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LAGRANGE PROBLEMS OF OPTIMAL CONTROL AND CONVEX SETS NOT CONTAINING ANY STRAIGHT LINE

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1. Introduction

In a previous paper [1] we proved that the usual sets $\tilde{\tilde{Q}}(x)$ of Lagrange problems of optimal control if bounded below have property (Q) if, and only if, they satisfy conditions similar to those of "weak seminormality" for free problems of the calculus of variations (see (4.iii) below).

In the present paper we prove that the same sets of $\tilde{\tilde{Q}}(x)$ if bounded below have property (Q) and contain no straight line if, and only if, they satisfy conditions similar to those for "seminormality" for free problems (see (4.vii) below).

These results will be used in successive papers in proving theorems of lower closure and existence theorems for optimal solutions.

In Section 2 we study real-valued functions $F(u)$, $u \in Q \subset E_m$, which are convex on a given fixed convex subset Q of E_m. Under suitable conventions we prove that these functions grow "more than linearly" as $|u| \to o$ with $u \in Q$ if, and only if, their graph does not contain any straight line. This statement, which had been proved by Tonelli and Turner for $Q = E_m$ (see [1] for references) requires a partially new proof in the present situation where Q is any convex subset of E_m, and F is only convex in Q, and hence may not be even continuous

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on the boundary of Q. The proof is based on a series of lemmas. The boundary points may actually present exceptional behavior as examples show.

In Section 3 we study real-valued functions $f_o(x,u)$, $u \in Q(z) \subset E_m$, which are convex in u on the convex subset Q(x) of E_m, and where Q(x) may depend on x. No continuity requirement is made on f_o, but only a suitable property of lower semicontinuity is assumed. Finally, in Section 4, we apply the results to the typical sets $\tilde{\tilde{Q}}(x)$ of Lagrange problems, by consistent use of the related real-valued functions $T(z;x)$, $z \in Q(x) \subset E_m$, which has been introduced in [1], and for which we proved there the properties which we require for f_o in the present paper.

2. Real-Valued Convex Functions on a Convex Set

Given a real-valued function F(u) on a subset Q of E_m, we shall often use the expression "limit of F(u) as $|u| \to \infty$ with $u \in Q$." This expression is clear in itself when Q is unbounded. Whenever Q is bounded, the limit above is taken to be $+\infty$. In the present section we shall prove the following

(2.i) <u>Theorem</u>. If F(u), $u \in Q$, is a real-valued convex function on the convex subset Q of E_n, then the following properties are equivalent:

1. There is a linear function $w(u) = \bar{r} + \bar{b} \cdot u$, $u \in E_n$, $\bar{r}$ real, $\bar{b} = (b_1,\dots,\bar{b}_n)$ real, with $F(u) \geq w(u)$ for all $u \in Q$, and $F(u) - w(u) \to +\infty$ as $|u| \to +\infty$, $u \in Q$.
2. For no two points $u_o \in Q$, $u_1 \in E_n$, $u_1 \neq o$, it occurs that $u_o + \lambda u_1 \in Q$ for all λ real, and

$$F(u_0) = 2^{-1}[F(u_0 + \lambda u_1) + F(u_0 - \lambda u_1)] \text{ for all } \lambda \geq 0. \qquad (1)$$

For $Q = E_n$ the function F is necessarily continuous, and then this theorem reduces to a well known statement proved by Tonelli for smooth functions and by Turner under sole hypotheses of continuity and convexity (see [1] for references). Note that if Q is bounded, or Q is unbounded but contains no straight line, then 2. is trivial, but 1. is not. In this situation statement (2.i) yields the following corollary which is interesting and far from trivial.

Corollary. If $F(u)$, $u \in Q$, is a real-valued convex function on a convex subset Q of E_m which contains no straight line, then 1. holds.

We shall prove first a series of lemmas, and then theorem (2.i).

(2.ii) If a convex set $Q \subset E_n$ does not contain any straight line, then clQ does not contain any straight line either.

Proof. First, let us assume that Q has at least an interior point, and, if possible, that clQ contains a straight line. It is not restrictive to assume that this straight line is the u_1-axis.

