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THE IMPLICIT FUNCTION THEOREM IN FUNCTIONAL ANALYSIS

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THE IMPLICIT FUNCTION THEOREM IN FUNCTIONAL ANALYSIS*

by

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We consider a functional $f(y,z)$, or $f\colon\ Y_0 x Z_0 \to F$, where F is a linear space, Y_0, Z_0 are subsets of linear spaces Y and Z of functions from an arbitrary space X into some Banach space E, and $Y \subset Z$, $Y_0 \subset Z_0$, (Y, Z Banach spaces, E finitely dimensional). The process usually associated with f may, or may not, involve a "loss of derivatives" in the terminology of J. Moser (Ref. 6). Under various sets of hypotheses we prove that there exists at least one function $\psi \in Y_0 \subset Z_0$ such that

$$f(\psi,\psi) = 0. \tag{1.1}$$

Under one of the sets of hypotheses taken into consideration, and for which the process usually associated with f involves no loss of derivatives, we prove a Theorem A (Section 2) one of whose corollaries (Section 5) includes a statement which we shall apply in a coming paper (Ref. 1) on hyperbolic partial differential equations. A slight restriction of the hypotheses guarantees (Theorem B) a local uniqueness of the function ψ above. A number of applications are made. The proof of Theorem A is based on the remark that, by the usual argument of the implicit function theorem in functional analysis (see, for instance Ref. 5, pp. 174-195), an element $y_z \in Y_0$ can be proved to exist, for every $z \in Z_0$, such that $f(y_z,z) = 0$, and that the ensuing map $\mathcal{T}\colon\ Z_0 \to Y_0$ restricted to Y_0 can be proved to satisfy Schauder's fixed point theorem. The usual argument is here given for locally convex topological vector spaces.

In the case in which the process usually associated with f involves an actual loss of derivatives, the previous remark still applies and an analogous theorem can be proved (Theorem C) provided full use is made of the implicit function theorem that J. Moser (Ref. 6) has recently proved—and applied to a number of problems. Also, use is made of other considerations of the papers of J. Nash and J. Schwartz (Ref. 7 and 10) which were point of departure for the results of J. Moser (Ref. 6). Applications, other than those mentioned above, will follow. Considerations in the line of J. Leray's are also included (Section 1, No. 8).

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SECTION 1. THE IMPLICIT FUNCTION THEOREM OF FUNCTIONAL ANALYSIS

1. THE EQUATION $f(y) = 0$.

Let Y be a Hausdorff, complete, locally convex topological vector space, and $f(y)$ a (not necessarily linear) functional $f\colon\ Y_0 \to F$ on a subset Y_0 of Y to a linear space F. We shall assume that an "approximate" solution y_0 to equation $f(y) = 0$ is known. Let $\{V\}$ be a base for the neighborhood systems of 0 in Y, each element V of which is balanced, absorbant, and convex, and assume that $V\in\{V\}$, $\lambda > 0$ implies $\lambda V\in\{V\}$. Let S be a balanced convex closed subset of Y with $y_0+S\subset Y_0$. We shall denote by $\mathcal{N}(B)$ the null space of an operator B. We shall need the hypothesis:

(G1) There are numbers $\mathcal{K}_0$, $\mathcal{K}$, $0 \le \mathcal{K}_0 < 1$, $0 \le \mathcal{K} \le 1-\mathcal{K}_0$, and linear operators $B\colon\ F \to Y$, $A\colon\ Y \to F$, with $\mathcal{N}(B) = 0$, such that $y_1, y_2\in y_0+S$, $y_1-y_2\in V\in\{V\}$, implies

$$B[f(y_1)-f(y_2)-A(y_1-y_2)]\in\mathcal{K}_0(S\cap V), \tag{1.2}$$

$$Bf(y_0)\in\mathcal{K}S,\quad BA \doteq I. \tag{1.3}$$

Let $T\colon\ S \to Y$ be the map defined by

$$Ty = y - Bf(y), \quad y\in S. \tag{1.4}$$

(i) Under hypothesis G1, there is one and only one element ψ in S with $f(\psi) = 0$, and ψ is a fixed point of T in S.

Proof of (i). By (1.3) and (1.4) we have

$$Ty_0 = y_0 - Bf(y_0),\ Bf(y_0)\in\mathcal{K}S. \tag{1.5}$$

For any two elements $y_1, y_2\in S$, by (1.2) and (1.4), and by taking $V = Y$, we have also

$$Ty_1 - Ty_2 = B[f(y_1)-f(y_2)-A(y_1-y_2)]\in\mathcal{K}_0S. \tag{1.6}$$

Finally, for every $y\in S$, by (1.5) and (1.6),

$$Ty - y_0 = [Ty-Ty_0] + [Ty-y_0] \in (\mathcal{K}+\mathcal{K}_0)S\subset S. \tag{1.7}$$

Thus T: $S \to S$. The remaining part of the proof is now rather standard. Let $y_{n+1} = Ty_n$, $n = 0,1,\dots$. Then $y_n \in S$ for all n. Let $V \in \{V\}$. Since V is absorbant, there is some $\lambda > 0$ such that $y_1 - y_2 \in \lambda V$, and $\lambda V \in \{V\}$. Let us prove that

$$y_{n+1} - y_n \in \mathscr{K}_0^n \lambda V, \quad n = 0,1,\dots . \tag{1.8}$$

This relation is true for $n = 0$. Assume (1.8) is true for $0,1,\dots, n-1$, and let us prove it for n. Indeed

$$y_{n+1} - y_n = B[f(y_n)-f(y_{n-1})-A(y_n-y_{n-1})] \in \mathscr{K}_0(\mathscr{K}_0^{n-1}\lambda V) = \mathscr{K}_0^n \lambda V.$$

Thus (1.8) is proved for every n. We have

$$\begin{aligned} y_{n+p} - y_n &= (y_{n+1}-y_n) + \dots + (y_{n+p}-y_{n+p-1}) \\ &\in \mathscr{K}_0^n \lambda V + \dots + \mathscr{K}_0^{n+p-1}\lambda V \in (1-\mathscr{K}_0)^{-1}\mathscr{K}_0^n \lambda V, \end{aligned} \tag{1.9}$$

where $\mathscr{K}_0 < 1$. Thus, given any $V \in \{V\}$, there is some $\bar{n}$ such that $n \geq \bar{n}$, $p \geq 1$ implies $(1-\mathscr{K}_0)^{-1}\mathscr{K}_0^n\lambda < 1/2$, and $y_{n+p}-y_n \in (1/2)V \subset (1/2)\bar{V} \subset V$. Thus $[y_n]$ is a Cauchy sequence in the topology of Y.

Since Y is complete, $\psi = \lim y_n$ as $n \to \infty$ exists, $\psi \in Y$, $\psi \in S$. Finally, (1.9) implies that $\psi - y_n$ belongs to the closure of $((1-\mathscr{K}_0)^{-1}\mathscr{K}_0^n\lambda)V$, and since the numerical factor is $< 1/2$ for $n \geq \bar{n}$, we conclude that $\psi - y_n \in (1/2)V$ for $n \geq \bar{n}$. Thus, for $n > \bar{n}$, we have

$$\begin{aligned} \psi - T\psi &= (\psi - y_n) + (y_n - T_{\ n-1}) + (Ty_n - T\psi) \\ &\in (1/2)V + \mathscr{K}_0(1-\mathscr{K}_0^{-1})\mathscr{K}_0^{n-1}\lambda V \subset (1/2)V + (1/2)V = V, \end{aligned} \tag{1.10}$$

where V is an arbitrary element of $\{V\}$. Since Y is Hausdorff, we conclude that $\psi - T\psi = 0$, or ψ is a fixed element of T. The uniqueness of ψ in S follows by standard argument. Indeed, if y, z are fixed points of T, then $y = T^n y$, $z = T^n z$ for every n, and, if $V \in \{V\}$, and λ is so chosen that $y - z \in \lambda V$, then $y - z = T^n(y-z) \in \mathscr{K}_0^n \lambda V$, for every n. We can take n so large that $\mathscr{K}_0^n \lambda < 1$, hence $y - z \in V$ for every $V \in \{V\}$. Since Y is Hausdorff, we have $y - z = 0$. Note that, from $y_1 - y_0 \in \lambda V$, we have deduced (1.9), hence, $y_{n+p} - y_0 \in (1-\mathscr{K}_0)^{-1}(\lambda V)$. This can be reworded by saying that, if $Ty_0 - y_0 \in V \in \{V\}$, then $\psi - y_0 \in (1-\mathscr{K}_0)^{-1}V$.

Now $\psi = T\psi$ implies, by (1.4), $Bf(\psi) = 0$, and since the null space $\mathscr{N}(B)$ of B is zero, we conclude $f(\psi) = 0$. Conversely, if $f(y) = 0$, $y \in S$, then $y = Ty$, and, by uniqueness, $y = \psi$.

2. CONTINUOUS DEPENDENCE OF ψ ON PARAMETERS

Let us assume that $f(y,z)$ depends on y as in No. 1 and on a parameter z varying in a subset Z_0 of a locally convex topological vector space Z, hence $f\colon\ Y_0 \times Z_0 \to F$. Let $\{W\}$ be a base for the neighborhood system of 0 in Z. Let Σ denote some closed subset of Z with $\Sigma \subset Z_0 \subset Z$.

We shall assume that for a fixed subset S of Y as in No. 1, and for fixed numbers $\varkappa_0$, ν as in No. 1, hypothesis (G1) holds for every $z \in Z$, and thus uniformly with respect to Σ. In particular, both operators A_z, B_z may depend on z. By (i), for every $z \in \Sigma$ there is a unique element $\psi_z \in S$ such that $f(\psi_z, z) = 0$, which is the fixed point of the map $T_z\colon\ S \to S$, defined by $T_z y = y - Bf(y,z)$, or $\psi_z = T_z \psi_z$, $\psi_z \in S$, $z \in W$. Let $\tau\colon\ \Sigma \to S$ be the map defined by $\psi_z = \tau z$, $z \in \Sigma$, $\psi_z \in S$. We shall need the hypothesis:

(G2) Given $V \in \{V\}$ there is some $W \in \{W\}$ such that z_1, $z_2 \in \Sigma$, $z_1 - z_2 \in W$, $y \in S$ imply

$$B_{z_1} f(y, z_1) - B_{z_2} f(y, z_2) \in V. \qquad (1.11)$$

The analogous hypothesis that, given $V \in \{V\}$ and any compact subset C' of Z, there is some $W \in \{W\}$ such that z_1, $z_2 \in \Sigma \cap C'$, $z_1 - z_2 \in W$, $y \in S$ imply (1.11), will be denoted by $(G2)_c$.

