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FUNCTIONAL ANALYSIS AND DIFFERENTIAL EQUATIONS

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1. THE SOLUTIONS OF AN EQUATION AS FIXED POINTS OF A SUITABLE MAP

We are concerning ourselves with an equation

$$Kz = 0 \qquad (1)$$

whose solutions are to be found in a Banach space S (or in a subset G of S), and we consider a projection operator P in S. If S_0 is the range of P, and S_1 the nullspace of P, then $S = S_0+S_1$, and every element $z \in S$ has the representation $z = (x,y)$, or $z = x+y$, with $x = Pz$, $y = (I-P)z$, I the identity operator in S. Obviously, equation (1) decomposes into the system of the two equations

$$PKz = 0 \quad , \quad (I-P)Kz = 0 . \qquad (2)$$

If D_0 denotes the projection of G on S_0, $D_0 = PG \subset S_0$, then we are seeking pairs $z = (x,y)$ with $x \in D_0$ and $z \in P^{-1}x$, such that $z = (x,y) \in G$, and z satisfies (2). We shall denote by $\|z\|$ the norm of z in S.

As in [1d] we shall assume $K = E-N$, where $E: S_E \to S$ is a linear operator nonnecessarily bounded with domain $S_E \subset S$, $N: S_N \to S$ an operator nonnecessarily linear with domain $S_N \subset S$, so that $K: S_E \cap S_N \to S$. Then, the given equation (1) takes the form

$$Ez = Nz ,$$

and the equivalent system (2) the form

$$PEz = PNz \quad , \quad (I-P)Ez = (I-P)Nz .$$

We may think of E as a linear differential operator and of S_E as the set of those elements $z \in S$ sufficiently smooth so that E can be applied.

We shall assume that a linear operator $H = S_1 \to S_1$ is known to exist, for which we require for a moment that

$$H(I-P)Ez = (I-P)z \text{ for all } z \in S_E .$$

In other words, we require here that $H: S_1 \to S_1$ is a partial left inverse of E. Then, for any solution z of $Kz = 0$, or $Ez = Nz$, we have

$$H(I-P)Ez = H(I-P)Nz ,$$

$$(I-P)z = H(I-P)Nz ,$$

or, finally,

$$z = Pz + H(I-P)Nz . \qquad (3)$$

In other words, any solution z of (1) is a fixed point $z = Tz$ of the map $T: S_N \to S$ defined by $T = P+H(I-P)N$, or $T = P+F$ with $F = H(I-P)N$.

Here the operator T has a relevant property, indeed T maps each fiber of P into itself, precisely

$$PTz = Pz \text{ for every } z \in S_N, \text{ or}$$

$$T: (P^{-1}x) \cap S_N \to P^{-1}x .$$

Indeed, for $z \in S_N$, we have

$$PTz = PPz + PH(I-P)Nz = Pz,$$

since P is idenpotent, and $H(I-P)Nz \in S_1$, the nullspace of P.

2. THE RELAXED EQUATION AND THE DETERMINING EQUATION

To simplify the exposition, let us assume that $G = D_0 \times D_1$, where both $D_0 \subset S_0$ and $D_1 \subset S_1$ are spheres, and that $G \subset S_N$.

If it happens that T maps each set $P^{-1}x \cap G$ into itself, and we know that $P^{-1}x \cap G$ is compact (or that $P^{-1}x \cap G$ is complete, and $T(P^{-1}x \cap G)$ is compact), then by Schauder's theorem there is at least one fixed point $z = Tz \in P^{-1}x \cap G$ for every $x \in D_0$.

Actually, by a suitable choice of P it is possible to make T a contraction in the sense that $\|Tz-Tz'\| \leq k\|z-z'\|$ for any two points $z, z' \in P^{-1}x \cap G$ of the same fiber of P, with $k < 1$ and $x \in D_0$ (see [1d], or no. 4 below). It should be noted that $T: P^{-1}x \cap G \to S$ is a contraction if and only if F is contraction. Indeed, if $z, z' \in P^{-1}x \cap G$, then $Tz = Pz+Fz$, $Tz' = Pz'+Fz'$, $Pz = Pz' = x$, and hence

$$\|Tz-Tz'\| = \|Fz-Fz'\| .$$

Whenever $T: P^{-1}x \cap G \to P^{-1}x \cap G$ is a contraction and into for every $x \in D_0$, then there is one and only one fixed point $z = Tz \in P^{-1}x \cap G$ for each $x \in D_0$, and we may well consider the map $\mathcal{T}: D_0 \to G$ which maps every $x \in D_0$ into the unique fixed element $z = Tz$ of T in $P^{-1}x \cap G$. The graph of $\mathcal{T}$ is a cross section of the fibers of P, and $\mathcal{T}$ itself can be thought of as a "lifting" operation from S_0 to S, namely, from D_0 to G_0.

