $u(t)$.

If A is not compact, but closed and contained in a slab $[t_0 \leq t \leq T,\ x \in E_n]$, t_0, T finite, then theorem I still holds if, in addition, we know that (a) $x^1 f_1 + \dots + x^n f_n \leq F[|x|^2 + 1]$ for all $(t,x,u) \in M$ and some constant $F \geq 0$, and (b) every trajectory in Ω contains at least one point $(t^*, x(t^*))$ on a given compact subset P of A (t^* may depend on $x(t)$). If A is not compact, not contained in a slab as above, but A is closed, then theorem I still holds if hypotheses (a), (b) are satisfied, and (c) $f_0(t,x,u) \geq \mu > 0$ for all $(t,x,u) \in M$ with $|t| \geq R$, for convenient constants $\mu > 0$, $R \geq 0$. Finally condition (a) can be replaced in either case by the hypotheses: (a') There are constants $G > 0$, $H \geq 0$ such that $f_0(t,x,u) \geq G|f(t,x,u)|$ for all $(t,x,u) \in M$ with $|x| \geq H$. Furthermore, when A is not compact but closed, both conditions $f_0 \geq \Phi(|u|)$, $|f| \leq C + D|u|$ can be replaced by the following set of conditions: (α) $f_0(t,x,u) \geq -L$ for all $(t,x,u) \in M$ and some constant $L \geq 0$; (β) $f_0(t,x,u) \geq \mu > 0$ for all $(t,x,u) \in M$ with $|t| \geq R$, and some constants $\mu > 0$, $R \geq 0$; (γ) for every compact subset A_0 of A there is a function Φ_0 as above and constants C_0, $D_0 \geq 0$ such that $f_0 \geq \Phi_0(|u|)$, $|f| \leq C_0 + D_0|u|$ for all $(t,x,u) \in M$ with $(t,x) \in A_0$ (where Φ_0, C_0, D_0 may depend on A_0).

Proof of Existence Theorem I. We have $\Phi(\zeta) \geq -M_0$ for some number $M_0 \geq 0$, hence $\Phi(\zeta) + M_0 \geq 0$ for all $\zeta \geq 0$, and $f_0(t,x,u) + M_0 \geq 0$ for all $(t,x,u)\epsilon M$. Let D be the diameter of A. Then for every pair $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, of Ω we have.

$$I[x,u] = \int_{t_1}^{t_2} f_0 dt \geq \int_{t_1}^{t_2} \Phi(|u|)dt \geq -DM_0 > -\infty.$$

Let $i = \text{Inf } I[u]$, where Inf is taken for all $u(t)\epsilon\Omega$. Then i is finite.

Let $x_k(t)$, $u_k(t)$, $t_{1k} \leq t \leq t_{2k}$, be a sequence of admissible pairs all in Ω, such that $I[x_k, u_k] \to i$ as $k \to \infty$. We may assume

$$i \leq I[x_k, u_k] = \int_{t_{1k}}^{t_{2k}} f_0(t, x_k(t), u_k(t))dt \leq i + k^{-1} \leq i + 1, \; k = 1,2,\dots.$$

Let us prove that the AC vector functions $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k = 1,2,\dots,$ are equiabsolutely continuous. Let $\epsilon > 0$ be any given number, and let $\sigma = 2^{-1}$ $(DM_0 + |i| + 1)^{-1}$.

Let $N > 0$ be a number such that $\Phi(z)/z > 1/\sigma$ for $z \geq N$. Let E be any measurable subset of $[t_{1k},t_{2k}]$ with meas $E < \eta = \epsilon/2N$. Let E_1 be the subset of all $t\epsilon E$ where $u_k(t)$ is finite and $|u_k(t)| \leq N$, and let $E_2 = E - E_1$. Then $|u_k(t)| \leq N$ in E_1, and $\Phi(|u_k|)/|u_k| \geq 1/\sigma$, or $u_k \leq \sigma\Phi(|u_k|)$, a.e. in E_2. Hence

$$\int_E |u_k(t)|dt = \left(\int_{E_1} + \int_{E_2}\right)|u_k(t)|dt$$

$$\leq N \text{ meas } E_1 + \sigma\int_{E_2} \Phi(|u_k(t)|)dt$$

$$\leq N \text{ meas } E + \sigma \int_{E_2} [\Phi(|u_k(t)|) + M_0]dt$$

$$\leq N\eta + \sigma \int_{t_{1k}}^{t_{2k}} [\Phi(|u_k(t)|) + M_0]dt$$

$$\leq N\eta + \sigma \int_{t_{1k}}^{t_{2k}} [f_0(t,x_k(t), u_k(t)) + M_0]dt$$

$$\leq N\eta + \sigma(DM_0 + |i| + 1)$$

$$\leq \epsilon/2 + \epsilon/2 = \epsilon . \qquad (2)$$

This proves that the vector functions $u_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k = 1,2,\ldots$, are equiabsolutely integrable. From here we deduce

$$\int_E |x_k'(t)|dt = \int_E |f(t,x_k(t),u_k(t)|dt \leq \int_E [A + B|u_k(t)|]dt$$

$$\leq A \text{ meas } E + B \int_E |u_k(t)|dt ,$$

and this proves the equiabsolute continuity of the vector functions $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k = 1,2,\ldots.$

Now let us consider the sequence of AC scalar functions $x_k^0(t)$ defined by

$$x_k^0(t) = \int_{t_{1k}}^{t} f_0(\tau,x_k(\tau),u_k(\tau))dt, \qquad t_{1k} \leq t \leq t_{2k} \qquad (3)$$

with

$$x_k^o(t_{1k}) = 0, \quad x_k^o(t_{2k}) = I[u_k,x_k] \to i \quad \text{as} \quad k \to +\infty .$$

$$i \le x_k^o(t_{2k}) \le i + k^{-1} \le i + 1 .$$

If $u_k^o(t) = f_o(t,x_k(t),u_k(t))$, $t_{1k} \le t \le t_{2k}$, then we define the functions $u_k^-(t)$, $u_k^+(t)$ as follows:

$$u_k^-(t) = u_k^o(t), \quad u_k^+(t) = 0 \quad \text{if} \quad u_k^o(t) \le 0 ,$$

$$u_k^-(t) = 0, \quad u_k^+(t) = u_k^o(t) \quad \text{if} \quad u_k^o(t) \le 0 .$$

Then $u_k^-(t) \le 0$, $u_k^+(t) \ge 0$ a.e. in $[t_1,t_2]$, and we define

$$y_k^-(t) = \int_{t_{1k}}^t u^-(t)dt, \quad . \quad y_k^+(t) = \int_{t_{1k}}^t u^+(t)dt, \qquad t_{1k} \le t \le t_{2k} , \qquad k = 1,2,\dots .$$

Since $-M_o \le u_k^-(t) = 0$, we have $-M_o(t_{2k} - t_{1k}) \le y_k^-(t) \le 0$, and the functions $y_k^-(t)$ are monotone nonincreasing and uniformly Lipschitzian with constant M_o. On the other hand, the functions $y_k^+(t)$ are nonnegative, monotone nondecreasing, and uniformly bounded since

$$0 \le y_k^+(t_{2k}) = (y_k^+(t_{2k}) + y_k^-(t_{2k})) - y_k^-(t_{2k}) = x_k^o(t_{2k}) - y_k^-(t_{2k})$$

$$\le i + 1 + M_o(t_{2k} - t_{1k}) \le DM_o + |i| + 1 .$$

By Ascoli's theorem we first extract a sequence for which $y_k^-(t)$ converges in the metric ρ toward a continuous function $Y^-(t)$, $t_1 \le t \le t_2$, and $Y^-(t)$ is then monotone nonincreasing Lipschitzian with constant M_o, and $Y^-(t_1) = 0$. Then we apply Helly's theorem to the sequence $y_k^+(t)$ and we perform a successive extraction so that the corresponding sequence of the $y_k^+(t)$ converges for every $t_1 < t < t_2$ toward a function $Y_0^+(t)$, $t_1 < t < t_2$, which is nonnegative, monotone nondecreasing, but not necessarily continuous. This function $Y_0^+(t)$ is defined say at t_1 only if ∞-many of these intervals t_{1k}, t_{2k} cover t_{1k}, and so is for t_2. If $Y_0^+(t)$ is not defined at t_1 ot t_2, we may define it at t_1 by taking $Y_0^+(t_1) = 0$, and at t_2 by continuity at t_2, because of its monotoneity. Thus $0 \leq Y_0^+(t) \leq DM_0 + i + 1$, $t_1 \leq t \leq t_2$.

Finally, $Y_0^+(t)$ admits of a unique decomposition $Y_0^+(t) = Y^+(t) + Z(t)$, $t_1 \leq t \leq t_2$, with $Y^+(t_1) = 0$, where both $Y^+(t), Z(t)$ are nonnegative monotone nondecreasing, where $Y^+(t)$ is AC, and $Z'(t) = 0$ a.e. in $[t_1 . t_2]$. Finally, if $Y(t) = Y^-(t) + Y^+(t)$, we see that $x_k(t)$, $t_1 \leq t \leq t_2$, converges for all $t_1 < t < t_2$, toward $Y(t) + Z(t)$, where $Y(t)$ is a (scaler) AC function, $DM_0 \leq Y(t) \leq DM_0 + |i| + 1$, $Y(t_1) = 0$. Let us prove that $Y(t_2) \leq i$. For the subsequence [k] we have extracted last, we have $t_{2k} \to t_2$, $x_k^0(t_{2k}) \to i$, $x_k(t_{2k}) = y_k^-(t_{2k}) + y_k^+(t_{2k})$. If $\bar{t}_2$ is any point, $t_1 < \bar{t}_2 < t_2$, $\bar{t}_2$ as close as we want to t_2, then $\bar{t}_2 < t_{2k}$ for all k sufficiently large (of the extracted sequence), since $t_{2k} \to t_2$. We can assume k so large that $\bar{t}_2 < t_{2k}$, $|\bar{t}_2 - t_{2k}| < 2|\bar{t}_2 - t_2|$. Then

$$|y_k^-(\bar{t}_2) - y_k^-(t_{2k})| \leq M_0|\bar{t}_2 - t_{2k}| \leq 2M_0|\bar{t}_2 \; t_2| \; .$$

Since $y_k^+(t)$ is nondecreasing, we have $y_k^+(t_2) \leq y_k^+(t_{2k})$, and finally

$$y_k^-(\bar{t}_2) + y_k^+(\bar{t}_2) \leq y_k^-(\bar{t}_2) + y_k^+(t_{2k})$$

$$\leq y_k^-(t_{2k}) + y_k^+(t_{2k}) + |y_k^-(\bar{t}_2) - y_k^-(t_{2k})|$$

$$\leq x^o(t_{2k}) + 2M_o|\bar{t}_2 - t_2| ,$$

where $x^o(t_{2k}) \to i$ as $k \to +\infty$, and $x^o(t_{2k}) < i + k^{-1}$. Hence

$$y_k^-(\bar{t}_2) + y_k^+(\bar{t}_2) < i + 2M_o|\bar{t}_2 - t_2| + k^{-1}.$$

As $k \to +\infty$ (along the extracted sequence), we have

$$Y^-(\bar{t}_2) + Y_o^+(\bar{t}_2) \leq i + 2M_o|\bar{t}_2 - t_2| .$$

or

$$Y^-(\bar{t}_2) + Y^+(\bar{t}_2) + Z(\bar{t}_2) \leq i + 2M_o|\bar{t}_2 - t_2|,$$

where the third term in the first member is ≥ 0. Thus

$$Y(\bar{t}_2) = Y^-(\bar{t}_2) + Y^+(\bar{t}_2) \leq i + 2M_o|\bar{t}_2 - t_2| .$$

As $\bar{t}_2 \to t_2 - 0$, we obtain $Y(t_2) \leq i$, since Y is continuous at t_2.

