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INTERVAL FUNCTIONS

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ADDENDUM V. INTERVAL FUNCTIONS

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We give below a brief but self-contained account of the Banach theorems on the derivatives of normal interval functions. For the sake of simplicity we shall deal with interval functions on the Euclidean xy-plane E_2, though arguments and results hold in any E_n.

V 1. NORMAL INTERVAL FUNCTIONS

We shall deal with a subset A of the xy-plane E_2, with certain collections $\{G\}$, $\{I\}$, $\{q\}$, $\{q\} \subset \{I\}$ of subsets of A and with a set function F(I) defined for all $I \in \{I\}$, $-\infty \leq F(I) \leq +\infty$. We shall refer to A as an open set of E_2; $\{G\}$ as the collection of all open sets $G \subset A$; $\{I\}$ as the collection of all closed intervals $I = [a \leq x \leq a+\ell,\ b \leq y \leq b+\ell']$, $I \subset A$; $\{q\}$ as the collection of all intervals $I \in \{I\}$ which are squares, i.e., $\ell = \ell'$, F(I) any real single-valued interval function defined for all $I \in \{I\}$. (See Remark 1.)

The function F(I), $I \in \{I\}$ is said to be subadditive, overadditive, additive in A if for every $I \in \{I\}$ and every finite subdivision S of I into (nonoverlapping) sets $I' \in \{I\}$ we have, respectively, $F(I) \leq$, or $\geq$, or $=$ $\sum F(I')$, where $\sum$ ranges over all $I' \in S$, where we use the usual conventions for the sum of real numbers (even $+\infty$ and $-\infty$) and where we suppose explicitly that never may happen to add $+\infty$ and $-\infty$. All subadditive, overadditive, and additive functions are said to be <u>normal</u> functions. Obviously the additive functions are both subadditive and overadditive; if F is subadditive [overadditive], then $-F$ is overadditive [subadditive]. We shall denote by $|A|$ the usual plane measure of any set $A \subset E_2$.

<u>Remark</u>. The collections $\{G\}$, $\{I\}$, $\{q\}$ we have mentioned may have meaning different from the ones we have designated. For instance, the set A above could be a closed domain, and $\{G\}$ the collection of all sets $G \subset A$ open in A. The theorems below are still valid even if the aforementioned collections have different meaning. For instance:

(1) The collection $\{I\}$ could be the collection $\{\pi\}$ of all simple closed polygonal regions $\pi \subset A$; or of all simple and not simple polygonal regions $\pi \subset A$.

(2) The collection q could be the collection of all intervals $I \in \{I\}$ with sides parallel to the axis and side lengths ℓ, ℓ' such that $1/\lambda \leq \ell/\ell' \leq \lambda$, where λ, $1 \leq \lambda < +\infty$, is any fixed number (coefficient of regularity).

(3) The collection $\{I\}$ could be the family $\{I\}_R$ of all intervals $I = [a \leq x \leq a+\ell,\ b \leq y \leq b+\ell'] \subset A$, where a, $a+\ell$ and b, $b+\ell'$ are elements of given sets $\{\xi\}$, $\{\eta\}$ of real numbers everywhere dense. In this case we could take for $\{q\}$ the collection of all $I \in \{I\}_R$ of a given coefficient of regularity $\lambda \geq 1$.

For other possible cases of some relevance we refer to the original paper by S. Banach [9b], or the books by S. Kempisty [63] and H. Hahn and A. Rosenthal [53].

V 2. BV AND AC INTERVAL FUNCTIONS

Given any function $F(I)$, $I \in \{I\}$, as in V 1, let S denote any finite system of nonoverlapping sets $I \in \{I\}$, $I \subset A$. By total variation of $F(I)$ in A we shall denote the number

$$V(A) = V(A,F) = \sup_{S} \sum_{I \in S} |F(I)|.$$

Since the same definition holds if we replace A by any of the sets $G \in \{G\}$, or $I \in \{I\}$, we have defined a set function $V(G)$, $G \in \{G\}$, or $V(I)$, $I \in \{I\}$. Obviously, we have $|F(I)| \leq V(I)$ and $V(I) \geq 0$ for all $I \in \{I\}$. Moreover, V is monotone increasing in $\{I\}$ and in $\{G\}$, that is, $I \subset I'$, $I, I' \in \{I\}$ implies $V(I) \leq V(I')$, and the same holds for the sets G. We shall denote the set function $V(I)$ as the total variation (set function) of $F(I)$.

The function $F(I)$ is said to be of bounded variation in A, or BV in A, if $V(A) < +\infty$. The function $F(I)$ is said to be absolutely continuous in A, or AC in A, if

(a) Given $\varepsilon > 0$ there is a $\delta > 0$ such that for all finite systems S of sets $I \in \{I\}$ we have $\sum |F(I)| < \varepsilon$, provided $\sum |I| < \delta$;

(b) $F(I)$ is additive in A.