The map $\mathcal{T}: D_0 \to G$ defined above is continuous. Indeed, for every two points $x, x' \in D_0$ and corresponding points $z = Tz = \ x$, $z' = Tz' = \ x'$, we have $z \in P^{-1}x$, $z' \in P^{-1}x'$, and

$$\|Fz-Fz'\| = \|Tz-Tz'\| \leq k\|z-z'\| \quad , \quad k < 1 ,$$

with $T = P+F$, and hence

$$\|z-z'\| = \|Tz-Tz'\| \leq \|Pz-Pz'\| + \|Fz-Fz'\| \leq \| x-x'\| + k\|z-z'\|$$

and

$$\|\mathcal{T}x-\mathcal{T}x'\| = \|z-z'\| \leq (1-k)^{-1}\|x-x'\| . \tag{5}$$

Let us assume for a moment that every fixed element $z = Tz$ of T satisfies

$$z \in S_E \quad , \quad PEz = EPz \quad , \quad EH(I-P)Nz = (I-P)Nz , \tag{6}$$

in other words, any such element is reasonably smooth, and H has also the property to be a right partial inverse of E as stated by (7). Then every fixed point $z = Tz$ of T satisfies the equation $(I-P)Kz = 0$, that is, the second of the two equations (2). Indeed

$$\begin{aligned}(I-P)Kz &= (I-P)(E-N)z = \\ &= Ez - PEz - (I-P)Nz \\ &= Ez - EPz - EH(I-P)Nz \\ &= E[z-Tz] = 0 .\end{aligned} \tag{7}$$

Thus, in case $T\colon P^{-1}x \cap G \to P^{-1}x \cap G$ is a contraction, then for every $x \in D_0$ the element $z = \mathcal{T}x$ of G satisfies the relaxed equation

$$Kz = PKz \quad , \quad \text{or } Kz = PK\mathcal{T}x . \tag{8}$$

Indeed

$$Kz = PKz + (I-P)Kz = PKz = PK\mathcal{T}x .$$

In other words, for every $x \in D_0$ there is an element $z = \mathcal{T}x$ of G, namely $z = Tz = \mathcal{T}x \in P^{-1}x \cap G$, satisfying $Kz = PKz$. Then, $z = \mathcal{T}x$ satisfies $Kz = 0$ if and only if the determining equation

$$PK\mathcal{T}z = 0 \tag{9}$$

is satisfied. This shows that the search for solutions $z \in G$ of $Kz = 0$ is reduced to the search for elements $x \in D_0 \subset S_0$ such that (9) is satisfied. Equation (9) was first encountered in perturbation problems for periodic solutions, or cycles, of ordinary differential equations, where it gave a new interpretation of Poincaré's bifurcation equation. Equation (9) is thus a functional analysis extension of Poincaré's bifurcation equation.

Often S_0 is one dimensional and D_0 is an interval, and then the equation $PK\mathcal{T}x = 0$ has certainly a solution in D_0 as soon as we know that, say $PK\mathcal{T}x$ has opposite signs at the end points of D_0.

Often S_0 is a finite dimensional space, and D_0 is a cell. Then the equation $PK\mathcal{T}x = 0$ has certainly a solution in D_0 as soon as we know that the topological degree $d(PK\mathcal{T}, D_0, 0)$ is nonzero.

Even if S_0 is infinite dimensional, as it does occur, the study of the determining equation has led to simple criteria for the existence of solutions to the original equation.

The method briefly summarized above first appeared in [1d,e] with details and applications, and was then reported in [3] and [5b].

3. A CONNECTION WITH GALERKIN'S APPROXIMATIONS

If S_o is finite dimensional, and $[\phi_1,\ldots,\phi_m]$ is a base for S_o, then

$$PK\mathcal{T}x = c_1(x)\phi_1 + \ldots + c_m(x)\phi_m ,$$

where the coefficients $c_1,\ldots,c_m$ depend on $x \in D_o$, and the determining equation $PK\mathcal{T}x = 0$ can be written in the form of the m equations

$$c_1(x) = 0 ,\ldots, c_m(x) = 0 , \quad x \in D_o . \tag{10}$$

These are similar to the equations for an m^{th} Galerkin approximate solution, with the difference that equations (10) yield an exact solution $z = \mathcal{T}x$ to the given equation $Kz = 0$.