We will apply below closure theorem II to an auxiliary problem, we shall now define. Let $\tilde{u} = (u^o,u) = (u^o,u^1,\dots,u^m)$, let $\tilde{U}(t,x)$ be the set of all $u \in E_{m+1}$ with $\tilde{u} = (u^1,\dots,u^m) \in U(t,x)$, $u^o \geq f_o(t,x,u)$, let $\tilde{x} = (x^o,x) = (x^o,x^1,\dots,x^n)$, let $\tilde{\tilde{f}} = \tilde{\tilde{f}}(t,x,u) = (\tilde{f}_o,f) = (\tilde{f}_o,f_1,\dots,f_n)$ with $\tilde{f}_o = u^o$. Thus $\tilde{\tilde{f}}$ depends only on $t,x,\tilde{u}$ (instead of $t,\tilde{x},\tilde{u}$), and U depends only on t,x, (instead of $t,\tilde{x}$). Finally we consider the differential system

$$d\tilde{x}/dt - \tilde{\tilde{f}}(t,x,u)$$

or

$$dx^o/dt = u^o(t), \qquad dx^i/dt + f_i(t,x,u), \qquad i = 1,\dots,n,$$

with the constraints

$$\tilde{u}(t) \in \tilde{U}(t,x(t)),$$

or

$$u^o(t) \geq f_o(t,x(t),u(t)), \qquad u(t) \in U(t,x(t)),$$

a.e. in $[t_1,t_2]$, besides $x^o(t_1) = 0$, and $[u,x] \in \Omega$. We have here the situation discussed in closure theorem II where $\tilde{x}$ replaces x, x replaces y, x^o replaces z, n + 1 replaces n, n replaces s, hence (n + 1) - n = 1 replaces n - s. For the new auxiliary problem the cost functional is

$$J[\tilde{u},\tilde{x}] = \int_{t_1}^{t_2} \tilde{f}_o du = \int_{t_1}^{t_2} u^o(t)\,dt = x^o(t_2)$$

Note that the set $\tilde{\tilde{Q}}(t,x) = \tilde{f}(t,x,\tilde{U}(t,x))$ of the new problem is the set of all $\tilde{z} = (z^o,z) \in E_{n+1}$ such that $z^o = u^o$, since $\tilde{f}_o = u^o$, $z = f(t,x,u)$, $u^o \geq f_o(t,x,u)$, $u \in U(t,x)$. Thus, the sets $\tilde{U}$, $\tilde{\tilde{Q}}$ for this auxiliary problem are the sets $\tilde{U}$, $\tilde{\tilde{Q}}$ considered at the beginning of this proof.

We consider now the sequence of trajectories $\tilde{x}_k(t) = [x_k{}^o(t), x_k(t)]$, $t_{1k} \leq t \leq t_{2k}$, for the problem $J[\tilde{u},\tilde{x}]$ corresponding to the control function $u(t) = [u_k{}^o(t), u_k(t)]$ with $u_k{}^o(t) = f_o(t,x_k(t),u_k(t))$, $u_k(t) \in U(t,x_k(t))$, and hence $\tilde{u}_k(t) \in \tilde{U}(t,x_k(t))$, $t_{1k} \leq t \leq t_{2k}$, $k = 1,2,\ldots$. The sequence $[\tilde{x}_k(t)]$ converges in the metric ρ toward the AC vector function $x(t)$, while $x_k{}^o(t) \to x^o(t)$ as $k \to +\infty$ for all $t \in (t_1,t_2)$, and $x^o(t) = Y(t) + Z(t)$, where $Y(t)$ is AC in $[t_1,t_2]$ and $Z'(t) = 0$ a.e. in $[t_1,t_2]$.

By closure theorem II we conclude that $X(t) = [Y(t),x(t)]$ is a trajectory for the problem. In other words, there is a control function $\tilde{u}(t)$, $t_1 \le t \le t_2$, $\tilde{u}(t) = (u^o(t), u(t))$, with

$$dY/dt = u^o(t) \ge f_o(t,x(t),u(t)), \qquad u(t) \in U(t,x(t)) ,$$

$$dx/dt = f(t,x(t),u(t)), \tag{4}$$

a.e. in $[t_1,t_2]$, and

$$i \ge Y(t_2) = J[\tilde{x},\tilde{u}] = \int_{t_1}^{t_2} u^o(t)\,dt . \tag{5}$$

First of all $[x(t),u(t)]$ is admissible for the original problem and hence belongs to Ω, since by hypothesis Ω is complete. From this remark, and relations (4) and (5) we deduce

$$i \le I[x,u] = \int_{t_1}^{t_2} f_o(t,x(t),u(t))\,dt \le \int_{t_1}^{t_2} u^o(t)\,dt \le i,$$

and hence all $\le$ signs can be replaced by = signs, $u^o(t) = f_o(t,x(t),u(t))$ a.e. in $[t_1,t_2]$, and $I[x,u] = i$. This proves that i is attained in Ω. Existence theorem I is proved in the case A is compact.

Let us assume now that A is not compact but closed, that A is contained in a slab $[t_o \le t \le T, -\infty < x^i < +\infty, \ i = 1,\dots,n, t_o, T$ finite$]$, and that the additional hypotheses (a) and (b) hold. If Z(t) denotes the scalar function $Z(t) = |x(t)|^2 + 1$, then condition $x^1 f_1 + \dots + x^n f_n \le C(|x|^2 + 1)$ implies $Z' \le CZ$, and hence, by integration from t^* to t, also

$$1 \leq Z(t) \leq Z(t^*) \exp C|t - t^*| .$$

Since $[t^*, x(t^*)] \epsilon P$ where P is a compact subset of A, then there is a constant N_o such that $|x| \leq N_o$ for every $x \epsilon P$, hence $1 \leq Z(t^*) \leq N_o^2 + 1$, and $1 \leq Z(t) \leq (N_o^2 + 1) \exp C(T - t_o)$. Thus, for $t_o \leq t \leq T$, $Z(t)$ remains bounded, and hence $|x(t)| \leq D$ for some constant D. We can now restrict ourselves to the consideration of the compact part A_o of all point (t,x) of A with $t_o \leq t \leq T$, $|x| \leq D$.

Thus, theorem I is proved for A closed and contained in a slab as above, and under the additional hypotheses (a), (b).

Let us assume that A is not compact, nor contained in any slab as above, but closed, and that hypotheses (a), (b), (c) hold. First, let us take an arbitrary element $\bar{x}(t)$, $\bar{u}(t)$ of Ω and let $j = I[\bar{x},\bar{u}]$. Then we consider an interval (a,b) of the t-axis containing the entire projection P_o of P on the t-axis, as well as the interval [-R,R]. Now let $\ell = \mu^{-1} [|j| + 1 + (b-a)M_o]$, and let [a',b'] denote the interval $[a - \ell, b + \ell]$. Then for any admissible pair (if any) $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, of the class Ω, whose interval $[t_1, t_2]$ is not contained in [a', b'], there is at least one point $t^* \epsilon [t_1, t_2]$ with $(t^*, x^*(t)) \epsilon P$, $a < t^* < b$, and a point $\bar{t} \epsilon [t_1, t_2]$ outside [a',b']. Hence $[t_1, t_2]$ contains at least one subinterval, say E, outside [a,b], of measure $\geq \ell$. Then $I[x,u] \geq \ell\mu - (b - a)M_o = |j| + 1 \geq i + 1$. Obviously, we may disregard all pairs $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, whose interval $[t_1, t_2]$ is not contained in [a',b']. In other words, we can limit ourselves to the closed part A' of all $(t,x) \epsilon A$ with $a' \leq t \leq b'$. We are now in the situation above, and theorem I is proved for any closed set A under the additional hypotheses (a), (b), (c).

Finally, we have to show that condition (a) can be replaced by condition: (a') There are numbers C, $D > 0$ such that $f_o(t,x,u) \geq C|f(t,x,u)|$ for all $(t,x,u) \in M$ with $|x| \geq D$. It is enough to prove theorem I under the hypotheses that A is closed and contained in a slab $t_o \leq t \leq T$, t_o, T as above, and hypotheses (a') and (b). First let us take D so large that the projection P* of P on the x-space is completely in the interior of the solid sphere $|x| \leq D$, and also so large that $D \geq T - t_o$. Let $\bar{u}(t), \bar{x}(t)$ be any arbitrary admissible pair contained in Ω, and let j denote the corresponding value of the cost functional. Let $L = C^{-1}[DM_o + |j| + 1]$, and let us take $D_o = D + L$. If any admissible pair u(t), x(t), $t_1 \leq t \leq t_2$, of Ω possesses a point $(t_o, x(t_o))$ with $|x(t_o)| \geq D_o$, then x(t) possesses also a point $(t^*, x(t^*)) \in P$, with $|x(t^*)| \leq D$. Thus, there is at least a subarc Γ: $x = x(t)$, $x(t)$, $t' \leq t \leq t''$, of x(t) along which $|x(t)| \geq D$ and $|x(t)|$ passes from the value D to the value $D_o = D + L$. Such an arc Γ has a length $\geq L$. If $E = [t_1, t_2] - [t', t'']$, then

$$I[x,u] = \int_{t_1}^{t_2} f_o dt = \left(\int_E + \int_{t'}^{t''} \right) f_o dt \geq - DM_o + \int_{t'}^{t''} C|f|dt$$

$$= -DM_o + C\int_{t'}^{t''} |dx/dt|dt = -DM_o + CL \geq |j| + 1 \geq i + 1 .$$

As before we can restrict ourselves to the compact part A_o of all points (t,x) of A with $t_o \leq t \leq T$, $|x| \leq D$. The case where A is closed, A is not contained in any slab as above, but conditions (a'), (b), (c) hold can be treated as before. Theorem I is thereby completely proved.