(ii) Under hypotheses (G1) (uniformly in $z \in \Sigma$), and (G2), $\tau\colon\ \Sigma \to S$ is uniformly continuous on Σ. Under $(G2)_c$, τ is uniformly continuous on every subset $C' \subset \Sigma$, C' any compact subset of Z.

Proof of (ii). Let $V \in \{V\}$, and $V_0 = 2^{-1}(1 - \varkappa_0)V$. By (G2) there is an element $W \in \{W\}$ such that z_1, $z_2 \in \Sigma$, $z_1 - z_2 \in W$, $y \in S$, imply

$$B_{z_1} f(y, z_1) - B_{z_2} f(y, z_2) \in V .$$

Let $y_{i,n+1} = T_{z_i} y_{in}$, $n = 0, 1, \ldots$, $i = 1, 2$, with $y_{10} = y_{20} = y_0$. Then

$$y_{i1} = T_{z_i} y_0 = - B_{z_i} f(y_0, z_i), \quad i = 1, 2,$$

and hence

$$y_{11} - y_{21} = B_{z_1} f(y_0, z_1) = B_{z_2} f(y_0, z_2) \in V_0 .$$

Let us prove that

$$y_{1n} - y_{2n} = T^n_{z_1} y_0 - T^n_{z_2} y_0 \in (1 + \nu_0 + \ldots + \nu_0^{n-1}) V_0, \quad n = 1, 2, \ldots . \qquad (1.12)$$

This is true for n = 0. Let us assume that (1.12) is true for 0,1,...,n, and let us prove it for n+1. Indeed

$$\begin{aligned} y_{1,n+1} - y_{2,n+1} &= - B_{z_1}f(y_{1n},z_1) + B_{z_2}f(y_{2n},z_2) + y_{1n} - y_{2n} \\ &= - B_{z_1}[f(y_{1n},z_1)-f(y_{2n},z_1)-A_{z_1}(y_{1n}-y_{2n})] \\ &\quad - B_{z_1}f(y_{2n},z_1) + B_{z_2}f(y_{2n},z_2), \end{aligned}$$

and then

$$y_{1,n+1} - y_{2,n+1} \in {}_o(1+ {}_o+\dots+ {}_o^{n-1})V_o + V_o = (1+ {}_o+\dots+ {}_o^{n})V_o.$$

Thus (1.12) is proved for every n. As $n \to \infty$, we obtain that $\psi_{z_1}-\psi_{z_2}$ belongs to the closure of $(1-{}_o)^{-1}V_o$, hence to the closure of $(1/2)V_o$, and hence to V. We have proved that, given $V\in\{V\}$ there is $W\in\{W\}$ such that $z_1, z_2\in\Sigma$, $z_1-z_2\in W$, imply $\psi_{z_1}-\psi_{z_2}\in V$. The first part of statement (ii) is thereby proved. The second part can be proved with obvious changes.

3. CONTINUITY OF ψ

Under the hypotheses of No. 1, let us assume that Y is a space of functions y: $X \to E$, where X is any metric space with distance function $\rho(x_1,x_2)$, that E is a locally compact Banach space (hence, finitely dimensional). If $(\phi_1,\dots,\phi_m)$ is any base for E, then $y = y_1\phi_1+\dots+y_m\phi_m$, $y_1,\dots,y_m$ real. Thus Y can be thought of as a space of vector valued functions. Let {C} be a given collection of compact subsets C of X, with the property that any compact subset C_o of X is contained in at least one element $C\in\{C\}$. We shall assume that Y is precisely a space of functions y: $X \to E$, with the topology of the uniform convergence on the compact subsets of X. Hence we can define the topology of Y by means of the seminorms

$$\|y\|_{CY} = \sup_{x\in C}\|y(x)\|_E, \quad C\in\{C\}. \tag{1.13}$$

It is equivalent to use the seminorms

$$\|y\|_{iCY} = \sup|y_i(x)|, \quad C\in\{C\}, \quad i = 1,\dots,m,$$

where | | denotes absolute value. It is possible that X be an open subset of a Euclidean space E_d, that each $y\in Y$ is of class C^N in X, that is, each y_i is continuous in Y together with all its partial derivatives, say $D^s y$, of all orders $s = 0,1,\ldots,N$, $D^o y = y$. Then we shall take the topology of the uniform convergence of y and all $D^s y$, $0 \le s \le N$, on each compact subset C of X. This topology is defined by the seminorms

$$\|y\|_{isCY} = \sup_{x\in C} |D^s y_i(x)|, \quad C\in\{C\},\ i = 1,\ldots,m,\ s = 0,1,\ldots,N.$$

or equivalently the seminorms

$$\|y\|_{CY} = \max_{s=0,1,\ldots,N} \sup_{x\in C} \|D^s y(x)\|_E, \quad C\in\{C\}. \tag{1.14}$$

(G3) For every $C\in\{C\}$ and $\xi > 0$, there is $\sigma > 0$ such that $x_1,x_2\in C$, $\rho(x_1,x_2)\le \sigma$, $y\in S$ imply

$$\|(Bf(y))(x_1)-(Bf(y))(x_2)-y(x_1)+y(x_2)\|_E \le \xi. \tag{1.15}$$

In the case $y\in C^N$ in X as above, we must replace (1.15) by

$$\|D^s(Bf(y))(x_1)-D^s(Bf(y))(x_2)-D^s y(x_1)+D^s y(x_2)\|_E \le \xi, \quad s = 0,1,\ldots,N. \tag{1.16}$$

(iii) Under hypothesis (G1) and (G3), the element ψ of (i) is (uniformly) continuous on each compact subset of X and continuous on X. If $y\in C^N$ in X as above, all $D^s\psi$, $0 \le s \le N$, are (uniformly) continuous on each compact subset of X, and continuous on X.

Proof of (iii). Let $C\in\{C\}$ and $\xi > 0$ arbitrary. Let $\sigma > 0$ be the number defined in (G3) in correspondence of C and the number ξ. Let x_1, $x_2\in C$, and $v_i = \psi(x_i)$, $i = 1,2$. Then $\psi = T\psi$, and

$$\|v_1-v_2\|_E = \|(T\psi)(x_1)-(T\psi)(x_2)\|_E = \|\psi(x_1)-(Bf(\psi))(x_1)-\psi(x_2)+(bf(\psi))(x_2)\|_E \le \xi.$$

This proves the uniform continuity of ψ on each $C\in\{C\}$. Since each compact subset of X in contained in some $C\in\{C\}$, the first part of (iii) is proved. The same if all $y\in Y$ are of class C^N as above. We shall only take $v_{si} = D^s\psi(x_i)$, $i = 1,2$, $s = 0,1,\ldots,N$. Since X is metric, hence Hausdorff, X satisfies the first axiom of countability and X is a k-space. Thus ψ, being continuous on each compact subset of X, is continuous on X. We now could assume that $f(y,z)$, or $f\colon\ Y_o x Z_o \to F$ depends on $y\in Y_o$ and on $z\in Z_o$ Z, Z parameter space, as in No. 2.

(iv) Under hypothesis (G1) (uniformly for $z \in W$), (G2)c and (G3), the function $\Phi(x,z)$, $x \in X$, $z \in \Sigma$, is uniformly continuous in any set $C \times C'$, C, C' compact subsets of X and Z, $C' \subset W$.

Proof of (iv). In No. 2 we have proved that given $\xi > 0$ and C', there is a $W \in \{W\}$ such that $z_1, z_2 \in C' \cap \Sigma$, $z_1 - z_2 \in W$, imply $\|\psi z_1 - \psi z_2\| \leq \xi$, and hence

$$\|\psi z_1(x) - \psi z_2(x)\|_E \leq \xi.$$

On the other hand, we have proved above that there is $\sigma' > 0$ such that $x_1, x_2 \in C$, $\rho(x_1,x_2) \leq \sigma'$ imply

$$\|\psi z_2(x_1) - \psi z_2(x_2)\|_E \leq \xi.$$

Thus $x_1, x_2 \in C$, $z_1, z_2 \in C' \cap \Sigma$, $\rho(x_1,x_2) \leq \sigma$, $z_1 - z_2 \in W$, imply

$$\|\Phi(x_1,z_1) - \Phi(x_2,z_2)\|_E$$

$$\leq \|\Phi(x_1,z_1) - \Phi(x_1,z_2)\|_E + \|\Phi(x_1,z_2) - \Phi(x_2,z_2)\|_E$$

$$= \|\psi z_1(x_1) - \psi z_2(x_1)\|_E + \|\psi z_2(x_1) - \psi z_2(x_2)\|_E \leq \xi + \xi = 2\xi.$$

This proves the uniform continuity of Φ on $C \times (C' \cap \Sigma)$, and hence the equicontinuity of the functions ψ_z: $C \to E$ (for each $C \in \{C\}$). The same reasoning holds when all $y \in Y$ are of Class C^N, and $\Phi(x,z)$, $\psi_z(x)$ are replaced by $D^s\Phi(x,z)$, $D^s\psi_z(x)$, $s = 0,1,\ldots N$.

SECTION 2. THE EQUATION $f(y,y) = 0$

1. HYPOTHESES

Let X be any metric space with distance function $\rho(x_1,x_2)$, let E be a locally compact Banach space (hence finite dimensional) with norm $\|e\|$. Let F be any linear space, let Y, Z be locally compact topological vector spaces of functions $y: X \to E$ and $z: X \to E$, with $Y \subset Z$, Y Hausdorff and complete. If $(\phi_1,\dots,\phi_m)$ is a base for Y, then $y = y_1\phi_1+\dots+y_m\phi_m$ for every $y \in Y$, and hence $y \in Y$ has a representation $y = (y_1,\dots,y_m)$ as a real vector-valued function on X. The same holds for $z \in Z$.

We shall take in Y the topology of uniform convergence on compacta.