Actually we may well expect that the exact solution $z = \mathcal{T}x$ is "close" to an m^{th} Galerkin approximation $x^{(m)} \in S_o$, or $x^{(m)} = \gamma_1(x)\phi_1 + \ldots + \gamma_m(x)\phi_m$. If we write

$$PKx^{(m)} = \Gamma_1(x)\phi_1 + \ldots + \Gamma_m(x)\phi_m ,$$

then $x^{(m)}$ is determined by the m Galerkin equations

$$\Gamma_1(x) = 0 ,\ldots, \Gamma_m(x) = 0 , \quad x \in D_o .$$

A closer inspection [1d,e] shows that indeed this is the case, and error bounds for $\|z-x^{(m)}\|$ $(z = \mathcal{T}x,\ x \in D_o,\ PK\mathcal{T}x = 0)$, are obtained by the analysis of the determining equation.

4. THE MAP T AS A CONTRACTION

It is typical of the method described in nos. 1 and 2 that the operator T can be made to be a contraction on each fiber of P as a consequence of spectral properties of the linear operator E, even if the nonlinear operator N is only Lipschitzian and nonnecessarily smooth. To see this let us assume here that S is a Hilbert space with inner product (z,z') and norm $\|z\| = (z,z)^{1/2}$. We shall also assume that the linear operator E possesses eigenvalues λ_i and eigenfunctions ϕ_i, say $E\phi_i+\lambda_i\phi_i = 0$, $i = 1,2,\ldots$, such that $|\lambda_1| \leq |\lambda_2| \leq \ldots$, $|\lambda_i| \to \infty$ as $i \to \infty$, and $[\phi_1,\phi_2,\ldots]$ is a complete orthonormal system in S. Then, every element $z \in S$ has a Fourier series

$$z = c_1\phi_1 + c_2\phi_2 + \ldots ,$$

with $c_i = (z,\phi_i)$, $i = 1,2,\ldots$, $\|z\| = (\sum_i c_i^2)^{1/2}$, and we define P by taking

$$Pz = c_1\phi_1 + \ldots + c_m\phi_m .$$

Then $[\phi_1,\ldots,\phi_m]$ is a basis in S_o and $[\phi_{m+1},\phi_{m+2},\ldots]$ is a complete orthonormal system in S_1. If we take m sufficiently large, then $\lambda_i \neq 0$ for all $i \geq m+1$. For every $z \in S_1$ we have $z = \sum_{i=m+1}^{\infty} c_i\phi_i$, and we may define $H: S_1 \to S_1$ by taking

$$Hz = -\sum_{i=m+1}^{\infty} \lambda_i^{-1} c_i \phi_i .$$

We showed in [1d] that this operator H possesses the properties required in nos. 1 and 2. In addition

$$\|Hz\| = \left(\sum_{i=m+1}^{\infty} \lambda_i^{-2} c_i^2\right)^{1/2} \leq |\lambda_{m+1}|^{-1}\|z\| ,$$

and thus H is a Lipschitzian operator in S_1, or

$$\|Hz-Hz'\| \leq k_o\|z-z'\| \quad , \quad z,z' \in S_1 ,$$

with constant $k_o = |\lambda_{m+1}|^{-1}$, which can be made as small as we wish by taking m sufficiently large.

Finally, if we assume that the restriction of N on G, say $N: G \to S$, is Lipschitzian of constant L, say

$$\|Nz-Nz'\| \le L\|z-z'\| \quad , \quad z,z' \in G ,$$

then we have, for $z,z' \in P^{-1}x \cap G$, $x \in D_o$,

$$\|Tz-Tz'\| = \|Fz-Fz'\| = \|H(I-P)(Nz-Nz')\| \le |\lambda_{m+1}|^{-1}L\|z-z'\| .$$

Thus, T is a Lipschitzian operator on $P^{-1}x \cap G$ for every $x \in D_o$ of constant $k = |\lambda_{m+1}|^{-1}L$, and we can always take m sufficiently large so as to have $k < 1$. Actually it turns out that k is already < 1 for rather small values of m, in a number of applications.