<u>Remark 1.</u> If the set

$$\tilde{Q}(t,x) = \tilde{f}[t,x,U(t,x)] = [\tilde{z} = (z^o,z) \mid \tilde{z} = \tilde{f}(t,x,u),\ u\epsilon U(t,x)]$$

$$= [\tilde{z} = (z^o,z) \mid z^o = f_o(t,x,u),\ z = f(t,x,u),\ u\epsilon U(t,x)] \subset E_{n+1}$$

is convex, then certainly the set $\tilde{\tilde{Q}}(t,x)$ of theorem I is convex also. On the other hand, trivial examples show that $\tilde{\tilde{Q}}(t,x)$ may be convex, when $Q(t,x)$ is not. This is actually the usual case in free problems of the calculus of variations (see remark 2 below). Thus, the requirement in theorem I that $\tilde{\tilde{Q}}(t,x)$ be convex for every (t,x) is a wide generalization of the analogous hypothesis concerning $\tilde{Q}(t,x)$ which is familiar in problems of optimal control. For these problems, Filippov's existence theorem is a particular case of theorem I.

<u>The theorem of A. F. Filippov [2].</u> As in theorem I, if $A = E_1 \times E_n$, if $\tilde{f}(t,x,u) = (f_o,f) = (f_o,f_1,\ldots,f_n)$ is continuous in M, if $U(t,x)$ is compact for every (t,x) in A, if $U(t,x)$ is an upper semicontinuous function of (t,x) in A, if $\tilde{Q}(t,x) = \tilde{f}(t,x,U(t,x))$ is a convex subset of E_{n+1} for every (t,x) in A, if conditions (a) and (c) are satisfied, and the class Ω of all admissible pairs for which $x(t_1) = x_1$, $x(t_2) = x_2$, t_1,x_1,x_2 fixed, t_2 undetermined, is not empty, then $I[x,u]$ has absolute minimum in Ω.

This statement is a corollary of theorem I. Indeed, under hypothesis (c) we can restrict A to the closed part A' of all $(t,x)\epsilon A$ with $a' \leq t \leq b'$, and $|x| \leq N$ for some large N. If M_o is the part of all (t,x,u) of M with $(t,x)\epsilon M$, then the hypothesis that $U(t,x)$ is compact and an uppersemicontinuous function of (t,x) in A_o certainly implies that $U(t,x)$ satisfies condition (U) in A_o and that M_o is compact (I, no. 4, (vi) and (vii)). Also, since $Q(t,x)$ is convex for every (t,x) by hypothesis, we deduce that $Q(t,x)$ is an uppersemicontinuous function of (t,x) and satisfies property (Q) (I, no. 4, (xiv) and (xiii)). Then $\tilde{\tilde{Q}}(t,x)$ certainly satisfies property (Q) by the remark 1 above. Finally, since M_o is compact, the growth condition $f_o \geq \Phi$ and the remaining condition $|f| \leq C + D|u|$ are trivially satisfied. Thus, all conditions of theorem I are satisfied, and Filippov's theorem is proved to be a particular case of Theorem I.

Remark 2. The analogous existence theorems of E. Roxin [8] and of L. Markus and E. B. Lee [9] are also essentially contained in Theorem I. For a detail on Roxin statement see Remark 4 below.

Remark 3. For free problems of the calculus of variations (no. 11 below) we have $m = n$, $U = E_n$, $\dot{f} = u$, hence

$$\tilde{Q}(t,x) = \tilde{f}[t,x,U(t,x)] = [\tilde{z} = (z^o,u) \,|\, z^o = f_o(t,x,u),\ u\epsilon E_n] \subset E_{n+1}$$

$$\tilde{\tilde{Q}}(t,x) = [\tilde{z} = (t^o,u) \,\Big|\, z^o \geq f_o(t,x,u),\ u\epsilon E_n] \subset E_{n+1}\ ,$$

the set $\tilde{Q}$ is convex if and only if f_o is linear in u, while $\tilde{\tilde{Q}}$ is convex if and only if f_o is convex in u. Thus condition $\tilde{\tilde{Q}}$ convex of theorem I reduces to the requirement f_o convex in u which is familiar for free problems in the calculus of variations. We shall prove in no. 11 that the Nagumo-Tonelli existence theorem for free problem is also a particular case of theorem I.

Remark 4. The condition $f_o \geq \Phi(|u|)$ with $\Phi(z)/z \to +\infty$ of the theorems above is said to be a growth condition on f_o. As it is well known such a condition (for f_o convex in u, and $U=E_m$) is equivalent to the condition that, for every $(t,x)\epsilon A$, we have we have $f_o(t,x,u)/|u| \to +\infty$ as $|u| \to +\infty$ [L. Tonelli, 9a]. On the other hand, it is known already for free problems, that if such a condition is not satisfied at the points $(t,x) \in A$ of even only one hyperplane $t = \bar{t}$, then the absolute minimum need not exist. For free problems, other additional conditions have been devised in such cases [9a].

Condition $x^1 f_1 + \dots + x^n f_n \leq C(|x|^2 + 1)$ of theorem I can be replaced by $x^1 f_1 + \dots + x^n f_n \leq \varphi(t)(|x|^2 + 1)$, where $\varphi(t) \geq 0$ is a fixed function of t which is L-integrable in any finite interval. The remark was made by E. Roxin [8] in analogous problems of optimal control.

Remark 5. For problems of optimal control where $U(t,x)$ is always compact we have given in [1bc] an existence theorem, say I*, similar to the Filippov's theorem above, where the condition "$\tilde{\tilde{Q}}$ convex" is replaced by the following requirement: $Q(t,x)$ is a convex subset of E_n, $f_o(t,x,u)$, $u \in U(t,x)$, is convex in u, and "the curvature of f is always small with respect to the convexity of f_o" (see [1b], or [1c] for a precise statement). Wherever this requirement implies the convexity of the set $\tilde{\tilde{Q}}$, then the theorem given in [1bc] becomes a corollary of Theorem I above. Also it should be pointed out that, whenever the relation $z = f(t,x,u)$ between $Q(t,x)$ and $U(t,x)$ can be inverted and $u = f^{-1}(t,x,z)$ is a continuous function of z in $Q(t,x)$, then the set $\tilde{\tilde{Q}}(t,x)$ can be represented by $\tilde{\tilde{Q}}(t,x) = [\tilde{z} = (z^o, z) \mid z^o \geq F(t,x,z),\ z \in Q(t,x)]$, where $F(t,x,z) = f_o(t,x,f^{-1}(t,x,z))$, and thus the requirement of the convexity of the set $\tilde{\tilde{Q}}$ reduces to the requirement of the convexity of the function $F(t,x,z)$ in z. Then the further requirement that $\tilde{\tilde{Q}}(t,x)$ satisfies property (Q) is certainly satisfied if, besides, F is quasi normally convex as proved in [I, no. 4, (xviii)]. We discussed in [1c] a case where the requirement of theorem (I*) implies the convexity of F in u, and correspondingly (I*) becomes a corollary of I. The simpler requirement: $Q(t,x)$ a convex subset of E_n and $f_o(t,x,u)$ convex in u, does not suffice for existence, as we prove in the following number.

8. EXAMPLE OF A PROBLEM WITH NO ABSOLUTE MINIMUM

The condition "$\tilde{\tilde{Q}}(t,x)$ convex for every $(t,x)\epsilon A$" of theorem I cannot be replaced by the simpler condition "$Q(t,x)$ convex for every $(t,x)\epsilon A$ and $f_o(t,x,u)$ convex in u," not even when A and all sets $U(t,x)$ are compact (that is, for Pontryagin problems). This is shown by the following example.

Let us consider the differential system:

$$x' = u(1 - v) + [2 - 2^{-1}(u - 1)^2]v ,$$

$$y' = [2 - 2^{-1}(u - 1)^2] (1 - v) + uv ,$$

with $t_1 = 0$, initial point (0,0), fixed target (0,1), and fixed control space $[-1 \leq u \leq 1,\ 0 \leq v \leq 1]$. If

$$z_1 = f_1 = u(1 - v) + [2 - 2^{-1}(u - 1)^2]v ,$$

$$z_2 = f_2 = [2 - 2^{-1}(u - 1)^2] (1 - v) + uv ,$$

we see that the segment $[v = 1, -1 \leq u \leq 1]$ is mapped by $f = (f_1,f_2)$ onto the arc of parabola $ABC = [z_1 = 2 - 2^{-1}(u - 1)^2,\ z_2 = u, -1 \leq u \leq 1]$, whose points $A = (0,-1)$, $B = (3/2,0)$, $C = (2,1)$ correspond to $u = -1,0,1$ respectively. The segment $[v = 0, -1 \leq u \leq 1]$ is mapped by f onto the arc $DEF = [z_1 = u,\ z_2 = 2-2^{-1}$

$(u - 1)^2$, $-1 \leq u \leq 1]$, whose points $-D = (-1,0)$, $E = (0,3/2)$, $F = (1,2)$ correspond to $u = -1,0,1$ respectively. Each segment $[u = c,\ 0 \leq v \leq 1]$ is mapped by f onto the segment joining the points corresponding to $(c,1)$ and $(c,0)$ on the two parabolas. Thus, the image $Q = f(U)$ of U is the convex body Q=(ABCFED) of the z_1,z_2-plane. Let us consider the cost functional

$$I = \int_{t_1}^{t_2} [x^2 + (y - t)^2 + v^2]dt .$$

For $k = 1,2,\ldots$, let $u_k(t)$, $v_k(t)$, $0 \leq t \leq 1$, be defined by taking $u_k(t) = -1$, $v_k(t) = -1$, $v_k(t) = 0$, or $u_k(t) = +1$, $v_k(t) = 0$, according as t belongs to the intervals $k^{-1}(i - 1) < t < k^{-1}(i - 1) + (2k)^{-1}$, or $k^{-1}(i - 1) + (2k)^{-1} < t < k^{-1}i$, $i = 1,2,\ldots,k$. Then the functions $x_k(t)$, $y_k(t)$, $0 \leq t \leq 1$, satisfy the differential equations $dx_k/dt = +1$, $dy_k/dt = 2$, or $dx_k/dt = -1$, $dy_k/dt = 0$, according as t belongs to one or the other of the two sets of intervals above. Then $x_k(t) \to x_o(t) = 0$, $y_k(t) \to y_o(t) = t$ uniformly in $0 \leq t \leq 1$ as $k \to \infty$. If C_k, C_o denote these trajectories, we say that $C_k \to C_o$.