Remark. In the definition of absolute continuity S. Banach [9b] requires only condition (a), but he adds condition (b) in the main statements.

L. Cesari [24e, pp. 216-17] proved by means of examples that conditions (a) and (b) are independent.

V 3. SOME PRELIMINARY STATEMENTS

(V 3.i) If $F(I)$ is any arbitrary interval function and $V(I)$ the corresponding total variation, then for every sequence G, G_i, $i = 1,2,\ldots$, of sets G, $G_i \in \{G\}$ (V 1) with $G_i \cap G_j = 0$, $i \neq j$, $\bigcup_i G_i \subset G$, we have $\sum_i V(G_i) \leq V(G)$.

(V 3.ii) If $F(I)$ is any arbitrary function, then the corresponding total variation $V(I)$, $I \in \{I\}$, is overadditive.

The proofs are left as exercises for the reader. Note that if a function $F(I)$ is nonnegative and overadditive, then $F(I) = V(I)$.

(V 3.iii) Each BV normal function $F(I)$ is the difference of two nonnegative overadditive function.

Proof. If $F(I)$ is overadditive, then $F(I) = [V(I) + F(I)] - V(I)$ where both $F+V$ and V are nonnegative overadditive. If $F(I)$ is subadditive, then $-F(I)$ is overadditive and $F(I) = V(I) - [V(I) - F(I)]$, where both V and $V-F$ are nonnegative and overadditive.

(V 3.iv) A function $F(I)$ satisfies (a) if and only if $V(I)$ satisfies (a).

(V 3.v) Any AC function $F(I)$ is BV, provided $|A| < +\infty$.

The proofs are left as exercises for the reader.

V 4. VITALI'S COVERING THEOREM

A subset E of E_2 is said to have parameter of regularity r(E) if r(E) is the supremum of the quotients $|E|/|J|$ when J denotes any square containing E. Thus $0 \le r(E) \le 1$, and $r(Q) = 1$ for every square Q. Also, we denote by $\delta(E)$ the diameter of a set E.

We shall now consider a fixed subset A of E_2 and a family $\mathcal{C} = \{E\}$ of closed sets with the following properties: (I) $r(E) \ge \alpha > 0$ for a given constant $\alpha > 0$; (II) for every point $p \in A$ there is a sequence $[E_k, k = 1,2,\ldots]$ of sets $E_k \in \mathcal{C}$ with $p \in E_k$, $k = 1,2,\ldots$, and $\delta(E_k) \to 0$ as $k \to \infty$.

(V 4.i) (a form of Vitali's covering theorem). Under hypotheses (I) and (II) there is a finite or countable collection $[E_i, i = 1,2,\ldots]$ of sets $E_i \in \mathcal{C}$ such that $E_i \cap E_j = \emptyset$ for $i \ne j$, and $|A - \cup_i E_i| = 0$. Moreover, given any $\varepsilon > 0$ there is a finite system $[E_1,\ldots,E_N]$ of sets $E_i \in \mathcal{C}$ such that $E_i \cap E_j = \emptyset$ for $i \ne j$, and $|A - \cup_{i=1}^N E_i| < \varepsilon$.

<u>Proof</u>. First, let us suppose that A is bounded, that is, contained in some fixed square $Q \subset E_2$. We may well assume that all sets E are also contained in Q. Now we define the sequence $[E_i, i = 1,2,\ldots]$ by the following induction process. For E_1 we take an arbitrary set $E_1 \in \mathcal{C}$. If $|A-E_1| = 0$ the construction is completed. Otherwise, we proceed to the choice of E_2 as follows. Suppose that $[E_1,\ldots,E_k]$ have been chosen. If $|A - \cup_{i=1}^k E_i| = 0$, the construction is completed. Otherwise, we denote by δ_k the supremum of the diameters of all sets $E \in \mathcal{C}$ which have no point in common with $\cup_{i=1}^k E_i$. Let us choose for E_{k+1} any one of these sets with diameter $E_{k+1} > \delta_k/2$.

Such a set must exist unless the sets $E_1,\ldots,E_k$ already cover A and in such a case they already form the requested sequence. Thus, we may well suppose that the induction process can be continued indefinitely.