5. A CONNECTION WITH THE LERAY-SCHAUDER FORMALISM

Once the structure mentioned above is well defined, that is, K, E, N, are given, $G = D_0 \times D_1$, and $T: P^{-1}x \cap G \to P^{-1}x \cap G$ is a contraction for every $x \in D_0$, so that $\mathcal{T}: D_0 \to G$ is defined by $z = Tz = \mathcal{T}x$, then S. A. Williams has shown [14] that it is possible to establish a theoretical connection with the Leray-Schauder theory. To do this he has introduced the maps $W: G \to S$ and $W' = G \to S$ defined by

$$Wz = Pz - PK\mathcal{T}Pz + H(I-\underline{P})Nz \quad , \quad z \in G ,$$

$$W'z = W\mathcal{T}Pz \quad , \quad z \in G .$$

As a consequence of the properties of the maps E, N, T, $\mathcal{T}$, Williams has shown that W' is completely continuous, that the fixed points of W' are the solutions of the original problem, and that the topological degree of $(PK\mathcal{T}, D_0, 0)$ coincides with the Leray-Schauder degree in Banach spaces $d_{LS}(I-W',G,0)$. Thus, the present approach gives rise to a situation where the Leray-Schauder theory theoretically applies through the map W' which in turn is defined in terms of the maps K, P, T, $\mathcal{T}$. This result parallels in the present situation, and hence even in the absence of smoothness properties for the operators (cf. no. 4) an analogous result established by J. Cronin in connection with her approach.

Williams has further discussed the question as to whether the topological degree of $(PK\mathcal{T}, D_0, 0)$ is invariant for suitable changes of P, say, if we replace $P: S \to S_0$ by another projector operator $P': S \to S_0'$ whose range S_0' is a linear subspace of S containing S_0, say $S_0 \subset S_0' \subset S$. The answer is affirmative, at least when S_0 is already sufficiently large. This answer is in harmony with Leray-Schauder theory where the topological index d_{LS} is actually defined by considering suitably finitely dimensional subspaces S_0 of S, and showing that the corresponding Brouwer topological degree defined in S_0 does not change by enlarging S_0, once S_0 is already sufficiently large.

6. A FEW EXAMPLES

In [1d] we considered the boundary value problem

$$x'' + x + \alpha x^3 = \beta t \ , \quad 0 \leq t \leq 1 \ , \quad x(o) = 1 \ , \quad x'(1) + hx(1) = 0 \ ,$$

and we took $m = \dim S_o = 1$. Then for $h = 1$, we found $k_o = 0.044$. For $\alpha = \beta = 1/2$ a first Galerkin approximation is $x^{(1)}(t) = -\,0.11721 \sin (2.0288t)$, $0 \leq t \leq 1$. By applying the considerations of the previous numbers we proved that the problem above has an exact solution $X(t)$, $0 \leq t \leq 1$, with $\|X-x^{(1)}\| \leq 0.0038$.

In [1e] we considered the boundary value problem

$$\ddot{x} + x^3 = \sin t \ , \quad x(o) = x(2\pi) \ , \quad \dot{x}(o) = \dot{x}(2\pi) \ ,$$

and we took $m = \dim S_o = 2$. We found $k_o = 0.04$, and a second Galerkin approximation $x^{(2)}(t) = 1.434 \sin t - 0.124 \sin 3t$, $0 < t \leq 2\pi$. By applying the considerations of the previous numbers we proved that the problem above has an exact solution $X(t)$, $0 \leq t \leq 2\pi$, with $\|X-x^{(2)}\| \leq 0.124$.

In [1j] we considered the boundary value problem (of nonlinear potential)

$$\begin{aligned} u_{xx} + u_{yy} &= g(x,y,u) \text{ for } (x,y) \in A = [x^2+y^2<1] \ , \\ u &= 0 \text{ for } (x,y) \in \partial A = [x^2+y^2 = 1] \ , \end{aligned} \tag{11}$$

where g is a given function continuous in u, measurable in x, y, and bounded for u bounded. We took $m = \dim S_o = 1$, and we found $k_o = 0.069$. By applying the considerations of the previous numbers we proved the existence of solutions $u(x,y)$ to problem (11) which are continuous in $A \cup \partial A$, with first order partial derivatives continuous in A and whose Laplacian $\Delta u = u_{xx}+u_{yy}$ (in the sense of the theory of distribution) is a measurable function in A. Essentially, we proved in [1j] the existence of at least one such solution provided g is not too large for $u = 0$ and does not grow too rapidly with $|u|$ as $|u| \to \infty$. For instance, for the case $u_{xx}+u_{yy} = f(x,y)|u|+h(x,y)$, f, g measurable $|f(x,y)| \leq \alpha$, $|h(x,y)| \leq \beta$, α, β finite, we proved that a solution exists for every f, g as above with $\alpha<3.02$ and any β. In these problems a first approximation solution of the form $u_o(x,y) = \gamma J_o(\lambda_{o1}\rho)$, $0 \leq \rho^2 = x^2+y^2 \leq 1$, is singled out, where λ_{o1} is the first positive zero of J_o and γ is a suitable constant. Error bounds for such a solution, that is, upper bounds for the norm $\|u-u_o\|$ of the difference between exact and approximate solutions have been given by C. D. Stocking in [13].