The question as to whether C_o is actually a trajectory, that is, whether there are admissible control functions $u_o(t)$, $v_o(t)$, $0 \leq t \leq 1$, whose corresponding trajectory is C_o can be answered in the affirmative because of the convexity of Q. Actually, the point $(\alpha_o,\beta_o) \in U$, $\alpha_o = 2 - \sqrt{5} = -0.23607$, $\beta_o = (11)^{-1}(4 - \sqrt{5}) = 0.16036$, is mapped by f into $(z_1 = 0,\ z_2 = 1)$, and thus $u_o(t) = \alpha_o$, $v_o(t) = \beta_o$, $0 \leq t \leq 1$, generate C_o. Now we have $x_k(t) \to 0$, $y_k(t) \to t$, uniformly in $[0,1]$ as $k \to \infty$, and $v_k(t) = 0$, hence $I[C_k] \to 0$ as $k \to \infty$. On the other hand $I[C_o] =$

$$\int_0^1 (o^2 + o^2 + \beta_o^2)\,dt = \beta_o^2 > 0.$$

Let us prove that I has no absolute minimum in the class Ω of all trajectories satisfying the differential equations, boundary conditions, and constraints above. Indeed, $I[C_k] \to 0$ shows that the infimum of $I[C]$ in Ω is zero, but this value cannot be attained by I in Ω. Indeed, $I[C] = 0$ implies $x = 0$, $y = t$, $v = 0$, and the first two relations alone imply $u = \alpha_o$, $v = \beta_o \neq 0$ a.e. in $[0,1]$, a contradiction. Thus I cannot attain the value zero in Ω.

In this example Q is a convex set, f_o is convex in (u,v), and even satisfies trivially the growth condition $f_o \geq \varphi$, since here U is a bounded set. Now let us prove that $\tilde{\tilde{Q}}$ is not convex. It is enough to verify this for $t = 0$, $x = 0$, $y = 0$. Then $\tilde{\tilde{Q}}$ is simply the set of all $z = (z_o, z_1, z_2)$ with $(z_1, z_2) \in Q$ satisfying the relation $z_o \geq f_o = v^2$, when z_1, z_2, u, v are related by $z_1 = f_1$, $z_2 = f_2$, $(u,v) \in U$. Now the segment $\tau = [v = 0, -1 \leq u \leq 1]$ is mapped by f onto the arc $\Gamma = (DEF) \subset Q$, and we have $f_o > 0$ in $Q - \Gamma$, $f_o = 0$ in Γ, and hence $\tilde{\tilde{Q}}$ convex would imply that Γ is a segment, and this is not the case. This proves that $\tilde{\tilde{Q}}$ is not a convex set.

9. ANOTHER EXISTENCE THEOREM FOR LAGRANGE PROBLEMS WITH UNILATERAL CONSTRAINTS

Existence theorem II. Let A be a compact subset of the tx-space $E_1 \times E_n$, and, for every $(t,x) \in A$, let $U(t,x)$ be a closed subset of the u-space E_m. Let $\tilde{f}(t,x,u) = (f_o, f_1, \ldots, f_n) = (f_o, f)$ be a continuous vector function on the set M of all (t,x,u) with $(t,x) \in A$, $u \in U(t,x)$. Assume that, for every $(t,x) \in A$, the set

$$\tilde{\tilde{Q}}(t,x) = [\tilde{z} = (z^o, z) \in E_{n+1} \,|\, z^o \geq f_o(t,x,u),\; z = f(t,x,u),\; u \in U(t,x)]$$

is convex, and that $U(t,x)$ satisfies property (U) and $\tilde{\tilde{Q}}(t,x)$ satisfies property (Q) in A. Let $\varphi(t)$ be a given function which is L-integrable in any finite interval such that $f_o(t,x,u) \geq \varphi(t)$ for all $(t,x,u) \in M$. Let Ω be a nonempty complete class of admissible pairs $x(t)$, $u(t)$ such that

$$\int_{t_1}^{t_2} |dx^i/dt|^p dt \leq N_i, \quad i = 1,\ldots,n, \tag{24}$$

for some constants $N_i \geq 0$, $p > 1$. Then the cost functional $I[u,x]$ has an absolute minimum in Ω.

If A is not compact, but closed and contained in a slab $[t_o \leq T \leq T, -\infty < x^i < +\infty, i = 1,\ldots,n, t_o, T \text{ finite}]$, then theorem II still holds under the additional hypothesis (b) after theorem I. If A is not compact, nor contained in any slab as above, but A is closed, then theorem II still holds under the additional hypotheses (b) and (c*): $f_o(t,x,u) \geq \varphi(t)$ for all $(t,x,u) \in M$ where $\varphi(t)$ is a given function which is L-integrable in any finite interval and $\int_0^{+\infty} \varphi(t)\,dt = +\infty$, $\int_{-\infty}^{0} \varphi(t)\,dt = +\infty$. Finally, if for some $i = 1,\ldots,n$, and any $N > 0$, there is some $N_i > 0$ such that $(x,u) \in \Omega$, $I[x,u] \leq N$ implies $\int_{t_1}^{t_2} |dx^i/dt|^p dt \leq N_i$, then the corresponding requirement (24) can be disregarded.

Proof of existence theorem II. We suppose A compact, hence necessarily contained in a slab $[t_o \leq t \leq T, t_o, T \text{ finite}, -\infty < x^i < +\infty, -i = 1,\ldots,n]$, and then $I[u,x] = \int_{t_1}^{t_2} f_o dt \geq -\int_{t_o}^{T} |\varphi(t)|\,dt$. This proves that the infinum i of $I[u,x]$ in Ω is necessarily finite. Let $u_k(t)$, $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k = 1,2,\ldots$, be a sequence of admissible pairs all in Ω with $I[u_k,x_k] \to i$. We may assume

$$i \leq I[x_k,u_k] = \int_{t_{1k}}^{t_{2k}} f_o(t,x_k(t),u_k(t))\,dt \leq i + 1/k \leq i + 1 . \qquad (25)$$

Then

$$\int_{t_{1k}}^{t_{2k}} |dx_k^i/dt|^p dt \leq N_i, \qquad i = 1,\ldots,n, \qquad k = 1,2,\ldots . \qquad (26)$$

By the weak compactness of L_p we conclude that there is some AC vector function $x(t) = (x^1,\ldots,x^n)$, $t_1 \leq t \leq t_2$, such that $t_{1k} \to t_1$, $t_{2k} \to t_2'$, $dx_k^i/dt \to dx^i/dt$ weakly in L_p, $x_k(t) \to x(t)$ in the ρ-metric. The proof is now exactly the same as for existence theorem I.

If A is <u>not</u> compact, but closed and contained in a slab as above, and condition (b) holds, then for every admissible pair $u(t)$, $x(t)$ of Ω we have

$$|x(t) - x(t^*)| = \left|\int_{t^*}^{t} dx/dt\,dt\right| \leq \left|\int_{t^*}^{t} dt\right|^{1/q} \left|\int_{t^*}^{t} |dx/dt|^p dt\right|^{1/p}$$

$$\leq |t - t^*|^{1/q} (N_1 + \ldots + N_n),$$

where $(t^*,x(t^*))$ belongs to a fixed compace subset P of A. Then $|x(t^*)| \leq N'$, $|t - t^*| \leq T - t_o$, and $|x(t)| \leq N''$ for some constants N', $N'' > 0$. Thus, we can limit ourselves to the compact part A_o of all points (t,x) of A with $t_o \leq t \leq T$, $|x| \leq N''$. If A is not compact, nor contained in any slab as above, but A is closed and conditions (b), (c) hold, then we can use the same argument as for existence theorem I.

Finally, we see that assumption (24) has been used only in (26) for a minimizing sequence u_k,x_k. Since for a minimizing sequence we see already in (25) that $I[u_k,x_k] \leq i + 1$, it is obvious that any relation (24) which is a consequence of a relation of the form $I \leq M$ need not be required among the assumptions of theorem II. Theorem II is thereby proved.

10. EXAMPLES

1. Let us consider the (free) Problem

$$I[x] = \int_{t_1}^{t_2} (1 + |x'|^2)dt = \text{minimum},$$

with $x = (x^1,\dots,x^n)$, in the class Ω of all absolutely continuous functions $x(t) = (x^1,\dots,x^n)$, $0 \leq t \leq t_2$, whose graph $(t,x(t))$ joins the point $(t_1 = 0, x(t_1) = (0,\dots,0))$ to a nonempty closed set B of the half-space $t_2 \geq 0$, $x \epsilon E_n$. This problem can be written as a Lagrange problem:

$$J[x,u] = \int_{t_1}^{t_2} (1 + |u(t)|^2)dt = \text{minimum} ,$$

$$dx^i/dt = u^i, \qquad i = 1,\dots,n ,$$

where $x(t) = (x^1,\dots,x^n)$, $u(t) = (u^1,\dots,u^n)$, $m = n$, $f_0 = 1 + |u|^2$, $f_i = u^i$, $i = 1,\dots,n$, and the control space $U(t,x)$ is fixed and coincides with the whole space E_n. Here $\tilde{\tilde{Q}}(t,x) = [(x,u)\,|\,z \geq 1 + |u|^2, u\epsilon E_n]$ is a fixed and convex subset of E_{n+1}. The conditions of theorem III are satisfied with $g = 1$, $g_0 = 1$, $\Phi(u) = \Phi(|u|) = |u|^2$, or $\Phi(z) = z^2$, $0 \leq z < +\infty$, A is the half space $A = [(t,x)|\ t \geq 0,\ x\epsilon E_n] \subset E_{n+1}$. Thus the problem above has an optimal solution.

2. The free problem

$$I[x] = \int_0^1 tx'^2 dt = \text{minimum}, \; x(0) = 1, \; x(1) = 0 \; ,$$

is known to have no optimal solution [6b]. The same problem can be written as a Lagrange problem with $m = n = 1$ in the form

$$J_1[x,u] = \int_0^1 tu^2 dt = \text{minimum}, \; x(0) = 1, \; x(1) = 0$$

$$dx/dt = u, \qquad u \epsilon E_1,$$

as well as in the form

$$J_2[x,u] = \int_0^1 t^3 u^2 dt = \text{minimum}, \qquad x(0) = 1, \; x(1) = 0 \; ,$$

$$dx/dt = tu \; , \qquad u \epsilon E_1 \; .$$

The relative sets $\approx{Q}(t,x)$ are here subsets of the $z^o z$-plane E_2. For the problem J_1 the sets $\approx{Q}$ satisfy condition (Q), but $f_o = tu^2$ does not satisfy the growth condition of Theorem I. For the problem J_2 the sets Q do not satisfy condition (Q). (We shall take into consideration the same sets under nos. 4 and 5 of Section 14 below).