To show that the sequence $[E_i, i = 1,2,\dots]$ covers A almost entirely, let $B = A - \cup_i E_i$, and assume, if possible, that $|B| > 0$. Because of condition (I) we can associate to each E_i a cube J_i such that $E_i \subset J_i$, $|E_i| \geq \alpha|J_i|$. Let $\tilde{J}_i$ be the cube concentric to J_i and of diameter five times the one of J_i. Now the series

$$\Sigma_i|\tilde{J}_i| \leq 5^2 \cdot \Sigma_i|J_i| \leq 5^2 \alpha^{-1} \Sigma_i|E_i| \leq 5^2 \alpha^{-1}|Q|$$

converges, and hence for some integer N we certainly have $\Sigma_{i=N+1}^{\infty} |\tilde{J}_i| < |B|$. Hence, there is some point p_o in B which belongs to none of the sets $\tilde{J}_i$ with $i \geq N+1$. On the other hand, $p_o \in B$, hence $p_o \in A$, but p_o is not contained in $\cup_{i=1}^{\infty} E_i$, in particular p_o is not contained in the closed set $\cup_{i=1}^{N} E_i$. Thus, $p_o \in A$ and has a positive distance from $\cup_{i=1}^{N} E_i$. Hence, by property (II) there is some set $E_o \in \mathcal{C}$ with $E_o \cap E_i = \emptyset$, $i = 1,\dots,N$.

Note that the convergence of the series above implies $|J_i| \to 0$ as $i \to \infty$, hence $\delta(J_i) = |J_i|^{1/2} \to 0$ as $i \to +\infty$, and hence $\delta(E_i) \to 0$ as $i \to \infty$.

Let us prove that we must have $E_o \cap E_i \neq \emptyset$ for some $i \geq N+1$. Indeed, in the contrary case, $E_o \cap E_i = \emptyset$ for all i, we first take any i with $i \geq N+1$ and $\delta_i/2 < \text{diam } E_i < \text{diam } E_o$, and we note, on the other hand, that since δ_i is the supremum of the diameters of all sets $E \in \mathcal{C}$ free of points of $E_1,\dots,E_{i-1}$, we have $\delta_i \geq \text{diam } E_o$, a contradiction.

Let i_o be the smallest integer with $E_o \cap E_i \neq \emptyset$. Then $E_o \cap E_i = 0$, $i = 1,2,\dots, i_o-1$, so that $\text{diam } E_o \leq \delta_{i_o-1}$. On the other hand, $E_o \cap E_i = \emptyset$

for $i = 1,2,\dots,N$, and hence $i_o \geq N+1$, and consequently p_o does not belong to $\tilde{J}_{i_o}$. Thus, we see that the set E_o has both points outside $\tilde{J}_{i_o}$ and points in the set J_{i_o}. Thus, diam $E_o \geq 2\,\delta_{i_o}$ while we have proved above that diam $E_o \leq \delta_{i_o-1}$, a contradiction. The hypothesis $|B| > 0$ has lead to a contradiction. We have proved therefore that $|B| = 0$, that is, $|A - \bigcup_{i=1}^{\infty} E_i| = 0$. The statement is proved for A bounded. If A is unbounded, we divide the plane E_2 into countably many squares, say of side-length one, and then we repeat the process above on each square.

Remark. The statement above is due to G. Vitali [112], who stated it for a family $\mathcal{C}$ of squares. The argument given above is due to S. Banach [9b]. Note that the statement is true under the following more general hypothesis replacing both (I) and (II): For each $p \in A$, there is a sequence $[E_n, n = 1,2,\dots]$ of sets $E_n \in \mathcal{C}$ with $p \in E_n$ and $r(E_n) \geq \alpha(p) > 0$ where $\alpha(p)$ may depend on p. For a proof see S. Saks [103].

V 5. DERIVATIVE OF AN INTERVAL FUNCTION AND DENSITY THEOREM

Given the interval function $F(I)$, $I \in \{I\}$, (V 1), let p_o be any point $p_o \in$ int A. Let q denote any closed square $q \in \{q\}$ with $p_o \in q$ and let $\delta = \delta(q)$ be the diameter of q. Then by upper and lower derivatives $\bar{F}'(p_o)$, $\underline{F}'(p_o)$ of $F(I)$ at the point p_o we denote the numbers

$$\underline{F}'(p_o) = \underline{\lim}_{\delta\to 0} \frac{F(q)}{|q|}, \qquad \bar{F}'(p_o) = \overline{\lim}_{\delta\to 0} \frac{F(q)}{|q|},$$

$-\infty \leq \underline{F}' \leq \bar{F}' \leq +\infty$. If $\underline{F}' = \bar{F}'$, then their common value is denoted the derivative $F'(p_o)$ of $F(I)$ at p_o.

(V 5.i) (lemma). Any set expressible as the union of a family of intervals is measurable.

Proof. Let $\mathcal{J}$ be any family of intervals I and let S be the union of all the intervals I of $\mathcal{J}$. Let $\mathcal{C}$ denote the family of all squares q each of which is contained in at least one interval I of $\mathcal{J}$. The set S is clearly covered by $\mathcal{C}$ and properties (I) and (II) of V 4 are satisfied with $\alpha = 1$. By Vitali's covering theorem (V 4.i) S is the union of countably many squares q and of a set of measure zero, and hence S is measurable.