7. PERTURBATION PROBLEMS FOR PERIODIC SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

The ideas of the previous numbers can be already traced in the papers by Cesari, J. K. Hale, R. A. Gambill, W. R. Fuller, H. R. Bailey in the years 1952-60 (see reports and summaries in [1a,b,c] and [5b]) on periodic solutions of perturbation problems for ordinary differential equations and systems (periodic or autonomous)

$$dz/dt = Az + \epsilon q(t,z) , \tag{12}$$

ϵ a small parameter, A an n x n constant matrix, $z = (z^1,\ldots,z^n)$, $q(t,z)$ periodic in t (or independent of t), measurable in t, and continuous, or Lipschitzian in z. It was shown in [1e] that the method used in those early papers is a particular case of the process described in nos. 1 and 2 above, and we can choose $m = 1$ since the numbers k, k_0 are now replaced by numbers $\bar{k}|\epsilon|$, $\bar{k}_0|\epsilon|$ which can be made less than one by taking $|\epsilon|$ sufficiently small.

For problems (12), periodic or autonomous, extensive existence theorems of periodic solutions, or cycles, have been proved, together with conditions for asymptotic, or asymptotic orbital stability. In particular J. K. Hale discovered by the method above simple criteria for the existence of families of given dimension k, $0 \le k \le n$, of periodic solutions [5a, 1a,b,c].

Under conditions of continuity only, the existence of solutions to the determining equations, and hence to the original problem, can be assured [1a] by the use of a C. Miranda's statement which gives an equivalent form of Brouwer's fixed point theorem. This statement concerns vector-valued continuous functions $F(z) = (F_1,\ldots,F_n)$ from a cube $C = [z = (z^1,\ldots z^n) | \; |z^i| \le L_i, \; i = 1,\ldots,n]$. If F_i has constants opposite signs on the sides $z^i = \pm L_i$ of C, then there is at least one point $z \in C$ where $F(z) = 0$. (C. Miranda, Boll. Un. Mat. Ital. 3, 1941, 5-7.)

Under Lipschitz conditions the method for these problems of periodic solutions as described in [1a,c] gives rise to a process of successive approximations which is reported in [4, pp. 308-317].

8. SQUARE NORM AND UNIFORM NORM

In problems as those of no. 6 it is advantageous to use the square norm because it affords the smallest value of the constants k, or k_0 (for instance, $k_0 = |\lambda_{m+1}|^{-1}$ under the hypotheses of no. 4) and thus very small values of m could be used (m = 1 or m = 2 in no. 6). Actually, in these problems both norms, the square norm and the uniform norm, have been used, since N (often a polynomial expression) may be Lipschitzian of a certain constant, say L, only if sup $|z|$ is not larger than some other constant R. On the other hand, the best values of k or k_0 may not be essential as, say, in perturbation problems (no. 7) as mentioned, or in association with Knobloch-type arguments as in no. 11 below.

H. W. Knobloch [6a] has given the necessary estimates for the use of the uniform norm only in questions involving periodic solutions of ordinary differential equations and the use of the method of nos. 1 and 2 above.

Recently J. Mawhin [8] has applied directly the formalism of nos. 1 and 2, with the use of a uniform norm only and Knobloch estimates, to the same perturbation problems of no. 7.

9. APPLICATIONS

P.A.T. Christopher [2a,b] has applied the method of the present paper to determine regions in the parameter space corresponding to stable harmonics and subharmonics for the nonhomogeneous Duffing equation:

$$\ddot{x} + b\dot{x} + c_1 x + c_3 x^3 = Q \sin \omega t .$$

A. M. Rodionov [12] has shown that the method of the present paper applies to nonlinear differential equations and systems with a fixed lag λ:

$$\dot{x} = g(t, x(t), x(t-\lambda)) .$$

C. Perello [10] has initiated the same type of analysis for functional differential equations

$$\dot{x} = g(t, x_t) ,$$

where g depends on t and on all values of $x(\tau)$ in an interval $t - \lambda \leq \tau \leq t$.