We shall denote by $\{C\}$ a given collection of compact subsets of X covering X, such that every compact subset C_o of X is contained in at least one element $c \in \{C\}$. To simplify our considerations we shall consider two alternate situations. Either X is any metric space, Y is a space of functions $X \to E$ which are bounded on each compact subset of X, and we then define the topology of Y by means of the seminorms

$$\|y\|_{CY} = \sup_{x \in C} \|y(x)\|_E, \quad X \in \{C\}. \tag{2.1}$$

Alternatively, X is an open subset of an Euclidean space E_d, and Y is a space of functions $y: X \to E$ possessing partial derivatives, say $D^s y$, of all orders $0 \leq s \leq N$, for a given N, which are bounded in every compact subset of X. Then we define the topology of Y by means of the seminorms

$$\|y\|_{CY} = \max_{0 \leq s \leq N} \sup_{x \in C} \|D^s y(x)\|_E, \quad C \in \{C\}. \tag{2.2}$$

Other analogous situations can be taken into consideration.

Let y_o be a particular element of Y. For each $C \in \{C\}$ let $b_C > 0$ be a given number, and let S denote the subset of Y defined by

$$S = S_b = [y \in Y \mid \|y\|_{CY} \leq b_C, \quad C \in \{C\}] \subset Y. \tag{2.3}$$

Thus, S is a balanced, convex, closed subset of Y. Let Σ denote a fixed subset of Z with $S \subset \Sigma$.

Let $f: S\times\Sigma \to F$ be a (not necessarily linear) functional, or $f = f(y,z)$ with $y\in S$, $z\in Z$, $f(y,z)\in F$. In Theorem A we shall prove under the set of hypotheses of this number that there is an element $\psi\in S\subset\Sigma$ such that $f(\psi,\psi) = 0$.

Concerning Z we need only to know that $Y\subset Z$, and the topology of Z is defined by means of seminorms $\|z\|_{CZ}$ with the following property: For every finite system C_s, $s = 1,\dots, M$, of elements $C_s\in\{C\}$ and corresponding arbitrary numbers $\sigma_s > 0$, there are numbers $\delta_s > 0$, $s = 1,\dots, M$, such that

$$y_1,y_2\in S,\ \|y_1-y_2\|_{C_sY} \leq \delta_s,\ s = 1,\dots, M, \text{ implies } \|y_1-y_2\|_{C_sZ} \leq \sigma,\ s = 1,\dots, M. \tag{2.4}$$

Let $\{V\}$, $\{W\}$ be bases for the system of neighborhoods of 0 in Y and of 0 in Z. We assume that each set V is of the form

$$V = [y\in Y \mid \|y\|_{C_sY} \leq \nu_s,\ \nu_s > 0,\ C_s\in\{C\},\ s = 1,\dots,M < +\infty] \tag{2.5}$$

and that the sets W have an analogous form with the seminorms of Z. We shall need the following hypotheses:

H1. There are numbers $\mathcal{H}_0, \mathcal{H}$, $0 \leq \mathcal{H}_0 < 1, \mathcal{H}_0 + \mathcal{H} \leq 1$, and for $z\in\Sigma$, linear operators B_z: $F \to Y$, A_z: $Y \to F$, with $\mathcal{N}(B) = 0$, such that y_1, $y_2\in Y_0+S$, $y_1-y_2\in V\in\{V\}$, implies

$$B_z[f(y_1,z)-f(y_2,z)-A_z(y_1-y_2)]\in\mathcal{H}_0(S\cap V), \tag{2.6}$$

$$B_zf(y_0,z)\in\mathcal{H}S,\quad B_zA_z = I. \tag{2.7}$$

H2. For every $V\in\{V\}$ there is a $W\in\{W\}$ such that z_1, $z_2\in\Sigma$, $y\in S$, $z_1-z_2\in W$ imply

$$B_{z_1}f(y,z_1) - B_{z_2}f(y,z_2)\in V. \tag{2.8}$$

H3. Given $\xi > 0$ and $C\in\{C\}$, there is $\sigma = \sigma(\xi,C) > 0$ such that $x_1,x_2\in C$, $\rho(x_1,x_2) \leq \sigma$, $y\in S$, $z\in\Sigma$ imply

$$\|(B_zf(y,z))(x_1)-(B_zf(y,z)(x_2)-y(x_1)+y(x_2)\|_E \leq \xi. \tag{2.9}$$

In the second alternative considered above we must replace (2.9) by

$$\|D^s(B_zf(y,z))(x_1)-D^s(B_zf(y,z))(x_2)-D^sy(x_1)+D^sy(x_2)\|_E \leq \xi,\quad s = 0,1,\dots,N. \tag{2.10}$$

We shall use below the following forms of Ascoli's and Thychonoff's theorems.

Ascoli's Theorem Let Y be the family of all continuous functions on a Hausdorff k-space X into a Hausdorff uniform space E and let Y have the topology of uniform convergence on compacta Then a subset F of Y is compact if and only if (i) F is closed in Y; (ii) the closure of F[x] is compact for every $x \in X$, and (iii) F is equicontinuous on each compact subset of X. (J. L. Kelley and A. P. Morse, see Ref. 2, pp. 234, No. 18).

Tychonoff's fixed point theorem. If Y is a locally convex topological vector space, M is a convex closed subset of Y, and t: $M \to M$ a continuous map of M into itself, then t admits of at least one fixed point in M (Ref. 11, and also 8c, p. 848).

2. THE EXISTENCE OF ψ

THEOREM A

Under hypotheses (H123) there is a continuous function ψ: $X \to E$, $\psi \in Y$, $\psi \in S$, such that $f(\psi,\psi) = 0$.

Proof of theorem A. Since X is metric, X is Hausdorff, satisfies the first axiom of countability, and is a k-space (Ref. 2, pp. 49-50, p. 120, No. 11, and p. 231, No. 13). The linear space E is Banach, hence Hausdorff, and uniform.

Since $Y \subset Z$, we denote by j: $Y \to Z$ the inclusion map, and by $\tilde{y}$, $\tilde{A}$ the images of an element $y \in Y$, or set $A \subset Z$ under j, that is $\tilde{y} = j(y)$, $\tilde{A} = j(A)$. Hypothesis $S \subset \Sigma$ can now be written more precisely in the form $j(S) = \tilde{S} \subset \Sigma$, and (2.4) becomes: given C and $\sigma > 0$ there is $\delta > 0$ such that $y \in Y$, $\|y\|_{CY} \leq \delta$ implies $\|\tilde{y}\|_{CZ} \leq \sigma$.

By Section 1 we know that, for every $z \in \Sigma$ there is exactly one element $\psi_z = \tau z \in S$ such that $f(\psi_z, z) = 0$, and τ : $\Sigma \to S$ is a continuous map (in the topologies of Z and Y).

By Section 1 we know that, for every C, $C \in \{C\}$, and $z \in C'$, the functions ψ_z: $X \to E$ are equicontinuous on C. First note that ψ_z is continuous on X, since ψ_z: $X \to E$ is continuous on each compact subset of X and X is Hausdorff and satisfies the first axiom of countability (Ref. 2, p. 225, No. 7). Secondly, if $C = C'$ and $z \in C$, the functions ψ_z: $C \to E$ admit of a "modulus of continuity," that is, there is some real-valued continuous monotone function $\mu C(\rho)$, $0 \leq \rho < +\infty$, with $\mu C(o) = 0$, such that $x_1, x_2 \in C$, $z \in C$, implies

$$\|\psi_z(x_1) - \psi_z(x_2)\|_E \leq \mu C[\rho(x_1,x_2)]. \qquad (2.11)$$

In the second alternative above we have to replace this by an analogous relation for each $D^s\psi$, $0 \leq s \leq N$.

Let us consider the family $\mathcal{W}$ of all elements $y \in Y$, $y \in S$, satisfying the same relations (2.11), that is, such that x_1, $x_2 \in C$ implies

$$\|y(x_1)-y(x_2)\|_E \leq \mu_C[\rho(x_1,x_2)]. \tag{2.12}$$

Obviously

$$\mathcal{W} \subset S \subset Y, \quad j(\mathcal{W}) = \tilde{\mathcal{W}} \subset \Sigma \subset Z$$

$$\tau: \tilde{\mathcal{W}} \to \mathcal{W}, \quad \tau_j: \mathcal{W} \to \mathcal{W}.$$

Now $\tau: \tilde{\mathcal{W}} \to \mathcal{W}$ is continuous in the topologies of Z and Y, precisely, we know, by Section 1, that given $V \in \{V\}$, there is some $W \in \{W\}$ such that z_1, $z_2 \in \mathcal{W}$, $z_1 - z_2 \in W$ implies $\psi_{z_1} - \psi_{z_2} \in V$. Actually, $z_1 - z_2 \in W$ means

$$\|z_1-z_2\|_{C_s Z} \leq \nu_s', \quad \nu_s' > 0, \quad C_s \in \{C\}, \qquad s = 1,\dots,M,$$

for certain elements $C_s \in \{C\}$ finite in number. By (2.4) there are new numbers $\nu_s > 0$, $s = 1,\dots,M$, such that $\|y_1-y_2\|_{C_s Y} \leq \nu_s$, $s = 1,\dots,M$, implies $\|\tilde{y}_1-\tilde{y}_2\|_{C_s Z} \leq \nu_s'$, $s = 1,\dots,M$, and finally $\psi_{\tilde{y}_1} - \psi_{\tilde{y}_2} \in V$. We shall denote by V_o the neighborhood of 0 in Y defined by $\|y\|_{C_s Y} \leq \nu_s$. We conclude that, given $V \in \{V\}$, we have determined $V_o \in \{V\}$ such that $y_1 - y_2 \in V_o$ implies $(\tau j)(y_1) - (\tau j)(y_2) \in V$. Thus, $\tau j: \mathcal{W} \to \mathcal{W}$ is a continuous map in the topology of Y.

Obviously, $\mathcal{W}$ is a closed subset of Y because of the topology we have chosen in Y. For every $x \in X$, the set $\mathcal{W}[x]$ is a subset of E contained in the sphere $\|e-y_o(x)\|_E \leq b_C$, where C is any of the elements $C \in \{C\}$ with $x \in C$. Since E is locally compact we conclude that $\mathcal{W}[x]$ has a compact closure. Finally, $\mathcal{W}$ is equicontinuous on each compact subset C of X. By Ascoli's Theorem, is compact. Finally, $\mathcal{W}$ is convex, and hence, by Tychonoff's fixed point theorem, τ has at least one fixed element in $\mathcal{W}$, say $\psi = \tau\psi$, and $f(\psi,\psi) = 0$. Theorem A is thereby proved.