The same free problem with an additional constraint

$$\int_0^1 x'^2 dt \leq N_o$$

where $N_o \geq 1$ is any constant, has an optimal solution by force of theorem II and subsequent remark. The optimal solution will depend on N_o. Note that $N_o \geq 1$ assures that the class Ω relative to the problem is not empty. Indeed for $x(t) = 1 - t$, we have $\int_0^1 x'^2 dt = 1$.

11. THE FREE PROBLEMS

If we assume $m = n$, $f_i = u_i$, $i = 1,\ldots,n$, $U(t,x) = E_m$, then the differential system reduces to $dx/dt = u$, and the cost functional to

$$I[u,x] = \int_{t_1}^{t_2} f_o(t,x(t),u(t))dt = \int_{t_1}^{t_2} f_o(t,s(t),x'(t))dt \; .$$

Then the problem under consideration in no. 6 reduces to a free problem (no differential system) where the integral is written in the form

$$I[x] = \int_{t_1}^{t_2} f_o(t,x(t),x'(t))dt \; , \qquad (1)$$

and the only constraint is now $(t,x(t)\epsilon A$ for all $t_1 \leq t \leq t_2$. Again, complete classes Ω of vector functions $x(t)$ can be defined by means of boundary conditions of the type $(t_1,x\ t_1)$, t_2, $x(t_2))\epsilon B$, where B is a closed subset of E_{2n+2} as in no. 6.

The Nagumo-Tonelli Theorem. If A is a compact subset of the tx-space $E_1 \times E_n$, if $f_o(t,x,u)$ is a continuous function on the set $M = A \times E_n$, if for every $(t,x)\epsilon A$, $f_o(t,x,u)$ is convex as a function of u in E_n, if there is a continuous scalar function $\varphi(\zeta)$, $0 \leq \zeta < +\infty$, with $\Phi(\zeta)/\zeta \to +\infty$ as $\zeta \to +\infty$, such that $f_o(t,x,u) \geq \Phi(|u|)$ for all $(t,x,u)\epsilon M$, then the cost functional (1) has an absolute minimum in any nonempty complete class Ω of absolutely continuous vector functions $x(t)$, $t_1 \leq t \leq t_2$, for which $f_o(t,z(t),x'(t))$ is L-integrable in $[t_1,t_2]$.

If A is not compact, but closed and contained in a slab $[t_o \leq t \leq T, x\epsilon E_n]$, t_o, T finite, then the statement still holds under the additional hypotheses (τ_1) $f_o \geq C|u|$ for all $(t,x,u)\epsilon M$ with $|x| \geq D$ and convenient constants $C > 0$, $D \geq 0$; (τ_2) every trajectory $x(t)$ of Ω possesses at least one point $(t^*, x(t^*))$ on a given compact subset $\underline{P}$ of A. If A is not compact, nor contained in a slab as above, but A is closed, then the statement still holds under the additional hypotheses (τ_1), (τ_2), and (τ_3) $f_o(t,x,u) \geq \mu > 0$ for all $(t,x,u)\epsilon M$ with $|t| \geq R$, and convenient constants $\mu > 0$ and $R \geq 0$.

Proof. First assume A to be compact. Then the set $\tilde{\tilde{Q}}(t,x)$ reduces here to the set of all $\tilde{z} = (z^o,z)\epsilon E_{n+1}$ with $z^o \geq f_o(t,x,z)$, $z\epsilon E_n$, where f_o is convex in z, and satisfies the growth condition $f_o \geq \varphi(|u|)$ with $\varphi(\zeta)/\zeta \to +\infty$ as $\zeta \to +\infty$. By the remark after lemma (xvi) of I, no. 4, f_o is normally convex in u, hence quasi normally convex, and, by lemma (xvii), part (β), of I, no. 4, $\tilde{\tilde{Q}}_o$ satisfies condition (Q) in A. Thus, all hypotheses of theorem I of no. 7 are satisfied. If A is closed but contained in a slab as above then the condition (a) of theorem I reduces to $u \cdot x = C(|x|^2 + 1)$ which cannot be satisfied since we have no bound on u. On the other hand, the condition (a') $f_o \geq C|f|$ for some $C > 0$ reduces here to requirement (τ_1) and condition (b) to requirement (τ_2). Finally, if A is not compact, nor contained in a slab as above, but A is closed, then requirement (c) of theorem I reduces to requirement (τ_3). All conditions of theorem I are satisfied, and the cost functional (1) has an absolute minimum Ω.

12. LAGRANGE PROBLEMS WITH f LINEAR IN u.

We shall consider now the case where all functions $f_i(t,x,u)$ $i = 1,\dots,n$, are linear in u, and the control space $U(t,x)$ is fixed and coincides with the total space E_m. Precisely, we shall consider the Lagrange problem.

$$I[x,u] = \int_{t_1}^{t_2} [g(t,x)\ \Phi(u) + g_0(t,x)]dt = \text{minimum}, \tag{1}$$

$$dx^i/dt = \sum_{j=1}^{m} g_{ij}(t,x)u^j + g_i(t,x), \quad i = 1,\dots,n, \tag{2}$$

where $x = (x^1,\dots,x^n)\epsilon E_m$, and $\Phi(u)$, $u\epsilon E_m$, is a convex function of u satisfying a growth condition as in Nagumo-Tonelli theorem. If $H(t,x)$ denotes the $n \times m$ matrix $(g_{ij}(t,x))$, and $h(t,x)$ the n-vector $(g_i(t,x))$, then the differential system (2) takes the form

$$dx/dt = H(t,x)u + h(t,x) \ .$$

The sets $Q(t,x)$, $\tilde{\tilde{Q}}(t,x)$ relative to the problem above are

$$Q(t,x) = [z \mid z = H(t,x)u + h(t,x),\ u\epsilon E_m] \subset E_n$$

$$\tilde{\tilde{Q}}(t,x) = [\tilde{z} = (z^0,z) \mid z^0 \geq g(t,x)\Phi(u) + g_0(t,x),\ z = H(t,x)u + h(t,x),\ u\epsilon E_n] \subset E_{n+1} \ . \tag{3}$$

Obviously, $Q(t,x)$ is a r-dimensional linear manifold in E_n where r is the rank of $H(t,x)$. We shall need a few lemmas concerning the sets $Q(t,x)$.

(i) If g is nonnegative, and Φ is nonnegative and convex, then both sets $Q(t,x)$, $\tilde{\tilde{Q}}(t,x)$ defined in (3) are convex for every $(t,x)\in A$.

Proof. We give the proof for $\tilde{\tilde{Q}}(t,x)$. Let $\tilde{\xi} = (\xi^o, \xi)$, $\tilde{\eta} = (\eta^o, \eta)$ be any two points of $\tilde{\tilde{Q}}(t,x)$, let $0 \leq \alpha \leq 1$, and $\tilde{z} = (z^o, z) = \alpha\tilde{\xi} + (1 - \alpha)\tilde{\eta}$. Then for some vectors $u, v \in E_m$ we have

$$\xi^o \geq g\Phi(u) + g_o, \qquad \xi = Hu + h,$$

$$\eta^o \geq g\Phi(v) + g_o, \qquad \eta = Hv + h,$$

$$\tilde{z} = \alpha\tilde{\xi} + (1 - \alpha)\tilde{\eta}, \qquad z^o = \alpha\xi^o + (1 - \alpha)\eta^o, \qquad z = \alpha\xi + (1 - \alpha)\eta .$$

If $w \in E_m$ denote the vector $w = \alpha u + (1 - \alpha)v$, we have

$$z = \alpha\xi + (1 - \alpha)\eta = \alpha(Hu + h) + (1 - \alpha)(Hv + h) =$$

$$= H(\alpha u + (1 - \alpha)v) + h = Hw + h,$$

$$z^o = \alpha\xi^o + (1 - \alpha)\eta^o \geq \alpha(g\Phi(u) + g_o) + (1 - \alpha)(g\Phi(v) + g_o)$$

$$= g(\alpha\Phi(u) + (1 - \alpha)\Phi(v)) + g_o \geq$$

$$\geq g\Phi(\alpha u + (1 - \alpha)v) + g_o = g\Phi(w) + g_o .$$

Thus, $\tilde{z} = (z^o, z) \in \tilde{\tilde{Q}}(t,x)$ and $\tilde{\tilde{Q}}(t,x)$ is convex.

(ii) If all functions Φ, g, g_o, g_{ij}, g_i are continuous, if $\Phi(u)$ is nonnegative and convex, and there is a function $\varphi(\zeta)$, $0 \leq \zeta < +\infty$, such that $\varphi(\zeta) \to +\infty$ as $\zeta \to +\infty$, and $\Phi(u) \geq \varphi(|u|)$ for all $u \in E_m$, if there is a neighborhood $N_\zeta(\bar{t},\bar{x})$ of $(\bar{t},\bar{x})$ where $g \geq \mu$ for some constant $\mu > 0$, then the set $\tilde{\tilde{Q}}(t,x)$ defined in (3) satisfies property (Q) at $(\bar{t},\bar{x})$.

Proof. We have to prove that $\tilde{\tilde{Q}}(\bar{t},\bar{x}) = \cap_\delta \mathrm{cl\ co}\ \tilde{\tilde{Q}}(\bar{t},\bar{x},\delta)$. It is enough to prove that $\cap_\delta \mathrm{cl\ co}\ \tilde{\tilde{Q}}(\bar{t},\bar{x},\delta) \subset \tilde{\tilde{Q}}(\bar{t},\bar{x})$ since the opposite inclusion is trivial. Let us assume that a given point $\tilde{z} = (\bar{z}^o,\bar{z}) \epsilon \cap_\delta \mathrm{cl\ co}\ \tilde{\tilde{Q}}(\bar{t},\bar{x},\delta)$ and let us prove that $\tilde{z} = (\bar{z}^o,\bar{z}) \epsilon \tilde{\tilde{Q}}(\bar{t},\bar{x})$. For every $\delta > 0$ we have $\tilde{z} = (\bar{z}^o,\bar{z}) \epsilon \mathrm{cl\ co}\ \tilde{\tilde{Q}}(\bar{t},\bar{x},\delta)$, and thus, for every $\delta > 0$, there are points $\tilde{z} = (z^o,z) \epsilon\ \mathrm{co}\ \tilde{\tilde{Q}}(\bar{t},\bar{x},\delta)$ at a distance as small as we want from $\tilde{z} = (z^o,z)$. Thus, there is a sequence of points $\tilde{z}_k = (z_k^{\prime},z_k) \epsilon\ \mathrm{co}\ Q(\bar{t},\bar{x},\delta_k)$ and a sequence of numbers $\delta_k > 0$ such that $\delta_k \to 0$, $\tilde{z}_k \to \tilde{z}$. In other words, for every integer k, there are some pair $(t_k^{\prime},x_k^{\prime})$, $(t_k^{\prime\prime},x_k^{\prime\prime})$, corresponding points $\tilde{z}_k^{\prime} = (z_k^{o\prime},z_k^{\prime}) \epsilon Q_k(t_k^{\prime},x_k^{\prime})$, $\tilde{z}_k^{\prime\prime} = (z_k^{o\prime\prime},z_k^{\prime\prime}) \epsilon Q_k(t_k^{\prime\prime},x_k^{\prime\prime})$, points $u_k^{\prime},u_k^{\prime\prime} \epsilon E_m$, and numbers α_k, $0 \leq \alpha_k \leq 1$, such that