10. EXTENSIONS OF THE PREVIOUS CONSIDERATIONS

The formalism of nos. 1 and 2 is based on certain properties of the operators, or axioms, and J. Locker [7] has proved that these properties are satisfied, for instance, for usual boundary value problems of nonlinear ordinary differential equations, when the underlying linear problem is selfadjoint. For the corresponding nonselfadjoint problems J. Locker has developed a theory which extends the one of the present paper, and whose axioms are satisfied for boundary value problems of ordinary differential equations. As an example, he gives numerical bounds for α and β in the nonlinear initial value problem

$$\ddot{x} + x + \alpha x^2 = \beta t \quad , \quad 0 \leq t \leq 2\pi \quad , \quad x(0) = 0 \ ,$$

under which the solution exists in $[0,2\pi]$. He also obtained estimates of the growth of the solution and error bounds.

J. K. Hale, S. Bancroft, and D. Sweet [5d] also have presented a formalism which extends the present one in a different direction, and established a connecting link with the previous work by D. C. Lewis, H. A. Antosiewicz, Jane Cronin, R. G. Bartle, and L. Nirenberg.

11. KNOBLOCH'S EXISTENCE THEOREMS FOR PERIODIC SOLUTIONS

For problems of periodic solutions of ordinary differential equations and systems (not of the perturbation type), say

$$\dot{z}^i = f_i(t,z) \quad , \quad i = 1,\dots,n \quad , \quad z = (z^1,\dots,z^n) \quad , \quad f_i(o,z) = f_i(T,z) \; ,$$

let us consider the space S of all continuous periodic vector functions with uniform norm. Every element $z \in S$ has a Fourier series

$$z(t) \sim 2^{-1}a_o + \Sigma_{i=1}^{\infty}(a_i \cos i\omega t + b_i \sin i\omega t) \; ,$$

$\omega = 2\pi/T$, T the period, where a_o, a_i, b_i are n-vectors. Let us define P by taking

$$Pz = 2^{-1}a_o + \Sigma_{i=1}^{m}(a_i \cos i\omega t + b_i \sin i\omega t) \; , \tag{13}$$

so that dim $S_o = n(2m+1)$. By taking m sufficiently large, the corresponding map T is a contraction and $\mathcal{T}$ is defined (H. W. Knobloch [6b]). Then, for every element $x \in S_o$ (a trigonometrical polynomial as in (13), we have

$$PK\mathcal{T}x = 2^{-1}\alpha_o + \Sigma_{i=1}^{m}(\alpha_i \cos i\omega t + \beta_i \sin i\omega t) \; ,$$

where α_o, α_i, β_i, $i = 1,\dots,m$, are functions of x, that is, of the coefficients a_o, a_i, b_i, $i = 1,\dots,m$, of the trigometric polynomial x. Then the determining equation $PK\mathcal{T}x = 0$ takes the form

$$\alpha_o = 0 \quad , \quad \alpha_i = 0 \quad , \quad \beta_i = 0 \quad , \quad i = 1,\dots,m \; , \tag{14}$$

namely a system of n(2m+1) equations in n(2m+1) unknowns. H. W. Knobloch has shown that, under hypotheses, a suitable distribution of signs of the n functions f_i, or associated functions, imply a typical distribution of signs for each of the n(2m+1) left hand members (components) of the determining equations (14) so that the existence of a solution $x \in S_o$ to the same equations, and hence the existence of a solution z to the original problem, can be deduced by the same C. Miranda's form of Brouwer's fixed point theorem mentioned in no. 7, and this no matter how large n and m are. H. W. Knobloch has used this analysis of the determining equation as a basis toward a qualitative discussion of the second order nonlinear differential equation

$$\ddot{x} = g(t,x,\dot{x}) \quad , \quad g(t,x,\dot{x}) = g(t+1,x,\dot{x}) \ , \tag{15}$$

where g is a real-valued function of t, x, $\dot{x}$, which is Lipschitzian in x, $\dot{x}$, periodic of period 1, and sectionally continuous in t. Two functions $\alpha(t)$, $\beta(t)$, $0 \le t \le 1$, (continuous with continuous first derivatives and sectionally continuous second derivatives) are said to be lower and upper solutions respectively provided

$$\alpha < \beta \quad , \quad -\ddot{\alpha} + g(t,\alpha,\dot{\alpha}) \le 0 \quad , \quad -\ddot{\beta} + g(t,\beta,\dot{\beta}) \ge 0 \ ,$$

for t in [0,1]. If α, β is such a pair, if Ω denotes the region $[0 \le t \le 1, \alpha(t) \le x \le \beta(t)]$, and there are two functions $\Phi(t,x)$, $\Psi(t,x)$ continuous and continuously differentiable in Ω, such that