If $n = 1$, then Q is contained on the same u_1-axis, and we may well assume that 0 is the interior point. If λ_1, λ_2 are the infimum and supremum respectively, of the real numbers u_1 with $u_1 \in Q$, then $-\infty \leq \lambda_1 < 0 < \lambda_1 \leq +\infty$, and at least one of the two numbers λ_1, λ_2 is finite. If, say, λ_1 is finite, hence $-\infty < \lambda_1 < 0 < \lambda_1 \leq +\infty$, then the points of the half straight line $(-\infty, \lambda_1)$ are not in Q, are not points of accumulation of points of Q and therefore are not in clQ, a contradiction.

If $n > 1$, it is not restrictive to assume that some point $p = (o,a,o,\ldots,o)$ with $a \geq o$ is interior to Q, and then all points in the direction of the u_2-axis of a neighborhood of p, say all points $u = (u_1,\ldots,u_n)$ with $a \leq u_2 \leq a + \delta$, $u_1 = u_3 = \ldots = o$, are interior to Q. For any ε, $o<\varepsilon<2^{-1}\delta(a+\delta)^{-1}$, and ξ real, $\xi \neq o$, the point $(u_1 = \varepsilon^{-1}\xi,\ u_2 = \ldots = u_n = o)$ is in clQ; hence, there are points $M = (u_1 = \varepsilon^{-1}\xi + \eta_1,\ u_2 = \eta_2,\ \ldots,\ u_n = \eta_n)$ which belong to Q with $\eta_1,\ldots,\eta_n$ as small in absolute value as we want. But then the points

$$N = (u_1 = -(1-\varepsilon)^{-1}\varepsilon\eta_1,\ u_2 = (1-\varepsilon)^{-1}(a + 2^{-1}\delta-\varepsilon\eta_2),\ u_j = -(1-\varepsilon)^{-1}\varepsilon\eta_j,\ j = 3,\ldots,n,$$

are all in Q, since $o < (1-\varepsilon)^{-1}(a+2^{-1}\delta) < a + \delta$, and provided we take M with $|\eta_1|,\ldots,|\eta_n|$ sufficiently small. Now the point $\varepsilon M + (1-\varepsilon)N$ is the point $(\xi, a + 2^{-1}\delta, o,\ldots,o)$ and belongs to Q. This proves that all points of the straight line $[u_1 \in E_1,\ u_2 = a+2^{-1}\delta,\ u_3 = \ldots = u_n = o]$ are in Q, a contradiction. We have proved (2.ii) in the case Q has an interior point.

If Q has no interior points, then let R denote the linear manifold of minimum dimension r, $o \leq r \leq n$, containing Q. If $r = o$ then Q reduces to a single point and so does clQ. If $r \geq 1$, then Rint Q is not empty. It is not restrictive to assume that R is the linear $(u_1,\ldots,u_r)$-subspace of E_n, so that Q has interior points with respect to $R = E_r$, and $Q \subset clQ \subset R$. We are now in the same conditions above with n replaced by r. Statement (2.ii) is thereby proved. This statement will be used only in Section 4.

Let Y denote the set of all vectors $v \in E_n$ with the property that $\text{Inf}\ (u \cdot v) > -\infty$ for all $u \in Q$.

Obviously $o \in Y$, and if $v \in Y$ and $k > o$, then kv also belongs to Y. Finally, if $v_1, v_2 \in Y$ and α real, $o \leq \alpha \leq 1$, then $\alpha v_1 + (1-\alpha)v_2 \in Y$. Thus, Y is a convex cone of vertex the origin.

(2.iii) If a convex set $Q \subset E_n$ does not contain any straight line, then the corresponding cone Y has interior points.