(V 5.ii) For every function F(I) the upper and lower derivatives $\bar{F}'(p)$, $\underline{F}'(p)$ are L-measurable in $A^o = \text{int } A$.

Proof. For every $a > 0$, let J denote the set of all points $p \in A^o$ where $\bar{F}'(p) > a$. Then, for each $p \in A^o$ there is a sequence $K_n(p)$, $n = 1,2,\ldots$, of squares $K_n(p)$ such that $p \in K_n(p)$, $K_n(p) \subset A^o$, $F(K_n(p))/|K_n(p)| \geq a-n^{-1}$, $|K_n(p)| < n^{-1}$, $n = 1,2,\ldots$. For each n let J_n denote the set $J_n = \cup K_n(p)$, where $\cup$ ranges over all points $p \in J$. Let $H = \cap_{n=1}^{\infty} J_n$. Obviously, $J \subset J_n$ for every n; hence $J \subset H$. Let now p be any point of H. Then $p \in J_n$ for all n and hence $p \in K_n(p_n)$ for at least one point $p_n \in J_n$. We have $F(K_n(p_n))/|K_n(p_n)| \geq a-n^{-1}$, and since $p \in K_n(p_n)$, we have $\bar{F}'(p) \geq a$. Thus, each point $p \in H$ is also a point of J, that is, $H \subset J$. Thereby we have proved that $H = J$. Now each set J_n satisfies the conditions of the lemma, and hence J_n is measurable. As a consequence, also $H = J$ is measurable and $\bar{F}'(p)$ is measurable. Analogously for $\underline{F}'(p)$.

(V 5.iii) (Density theorem). If A is any subset of E_2, and for any interval $I \subset E_2$ we denote by $F(I) = |A \cap I|$ the outer measure of $A \cap I$, then $F(I)$ is an interval function, the derivative $F'(p)$ exists a.e. in E_2, $0 \leq F'(p) \leq 1$, and $F'(p) = 1$ a.e. in A. If A is measurable and $\chi(p)$ denotes the characteristic function of A, then $F'(p) = \chi(p)$ a.e. in E_2.

Remark. This is a form of Lebesgue's density theorem. However, the statement holds also in a stronger form (S. Saks [103], p. 129) with $F'(p)$ computed as the limit of $|H \cap I|/|I|$ with $p \in I$, I any interval with sides parallel to the axes and no regularity condition, and the limit is taken as $\delta(I) \to 0$.

Proof of (V 5.iii). It is not restrictive to assume A contained in a closed square Q and all I contained in some larger square Q', $Q \subset \text{int } Q'$. By set function theory there is a measurable set H with $H \supset A$, $|H| = |A|$, $|H \cap M| = |A \cap M|$ for every measurable set M (see, e.g., H. Hahn and A. Rosenthal [53], p. 96, (8.2.5)). If A is measurable, we can take $H = A$. Since H is measurable, then $F(I) = F_A(I) = |A \cap I| = |H \cap I|$ is an additive nonnegative BV and AC interval function for $I \subset Q'$, $0 \leq F(I) \leq |I|$, and its lower and upper derivatives $\underline{F}'(p)$ and $\bar{F}'(p)$ are measurable, $0 \leq \underline{F}'(p) \leq \bar{F}'(p) \leq 1$. Let α, ε be arbitrary numbers, $0 \leq \alpha < 1$, $\varepsilon > 0$. Let H_α denote the set of all points $p \in H_\alpha$ where $0 \leq \underline{F}'(p) \leq \alpha$. Then H_α is measurable, and there are two sets K closed, G open, such that $K \subset H_\alpha \subset G$, $|G-K| < \varepsilon$. For every point $p \in K$ there are squares q of diameter as small as we want and with $p \in q$ such that $|q \cap H_\alpha| \leq |q \cap H| \leq (\alpha+\varepsilon)|q|$. We may well assume

$q \subset G$. By Vitali's covering theorem (V 4.i) there are countably many disjoint squares q_j such that $|K - \cup_j q_j| = 0$, $q_j \subset G$, $j = 1,2,\ldots$. Then

$$|H_\alpha| \leq |K| + \varepsilon \leq |K - \cup_j q_j| + \Sigma_j |q_j \cap K| + \varepsilon$$

$$\leq \Sigma_j |q_j \cap H_\alpha| + \varepsilon \leq (\alpha+\varepsilon) \Sigma_j |q_j| + \varepsilon$$

$$\leq (\alpha+\varepsilon)|G| + \varepsilon \leq (\alpha+\varepsilon)(|H_\alpha| + \varepsilon) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain $|H_\alpha| \leq \alpha|H_\alpha|$ with $0 \leq \alpha < 1$, and hence $|H_\alpha| = 0$. Thus, $\alpha \leq \underline{F}'(p) \leq \bar{F}'(p) \leq 1$ a.e. in H, for every $0 \leq \alpha < 1$, and hence $\underline{F}'(p) = \bar{F}'(p) = 1$ a.e. in H. This proves that $F'(p) = 1$ a.e. in H, and hence the same holds for almost all point $p \in A$.