3. THE UNIQUENESS OF ψ

We need the further hypothesis

(H4) There is a number $\mathcal{H}$, $0 \leq \mathcal{H}' < 1$, such that $y \in S$, z_1, $z_2 \in S$, $z_1 - z_2 \in V \in \{V\}$ imply

$$B_{z_1}f(y,z_1) - B_{z_2}f(y,z_2) \in (1-\mathcal{K}_0)\mathcal{K}' V \tag{2.13}$$

THEOREM B

Under hypotheses (H1234) there is one and only one function ψ: $X \to E$, $\psi \in S$, such that $f(\psi,\psi) = 0$.

Proof of Theorem B. For every $z \in \Sigma$, the element $\psi_z = \mathcal{T} z$ is the fixed element of the mapping T_z corresponding to (1.4) of Section 1, or

$$T_z y = y - B_z f(y,z). \tag{2.14}$$

Since $S \subset \Sigma$, we take elements z belonging to S (properly, to the image of S in Σ under the inclusion map j: $S \to \Sigma$). For $z_1, z_2 \in S$, $\psi_i = \psi_{z_i} = (\mathcal{T} j) z_i$, $y_i = T_{\tilde{z}_i} y_i = \psi_{z_i}$, $i = 1,2$, $z_1 - z_2 \in V \in \{V\}$, we have

$$\begin{aligned}\psi_1 - \psi_2 &= (\mathcal{T} j)z_1 - (\mathcal{T} j)z_2 = T_{z_1}\psi_1 - T_{z_2}\psi_2 \\ &= - B_{\tilde{z}_1} f(\psi_1,\tilde{z}_1) + B_{\tilde{z}_2} f(\psi_2,\tilde{z}_2) + \psi_1 - \psi_2 \\ &= - B_{\tilde{z}_1}[f(\psi_1,\tilde{z}_1) - f(\psi_2,\tilde{z}_1) - A_{\tilde{z}_1}(\psi_1-\psi_2)] \\ &\quad - B_{z_1} f(\psi_2,\tilde{z}_1) - B_{z_2} f(\psi_2,z_2).\end{aligned}$$

By (H14) we have

$$\psi_1 - \psi_2 \in \mathcal{K}_0 V + (1-\mathcal{K}_0)\mathcal{K}' V = \lambda V, \tag{2.15}$$

where $\lambda = \mathcal{K}_0 + (1-\mathcal{K}_0)\mathcal{K}' < 1$. Now, for every $C \in \{C\}$, let $\nu_C = \|\psi_1-\psi_2\|_{CY}$. Note that V is a set of the form

$$V = [y \in Y \mid \|y\|_{C_s} \le \nu_s,\ \nu_s > 0,\ C_s \in \{C\},\ \ s = 1,\dots,M < +\infty] \subset Y.$$

Hence, $z_1 - z_2 \in V$ implies also $z_1 - z_2 \in V'$ where V' is the same set V above where the numbers ν_s are replaced by numbers $\nu_s + \epsilon_s = \nu_{C_s} + \epsilon_s$, $\epsilon_s > 0$ arbitrary, $s = 1,\dots,M$. Then (2.15) implies

$$\|\psi_1-\psi_2\|_{C_s Y} < \lambda(\|z_1-z_2\|_{C_s Y} + \epsilon_s), \quad s = 1,\dots,M.$$

The numbers ϵ_s being arbitrary, we conclude that

$\|\psi_1-\psi_2\|_{C_sY} \leq \lambda\|z_1-z_2\|_{C_sY}$, $s = 1,\ldots,M$.

This relation essentially says that $\mathcal{T}j$: $S \to S$ has a contraction type property, and this definitely excludes that $\mathcal{T}$ can have more than one fixed point.

Now assume, if possible, that $f(y,y) = 0$ has two solutions $z_1,z_2\epsilon S$. Then, from (2.14), we would deduce $z_i = T_{z_i}z_i$, $i = 1,2$. Since, for every $z\epsilon S$ the equation $y = T_z y$ has only one solution $y = \psi_z$ in S, we conclude that $z_i = (\mathcal{T}j)z_i$, $z_i\epsilon S$, $i = 1,2$, hence $z_1 = z_2$ by (2.15). Theorem B is thereby proved.

4. A LERAY'S TYPE THEOREM

Let us assume that $G(y,z,\lambda)$ depends on y and z as in No. 1 and on a real parameter λ varying in a finite interval $J = [\lambda_0 \leq \lambda \leq \lambda_1]$. We shall assume that it satisfies the properties H123 for every $\lambda\epsilon J$ and uniformly with respect to λ. Thus, for every $\lambda\epsilon J$ there are sets W_λ and $S_{b\lambda}$, and a map $\mathcal{T}$: $W_\lambda xJ \to S_\lambda$, or $y = \mathcal{T}(z,\lambda)$ with $\lambda\epsilon J$, $z\epsilon W_\lambda$, $y\epsilon S_{b\lambda}$, and $G(\mathcal{T}(z,\lambda),z,\lambda) = 0$ for every $\lambda\epsilon J$, $z\epsilon W_\lambda$. Thus the equation $G(y,y,\lambda) = 0$ was actually reduced in No. 7 to the equation $y-\mathcal{T}(y,\lambda) = 0$ as in Leray's Theory, and $\mathcal{T}$ is actually a compact map on W_λ, since it transforms (bounded) sequences in W_λ into compact sequences of elements of $S_{b\lambda}$. In addition $Y\subset Z$, and $S_{b\lambda}\subset W_\lambda$ as in Leray's Theory. Nevertheless we do <u>not</u> assume that Y, Z are Banach spaces, we do <u>not</u> assume that bounded subsets of Y are compact subsets of Z, and we do <u>not</u> assume that G has a Fréchet differential as a function of (y,z,λ), but as a function of y only.

Whenever Y and Z are Banach spaces (as in the sections below), then for every λ the topological degree of the map $y-\mathcal{T}(y,\lambda)$: $S_{b\lambda} \to Y$ can be defined at every point $y\epsilon Y$ which is not the image of points of the boundary $\partial S_{b\lambda}$ of $S_{b\lambda}$.

Let us now assume with Leray that

L1. $y - \mathcal{T}(y,\lambda) = 0$ for every $\lambda\epsilon J$ and $y\epsilon S_{b\lambda}$.

L2. For $\lambda = \lambda_0$ the topological degree of the map $y - \mathcal{T}(y,\lambda_0)$ is known and is a number $m \neq 0$.

Then we have

THEOREM C

Under the hypotheses above, then (i) the topological degree of $y-\mathcal{T}(y,\lambda)$: $S_{b\lambda} \to S_{b\lambda}$ is the same for every $\lambda\epsilon J$; (ii) there is a continuum subset C of $\bigcup_\lambda S_{b\lambda}$ containing at least one element (y,λ) with $y-\mathcal{T}(y,\lambda) = 0$ for every $\lambda \in J$.

SECTION 3. MOSER'S THEOREM FOR THE EQUATION $f(y) = 0$

1. HYPOTHESES

Let X be an open subset of a Euclidean space E_d, let Y be a Banach space of functions $y: X \to E$ with partial derivatives $D^s y$ of all orders $0 \leq s \leq m$. Let $f(y)$ be a (nonlinear) functional defined in a subset Y_o of Y, or $f: Y_o \to E$. The existence of an exact solution y of the equation $f(y) = 0$ in a neighborhood of an approximate solution y_o was proved by J. Moser (Ref. 6) when the usual process involves a "loss of derivatives." We restate below, with few changes, hypotheses, statement, and proof of Moser's Theorem (i), both for the convenience of the reader and in order to emphasize the "uniform" character of the statement. This will enable us to complete (i) with statements (ii) and (iii) of continuous dependence of solutions upon parameter and uniform continuity on X, which are analogous to those of Section 1, and then to obtain (Section 4) existence theorems for solutions of equations of the form $f(y,y) = 0$ analogous to those of Section 2.

Let Y_u, $u = 0,1,\ldots,Y_\infty$ be linear spaces of functions $y: X \to E$ bounded in X and possessing bounded partial derivatives, say for the sake of simplicity $D^s y: X \to E$, of all orders $0 \leq s \leq u$, and $0 \leq x < +\infty$ respectively, and $Y_\infty \subset Y_{u+1} \subset Y_u$. In each Y_u, $u = 0,1,\ldots,$ we take the norm

$$\|y\|_u = \max_{0 \leq s \leq u} \sup_{x \in X} \|D^s y(x)\|_E, \quad y \in Y_u. \tag{3.1}$$

Let T_N be a smoothing operator $T_N: Y_u \to Y_\infty$ depending on the real parameter $N \geq 1$ such that, if $R_N = I - T_N$ (I identity operator, R_N error), we have

(Koa) $\|T_N y\|_{u+v} \leq hN^{v+\delta}\|y\|_u$, $u,v \geq 0$, $y \in Y_u$, $T_{N_y} \in Y_\infty \subset Y_{u+v}$

(Kob) $\|R_N y\|_u \leq hN^{-v+\delta}\|y\|_{u+v}$, $u \geq 0$, $v \geq \delta$, y, $R_N y \in Y_{u+v}$,

for given constants $h > 0$, $\delta \geq 0$, δ integer which are independent of y,u,v.

Below, r, α, β, $\gamma \geq 0$ denote fixed integers, $\alpha+\beta \leq r$, $\gamma \leq r$, and we assume that Y_r is a Banach space.

Let $f(y)$ be a (not necessarily linear) functional defined in a subset Y_r' of Y_r with values in $Y_{r-\alpha}$, or $f: Y_r' \to Y_{r-\alpha}$. We shall also assume that $f: Y_r' \cap Y_s \to Y_{s-\alpha}$ for every $s \geq r$. Thus, f involves the "loss" of α derivatives.

As before, we shall denote by $y_o \epsilon Y_r'$ an approximate solution of the equation $f(y) = 0$. In (K1b) we shall suppose y_o sufficiently smooth, that is, $y_o \epsilon Y_s$ for some $s \geq r$ sufficiently large, and in (K1c) we shall suppose the error $f(y_o)$ sufficiently small. Also, we shall introduce in (K1a) operators L and A (L a right inverse of A as stated in (K1a)), each L and A involving the loss of β and α derivatives respectively.