Proof. First, let us assume that Q has at least an interior point. It is not restrictive to suppose that the origin is an interior point of Q. Then there is some sphere σ of center the origin completely contained in Q. Suppose, if possible that Y has no interior points. Then Y must be contained in an (n-1)-dimensional subspace E_{n-1}, and it is not restrictive to assume that E_{n-1} is the $(u_1 \ldots u_{n-1})$-space. Let Λ_n be the linear set of all real numbers ξ such that $(o, \ldots, o, \xi) \in Q$. Then Λ_n is a convex linear set, and $\xi = o$ is an interior point of Λ_n. If λ_n and λ_n' denote the infimum and the supremum of all numbers $\xi \in \Lambda_n$, then $-\infty \leq \lambda_n < o < \lambda_n \leq \neq \infty$, but at least one of the numbers λ_n, λ_n' must be finite (otherwise the entire u_n-axis would belong to Q). We may well assume that say, λ_n is finite, hence $-\infty < \lambda_n < o < \lambda_n' \leq +\infty$. Then the half straight line ℓ_n^- of all points $(o, \ldots, o, \xi)$ with $\xi < \lambda_n$ is outside Q, and there must be a plane π separating ℓ^- from Q in E_n,

$$\pi: \qquad \alpha_1 u_1 + \ldots + \alpha_{n-1} u_{n-1} + \alpha_n(u_n - \lambda_n) = 0 ,$$

with $z(u) \geq o$ for all $u \in Q$, $z(u) \leq o$ for all $u \in \ell^-$. We must have $\alpha_n \neq o$ since, otherwise we would have $o \in \pi$, and π would separate some pairs of points of σ which certainly are both in Q. Also, if $\alpha_n \neq o$, then we must have $\alpha_n > o$

since ℓ^- contains points $(o,\ldots,o,\xi)$ with $\xi < \lambda_n$ as large in absolute value as we want. Thus, π can be written in the form $u_n = \lambda_n + \beta_1 u_1 + \ldots + \beta_{n-1} u_{n-1}$, and $u_n \geq \lambda_n + \beta_1 u_1 + \ldots + \beta_{n-1} u_{n-1}$ for all $u \in Q$. In other words, $v_n = (-\beta_1,\ldots,-\beta_{n-1},1) \in Y$. A contradiction, since $Y \subset E_{n-1}$, that is, Y is made up only of vectors $(u_1,\ldots,u_{n-1},o)$. We have proved (2.iii) when Q has interior points.

If Q has no interior points, then let R denote the linear manifold of minimum dimension r, $o \leq r \leq n$, containing Q. If $r = o$ then Q is reduced to a single point, and obviously $Y = E_n$. If $r \geq 1$, then Rint Q is not empty. It is not restrictive to assume that R is the linear $(u_1 \ldots, u_r)$-subspace E_r of E_n, so that Q has now interior points with respect to $R = E_r$. The set Y_o of all vectors $v = (v_1,\ldots,v_2,o,\ldots,o)$ with $\operatorname{Inf}_{y \in Q}(y \cdot v) > -\infty$, now has interior points in E_r, and obviously $Y = Y_o \times E_{n-r}$ has interior points in E_n. Statement (2.iii) is thereby proved.

When dealing with properties of convex sets which are invariant with respect to affine transformations in E_n, we shall freely change systems of coordinates.

(2.iv) If a convex set $Q \subset E_n$ does not contain a line, then there is a system of coordinates $v_1,\ldots,v_n$ in E_n such that $Q \subset K = [v | v_i \geq o,\ i = 1,\ldots,n]$.

Proof. Since Y has interior points, then Y contains a system of n independent unit vectors $e_i = (e_{i1},\ldots,e_{in})$, with $\operatorname{Inf}_{u \in Q}(u \cdot e_i) = \lambda_i > -\infty$, $i = 1,\ldots,n$. Then the n relations $v_i = e_{i1}u_1 + \ldots + e_{in}u_n - \lambda_i$, $i = 1,\ldots,n$, define a new system of coordinates $v_1,\ldots,v_n$ in E_n, and $v_i \geq o$ for all $v = (v_1,\ldots,v_n) \in Q$.

Remark. Note that (2.ii), of which we had given a direct proof, is now a corollary of (2.iv).

(2.v) If $F(u)$, $u \in Q$, is a real-valued convex function on a convex subset Q of E_n which contains no straight line, then there is a linear function $w(u) = \bar{r} + \bar{b} \cdot u$, $u \in E_n$, $\bar{r}$, $\bar{b} = (\bar{b}_1,\dots,\bar{b}_n)$ real, such that $F(u) \geq w(u)$ for all $u \in Q$, and $f(u) - w(u) \to +\infty$ as $|u| \to +\infty$ with $u \in Q$.