Let us assume A measurable, hence H = A. Then B = Q-A is also measurable, and for every interval I we have $F_A(I) + F_B(I) = |I|$. Hence $F'_A(p) + F'_B(p) = 1$ a.e. in Q with $F'_A(p) = 1$ a.e. in A, and $F'_B(p) = 1$ a.e. in B, hence $F'_A(p) = 0$ a.e. in B, and $F'_B(p) = 0$ a.e. in A.

We have proved that $F'_A(p) = \chi(p)$ a.e. in Q.

(V 5.iv) Each BV normal function F(I) has derivative F'(p) almost everywhere in $A^o = \text{int } A$.

Proof. By (V 3.iii) we can suppose F overadditive and nonnegative. Thereby $0 \leq \underline{F}' \leq \bar{F}' \leq +\infty$ everywhere in A^o. By (V 3.iii) the set $J \subset A$ of all points $p \in A^o$ where $\underline{F}' < \bar{F}'$ is measurable. Suppose, if possible, $|J| > 0$. If (u_n, v_n), $n = 1,2,\ldots$, is any sequence containing all couples of rational numbers $0 < u_n < v_n < +\infty$, and H_n is the subset of J of all $p \in H$ where

$0 \leq \underline{F}' < u_n < v_n < \bar{F}' \leq +\infty$, then $J = \bigcup_n H_n$. As a consequence there is at least one n such that $|H_n| = m > 0$. By (V 5.iii) there is in H at least one point $p_o \in H$ of outer density one. If q is any square with $p_o \in q$, $q \subset A^o$, we have $\lim|qH|/|H| = 1$, $\underline{\lim}\ F(q)/|q| < u$ as $\delta(q) \to 0$. Therefore, given $\varepsilon > 0$, there is a square $q = K$ with the following properties: (α) $p_o \in K$, $K \subset A^o$, $|K \cap H|/|K| > 1-\varepsilon$, $F(K)/|K| < u$. For every point $p \in K^o \cap H$, let us denote by q any square such that $p \in q$, $q \subset A^o$; thus $\overline{\lim}\ F(q)/|q| > v$ as $\delta(q) \to 0$. Therefore, there is a sequence $q_k(p)$, $k = 1,2,\ldots$, of squares with $p \in q_k(p)$, $q_k(p) \subset K^o$, $\delta[q_k(p)] \to 0$ as $k \to \infty$, $F(q_k)/|q_k| > v$. By Vitali's covering theorem (V 4.i), there is a finite system of nonoverlapping squares $K_1,\ldots,K_N$ such that (β) $K_i \subset K \subset A^o$, $F(K_i)/|K_i| > v$, $i = 1,\ldots,N$, $\Sigma_i|K_i| > |K \cap H| - \varepsilon|K|$. The system $\{K_i,\ i = 1,\ldots,N\}$ can now be completed by some system of nonoverlapping intervals $[J_j,\ j = 1,\ldots,M]$ in such a way that $[K_i] \cup [J_j]$ is a subdivision of K into a finite system of nonoverlapping intervals. Since F is nonnegative and overadditive, and because of (α) and (β), we have now successively

$$u|K| > F(K) \geq \Sigma_i\ F(K_i) + \Sigma_j\ F(I_j) \geq \Sigma_i\ F(K_i)$$

$$\geq \Sigma_i\ v|K_i| > v|K \cap H| - \varepsilon v|K|$$

$$> (1-2\varepsilon)\, v\, |K|.$$

As a consequence $u > (1-2\varepsilon)v$, when ε is any positive number, and hence $u \geq v$, a contradiction. Thus it is proved that $|J| = 0$, and thereby (V 5.iv) is proved.

(V 5.v) For each BV normal function the set of points $p \in A^{o}$ where $\bar{F}' = +\infty$, or $\underline{F}' = -\infty$ has measure zero.