$$\tilde{z}_k = \alpha_k \tilde{z}_k^{\prime} + (1 - \alpha_k)\tilde{z}_k^{\prime\prime} ,$$

$$z_k^o = \alpha_k z_k^{o\prime} + (1 - \alpha_k) z_k^{o\prime\prime} , \qquad z_k = \alpha_k z_k^{\prime} + (1 - \alpha_k) z_k^{\prime\prime} ,$$

$$z_k^o \geq g(t_k^{\prime},x_k^{\prime})\Phi(u_k^{\prime}) + g_o(t_k^{\prime},x_k^{\prime}) , \qquad z_k^{\prime} = H(t_k^{\prime},x_k^{\prime})u_k^{\prime} + h(t_k^{\prime},x_k^{\prime}) ,$$

$$z_k^{o\prime\prime} \geq g(t_k^{\prime\prime},x_k^{\prime\prime})\Phi(u_k^{\prime\prime}) + g_o(t_u^{\prime\prime},x_k^{\prime\prime}) , \qquad z_k^{\prime\prime} = H(t_k^{\prime\prime},x_k^{\prime\prime})u_k^{\prime\prime} + h(t_k^{\prime\prime},x_k^{\prime\prime}) , \quad (4)$$

and such that $t_k^{\prime} \to \bar{t}$, $x_k^{\prime} \to \bar{x}$, $t_k^{\prime\prime} \to t$, $x_k^{\prime\prime} \to \bar{x}$, $\tilde{z}_k \to \tilde{z}$, $z_k^o \to \bar{z}^o$, $z_k \to \bar{z}$ as $k \to \infty$. Obviously $g_o(t,x)$ is bounded in $N_\delta(t,x)$, say $g_o(t,x) \geq - G$ for $G \geq 0$.

The second relation (4) shows that of the two numbers $z_k^{o\prime}, z_k^{o\prime\prime}$ one must be $\leq z_k^o$. It is not restrictive to assume that $z_k^{o\prime} \leq z_k^o$ for all k. Then the fourth relation (4) yields

$$z_k^o \geq z_k^{o\prime} \geq g(t_k^{\prime},x_k^{\prime})\Phi(u_k^{\prime}) + g_o(t_k^{\prime},x_k^{\prime})$$

$$\geq \mu\Phi(u_k') - G,$$

where $z_k^o \to z^o$, and hence $[z_k^o]$ is a bounded sequence. This shows that $\Phi(u_k') \leq \mu^{-1}(G + z_k^o)$, hence $[\Phi(u_k)]$ is a bounded sequence, and finally $[u_k']$ is a bounded sequence because of the property of growth of Φ. We can select a subsequence, say still $[u_k']$, which is convergent, say $u_k' \to \bar{u}' \epsilon E_m$ as $k \to \infty$. The sequence $[\alpha_k]$ is also bounded, hence we can further select a subsequence, say still $[\alpha_k]$, for which $[\alpha_k]$ is also convergent. Thus $u_k' \to u'$, $\alpha_k \to \bar{\alpha}$ as $k \to \infty$. Let $u_k \epsilon E_m$ be the point $u_k = \alpha_k u_k' + (1 - \alpha_k)u_k''$. Then

$$\begin{aligned}
z_k &= \alpha_k z_k' + (1 - \alpha_k) z_k'' = \\
&= \alpha_k[H(t_k',x_k')u_k' + h(t_k',x_k')] + (1 - \alpha_k)[H(t_k'',x_k'')u_k'' + h(t_k'',x_k'')] = \\
&= H(t_k'',x_k'')[\alpha_k u_k' + (1 - \alpha_k)u_k''] + h(t_k'',x_k'') + \\
&+ \alpha_k\{[H(t_k',x_k') - H(t_k'',x_k'')]u_k' + [h(t_k',x_k') - h(t_k'',x_k'')]\} \\
&= H(t_k'',x_k'')u_k + h(t_k'',x_k'') + \Delta_k,
\end{aligned} \tag{5}$$

$$\begin{aligned}
z_k^o &\geq \alpha_k z_k^{o\prime} + (1 - \alpha_k) z_k^{o\prime\prime} = \\
&= \alpha_k[g(t_k',x_k')\Phi(u_k') + g_o(t_k',x_k')] + (1 - \alpha_k)[g(t_k'',x_k'')\Phi(u_k'') + g_o(t_k'',x_k'')] \\
&= g(t_k'',x_k'')[\alpha_k\Phi(u_k') + (1 - \alpha_k)\Phi(u_k'')] + g_o(t_k'',x_k'') + \\
&+ \alpha_k\{[g(t_k',x_k') - g(t_k'',x_k'')]\Phi(u_k') + [g_o(t_k',x_k') - g_o(t_k'',x_k'')]\} \\
&\geq g(t_k'',x_k'')\Phi(u_k) + g_o(t_k'',x_k'') + \Delta_k^o.
\end{aligned}$$

Obviously $\Delta_k \to 0$, $\overset{o}{\Delta}_k \to 0$, $h(t_k'', x_k'') \to h(\bar{t}, \bar{x})$, $g_o(t_k'', x_k'') \to g_o(\bar{t}, \bar{x})$. Since $g(t_k'', x_k'') \geq \mu$, we conclude as before that $[\Phi(u_k)]$ is a bounded sequence, and so is $[u_k]$, hence we can further select a convergent subsequence, say still $[u_k]$, with $u_k \to \bar{u}$. Relations (5) yield now as $k \to \infty$,

$$\bar{z} = H(\bar{t}, \bar{x})\bar{u} + h(\bar{t}, \bar{x}) ,$$

$$z^o \geq g(\bar{t}, \bar{x})\Phi(\bar{u}) + g_o(\bar{t}, \bar{x}) .$$

Thus, $\tilde{z} = (z^o, z) \in \tilde{\tilde{Q}}(\bar{t}, \bar{x})$, and statement (ii) is proved.

<u>Remark.</u> Here are a few examples of linear problems and corresponding sets $Q(t,x)$ and $\tilde{\tilde{Q}}(t,x)$.

1. Take $m = 1$, $n = 2$, $U = E_1$, let $u \in E_1$ be the control variable, and take $\Phi(u) = 1$, $g = 1$, $g_o = 0$, $g_{11} = 1$, $g_{12} = g_{21} = 0$, $g_{22} = t$. Then the sets Q and $\tilde{\tilde{Q}}$ depend on t, $-1 \leq t < +1$, and

$$Q(t) = [z = (z^1, z^2) \mid z^1 = u,\ z^2 = tu,\ -\infty < u < +\infty] =$$

$$= [z = (z^1, z^2) \mid z^2 = tz^1,\ -\infty < z^1 < +\infty] \subset E_2 ,$$

$$\tilde{\tilde{Q}}(t) = [\tilde{z} = (z^o, z^1, z^2) \mid z^o \geq 1,\ z^2 = tz^1,\ -\infty < z^1 < +\infty] \subset E_3 .$$

Each set $Q(t)$ is a straight line in E_2 of slope t, and for each $\zeta > 0$, the set $Q(0,\delta)$ contains both lines $z^2 = \pm\delta z^1$, and the convex hul of $Q(0,\delta)$ coincides with the whole plane E_2. Thus $Q(0)$ is the z^1-axis and $\bigcap_\delta \mathrm{cl\,co}\, Q(0,\delta)$ is the whole z^1z^2-plane. The set $Q(t)$ does not satisfy property (Q) at $t = 0$, and the same holds for $\tilde{\tilde{Q}}(t)$. Here $\Phi = 1$ does not satisfy the growth condition requested in (ii).

2. Take $m = 1$, $n = 2$, $U = E_1$, let $u \in E_1$ be the control variable, and take $\Phi(u) = |u|$, $g = |t|$, $g_0 = 0$, $g_{11} = 1$, $g_{12} = g_{21} = 0$, $g_{22} = t$. Then again the sets Q and $\tilde{\tilde{Q}}$ depend on t only, $g\Phi = |tu| = z^2$,

$$Q(t) = [z = (z^1,z^2) \,|\, z^2 = tz^1, \; -\infty < z^1 < +\infty] \subset E_2 ,$$

$$\tilde{\tilde{Q}}(t) = [\tilde{z} = (z^0,z^1,z^2) \,|\, z^0 \geq |z_2|, \; t_2 = tz^1, \; -\infty < t < +\infty] \subset E_3 .$$

As before, the set $Q(t)$ does not satisfy property (Q) at $t = 0$. Analogously, for any $\zeta > 0$, and $-\zeta \leq t \leq \zeta$, we see that

$$\tilde{z}' = (z^{0\prime},z^{1\prime},z^{2\prime}) = (1,\delta^{-1},1) \in \tilde{\tilde{Q}}(\delta) ,$$

$$\tilde{z}'' = (z^{0\prime\prime},z^{1\prime\prime},z^{2\prime\prime}) = (1,-\delta^{-1},1) \in \tilde{\tilde{Q}}(-\delta) ,$$

and, for $\alpha = 1/2$, also

$$\tilde{z} = \alpha\tilde{z}' + (1 - \alpha)\tilde{z}'' = (z^0,z^1,z^2) = (1,0,1) \in \text{co } \tilde{\tilde{Q}}(0,\delta) .$$

Hence,

$$\tilde{z} = (1,0,1) \in \cap_\delta \text{clco } Q(0,\text{co}), \qquad z = (1,0,1) \notin \tilde{\tilde{Q}}(0),$$

and $\tilde{\tilde{Q}}(t)$ does not satisfy property (Q) at $t = 0$. Here g does not satisfy the condition $g \geq \mu > 0$ requested in (ii).

3. Take $m = 1$, $n = 2$, $U = E_1$, let $u \in E_1$ be the control variable, and take $\Phi(u) = |u|$, $g = 1$, $g_0 = 0$, $g_{11} = 1$, $g_{12} = g_{21} = 0$, $g_{22} = t$. Then again the sets Q and $\tilde{\tilde{Q}}$ depend on t only, and

$$Q(t) = [z = (z^1,z^2) \,|\, z^2 = tz^1, \; -\infty < z^1 < +\infty] \subset E_2 ,$$

$$\tilde{\tilde{Q}}(t) = [\tilde{z} = (z^0, z^1, z^2) | z^0 \geq |z^1|,\ z^2 = tz^1,\ -\infty < z^1 < +\infty] \subset E_3 .$$

As before, $Q(t)$ does not satisfy property (Q), while $\tilde{\tilde{Q}}(t)$ does satisfy property (Q) at every t because of statement (ii).