An equivalent form of this statement is obtained by extending F to all of E_n by taking $F = +\infty$ in $E_n - Q$, and then the contention of (2.v) becomes $F(u)-w(u) \geq o$ for all $u \in E_n$, and $F(u)-w(u) \to +\infty$ as $|u| \to +\infty$ with $u \in E_n$. Statement (2.v) is trivial whenever Q is bounded. Note that (2.v) is precisely the corollary to (2.i) stated at the beginning.

Proof of (2.v). First, let us choose in E_n a system of coordinates $v_1,\dots,v_n$ such that $Q \subset K = [v | v_i \geq o,\ i = 1,\dots,n]$. Let R denote the linear manifold of minimum dimensions r, $o \leq r \leq n$, containing Q. If $r = o$ then Q reduces to a single point $\bar{v}$, and we take $w(v) = \bar{r} = F(\bar{v})$, $b = (o,\dots,o)$. If $1 \leq r \leq n$, then RintQ is not empty, and if $v \in$ RintQ, then F possesses a supporting plane $z(v) = F(\bar{v}) + b \cdot (v - \bar{v})$, $b = (b_1,\dots,b_n)$, such that $F(v) \geq z(v)$ for all $v \in Q$. Finally, let $\varepsilon > o$ arbitrary, and let $w(v) = z(v) - \varepsilon(v_1 + \dots + v_n)$, or $w(v) = (F(\bar{v}) - b \cdot \bar{v}) + (b-\varepsilon) \cdot v$, where $b-\varepsilon$ stands for the real vector $(b_1-\varepsilon,\dots, b_n-\varepsilon)$. Then

$$F(v) - w(v) = F(v) - z(v) + \varepsilon(v_1 + \dots + v_n) \geq \varepsilon(v_1 + \dots + v_n),$$

where $v \in Q$, and hence $v_i \geq o$, $i = 1,\dots,n$. To prove that $F(v) - w(v) \to +\infty$

as $v \to +\infty$, $v \in Q$, we have only to take an arbitrary $N > o$, and note that, for $|v| > \varepsilon^{-1} nN$, $v \in Q$, certainly one of the coordinates v_i must be $> \varepsilon^{-1} N$, while the others are $\geq o$. Hence $F(v)-w(v) \geq N$ for $|v| > \varepsilon^{-1} nN$, $v \in Q$.

(2.vi) If Q is a convex subset of E_n possessing interior points, and $u \in \text{int } Q$, $\bar{u} \in E_n$, $\bar{u} \neq o$, and $u + \lambda\bar{u} \in Q$ for all real λ, then $v + \lambda\bar{u} \in Q$ for all $v \in \text{int } Q$ and all real λ.

Proof. For every ξ real, and ε, $o < \varepsilon < 1$, sufficiently small, the points $M = u + \varepsilon^{-1}\xi u$ and $N = (1-\varepsilon)^{-1}(v-\varepsilon u)$ certainly belong to Q. Hence $\varepsilon M + (1-\varepsilon)N \in Q$, that is, the point

$$\varepsilon M + (1-\varepsilon)N = \varepsilon(u + \varepsilon^{-1}\xi u) + (1-\varepsilon)(1-\varepsilon)^{-1}(v-\varepsilon u) = v + \xi\bar{u}$$

belongs to Q. In other words, the entire straight line $v + \lambda\bar{u}$, λ real, belongs to Q. Since $v \in \text{int } Q$, all points of this straight line are interior points of Q.

(2.vii) If Q is a convex subset of E_n, and $u \in \text{Rint}Q$, $\bar{u} \in E_n$, and $u + \lambda\bar{u} \in Q$ for all real λ, then $v + \lambda\bar{u} \in \text{Rint}Q$ for all $v \in \text{Rint}Q$ and all real λ.

The proof is the same as above by considering the linear manifold R of minimum dimension r containing Q, $o \leq r \leq n$.