We shall denote by S_b the subset of Y_r defined by $\|y-y_o\|_r \leq b$ for some $b > 0$. Let K be a given constant $1<k<2$, and take

$$2s = k(\alpha+\beta+\delta), \quad t = k(2s+\beta+\gamma+\delta), \tag{3.2}$$

and let us choose first $\mu > 0$, and then an integer $\lambda \geq 0$, such that

$$\begin{gathered} 2s+(k-2)\mu \leq 0, \quad (t+k^2\mu)+(1-k)\lambda \leq -\lambda \\ 2s+(1-k)\lambda < 0, \quad \lambda \geq \alpha+\beta+\gamma+\delta, \end{gathered} \tag{3.3}$$

(K1a) The integers α, β, γ, δ, $r \geq 0$ and the constant $b > 0$ are so chosen that $S_b \subset Y_r'$, and, for every $y \epsilon S_b \cap Y_{r+\lambda}$, there are linear operators

$L(y)$: $Y_{r+\lambda-\alpha} \rightarrow Y_{r+\lambda-\alpha-\beta}$,

$A(y)$: $Y_s \rightarrow Y_{s-\gamma}$, $s = r+\lambda-\alpha-\beta$, $s = r+\gamma-\alpha$, $s = r$,

such that

$$A(y)L(y)z = z, \quad y \epsilon S_b \cap Y_{r+\lambda}, \quad z \epsilon Y_{r+\lambda-\alpha}, \tag{3.4}$$

$$\|f(y+w)-f(y)-A(y)w\|_{r-\alpha} \leq h_o \|w\|_r^2, \quad y, y+w \epsilon S_b \cap Y_{r+\lambda}, \tag{3.5}$$

$$\|L(y)z\|_{r-\alpha-\beta} \leq h_o \|z\|_{r-\alpha}, \quad y \epsilon S_b \cap Y_{r+\lambda}, \quad z \epsilon Y_{r+\lambda-\alpha}, \tag{3.6}$$

$$\|A(y)z\|_{r-\alpha} \leq h_o \|z\|_{r+\gamma-\alpha}, \quad y \epsilon S_b \cap Y_{r+\lambda}, \quad z \epsilon Y_{r+\gamma-\alpha}, \tag{3.7}$$

$$\|A(y)z\|_{r-\gamma} \leq h_o \|z\|_r, \quad y \epsilon S_b \cap Y_{r+\lambda}, \quad z \epsilon Y_r . \tag{3.8}$$

(K1b) $y_o \epsilon K_{r+\lambda}$, and $y \epsilon S_b \cap Y_{r+\lambda}$, $\|y-y_o\|_{r+\lambda} \leq M$, $M \geq 1$ imply

$$\|L(y)f(y)\|_{r+\lambda-\alpha-\beta} \leq h_o M. \tag{3.9}$$

Let us take

$$c \;=\; \max\,[h^2h_3^3,\ hh_o^2,\ h^2h_o^2,\ h,\ h_o,\ b]. \qquad (3.10)$$

We choose an arbitrary number $N_o > 1$ such that

$$N_o \geq 4c^2,\ 2N_o^{-\mu} \leq 1,\ 2N_o^{\lambda(1-k)} \leq 1,\ 2hh_oN_o^{2s+(1-k)\lambda} \leq 1,\ (2hh_o)^{-1}N_o^{-2s} \leq 1, \qquad (3.11)$$

$$\|L(y_o)f(y_o)\|_{r+\lambda-\alpha-\beta} \leq h_oN_o^{\lambda}\ .$$

(Klc) $\|f(y_o)\|_{r-\alpha} \leq \min\,[(2hh_o)^{-1}N_o^{-2s}b,\ (2chh_o)^{-1}N_o^{-2s}N^{-\mu}]$

2. MOSER'S THEOREM (Ref. 6)

(i) Under hypotheses (KOab), (Klabc) there is at least one function $\psi\epsilon S_b$, $\psi\epsilon Y_r$, such that $f(\psi) = 0$. There is a method of successive approximations y_o, $y_1,\ldots$, starting at y_o which converges in Y_r toward a function ψ as above.

Proof of Moser's Theorem.

Take $N_{n+1} = N_n^k$, $n = 0,1,\ldots$, and

$$y_{n+1} \;=\; y_n - T_{N_{n+1}}\,L(y_n)f(y_n),\ n = 0,1,\ldots\ . \qquad (3.12)$$

By induction we prove that for every n we have

$$y_n\epsilon Y_{r+\lambda},\ \|y_n-y_o\|_r < b,\ \|y_n-y_o\|_{r+\lambda} < N_n^{\lambda},\ \|L(y_n)f(y_n)\|_{r+\lambda-\alpha-\beta} \leq h_oN_n^{\lambda}\ . \qquad (3.13)$$

We use the notations $L_n = f(y_n)$, $A_n = A(y_n)$, $f_n = f(y_n)$, $v_n = -L_nf_n$.

The first three relations (3.13) are obvious for n = 0, and the fourth one is true because of (3.9). Assume $y_\nu\epsilon Y_{r+\lambda}$, $\|y_\nu-y_o\|_r < b$ for r = 0,1,...,n. Then

$$y_{n+1} - y_n\epsilon Y_\infty,\ y_{n+1}\epsilon Y_{r+\lambda},\ f_n\epsilon Y_{r+\lambda-\alpha},\ v_n\epsilon Y_{r+\lambda-\alpha-\beta},$$

$$R_{N_n}v_n\epsilon Y_{r+\lambda-\alpha-\beta},\ A_nv_n \;=\; -\,A_nL_nf_n \;=\; -\,f_n\epsilon Y_{r+\lambda-\alpha}.$$

Then, by (Koa) with $v = \alpha+\beta$, and $u = r-\alpha-\beta$,

$$\|y_{n+1}-y_n\|_r = \|T_{N_{n+1}}v_n\|_r \le hN_{n+1}^{\alpha+\beta+\delta}\|v_n\|_{r-\alpha-\beta} = hN_n^{2s}\|L_nf_n\|_{r-\alpha-\beta}, \tag{3.14}$$

$$\|y_{n+1}-y_n\|_{r+\lambda} = \|T_{N_{n+1}}v_n\|_{r+\lambda} \le hN_{n+1}^{\alpha+\beta+\delta}\|v_n\|_{r+\lambda-\alpha-\beta} = hN_n^{2s}\|L_nf_n\|_{r+\lambda-\alpha-\beta}, \tag{3.15}$$

where 2s is defined by (3.2). By (3.13) we have now

$$\|y_{n+1}-y_n\|_r \le hh_oN_n^{2s}\|f_n\|_{r-\alpha}. \tag{3.16}$$

On the other hand,

$$y_n-y_{n-1} = T_{N_n}v_n = v_{n-1} - R_{N_n}v_{n-1}, \tag{3.17}$$

$$f_{n-1} + A_{n-1}v_{n-1} = f_{n-1} - f_{n-1} = 0, \tag{3.18}$$

$$f_{n-1} + A_{n-1}(y_n-y_{n-1}) = f_{n-1} + A_{n-1}v_{n-1} - A_{n-1}R_{N_n}v_{n-1} = A_{n-1}R_{N_n}v_{n-1}. \tag{3.19}$$

By (3.5), (3.17), (3.7),

$$\begin{aligned}\|f_n\|_{r-\alpha} &= \|f(y_{n-1}+(y_n-y_{n-1}))\|_{r-\alpha} \\ &\le \|f_{n-1}+A_{n-1}(y_n-y_{n-1})\|_{r-\alpha} + h_o\|y_n-y_{n-1}\|_r^2 \\ &\le \|A_{n-1}R_{N_n}v_{n-1}\|_{r-\alpha} + h_o\|y_n-y_{n-1}\|_r^2 \\ &\le h_o\|R_{N_n}v_{n-1}\|_{r-\alpha+\gamma} + h_o\|y_n-y_{n-1}\|_r^2 .\end{aligned}$$

By (Kob) with $u = r-\alpha+\gamma$, $v = \lambda-\beta-\gamma$,

$$\|f_n\|_{r-\alpha} \le hh_oN_n^{-\lambda+\beta+\gamma+\delta}\|v_{n-1}\|_{r+\lambda-\alpha-\beta} + h_o\|y_n-y_{n-1}\|_r^2. \tag{3.20}$$

By (K1c), (3.6), (3.11),

$$\|y_1-y_0\|_r \le hh_oN_0^{2s}\|f_0\|_{r-\alpha} \le \min[b,(2c)^{-1}N^{-\mu}], \tag{3.21}$$

$$\|y_1-y_0\|_{r+\lambda} \le hh_0N_0^{2s}\|f_0\|_{r+\lambda-\alpha-\beta} \le hh_0^2N_0^{\lambda+2s} = (hh_0^2N_0^{2s+(1-k)\lambda})N_1^{\lambda} < N_1^{\lambda}, \quad (3.22)$$

$$\|v_1\|_{r+\lambda-\alpha-\beta} \le h_0N_1. \quad (3.23)$$

Thus, relations (3.13) are proved for n = 0 and n = 1. Assume that they are proved for $0,1,\dots,n$. Then $y_{n+1}\epsilon Y_{r+\alpha}$ as proved above, and by (3.15), (3.13)

$$\|y_{n+1}-y_n\|_{r+\lambda} \le hN_n^{2s}\|v_n\|_{r+\lambda-\alpha-\beta} \le hh_0N_n^{2s+\lambda} = (hh_0N_0^{2s+(1-k)\lambda})N_{n+1}^{\lambda} < N_{n+1}^{\lambda}$$

$$\begin{aligned} \|y_{n+1}-y_0\|_{r+\lambda} &\le \sum_{\nu=0}^{n}\|y_{\nu+1}-y_\nu\| \le \sum_{\nu=0}^{n} N_\nu^{\lambda} \\ &\le N_{n+1}^{\lambda}N_0^{(1-k)\lambda}[1-N_0^{(1-k)\lambda}]< N_{n+1}^{\lambda}, \end{aligned} \quad (3.24)$$

$$\|v_{n+1}\|_{r+\lambda-\alpha-\beta} = \|L_{n+1}f_{n+1}\|_{r+\lambda-\alpha-\beta} \le h_0N_{n+1}^{\lambda}.$$