4. Take $m = n = 1$, $U = E_1$, let $u \epsilon E_1$ be the control variable, and take $\Phi(u) = u^2$, $g = t$, $g_0 = 0$, $g_{11} = t$, $G_1 = 0$. Then

$$Q(t) = [z | z = tu,\ -\infty < u < +\infty] \subset E_1 ,$$

$$\tilde{\tilde{Q}}(t) = [\tilde{z} = (z^0, z) | z^0 \geq tu^2,\ z = tu,\ -\infty < u < +\infty] \subset E_2 .$$

How $Q(0)$ is reduced to the single point $z = 0$, while $Q(t)$ for every $t \pm 0$ coincides with E_1. Thus $Q(t)$ does not satisfy roperty (Q) at $t = 0$. On the other hand $\tilde{\tilde{Q}}(0)$ is the half straight line $[z^0 \geq 0,\ z^1 = 0]$, while $\tilde{\tilde{Q}}(t)$ for $t \neq 0$ is the set $Q(t) = [z^0 \geq t^{-1}z^2,\ -\infty < z < +\infty]$. Obviously, $\tilde{\tilde{Q}}$ satisfies property (Q) at $t = 0$ (and at every t as well).

5. Take $m = n = 1$, $U = E_1$, let $u \epsilon E_1$ be the control variable, and take $\Phi(u) = u^2$, $g = t^3$, $g_0 = 0$, $g_{11} = t$. Then

$$Q(t) = [z | z = tu,\ -\infty < u < +\infty] \subset E_1 ,$$

$$\tilde{\tilde{Q}}(t) = [z = (z^0, z) | z^0 \geq t^3u^2,\ z = tu,\ -\infty < u < +\infty] \subset E_2 .$$

Here $Q(t)$ as in no. 4 does not satisfy property (Q) at $t = 0$. Also $\tilde{\tilde{Q}}(0) = [z^0 \geq 0,\ z^1 = 0]$ while $\tilde{\tilde{Q}}(t)$ for $t \neq 0$ is the set $\tilde{\tilde{Q}}(t) = [z^0 \geq tz^2,\ -\infty < z < +\infty]$, and cl co $\tilde{\tilde{Q}}(0, \delta)$ is the entire half plane $[z^0 \geq 0,\ -\infty < z^1 < +\infty]$. Thus, neither Q nor $\tilde{\tilde{Q}}$ satisfy property (Q) at $t = 0$.

We shall denote by $r(t,x)$ the rank of the $n \times m$ matrix $(g_{ij}(t,x))$. Then $0 \leq r(t,x) \leq \min [m,n]$.

(iii) If all functions $g_{ij}(t,x)$ are continuous, then $r(\bar{t},\bar{x}) \leq \underline{\lim}\, r(t,x)$ as $(t,x) \to (\bar{t},\bar{x})$.

The proof is a straightforward consequence of the continuity hypotheses. The statement below shows that a necessary condition for $Q(t,x)$ to satisfy (Q) at $(\bar{t},\bar{x})$ is that $r(t,x)$ is constant in a neighborhood of $(\bar{t},\bar{x})$, and this explains why the set Q of the examples 4 and 5 does not satisfy property (Q). On the other hand, the condition is not sufficient, as the sets Q of the examples 1, 2, 3 show since in these examples $r = 1$ is constant.

(iv) If all functions g_{ij}, g_i are continuous in A, then a necessary condition in order that the set $Q(t,x)$ satisfies condition (Q) at $(\bar{t},\bar{x})$ is that $r(\bar{t},\bar{x}) = \lim r(t,x)$ as $(t,x) \to (\bar{t},\bar{x})$ (thus, there is a neighborhood $N_\delta(\bar{t},\bar{x})$ of $(\bar{t},\bar{x})$ with $r(t,x) = r(\bar{t},\bar{x})$ for every $(t,x) \in N_\delta(\bar{t},\bar{x})$. If $Q(t,x)$ satisfies condition (Q) in A, then $r(t,x)$ is a constant.

Proof. Suppose that $r(\bar{t},\bar{x}) = r$ is not the limit of the (integral-valued) function $r(t,x)$ as $(t,x) \to (\bar{t},\bar{x})$. Since $r(\bar{t},\bar{x}) = r \leq \underline{\lim}\, r(t,x)$ we must have $r(\bar{t},\bar{x}) = r < r + 1 \leq \overline{\lim}\, r(t,x)$. There is, therefore, a sequence (t_k,x_k), $k = 1,2,\ldots$, with $t_k \to \bar{t}$, $x_k \to \bar{x}$, and $r + 1 \leq r_k = r(t_k,x_k) \leq \min [m,n]$. The image of $U = E_m$ under the mappings $H(t_k,x_k)u + h(t_k,x_k)$ and $H(\bar{t},\bar{x})u + h(\bar{t},\bar{x})$ are, therefore, linear manifolds of E_n, say $Q(t_k,x_k)$ of dimensions $r_k \geq r + 1$, and $Q(\bar{t},\bar{x})$ of dimension r. The images of $u = 0$ in $Q_k(t_k,x_k)$ and $Q(\bar{t},\bar{x})$ are the points $z_k = h(t_k,x_k)$, $\bar{z} = h(\bar{t},\bar{x})$. Let $\eta_1,\ldots,\eta_r$ be r orthonormal vectors in E_n such that

$$Q(t,x) = [z\epsilon E_n | z = \bar{z} + \xi_1\eta_1+\ldots+\xi_r\eta_r,\ \xi_1\ldots,\xi_r \text{ real}] ,$$

and let us complete $\eta_1,\ldots,\eta_r$ into a system of n orthonormal vectors $\eta_1,\ldots,\eta_r, \eta_{r+1},\ldots,\eta_n$. For every k, there are systems of r_k orthonormal vectors $\eta_{1k},\ldots,\eta'_{r_k k}$ of E_n such that

$$Q(t_k,x_k) = [z\epsilon E\ | z = z_k + \xi_1\eta'_{1k} +\ldots+ \xi_{r_k}\eta'_{r_k k},\ \xi_1,\ldots,\xi_{r_k} \text{ real}].$$

Since $h(t_k,x_k) \to h(\bar{t},\bar{x})$, $H(t_k,x_k) \to H(\bar{t},\bar{x})$, we can select $\eta'_{1k},\ldots,\eta'_{r_k k}$ so that, together with $z_k \to \bar{z}$, we have also

$$\eta'_{ik}\cdot\eta_i \to 1,\ i = 1,\ldots,r ,$$

$$\eta'_{jk}\cdot\eta_i \to 0,\ j \neq i,\ j = 1,\ldots,r_k \text{ as } k \to \infty .$$

If we take $\xi_1 = \ldots = \xi_r = 0$, $\xi_{r+1} = 1$, $\xi_{r+2} = \ldots = \xi_n = 0$, then the point $z'_k = z_k + \eta_{z+1,k}\ \epsilon Q(t_k,x_k)$. It is not restrictive to assume that for all k we have

$$|z_k - \bar{z}| < 1/4,\ |\eta_{jk}\cdot\eta_i| < 1/4n,\ j \neq i,\ j = 1,\ldots,r_k .$$

Then

$$\eta_{r+1,k} = \sum_{i=1}^{n} (\eta_{r+1,k}\cdot\eta_i)\eta_i = \left(\sum_{i=1}^{r} + \sum_{i=r+1}^{n}\right) (\eta_{r+1,k}\cdot\eta_i)\ \eta_i = \eta' + \eta'',$$

$$|\eta'| = |\sum_{i=1}^{r} (\eta_{r+1,k}\cdot\eta_i)\eta_i| \leq \sum_{i=1}^{r}|\eta_{r+1,k}\cdot\eta_i| \leq r(1/4n) \leq 1/4 ,$$

$$|\eta''| = |\eta_{r+1,k} - \eta'| \geq |\eta_{r+1,k}|-|\eta'| \geq 1 - 1/4 = 3/4 .$$

Finally,

$$|z_k' - z| = |(z_k + \eta_{r+1,k}) - \bar{z}| \leq |\eta_{r+1,k}| + |z_k - \bar{z}| \leq 1 + 1/2 = 3/2 ,$$

and, for every $z \in Q(\bar{t},\bar{x})$, also

$$|z_k' - z| = |(z_k + \eta_{r+1,k}) - (\bar{z} + \xi_1\eta_1 + \ldots + \xi_r\eta_r)|$$

$$= |\sum_{i=1}^{n} (\eta_{r+1,k}\cdot\eta_i)\eta_i + \sum_{i=1}^{n} \xi_i\eta_i + (z_k - \bar{z})|$$

$$= |\sum_{i=1}^{r} (\eta_{r+1,k}\cdot\eta_i - \xi_i)\eta_i + \sum_{i=r+1}^{n} (\eta_{r+1,k}\cdot\eta_i)\eta_i| - |z_k - \bar{z}|$$

$$\geq |\sum_{i=r+1}^{n} (\eta_{r+1}\cdot\eta_i)\eta_i| - |z_k - \bar{z}| = \eta'' - |z_k - \bar{z}|$$

$$\geq 3/4 - 1/4 + 1/2 .$$

Thus,

$$|z_k' - \bar{z}| \leq 3/2 , \qquad \text{dist } (z_k', Q(\bar{t},\bar{x})) \geq 1/2 .$$

The sequence $[z_k']$ is bounded, hence, it contains a convergent sequence, say still $[z_k']$ with $z_k' \to z' \in E_n$, and

$$|z' - z| \leq 3/4 , \qquad \text{dist } (z', Q(t,x)) \geq 1/2 .$$

Finally, for every k there is a $u_k \in U = E_m$ such that $z_k' = H(t_k,x_k)u_k + h(t_k,x_k)$, or $z_k' \in Q(t_k,x_k)$, with $z_k \to z'$. Then $z' \in \text{cl co } Q(\bar{t},\bar{x},\delta)$ for every $\delta > 0$, and hence

$$z' \in \cap_\delta \text{cl co } Q(\bar{t},\bar{x},\delta), \qquad z' \notin Q(\bar{t},\bar{x}).$$

We have proved that $Q(t,x)$ does not satisfy property (Q) at $(\bar{t},\bar{x})$, a contradiction. This proves that, if $Q(t,x)$ satisfies property (Q) at $(\bar{t},\bar{x})$, then $r(\bar{t},\bar{x}) = \lim r(t,x)$ as $(t,x) \to (\bar{t},\bar{x})$. The necessity of the condition is thereby proved.