$$\Phi(t,\alpha) \le \dot{\alpha} \le \Psi(t,\alpha) \quad \Phi(t,\beta) \le \dot{\beta} \le \Psi(t,\beta) \ ,$$

$$\Phi(x,0) = \Phi(x,T) \quad , \quad \Psi(x,0) = \Psi(x,T) \ ,$$

and such that both expressions

$$\Phi_x\Phi + \Phi_t - g(t,x,\Phi), \text{ and } \Psi_x\Psi + \Psi_t - g(t,x,\Psi)$$

have constant signs in Ω, then (15) possesses a periodic solution x(t) satisfying

$$\alpha(t) \le x(t) \le \beta(t) \quad , \quad \Phi(t,\alpha(t)) \le \dot{x}(t) \le \Psi(t,\beta(t))$$

for all t in [0,1] (H. W. Knobloch [6b]).

A corollary of this statement states that if α, β is a pair of periodic lower and upper solutions, and $g(t,x,\dot{x}) \le C|\dot{x}|^2$ for some constant C and all t, x, $\dot{x}$ with $(t,x) \in \Omega$ and $|\dot{x}|$ sufficiently large, then again (15) possesses a periodic solution of period 1. H. W. Knobloch has proved other statements of the same type, based on the process mentioned in the previous numbers and topological considerations [5b].

In successive papers H. W. Knobloch [6c] has also proved, by the same process, comparison and oscillation theorems for second order nonlinear equations (15), which extend Sturm theory for linear equations. Some of the results give general form to statements proved earlier by M. Carthright for the van der Pol equation only.

12. PERIODIC SOLUTIONS OF NONLINEAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

The question of the periodic solutions of the hyperbolic partial differential equation

$$u_{xy} = g(x,y,u,u_x,u_y) , \tag{16}$$

can be posed in two ways, either we assume g periodic in x only and we ask for solutions u periodic in x of the same period and existing in a strip, say $-a \leq y \leq a$, $-\infty < x < +\infty$ (problem in a "cylinder"), or we assume g periodic in both x and y of a given period T and we ask for solutions u periodic in both x and y of the same period, hence existing in the whole xy-plane (problem on a "torus"). To simplify the exposition we limit ourselves to the second case.

The process described in the nos. 1-5 above has been applied by Cesari [1f,g,h,i] to this problem, taking for S the space of all continuous functions u(x,y) periodic of period T in x and y, and continuous with their first order partial derivatives, and uniform norm, and taking for S_0 the subspace of the functions u of the form $u(x,y) = h(x)+k(y)$. Then, the projection operator P is defined by taking $Pu(x,y) = u(x,0)+u(0,y)-u(0,0)$, and S_0 is now infinite dimensional. This choice leads to the following existence statement and a very simple interpretation of the relaxed problem: If g is continuous in its arguments, periodic in x and y of period T, and Lipschitzian in u_x, u_y of constants b_1, b_2 satisfying

$$2Tb_1 < 1 \quad , \quad 2Tb_2 < 1 , \tag{17}$$

then for every two functions $u_0(x)$, $v_0(y)$ continuous with u_0', v_0', periodic of period T, and $u_0(0) = v_0(0)$, there exists at least one solution u(x,y) periodic in x and y of period T continuous with u_x, u_y, u_{xy}, satisfying the relaxed (Darboux type) boundary value problem

$$u_{xy} = g(x,y,u,u_x,u_y) - m(y) - n(x) - \mu ,$$

$$u(x,0) = u_0(x) = u(x,T) \quad , \quad v(0,y) = v_0(y) = v(T,y) .$$

(See [1h] for details.) Here m, n are suitable continuous functions and μ a constant, m, n, μ depending on u_o, v_o, and of course

$$m(y) = T^{-1} \int_0^T g\, dx \quad , \quad n(x) = T^{-1} \int_0^T g\, dy \quad , \quad \mu = T^{-2} \int_0^T \int_0^T g\, dx\, dy \, .$$

In other words, for every Darboux type boundary condition there is a periodic solution to the relaxed problem. This relaxed problem is "well posed". Indeed, if g is Lipschitzian in u, u_x, u_y of constants b_o, b_1, b_2 satisfying relations analogous to (17), then the solution u(x,y) is unique and depends continuously on u_o, v_o in the proper uniform topology. The determining equation is now a functional equation (since S_o is ∞-dimensional). Indeed, the solution u(x,y) above of the relaxed problem is actually a solution to the original problem (16) provided the functions u_o, v_o are so chosen that

$$m(y) = 0 \quad , \quad n(x) = 0 \quad , \quad \mu = 0$$

(determining equation). Very simple criteria can be given (see [1h]) for the existence of a solution to the determining equation (and hence for the original problem (16)) for perturbation equations of the form

$$u_{xy} = \psi(x,y) + Cu + \psi_1(y)u_x + \psi_2(x)u_y + \epsilon g(x,y,u,u_x,u_y) \, , \qquad (18)$$

or of the form

$$u_{xy} = \epsilon\, [\psi(x,y)+Cu+\psi_1(y)u_x+\psi_2(x)u_y] + \epsilon^2 g(x,y,u,u_x,u_y) \, , \qquad (19)$$