Finally, (3.14), (3.6), (3.20) yield

$$\begin{aligned} \|y_{n+1}-y_n\|_r &\le hh_0N_n^{2s}[hh_0N_n^{-\lambda+\beta+\gamma+\delta}h_0N_{n-1}^{\lambda}+h_0\|y_n-y_{n-1}\|_r^2] \\ &\le c[N_n^{2s}\|y_n-y_{n-1}\|_r^2+N_n^{-\lambda}N_{n-1}^{\lambda+t}], \end{aligned} \quad (3.25)$$

where c and t are given by (3.2) and (3.12). If δ_n denotes $\delta_n = N_n^{\mu}\|y_n-y_{n-1}\|_r$ then (3.25) becomes

$$\delta_{n+1} \le c[N_n^{2s+(k-2)\mu}\delta_n^2+N_{n-1}^{(t+k^2\mu)+(1-k)\lambda}],$$

where the exponents are ≤ 0 because of (3.3). Hence

$$\delta_{n+1} \le c[\delta_n^2+N_n^{-1}], \quad n = 1,2,\dots ,$$

where $\delta_1 = N^{\mu}\|y_1-y_0\|_r \le (2c)^{-1}$, $N \ge 4c^2$ by (3.11) and (3.21). By induction we prove that $\delta_n \le (2c)^{-1}$ for all n, and hence

$$\|y_{n+1}-y_n\|_r \le (2c)^{-1}N_n^{-\mu},$$

$$\|y_{n+1}-y_0\|_r \le \sum_{\nu=1}^{n}\|y_{\nu+1}-y_\nu\|_r \le (2c)^{-1}\sum_{\nu=1}^{n}N_n^{-\mu} < c^{-1} \le b.$$

Thus, all relations (3.13) are proved for n+1 and, therefore, for every n. Also,

$$\|y_{n+p}-y_n\|_r \leq \sum_{\nu=n}^{n+p-1} \|y_\nu - y_{\nu-1}\|_r < c^{-1} N_n^{-\mu} ,$$

and, therefore,

$$\psi = \lim_{n\to\infty} y_n$$

is an element of Y_r and $\psi \epsilon S_b$. We have

$$\|\psi - y_n\|_r \leq c^{-1} N_n^{-\mu}, \tag{3.26}$$

$$\begin{aligned} \|f(y_n)\|_{r-\alpha} &\leq h^2 h_o^2 N_{n-1}^{2s+(1-k)\lambda} + h h_o^2 \|y_n - y_{n-1}\|_r^2 \\ &\leq c N_{n-1}^{2s+(1-k)\lambda} + N_{n-1}^{-\mu} \end{aligned} \tag{3.27}$$

On the other hand, for every n,

$$\begin{aligned} \|f(\psi)\|_o &= \|f(y_n+(\psi-y_n))\|_o \\ &= \|f(y_n+(\psi-y_n) - f_n - A_n(\psi-y_n) + f_n + A_n(\psi-y_n)\|_o \\ &\leq \|f(y_n+(\psi-y_n)-f_n-A_n(\psi-y_n)\|_o + \|f_n\|_o + \|A_n(\psi-y_n)\|_o \\ &\leq \|f(y_n+(\psi-y_n)-f_n-A_n(\psi-y_n)\|_{r-\alpha} + \|f_n\|_{r-\alpha} + \|A_n(\psi-y_n)\|_{r-\gamma} \\ &\leq h_o\|\psi-y_n\|_r^2 + \|f_n\|_{r-\alpha} + h_o\|\psi-y_n\|_r \\ &\leq c^{-1}N_n^{-2\mu} + cN_{n-1}^{2s+(1-k)\lambda} + N_{n-1}^{-\mu} + N_n^{-\mu} . \end{aligned}$$

Hence, $\|f(\psi)\|_o = 0$, or $f[\psi(x)] \equiv 0$ for every $x \epsilon X$, and finally $f(\psi) = 0$. Theorem (i) is thereby proved.

3. CONTINUITY OF ψ

The continuity of the function ψ of Moser's theorem can be assured under a very simple set of additional hypotheses:

(K2a) The functions $y_o(x)$, $D^s y_o(x)$, $s = 0,1,\dots,r$, or $X \to E$, are uniformly continuous in the topologies of X and E, that is, given $\xi > 0$, there is $\sigma > 0$ such that x, $x' \in X$, $\rho(x,x') \leq \sigma$ imply

$$\|D^s y_o(x) - D^s y_o(x')\|_E \leq \xi, \quad s = 0,1,\dots,r. \qquad (3.28)$$

(K2b) For every $y \in S_b \cap Y_{y+\lambda}$, the functions $L(y)f(y)(x)$, $D^s L(y)f(y)(x)$, $s = 0,1,\dots,r-\alpha$, or $X \to E$ are uniformly continuous, that is, given $\epsilon > 0$ and $y \in S_b x Y_{r+\lambda}$, there is $\sigma' > 0$ (which may depend on y) such that $x,x' \in X$, $\rho(x,x') \leq \sigma'$ imply

$$\|D^s L(y)f(y)(x) - D^s L(y)f(y)(x')\|_E, \quad s = 0,.,\dots,r-\alpha. \qquad (3.29)$$

The same hypotheses when the uniform continuity is required only on every compact subset C of X, will be denoted by $(K2a)_c$, $(K2b)_c$. Also, we shall add to the properties (Koa, (Kob) of the smoothing process T_{N_n} the following one:

(Koc) If $\omega(\rho)$, $0 \leq \rho < \infty$, is a real-valued continuous monotone function with $\omega(o) = 0$, and

$$\|D\ y(x) - D\ y(x')\| \leq \omega[\rho(x,x')], \quad x,x' \in X, \ 0 \leq s \leq r-v,$$

then

$$\|D^s T_N y(x) - D^s T_N y(x')\|_E \leq hN^{\nu+\delta} \omega[\rho(x,x')], \quad x,x' \in X, \ 0 \leq s \leq r. \qquad (3.30)$$

(ii) Under hypotheses (K1abc) and (K2abc), the functions $\psi(x)$, $D^s\psi(x)$, $s = 0,1,\dots,r$, are uniformly continuous on X. If (K2ab) is replaced by $(K2ab)_c$, then $\psi(x)$, $D^s\psi(x)$, $s = 0,1,\dots,r$, are uniformly continuous on each compact subset C of X.

Let us prove first that the functions $y_1(x)$, $D^s y_1(x)$, $s = 0,1,\dots,r$, are uniformly continuous function of x in X. Indeed by (K2b) there is a function $\omega(\rho)$ as in (K2c) such that $x,x' \in X$, $s = 0,1,\dots,r-\alpha-\beta$, imply

$$\|D^s L(y_o)f(y_o)(x) - D^s L(y_o)f(y_o)(x')\|_E \leq \omega[\rho(x,x')],$$

Then given $\xi > 0$, let σ be a number as in (K2a), and σ' a number such that $\omega(\sigma') \leq \xi$. Then for $x,x' \in X$, $\rho(x,x') \leq \sigma'' = \min[\sigma,\sigma']$, we have, for $L_o = f(y_o)$, $f_o = f(y_o)$, and $s = 0,1,\dots,r$,

$$\begin{aligned}\|D^s y_1(x)-D^s y_1(x')\|_E &= \|D^s y_o(x)-D^s y_o(x')+D^s T_{N_o}(L_o f_o)(x)-D^s T_{N_o}(L_o f_o)(x')\|_E \\ &\le \|D^s y_o(x)-D^s y_o(x')\|_E + \|D^s T_{N_o}[(L_o f_o)x-(L_o f_o)x']\|_E \\ &\le \xi + hN_o^{\alpha+\beta+\delta}\ \omega[\rho(x,x')] \\ &\le \xi + hN_o^{\alpha+\beta+\delta}\ \omega(\sigma') \le \xi[1+hN_o^{\alpha+\beta+\delta}] .\end{aligned}$$

This proves the uniform continuity of $y_1(x)$, $D^s y_1(x)$, $x\epsilon X$, $s = 0,1,\dots,r$, in X. By induction, we can prove that, for each n, the functions $y_n(x)$, $D^s y_n(x)$, $x\epsilon X$, $s = 0,1,\dots,r$, are uniformly continuous on X (uniformly in X, not uniformly with respect to n). Since

$$\psi = y_o + \sum_{\nu=0}^{\infty}(y_{\nu+1}-y_\nu),$$

where $\|y_{\nu+1}-y_\nu\|_r \le \epsilon_\nu$, ϵ_ν numbers independent of n with $\sum\epsilon_\nu < +\infty$, we conclude that the functions $\psi(x)$, $D^s\psi(x)$, $x\epsilon X$, $s = 0,1,\dots,r$, are uniformly continuous on X.

4. CONTINUOUS DEPENDENCE OF ψ ON PARAMETERS

Let us assume now that f depends also on a parameter z varying in a topological space Z_o. To simplify we may assume that Z_o is a subset of a metric space with distance function Δ. We assume that all hypotheses of No. 1, in particular (K1abc) holds for every $z\epsilon Z_o$ and uniformly, that is, with the same constants $r,\alpha,\dots,c,N_o$. We may assume that $y_o(z)$ be either a constant, or vary with z in Z_o. Then the element $\psi\epsilon S_b$ determined univocally by the method of successive approximations described above is a continuous function of z. To prove this we need only the simple assumption of uniform continuity:

(K3) Given $\xi > 0$ there is a $\sigma > 0$ such that $z, z'\epsilon Z_o$, $y,y'\epsilon S_b$, $\|z-z''\|_Z < \sigma$, $\|y-y'\|_r \le \sigma$ imply

$$\|L(y,z)f(y,z)-L(y',z')f(y',z')\|_{r-\alpha-\beta} \le \sigma,$$

$$\|y_o(z)-y_o'(z)\|_r \le \sigma.$$

The same hypothesis when z,z' are restricted to belong to the same compact subset C of Ω, (C arbitrary), will be denoted by $(K2)_c$.

(iii) Under hypotheses (K1abc) and (K3), the element ψ determined by the process of successive approximations (3.12) is a uniformly continuous function of z in Z_0 (in the topologies of Y_r and Z). Under hypotheses (K1abc) and $(K2)_c$, ψ is a uniformly continuous function of z on every compact subset C of Z.

The proof is the same as above.

Under hypotheses (K1abc), and (K2abc) uniformly in z, and (K3a) uniformly in x, we could now assume that the functions $\psi(x,z)$, $D^s\psi(x,z)$, s = 0, 1,...,r, or $X \times Z_0 \to E$, are uniformly continuous on $X \times Z_0$.

SECTION 4. FURTHER RESULTS ON EQUATION $f(y,y) = 0$

1. HYPOTHESES

Let the spaces Y_u, $u = 1,2,\ldots$, be the same normed linear spaces and Y_∞ a linear space, $Y_\infty \subset Y_{u+1} \subset Y_u \subset Y_o$, as in No. 1 of Section 3. Let T_N, R_N be the same smoothing and error operators as in No. 1 of Section 3, with constants $h > 0$, and integer $\delta \geq 0$.