Proof. By (V 3.iii) we can suppose F nonnegative and overadditive. We have only to prove that the set J of the points $p \in A^{o}$ where $F' = +\infty$ has measure zero. By (V 3.iii) we know that J is measurable. Suppose $|J| = m > 0$, if possible. For each point $p \in J$ and any number $a > 0$ there is a sequence $q_n(p)$, $n = 1,2,\dots$, of squares such that $p \in q_n(p)$, $\delta[q_n(p)] \to 0$, $F(q_n(p))/|q_n(p)| > a$. By Vitali's theorem (V 4.i), given $\varepsilon > 0$, there is a finite system of nonoverlapping squares K_i, $i = 1,\dots,N$, such that (α) $K_i \subset A$, $F(K_i)/|K_i| > a$, $\sum|K_i| > |J| - \varepsilon$, [or $> \varepsilon^{-1}$ if $|J| = +\infty$]. Consequently we have $+\infty > V(A) \geq \sum_i(K_i) > \sum_i a|K_i| > a[|J|-\varepsilon]$, (or $> a\,\varepsilon^{-1}$), a contradiction, since a, ε are arbitrary numbers. Thus we have proved that $|J| = 0$ and (V 5.v) is proved.

(V 5.vi) (Derivative of the indefinite integral). If a is any measurable subset of E_2, if $g(p)$, $p \in A$, an L-integrable function in A, and $F(I)$ the interval function defined by $F(I) = \int_{A \cap I} g(p)dp$, $I \subset E_2$, then $F'(p)$ exists a.e. in A, and $F'(p) = g(p)$ a.e. in A.

Proof. (a) Let us suppose first $g(p)$ bounded. It is not restrictive to suppose g nonnegative and hence $0 \leq g(p) \leq L$ for all $p \in A$ and some $L > 0$. For every integer k and $i = 0,1,\dots,k-1$, let H_{ki} be the subset of all $p \in A$ where $iL/k \leq f(p) < (i+1)L/k$ for $i = 0,1,\dots,k-2$, and $iL/k \leq f(p) \leq (i+1)L/k$ for $i = k-1$. Let $\chi_{ki}(p)$ denote the characteristic function of H_{ki}. Then

$$\sum_{i=0}^{k-1} iLk^{-1} |H_{ki} \cap I| \leq F(I) \leq \sum_{i=0}^{k-1} (i+1)Lk^{-1} |H_{ki} \cap I|.$$

Since F(I) is obviously additive, BV, AC, and nonnegative, the derivative F'(p) exists a.e. in A, and hence, by force of (V 5.iv),

$$\sum_{i=0}^{k-1} iLk^{-1}\chi_{ki}(p) \leq F'(p) \leq \sum_{i=0}^{k-1} (i+1)Lk^{-1} \chi_{ki}(p)$$

again a.e. in A. Since f(p) verifies the same inequalities, we have

$$|F'(p) - f(p)| \leq Lk^{-1}$$

a.e. in A. Since k is arbitrary, we have F'(p) = f(p) a.e. in A.

(b) If g is unbounded but still L-integrable in A, then the statement is an immediate consequence of a Banach's theorem which is proved in (V 6) below.

V 6. BANACH'S THEOREMS

(V 6.i) Theorem. For each BV normal function F(I) the derivative F'(p) exists a.e. in A^o and is L-integrable in A^o. If F(I) is overadditive and nonnegative, then $V(A^o) \geq \int_{A^o} F'(p)dp$, and for every $I \in \{I\}$, also V(I) = $F(I) \geq \int_I F'(p)dp$.

Proof. By (V 3.iii) we can suppose F(I) overadditive and nonnegative in both parts of the theorem. Let Q denote any closed square of the p-plane E_2 and m any positive number. Let g(p), $p \in A^o$, be the function defined as follows: g(p) = m if F'(p) exists and F'(p) > m; g(p) = F'(p) if F'(p) exists and $0 \leq F'(p) \leq m$; g(p) = 0 otherwise. Thus, g(p), $p \in A$, is bounded and measurable in A, and $g(p)$ is also L-integrable in the bounded set $Q \cap A^o$. Let $U(I)$ be the additive interval function $U(I) = \int_I g\, dp$. For almost all points $p \in Q^o \cap A^o$ we have $\lim U(q)/|q| = g(p)$; $\lim F(q)/|q| \geq g(p)$ as $\delta(q) \to 0$ and q denotes any square with $p \in q$, $q \subset A^o$. The first statement was proved in (V 5.v, (a)); the second one is a consequence of (V 5.ii) and of the definition of g. Let J denote the set of the points $p \in Q^o \cap A$ which have both properties; thus, $|J| = |Q \cap A^o|$. Given $\varepsilon > 0$, for each point $p \in J$ there is a sequence of squares $q_n(p)$, $n = 1,2,\dots$, such that $U[q_n(p)]/|q_n(p)| < g(p) + \varepsilon$; $F(q_n(p))/|q_n(p)| \geq g(p) - \varepsilon$; $\delta[q_n(p)] \to 0$ as $n \to \infty$. By Vitali's covering theorem, applied to the set $J \subset Q^o \cap A^o$, there is a finite family $[p_i, i = 1,\dots,N]$ of points p_i and corresponding system of nonoverlapping squares K_i such that (α) $p_i \in K_i$, $K_i \subset Q \cap A^o$, $U(K_i)/|K_i| < g(p_i) + \varepsilon$, $F(K_i)/|K_i| \geq g(p_i) - \varepsilon$, $i = 1,\dots,N$, and (β) $|Q \cap A^o - \cup_i K_i| < m^{-1}\varepsilon$. By ($\alpha$) and ($\beta$) we have now, successively, with $S = Q \cap A^o - \cup_i K_i$,