ϵ a small parameter. These results, for the relaxed problem, as well as for the original problem, can be completed with smoothness theorems [1i], and transferred to the nonlinear wave equation

$$u_{xx} - u_{yy} = g(x,y,u,u_x,u_y) \, , \qquad (20)$$

by usual transformations [1h,i].

J. K. Hale [5c] has studied the same problems for the nonlinear wave equations choosing as operator P the projection of S into the space of the periodic solutions of the homogeneous problem $u_{xx}-u_{yy} = 0$, and has obtained corresponding results for the relaxed problem, and ensuing perspicuous criteria for the exact problem for equations (18) and (19), some of which extend or parallel previous results of Rabinovitch and O. Vejvoda. In a sense, analogous results have been proved for periodic solutions in a strip, and Cesari [1g] has shown that criteria for the solutions of the original problem can be obtained by a novel implicit function theorem of the hereditary type based on functional analysis. Well within the frame of the present method A. K. Aziz (Proc. Amer. Math. Soc. 17, 1966), by more stringent estimates, slightly reduced requirement (17) in the existence theorems for the relaxed problem.

Again by the method of the present paper D. Petrovanu [10a] has recently shown that the analysis mentioned above for the periodic solutions of the hyperbolic equation in the plane can be extended to the periodic solution of the equation

$$u_{xyz} = g(x,y,z,u,u_x,u_y,u_z,) ,$$

with g also periodic in x, y, z. D. Petrovanu [10b] has proved also existence theorems for the relaxed problems, both in a strip and on a torus, for the periodic solutions of the Tricomi system of equations

$$u_x = g(x,y,u,v) \quad , \quad v_y = h(x,y,u,v) ,$$

u an m-vector and v an n-vector.

Very recently A. Naparstek [9] has discussed the nonlinear wave equation

$$u_{xx} - \sigma^2 u_{yy} = \epsilon g(x,y,u) \qquad (21)$$

on a torus by the same method of the present paper. Here ϵ is a small parameter, g is periodic in x and y of period, say, $T = 2\pi$, the solutions u, also periodic in x and y of period 2π, are sought in suitably Sobolev spaces on the torus, and the hypotheses on g are also expressed in terms of measureability and belonging to suitable Sobolev spaces.

Above, $\sigma > 0$ is a fixed number, and two cases must be distinguished: σ rational, and σ irrational. In the first case the homogeneous equation $u_{xx}-\sigma^2 u_{yy} = 0$ has ∞- many well known and independent (periodic) solutions on the torus, while for σ irrational the same equation has only constant solutions. For σ irrational also questions of small divisors occur. If C denotes the class of all positive irrational numbers σ such that $m^2-\sigma^2 n^2$ is bounded away from zero when m, n describe all possible pairs of not both zero integers, then C is known to be countable and everywhere dense in $[0,\infty)$. For the two cases 1. σ rational, 2. σ

irrational, $\sigma \in C$, the projection operator P is chosen as the projection of the doubly periodic functions into the subspace S_o of the periodic solutions of the homogeneous equation $u_{xx}-\sigma^2 u_{yy} = 0$. Thus, S_o is ∞- many dimensional for σ rational, and one-dimensional for σ irrational, $\sigma \in C$. For both cases A. Naparstek has given existence theorems for the relaxed problems, together with uniqueness and continuous dependence statements, and has given criteria for the existence of solutions to the original problem.

For σ rational the determining equation is a functional equation. Two sets of different considerations are used for its discussion. One is based on the existence of a Gateaux nonzero differential at $\epsilon = 0$, and the use of implicit function theorems of functional analysis. Another set of considerations concerns the interpretation of the first member of the determining equation as an operator, for which conditions are given for it to be monotone in the sense of G. J. Minty and F. E. Browder. From both A. Naparstek deduces criteria for the existence of doubly periodic solutions to the original problem (21). A. Naparstek also has extended these results to equations similar to (21) containing the first order partial derivatives u_x, u_y.

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