13. EXISTENCE THEOREMS FOR LAGRANGE PROBLEMS WITH f LINEAR IN u.

Existence Theorem III. Let us consider the Lagrange problem

$$I[x,u] = \int_{t_1}^{t_2} [g(t,x)\Phi(u) + g_0(t,x)]dt = \text{minimum}, \tag{1}$$

$$dx^i/dt = \sum_{j=1}^{m} g_{ij}(t,x)u^j + g_i(t,x),\ i = 1,\dots,n, \tag{2}$$

where $x = (x^1,\dots,x^n)\epsilon E_n$, $u = (u^1,\dots,u^m)\epsilon E_m$, and $\Phi(u)$ is a continuous nonnegative convex function of u. Assume that there is some continuous function $\varphi(\zeta)$, $0 \leq \zeta < +\infty$, with $\varphi(\zeta)/\zeta \to +\infty$ as $\zeta \to +\infty$ and $\Phi(u) \geq \varphi(|u|)$ for every $u\epsilon E_m$. Assume that all functions $g(t,x)$, $g_0(t,x)$, $g_{ij}(t,x)$, $g_i(t,x)$ are continuous in $A = E_1 \times E_n$, and that

$$g \geq \mu > 0,\ g_0 \geq \mu > 0, \sum_{ij} |g_{ij}| \leq Cg, \sum_{ij} |g_{ij}| + \sum_i |g_i| \leq Cg,$$

for some constants $\mu > 0$, $C \geq 0$, and all $(t,x)\epsilon A$. Let Ω be the class of all pairs $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, $x(t)$ absolutely continuous, $u(t)$ measurable, satisfying (2) a.e., and such that the graph $(t,x(t))$ joins the fixed point $(t_1 = 0,\ x(t_1) = (0,\dots,0))\epsilon A$ to a given closed subset B of the half-space $t \geq 0$, $x\epsilon E_n$ in A. Then the Lagrange problem (1), (2) has an optimal solution in Ω.

The functions $\Phi(u) = \varphi(|u|) = |u|^p$, $u \in E_m$, $p > 1$, as well as $\Phi_c(u) = \varphi(|u|) = 0$ for $|u| \leq C$, $\Phi_c(u) = \varphi(|u|) = |u|^p - C^p$ for $|u| \geq C$, certainly satisfy the requirement for Φ.

Proof. By lemmas (i) and (ii) of no. 14 the set $\approx{Q}(t,x)$ is convex for every (t,x) and satisfies condition (Q) in A. The set $U = E_m$ is fixed, closed, and obviously satisfies condition (U). Also $f_o(t,x,u) = g(t,x)\Phi(u) + g_o(t,x)$ and hence $f_o \geq \mu\Phi(u) \geq \mu\varphi(|u|)$, $f_o \geq g_o \geq \mu$, where $\mu > 0$, and hence both the growth condition for f_o and the conditions (c) and (d) of Theorem I are satisfied. Now if A_o is any compact subset of $A = E_1 \times E_n$, then the continuous functions g_{ij}, g_i are bounded in A_o, say $|g_{ij}| \leq C_o$, $|g_i| \leq C_o$, (where C_o depends on A_o), and

$$|f| = |Hu + h| \leq |H||u| + |h| \leq n^2C_o|u| + nC_o$$

for all $(t,x) \in A_o$. Thus condition (β) of Theorem I is also satisfied. Condition (b) is satisfied since the initial point $(t_1, x(t_1))$ is fixed. Let us prove that condition (a') is satisfied. Indeed $\varphi(\zeta)/\zeta \to +\infty$ as $\zeta \to +\infty$, hence $\varphi(\zeta)/\zeta \geq 1$ for all $|\zeta| \geq D$ and some constant $D \geq 0$. Then for $|u| \geq D$ we have $|u| \leq \varphi(|u|)$, and hence $|u| \leq D + \varphi(|u|)$ for all $u \in E_m$. Now for all $(t,x) \in A = E_1 \times E_n$ and $u \in E_m$ we have

$$\begin{aligned} |f| &= |Hu + h| \leq |H||u| + |h| \leq |H|(D + \varphi(|u|)) + |h| \\ &= |H|\,\varphi(|u|) + (D|H| + |h|) \leq \\ &\leq Cg\Phi(u) + (D + 1)Cg_o = \\ &\leq C(D + 1)(g\Phi(u) + g_o) = C(D + 1)f_o \,. \end{aligned}$$

Thus $f_o \geq C^{-1}(D + 1)^{-1} |f|$ for all $(t,x,u) \in E_1 \times E_n \times E_m$. All conditions of theorem I are satisfied, and the Lagrange problem I has an optimal solution.

Existence Theorem IV. Let us consider the functional

$$I[x,u] = \int_{t_1}^{t_2} [g(t,x)\Phi(u) + g_o(t,x)]dt \tag{1}$$

with differential equations

$$dx^i/dt = \sum_j g_{ij}(t,x)u^i + g_i(t,x), \; i = 1,\ldots,n, \tag{2}$$

or

$$dx/dt = H(t,x)u + h(t,x),$$

where $x = (x^1,\ldots,x^n) \in E_n$, $u = (u^1,\ldots,u^m) \in U = E_m$, where H is the n x m matrix (g_{ij}), where h is the n-vector (g_i), and where $\Phi(u)$ is a continuous nonnegative convex function of u. Assume that all functions $g(t,x)$, $g_o(t,x)$, $g_{ij}(t,x)$, $g_i(t,x)$ are continuous in $A = E_1 \times E_n$, and that

$$g(t,x) \geq 0, \; g_o(t,x) \geq -G_o \text{ for all } (t,x) \in A = E_1 \times E_n ,$$

$$g_o(t,x) \geq \mu > 0 \text{ for all } (t,x) \in A_2 E_1 \times E_n \text{ with } |t| \geq D_o ,$$

for some constants $\mu > 0$, $G_o \geq 0$, $D_o \geq 0$. Assume that the (convex) set

$$\tilde{\tilde{Q}}(t,x) = [\tilde{z} = (z^o,z) \,|\, z^o \geq g\Phi(u) + g_o, \; z = Hu + h, \; u \in U = E_m] \subset E_{n+1}$$

satisfies condition (Q) in A. Let Ω be the class of all pairs $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, $x(t)$ absolutely continuous, $u(t)$ measurable, satisfying (2) a.e., and such that the graph $(t,x(t))$ joins the fixed point $(t_1 = 0, x(t_1) =$

$(0,\ldots,0)) \epsilon E_1 \times E_n$ to a given closed subset B of the half-space $t \geq 0$, $x \epsilon E_n$ in $E_1 \times E_n$ and such that

$$\int_{t_1}^{t_2} |dx^i/dt|^p dt \leq N_i, \quad i = 1,\ldots,n, \tag{6}$$

for some constants $p > 1$, $N_i \geq 0$. If Ω is not empty then the Lagrange problem above has an optimal solution in Ω.

The functions $\Phi_c(u) = 0$ for $|u|$, $\Phi(u) = |u|^p$, $p \geq 1$, as well as $\Phi_c(u) = 0$ for $|u| \leq c$, $\Phi(u) = |u|^p - c^p$ for $|u| \geq c$, $p \geq 1$, all satisfy the requirements above for Φ. The requirement $g_o \geq \mu > 0$ can be disregarded if B is contained in a slab $[0 \leq t < T, x \epsilon E_n]$, T finite. Any requirement (6) which is consequence of a relation $\int_{t_1}^{t_2} (g\Phi + g_o)dt \leq N_o$ can be disregarded.

Proof. By (i) of no. 14 the set $\tilde{\tilde{Q}}(t,x)$ is convex for every (t,x) in A. All conditions of Theorem II of no. 7 are satisfied, and thus IV is a corollary of II.

Remark. The requirement concerning $\tilde{\tilde{Q}}(t,x)$ of Theorem IV is certainly satisfied if we assume that

(α) $g(t,x) \geq \mu > 0$ for all $(t,x) \epsilon A = E_1 \times E_n$,

(β) there exists a nonnegative convex function $\varphi(\zeta)$, $0 \leq \zeta < +\infty$, with $\varphi(\zeta) \to +\infty$ as $\zeta \to +\infty$ and $\Phi(u) \geq \varphi(|u|)$ for all $u \epsilon U < E_m$.

Indeed, by statements (i) (ii) of no. 14 the convex set $\tilde{\tilde{Q}}(t,x)$ satisfies property (Q) in A.

Content. 1. The problem. I, p.3. - 2. The space of continuous vector

REFERENCES

1. L. Cesari, (a) Semicontinuità e convessità nel calcolo delle variazioni Annali Scuola Normale Sup. Pisa, 14, 1964, 389-423. (b) Un teorema di esistenza in problemi di controlli ottimi. Ibid., 1965, 35-78. (c) An existence theorem in problems of optimal control, J. SIAM Control, to appear. (d) Existence theorems in problems of Lagrange and optimal control. J. SIAM Control, to appear. (e) Existence Theorems for weak and usual optimal solutions in Lagrange problems with unilateral constraints. I: Closure theorems. To appear.

2. A. F. Filippov, On certain questions in the theory of optimal control, Vestnik Moskow. Univ. Ser. Mat. Mech. Astr. 2, 1959, 25-32 (Russian). English translation in J. SIAM Control, (A) 1, 1962, 76-84.

3. R. V. Gamkrelidze, On sliding optimal regimes. Dokl. Akad. Nauk SSSR 143, 1962, 1243-1245(Russian). English translation in Soviet Math. Doklady 3, 1962, 390-395.

4. L. Markus and E. B. Lee, Optimal control for nonlinear processes, Arch. Rational Mech. Anal. 8, 1961, 36-58.

5. M. Naguno, Uber die gleichmässige Summierbarkeit und ihre Anwendung auf ein Variation Problem. Japanese Journ. Math. 6, 1929, 178-182.

6. L. S. Pontryagin, Optimal control processes. Uspekhi Mat. Nauk 14, 1(85), 1959, 3-20. English translation in Automation Express 2, no. 1, 1959, 26-30.

7. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, The Mathematical Theory of Optimal Processes, Gosudarst, Moscow 1961. English translations: Interscience 1962; Pergamon Press 1964.

8. E. Roxin, The existence of optimal controls, Mech. Math. J. 9, 1962, 103-119.

9. L. Tonelli, (a) Sugli integrali del calcolo delle variazioni in forma ordinaria, Annali Scuola Normale Sup. Pisa (2) 3, 1934, 401-450. Opere scelte, Cremonese, Roma 1962, 3, 192-254; (b) Un teorema di calcolo delle variazioni, Zanichelli, Bologna 1921-23, 2 vols.

10. L. Turner, The direct method in the calculus of variations, A Ph.D. thesis at Purdue University, Lafayette, Indiana, August 1957.