$$\begin{aligned} \int_J g(p)dp &= \sum_i U(K_i) + \int_S g(p)dp \\ &\leq \sum_i [g(p_i) + \varepsilon] |K_i| + m(m^{-1}\varepsilon) \\ &\leq \sum_i F(K_i) + 2\varepsilon \sum_i |K_i| + \varepsilon \\ &\leq V(A^o) + \varepsilon |A^o \cap Q| + \varepsilon, \end{aligned}$$

where $\varepsilon > 0$ is any number. As a consequence we have $\int_J g(p)dp \leq V(A^o)$, where $J \subset Q \cap A^o$, $|J| = |Q \cap A^o|$. Hence $\int_{Q \cap A^o} g(p)dp \leq V(A^o)$, where Q denotes any square of the p-plane E_2. If Q is made to invade E_2 we have also $\int_{A^o} g(p)dp$

$\leq V(A^o)$, where g depends upon the number $m > 0$. As $m \to +\infty$, we have $\int_{A^o} F'(p)dp \leq V(A^o)$. Theorem (V 6.i) is thereby proved.

(V 6.ii) Theorem. If $F(I)$, $I \in \{I\}$, is a nonnegative, overadditive, BV interval function, then a necessary and sufficient condition in order that $V(A^o) = \int_{A^o} F'(p)dp$ is that $F(I)$ is AC in A^o. In these conditions we have also $V(I) = F(I) = \int_I F'(p)dp$ for all $I \in \{I\}$.

Proof. By Theorem (V 6.i), the function $F'(p)$ is defined a.e. in A^o and is nonnegative and L-integrable in A^o; hence the interval function $U(I) = \int_I F'(p)dp$ is nonnegative, additive and AC in A^o. By Theorem (V 6.i), we have also $V(I) = F(I) \geq U(I)$ for every $I \in \{I\}$.

Necessity. Suppose $V(A^o) = \int_{A^o} F'dp$ and let K be any closed interval, $K \in \{I\}$, $K \subset A^o$. Let $[M_n, n = 1,2,\ldots]$ be any sequence of sets such that $K \subset M_n \subset A^o$, $M_n^o \uparrow A^o$ as $u \to \infty$, and each M_n is a finite sum of nonoverlapping intervals I. Then for each n we can divide M_n into a finite system S_n of intervals I one of which is K. By Theorem (V 6.i), we have $V(K) = U(K) + \sigma$, $\sigma \geq 0$, $V(I) \geq U(I)$ for every $I \in S_n$. Hence, if Σ denotes any sum ranging over all $I \in S_n - (K)$, we have $V(A^o) \geq V(M_n) \geq V(K) + \Sigma\, V(I) \geq U(K) + \sigma + \Sigma\, U(I) = \sigma + \int_{M_n} F'(p)dp$. As $n \to \infty$, the last integral approaches $\int_{A^o} F'(p)dp = V(A^o)$; hence $V(A^o) \geq \sigma + V(A^o)$, $\sigma \geq 0$, and finally $\sigma = 0$. Therefore, $V(K) = F(K) = U(K)$ and this relation holds for all intervals $K \in \{I\}$, $K \subset A^o$. Thus, $F(K)$ as well as $V(K)$ are additive and AC in A^o.

Sufficiency. Suppose $F(I)$ be AC in A^o. Given $\varepsilon > 0$ let $\delta > 0$ be the number defined defined by condition (a) of (V 2). Let K be any closed interval $K \subset A^{o}$, $K \in \{I\}$. Almost all points $p \in K$ have the following properties: $p \in K^{o}$, $F'(p)$ exists, and $0 \leq F'(p) < +\infty$; $\lim U(q)/|q| = F'(p)$ as $\delta(q) \to 0$ and q is any closed square with $p \in q$, $q \subset A^{o}$. Let J denote the set of all points $p \in K$ which have the properties above; thus $|J| = |K|$. For each point $p \in J$ there is a sequence $q_n(p)$, $n = 1,2,\ldots$, of squares such that $p \in q_n(p)$; $q_n(p) \subset K$, $U(q_n(p))/|q_n(p)| > F'(p) - \varepsilon$; $F(q_n(p))/|q_n(p)| < F'(p) + \varepsilon$; $\delta[q_n(p)] \to 0$ as $n \to \infty$. By Vitali's covering theorem applied to the set J, there exists a finite system of points $p_i \in J$, $i = 1,\ldots,N$, and a corresponding system of nonoverlapping squares K_i such that $p_i \in K_i$, $K_i \subset K$, $U(K_i)/|K_i| > F'(p_i) - \varepsilon$, $F(K_i)/|K_i| < F'(p_i) + \varepsilon$, $i = 1,\ldots,N$, $|K - \cup_i K_i| = |J - \cup_i K_i| < \delta$. Let us denote by $[I_j, j = 1,\ldots,M]$ any finite system of intervals such that $[K_i] \cup [I_j]$ is a finite subdivision of K into overlapping intervals. Thus $\sum_j |I_j| < \delta$, and, by (a), $\sum_j F(I_j) < \varepsilon$. On the other hand, by condition (b) of (V 2), we have $U(K) = \sum_i U(K_i) + \sum_j U(I_j)$, $F(K) = \sum_i F(K_i) + \sum_j F(I_j)$. We obtain now