Let $r,\alpha,\beta,\gamma \geq 0$ be integers as in No. 1 of Section 3, $\alpha+\beta \leq r$, $\gamma \leq r$, and assume that Y_r be a Banach space. Let $r',\alpha' \geq 0$ be new integers with $0 \leq r-r' \leq \alpha-\alpha'$. Then $0 \leq r-\alpha \leq r'-\alpha'$, $r' \leq r$, and $Y_{r'} \supset Y_r$. Also, if $y \epsilon Y_r$, then $y \epsilon Y_{r'}$, and

$$\|y\|_r \geq \|y\|_{r'} \tag{4.1}$$

Let $f(y,z)$ denote a (not necessarily linear) functional defined for all y of a subset Y' of Y_r and all z of a subset Z' of $Y_{r'}$ with $Y' \subset Z'$, and values in $Y_{r-\alpha}$, or f: $Y' x Z' \to Y_{r-\alpha}$. We shall also assume $f:(Y' \cap Y_{r+\ell}) x (Z' \cap Y_{r+\ell}) \to Y_{r+\ell-\alpha}$ for all $\ell \geq 0$. Actually, we need this last statement only for all $0 \leq \ell \leq \gamma$, where γ is the integer which is defined below. Thus f involves the "loss" of α derivatives on y.

Let us denote by y_o a given element of Y', hence $y_o \epsilon Y' \subset Z'$. Let S_b the usual set $S_b = [y \epsilon Y_r \| \|y-y_o\|_r \leq b]$, $b > 0$, for which we shall assume $S_b \subset Y' \subset Y_r$. We shall consider a subset Σ of Y_r', with S_b Σ and $\Sigma \subset Z' \subset Y_r'$. We shall denote by $r,\alpha,\beta,\gamma,\delta,\mu,\lambda,h,h_o,c,N_o$ constants as in No. 1 of Section 3, and we assume

(L1abc). The same as (K1abc) for every $z \epsilon \Sigma \cap W'_{r+\lambda}$.

(L2abc). The same as (K2abc) for every $z \epsilon \Sigma \cap W'_{r+\lambda}$ and uniformly in $\Sigma \cap W_{r+\lambda}$.

(L3). The same as (K3) for every $z \epsilon \Sigma \cap W_r$.

Theorem C. Under hypotheses (L1abc)(L2abc)(L3), there is at least one function $\psi \epsilon S_b$ such that $f(\psi,\psi) = 0$.

Proof of Theorem C. For every $z \epsilon \Sigma \cap W_{r'+\lambda}$ there is a unique element $\psi_z = \mathcal{T} z$ which is the fixed point of the map T_z, on $\psi_z = T_z \psi_z$, $\psi_z \epsilon S$, as in Section 3. The map $\mathcal{T}$: $\Sigma \cap W_{r'+\lambda} \to S_b$ so defined is continuous in the topologies of Y_r and $Y_{r'}$. Thus, we can extend $\mathcal{T}$ to all of Σ in the topology of Y_r and $Y_{r'}$, into a map $\mathcal{T}$: $\Sigma \to S$ which is continuous in these topologies.

Since $X \subset \Sigma$, we have also $\mathcal{T}$: $S \to S$, and relation (K3) assures that $\mathcal{T}$: $S \to S$ is a continuous map also when we use the topology of Y for both S as a domain and range. The hypotheses (L) are worded so that the functions ψ_z of the range of $\mathcal{T}$ are uniformly continuous in X together with all their partial derivatives $D^s y$, $0 \leq s \leq N$. In other words, the functions ψ_y with their partial derivatives $D^s \psi_y$, $0 \leq s \leq N$, admit of a unique "modulus of continuity" $\mu(\rho)$ in X. Moreover X is a bounded open subset of E_d, hence $\bar{X}$ is compact, and by Ref. 12, the functions ψ_z: $X \to E$ and their partial derivatives $D^s \psi_z$, $0 \leq s \leq N$, admit of a continuous extension in $\bar{X}$ with the same modulus of continuity μ, and we shall use for them the same notation ψ_z, $D^s \psi_z$. Finally, we define the class $\mathcal{W}$ of functions y: $\bar{X} \to E$, with $y \in S$, having the same modulus of continuity μ. Then $\mathcal{W}_j$: $\mathcal{W} \to \mathcal{W}$, and $\mathcal{W}$ is a complete class in the topology of Y by the same Ascoli Theorem mentioned in No. 2 of Section 2 since $\bar{X}$ now is compact. The existence of ψ follows now from Tychonoff's Theorem of No. 2 of Section 2 (as well as Schauder's Theorem).

SECTION 5. SIMPLE APPLICATIONS TO THE EQUATION $f(x,y(x),y) = 0$

1. FIRST APPLICATION

In the situation considered in Section 2, when $N = 0$, and $f(y,z)$ depends only on the evaluation $y(x)$ of y at x with $y \in Y$, and on the function $z \in X$, it is convenient to deduce from (A) and (B) statements which are much easier to apply, though they are more restrictive.

Let X be a compact matrix space with distance function ρ, let E, E_n be a Euclidean space, and $Y = Z$ the space of all continuous (bounded) functions

$y: X \to E_n$, or $y = (y_1,\dots,y_n)$, with norm

$$\|y\| = \|y\|_U = \max_{x \in X} \|y(x)\|_{E_n}$$

Let $y_o = (y_{1o},\dots,y_{no})$ be a point of E_n, S the sphere of center y_o and radius $b > 0$, Σ the set of all functions $y \in Y$ with values $y(x) \in S$. Let $f(x,y,z)$, or $f: X \times S \times \Sigma \to E_n$ be a given functional. In this situation, the Fréchet differential of f with respect to y is an $n \times n$ matrix $A(x,y,z)$ depending on x and z, so that, if we replace y by any given function $y \in Y$, we obtain a matrix $A(x,y(x),z)$, depending on x and z only. Instead of the operators A, L of Sections 1 and 2 we may take matrices A_{xz}, A_{xz}^{-1}. We shall denote by y_o also the constant function $y_o(x) = y_o$, $x \in X$.

Note that, with these conventions, the subset of functions $y \in Y$ denoted in Section 2 by S, coincides with Σ.

(H1) There are constants $M > 0$, $\gamma \geq 0$, $\mathscr{H} \geq 0$, with $M_\gamma < 1, \mathscr{H} \leq 1-M_\gamma$, such that

$$\|f(x,y_1,z)-f(x,y_2,z)-A_{xz}(y_1-y_2)\|_E \leq \gamma\|y_1-y_2\|_E \tag{5.1}$$

$$\|f(x,y_o,z)\|_E \leq b\mathscr{H}\|M, \quad \|A_{xz}^{-1}\| \leq M, \tag{5.2}$$

for all $x \in X$, $z \in \Sigma$, $y_1,y_2 \in S$.

Hypothesis (5.2) can be further replaced by the hypotheses

$\mathscr{H}_1 \geq 0, \mathscr{H}_2 \geq 0, \mathscr{H}_1 + \mathscr{H}_2 \leq \mathscr{H} \leq 1-M_\gamma$,

$$\|f(x,y_0(x),y_0)\|_E \le b\chi_1/M, \tag{5.3}$$

$$\|f(x,y_0(x),z)-f(x,y_0(x),y_0)\|_E \le b\,\mathscr{K}_2/M. \tag{5.4}$$

for all $x\epsilon X$, $z\epsilon\Sigma$. Then (5.3) states a bound for the admissible error to which the "approximate" solution $y_0(x) = y_0$ satisfies the equation $f(x,y(x),y) = 0$, and (5.4) states a bound for the admissible deviation of $f(x,y_0(x),z)$ from $f(x,y_0(x),y_0)$ when z describes Σ.

(H2)' $\|f(x,y,z)\|_E \le P$, $x\epsilon X$, $y\epsilon S$, $z\epsilon\Sigma$.

(H3)' Given $\xi > 0$ there is $\sigma > 0$ such that $x_1,x_2\epsilon X$, $y\epsilon S_b$, $z_1,z_2\epsilon\Sigma$, $\rho(x_1,x_2)\epsilon\sigma$, $\|z_1-z_2\|_Y \le \sigma$ imply

$$\|f(x_1,y,z_1)-f(x_2,y,z_2)\|_E \le \xi, \tag{5.5}$$

$$\|A^{-1}_{x_1z_1}-A^{-1}_{x_2z_2}\| \le \xi. \tag{5.6}$$

As a corollary of Theorem A we have

Theorem D. Under hypotheses (H123)' there is at least one continuous function ψ: $X \to E$, such that

$$\psi(x)\epsilon S,\ f(x,y(x),y) = 0,\ x\epsilon X. \tag{5.7}$$

Also, we shall need the hypothesis

(H4)' There is a number $\mathscr{K}'$, $0 \le \mathscr{K} < 1$, such that $x\epsilon X$, $y\epsilon S$, $z_1,z_2\epsilon\Sigma$, imply

$$\|f(x,y,z_1)-f(x,y,z_2)\|_E \le ((1-M_\gamma)\,\mathscr{K}/M)\|z_1-z_2\|_Y \tag{5.8}$$

Then, as a corollary of Theorem B, we have

Theorem E. Under hypotheses (H1234)' there is one and only one continuous function ψ: $X \to E$, such that (5.7) holds.

Remark 1. Under hypotheses (H123)' of Theorem D, if we know that for some $x_0\epsilon X$ we have $f(x_0,y_0,z) = 0$ for all $z\epsilon\Sigma$, then, among the functions ψ satisfying (5.7) there is at least one ψ with $\psi(x_0) = y_0$.

2. SECOND APPLICATION

As an application of Theorem D we consider a functional $f = (f_1,\dots,f_n)$, or $f(t,y;z(\xi),\ 0 \le \xi < t)$ with values in E_n, depending on the real variable t, $0 \le t \le t_1$, the real vector $y \in E_n$ in some set $Y_0 \subset E_n$, and the values in E_n taken by a function $z\colon [0,t_1] \to E_n$ of some family Y_0 in the interval $0 \le \xi \le t$. As a corollary of Theorem D we shall prove under conditions the existence of at least one function

$\psi(t)$, $0 \le t \le a$, $a > 0$ sufficiently small so that

$$f(t,\psi(t);\ \psi(\xi),\ 0 \le \xi \le t) = 0, \quad t \in I = [0,a]. \tag{5.9}$$

For the sake of simplicity we write $f(t,y,z)$, and then (5.9) takes the form $f(t,\psi(t),\psi) = 0$.

Let $\mu = (\mu_1,\dots,\mu_n)$ be a point of E_n, and Y_0 the sphere in E_n of center μ and radius b_0 for some $b_0 \ge 0$. Let Z_0 be the family of all continuous functions $z(\xi)$, $0 \le \xi \le a$, with values in Y_0 and some a_0, $0 < a_0 \le t_1$. Suppose $f\colon IxY_0xZ_0 \to E_n$. We shall assume that, for $t = 0$, f does <u>not</u> depend on z, and we may write $f(0,y,\cdot)$. As in the implicit function theorem we shall assume $f(0,\mu,\cdot) = 0$. We shall take for the continuous functions $z\colon I \to E_n$ the norm $\|z\| = \max|z(\xi)|$ for $\xi \in I$.

Corollary 1. If $f\colon IxY_0xZ_0 \to E_n$ is bounded and continuous in IxY_0xZ_0, and possesses first order partial derivatives $a_{ij}(t,y,z) = \partial f_i/\partial y_j$, $i,j = 1,\dots,n$, which also are bounded and continuous on IxY_0xZ_0, if $f(0,\mu,\cdot)=0$, and det $A_0=0$, with $A_0 = [a_{ij}(0,\mu,\cdot)]$, then there is some $a > 0$, $b > 0$, and a continuous function $\psi(t)$, $0 \le t \le a$, such that $\psi(o) = \mu$, $|\psi(t)-\mu| \le b$, and $f(t,\psi(t),\psi)=0$ for all $0 \le t \le a$.

Proof. Obviously, $a_{ij}(0,y,\cdot)$ does not depend on the third argument. Let us take for y_0 the constant function $y_0(t) = y_0$, $0 \le t \le a$. We have

$$f_i(t,y_2,z)-f_i(t,y_1,z)-A_{0i}(y_2-y_1)$$
$$= \int_0^1 \textstyle\sum_j [a_{ij}(t,y_1+\tau(y_2-y_1),z)-a_{ij}(0,\mu,\cdot)](y_{2j}-y_{1j})d\tau,$$

where A_{0i} is the ith rate of A_0, and $y_1 = (y_{11},\dots,y_{1n})$, $y_2 = (y_{21},\dots,y_{2n})$. Let $M = \|A^{-1}\|$ be the spectral norm of A^{-1}, and take $\gamma = 1/2\ M$, so that $M\gamma = 1/2$. Then, take b and a, $0 < b \le b_0$, $0 < a \le a_0$, sufficiently small so that

$$|a_{ij}(t,y_1+\tau(y_2-y_1),z)-a_{ij}(0,\mu,\cdot)| \le 1/2\ n^2$$

for all $0 \leq t \leq a$, all y_1, y_2 with $|y_i - \mu| \leq b$, $i = 1,2$, and all z with $|z(t)-y_o(t)| \leq b$, $0 \leq t \leq a$. Then, by Schwarz inequality,

$$|f_i(t,y_2,z)-f_i(t,y_1,z)-A_{oi}(y - y_1)| \leq (2n)^{-1}|y_2-y_1|, \; i = 1,\ldots,n,$$

$$|f(t,y_2,z)-f(t,y_1,z)-A_o(y_2-y_1)| \leq 2^{-1}|y_2-y_1|.$$

Finally, we may restrict a further, if needed, so that

$$|f(t,\mu,z)-f(0,\mu,\cdot)| \leq b/2M, \; 0 \leq t \leq a.$$

Then $|f(t,\mu,z)| \leq b/2M = (1-M\gamma)b/M$. All conditions of Theorem D are satisfied for the restricted interval $X = I = [0,a]$.

We shall need the previous statement in Ref. 1. Concerning the uniqueness of the function ψ we deduce, from Theorem B, the

Corollary 2. Under the hypotheses of the previous corollary, if there are numbers a', b', $0 < a' \leq a$, $0 < b' \leq b$, such that $0 \leq t \leq a$, $|y-\mu| \leq b'$, $|z_1(t)-\mu| \leq b'$, $|z_2(t)-\mu| \leq b'$ for $0 \leq t \leq a'$, imply

$$|f(t,y,z_1)-f(t,y,z_2)| \leq (1/2M) \max_{0\leq\xi\leq t} |z_1(\xi)-z_2(\psi)|, \tag{5.10}$$

the function $\psi(t)$, $0 \leq t \leq a'$, of the previous corollary is unique.

3. A PARTICULAR CASE

As a particular case of the previous corollaries we may consider functions $f,g\colon I \times E_n \times E_n \to E_n$, and the functional equation in $\psi\colon I \to E_n$,

$$f(t,\psi(t), \int_0^t g(t,\tau,\psi(\tau))d\tau) = 0. \tag{5.11}$$

Here we assume that $f(t,u,v) = (f_1,\ldots,f_n), g(t,u,v) = (g_1,\ldots,g_n)$ are continuous in a set $0 \leq t \leq a_o$, $|u-u_o| \leq b_o$, $|v-u_o| \leq b$, that $|g| \leq N$ in this set and the partial derivatives $\partial f_i/\partial y_j$ exist and are continuous, and that $\det A_o \neq 0$, $A_o = f_u(o,u_o,o) \neq 0$. If $M = \|A_o^{-1}\|$, $f(o,u_o,o) = 0$, and $\mathscr{B}$, γ are chosen so that $M\gamma < 1$, $0 < \mathscr{B} = 1 - M\gamma$, then in a set $0 \leq t \leq a$, $|u-u_o| \leq b$, $|v-v_o| \leq b$, and a, b sufficiently small the relations

$|f(t,u_1,v)-f(t,u_z,v)-A_0(u_1-u_2)| \leq \gamma|u_1-u_2|,$

$f(t,u_0,v)\| \leq \mathscr{B}b/M,$

$|f(t,u_0,\int_0^t g(t,\tau,z(\tau))d\tau)| \leq \mathscr{B}b,$

are satisfied for $|u_1-u_0|$, $|u_2-u_0|$, $|v-u_0| \leq b$, and all continuous functions $z(\tau)$, $0 \leq \tau \leq a$, with values $|z(\tau)-u_0| \leq b$. Then the existence of at least one solution to (5.11) follows from corollary 1.

4. A KNOWN PARTICULAR CASE

For $f = u-u_0-v$ (5.11) reduces to the equation

$$\psi(t) = u_0 + \int_0^t g(t,\tau,\psi(\tau))d\tau. \tag{5.12}$$

Then we can take $A = I$, $M = 1$, $\gamma = 0$, $\mathscr{B} = 1$, and a solution in the small certainly exists.

The differential system with initial condition

$$dy/dt = G(t,y), \quad y(o) = u_0$$

reduces to the equation

$$\psi(t) = u_0 + \int_0^t G(\tau,\psi(\tau))d\tau$$

of the type (5.12). The usual example $dy/dt = y^{1/2}$, $u_0 = 0$, for which the solution is not unique, shows that the conditions of Theorems A and D do not assure uniqueness. The existence analysis of the problems of the present number in terms of fixed point theorems is Schauder's.

5. ANOTHER PARTICULAR CASE

Instead of (5.11), we may consider the integral equation

$$f(x,\psi(x), \int_V g(x,\xi,\psi(\xi))d\xi) = 0 \tag{5.13}$$

where $x = (x_1,\ldots,x_m)\epsilon E_m$, $\psi(x)\epsilon E_n$, $f(o,u_0,o) = 0$, and V is the variable sphere

$$V = [\xi \in E_m | \|\xi\| \leq |x|]$$

Analogous results hold for the integral equation (5.13) where $V = V(x,h)$ is the variable sphere

$$V = V(x,h) = [\xi \in E_m | \ |\xi - x| \leq h]$$

and $h > 0$ is a fixed number sufficiently small.

6. ANOTHER APPLICATION

As an application of Theorem D, let us consider the integral equation

$$f(t,\psi(t),\int_{t_0}^{t_1} g(t,\tau,\psi(\tau))d\tau) = 0 \qquad (5.14)$$

where $I = [t_0 \leq t \leq t_1]$ is a fixed interval of the real axis, $f(t,u,v) = (f_1,\ldots,f_n)$, $g(t,u,v) = (g_1,\ldots,g_n)$ are continuous functions f,g: $I x E_n x E_n \to E_n$, and an "approximate" solution $z_0(t)$, $t \in I$, is known, with values in the sphere $S = [u \in E_n | \ |u-u_0| \leq b]$ for some $b > 0$. We shall assume that f and g are continuous in the set $H = IxSxS$, and f has continuous first order partial derivatives. If f_u is the nxn matrix of these derivatives, assume that set $f_u \neq 0$ in H. Let $A_0 = f_u(t,z_0(t),v)$, and assume that

$$|f(t,u_1,v)-f(t,u_2,v)-A_0(u_1-u_2)| \leq \gamma|u_1-u_2|,$$

$$|g(t,\tau,u)| \leq N, \ \|f_u^{-1}(t,u,v)\| \leq M,$$

$$|f(t,z_0(t),\int_{t_0}^{t_1} g(t,\tau,z_0(\tau))d\tau| \leq b\mathscr{K}_1/M,$$

$$|f(t,z_0(t),v)-f(t,z_0(t),\int_{t_0}^{t_1} g(t,\tau,z_0(\tau))d\tau) \leq b\mathscr{K}_2/M$$

for all $t,\tau \in I$, $u,v \in S$, where $\gamma, \mathscr{K}_1, \mathscr{K}_2 \geq 0$, $M\gamma < 1$, $\mathscr{K}_1 + \mathscr{K}_2 \leq 1 - M\gamma$. Then, by Theorem D, there is at least one solution ψ for equation (5.14).

As particular cases of (5.14), we may consider the Uryshon equation

$$\psi(t) = h(t) + \int_{t_0}^{t_1} g(t,\tau,\psi(\tau))d\tau,$$

and the Harmerstein equation

$$\psi(t) = h(t) + \int_{t_0}^{t_1} g(t,\tau)k(\psi(\tau))d\tau.$$

7. GENERALIZATION

Instead of (5.14) we may consider the equation

$$f(x,\psi(x),\int_{G_0} g(x,\xi,\psi(\xi))d\xi) = 0, \quad x \in G_0,$$

where $x = (x_1,\ldots,x_n) \in G_0 \subset E_m$, f,g: $G_0 x E_n x E_n \to E_n$, where G_0 is a compact subset of E_m. The conditions for existence are analogous.

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