$$U(K) \geq \sum_i U(K_i) \geq \sum_i |K_i| \, [F'(p_i) - \varepsilon]$$

$$\geq \sum_i |K_i| \, (F'(p_i) + \varepsilon) - 2\varepsilon \, |K| > \sum_i F(K_i) - 2\varepsilon \, |K|$$

$$= [F(K) - \sum_j F(I_j)] - 2\varepsilon \, |K| \geq F(K) - \varepsilon - 2\varepsilon \, |K|.$$

Since ε is any positive number we have $U(K) \geq F(K)$, while, by Theorem (V 6.i), $F(K) \geq U(K)$. Hence, $U(K) = F(K)$ and this relation holds for every interval $K \subset A^{o}$, $K \in \{I\}$. Let ε by again any positive number. By defintion of $V(A^{o})$ there exists a finite system S of nonoverlapping intervals $I \subset A^{o}$ such that

$0 \leq V(A^o) - \sum F(I) < \varepsilon$, where $\sum$ ranges over all $I \in S$. Hence, $V(A^o) \leq \sum F(I) + \varepsilon \leq \int_{\sum I} F'dp + \varepsilon \leq \int_{A^o} F'dp + \varepsilon$, and finally $V(A^o) \leq \int_{A^o} F'(p)dp$ since ε is any positive number. On the other hand, by Theorem (V 6.i), we have $V(A^o) \geq \int_{A^o} F'dp$; hence, $V(A^o) = \int_{A^o} F'(p)dp$. Theorem (V 6.ii) is thereby proved.

An interval function $S(I)$ is said to be singular if $S'(p)$ exists and $= 0$ a.e. in A.

(V 6.iii) (Lebesgue's decomposition theorem). Every BV normal function $F(I)$ has a unique decomposition $F(I) = C(I) + S(I)$, where $C(I)$ is AC and $S(I)$ is singular. If $F(I) \geq 0$, then both $C(I)$ and $S(I)$ are nonnegative.

Proof. It is not restrictive to assume F nonnegative. By force of (V 5.ii), we know that $F'(p)$, $p \in A$, exists a.e. and is nonnegative and L-integrable in A. We shall take

$$C(I) = \int_{A \cap I} F'(p)dp.$$

Then $C(I)$ is AC in A, by (V 6.i) $F(I) \geq C(I) \geq 0$ for every I, and by (V 5.v) $C'(p) = F'(p)$ a.e. in A. Then $S(I) = F(I) - C(I) \geq 0$ and $S'(p) = F'(p) - C'(p) = 0$ a.e. in A. We have proved the existence of the decomposition $F = C+I$. To prove uniqueness note that, if $F = C_1 + S_1$, $F = C_2 + S_2$, then $C_1 - C_2 = S_1 - S_2$, where $C_1 - C_2$ is AC, $S_1 - S_2$ is singular, and $C_1' - C_2' = S_1' - S_2' = 0$. By force of (V 6.ii) $C_1 - C_2$ is the indefinite integral of its derivative, hence $C_1 - C_2 = 0$, and $S_1 - S_2 = 0$.

Bibliographical Notes. The theory above is due to S. Banach [9b]. For numerous application of normal functions of intervals, in particular for questions of differentiation, we refer to S. Banach's paper just mentioned and to the books of S. Saks [103], L. Tonelli [109h], S. Kempisty [63], C. Carathéodory [21b], and L. Cesari [24c].

We mention here that the concept of normal functions with related properties has been widely extended by L. Cesari [24fg] (quasi additive, quasi subadditive set functions), with applications to the theory of integration, in particular to the concept of Weierstrass integral in surface area theory. The same concept of quasi additive set functions with values in locally convex topological vector spaces have been proved to have far reaching applications by G. Warner [115ab]. The same concept has been developed in different directions also by T. Nishiura [85b] and J. C. Breckenridge [15ac].