(2.viii) If Q is an arbitrary convex subset of E_n which may contain straight lines, then there are integers $s \geq o$, $\sigma \geq o$, $s + \sigma \leq n$, and a decomposition $E_n = E_s \times E_\sigma \times E_{n-s-\sigma}$ such that $Q \subset E_s \times E_\sigma$, and there is a convex subset Q_o of E_s, open in E_s and not containing any straight line, such that

$$Q_o \times E_\sigma \subset Q \subset clQ_o \times E_\sigma \qquad (2)$$

Note that Q_o may be reduced to a single point, and in that case we take $s = o$.

Proof. As usual, let R be the linear manifold of minimum dimension containing Q, $o \leq r \leq n$. If $r = o$, then Q is reduced to a single point, and we take $s = o$, $\sigma = o$, $Q = Q_o$. Let us assume now that $r \geq 1$, so that $RintQ \neq \emptyset$. It is not restrictive to assume that R is the $(u_1,\dots,u_r)$-space E_r. Let Z be the set of all vectors $\bar{u}$ such that $u + \lambda\bar{u} \in RintQ$ for all real λ, where $u \in Rint\ Q$. By (2.vii) we know that Z does not depend on the particular point $u \in Rint\ Q$. Thus, Z is a linear space $Z = E_\sigma$ of E_r of some dimension σ, $o \leq \sigma \leq r$. Now let E_s be the complementary space of E_σ in E_r, so that $E_r = E_s \times E_\sigma$, $s \geq o$, $s + \sigma = r$. The intersection $Q \cap E_s$ is a convex set, and we take $Q_o = int(Q \cap E_s)$, so that $Q_o \subset Q \cap E_s \subset cl\ Q_o$, $RintQ = Q_o \times E_\sigma$, and (2) follows.

Proof of (2.i). (a) Let us prove that 1 implies 2. It is enough to prove that property 1 and the negation of 2 together lead to a contradiction. Thus, we assume that 1 holds, and that there exist points $u_o \in Q$, $u_1 \in E_n$, $u_1 \neq o$, as in property 2. Then, all points $u_o + \lambda u_1$, λ real, are points of Q; hence, $u_o \in RintQ$. By (5.iv) of [1], F possesses a supporting plane $z(u) = r + b\cdot u$, $u \in E_n$, at u_o; thus r and $b = (b_1,\dots,b_n)$ are real, $F(u) \geq z(u)$ for all $u \in Q$, and $F(u_o) = z(u_o) = r + b\cdot u_o$. Besides $F(u_o + \lambda u_1) \geq r + b\cdot(u_o + \lambda u_1)$, $F(u_o - \lambda u_1) \geq r + b(-\lambda u_1)$ for all $\lambda \geq o$. The proof proceeds now exactly as the proof of (5.x) in [1], and will not be repeated.

(b) Let us prove that 2 implies 1. First, let us assume that $o \in RintQ$, and that $F(u) \geq o$ for all $u \in Q$ with $F(o) = o$. Let $E_n = E_s \times E_\sigma \times E_{n-s-\sigma}$ be the decomposition of E_n according to (2.viii) so that $Q_o \times E_\sigma \subset Q \subset \mathrm{cl}Q_o \times E_\sigma$, where Q_o is a convex subset of E_s not containing any straight line and open in E_s. Let us choose $(v_1 \dots v_n)$-coordinates in E_n so that E_s is the $(v_1 \dots v_s)$-space; E_σ is the $(v_{s+1} \dots v_{s+\sigma})$-space, and $E_{n-s-\sigma}$ is the $(v_{s+\sigma+1} \dots v_n)$-space in E_n, with $s \geq o$, $\sigma \geq o$. Also, according to (2.iv), let us choose $(v_1 \dots v_s)$-coordinates in E_s so that $Q_o \subset K = [(v_1, \dots, v_s) | v_i \geq o, i = 1, \dots, s]$. Let $\bar{e}_1, \dots, \bar{e}_n$ be the unit vectors of the v-space E_n defined by $\bar{e}_i = (e_{is}, s = 1, \dots, n)$, $i = 1, \dots, n$, with $e_{ii} = 1$, $e_{is} = o$ for $s \neq i$.

Let T be the set of all real vectors $b = (b_1, \dots, b_n)$ for which there is some real number r such that $F(v) \geq r + b \cdot v$ for all $v \in Q$. Let us prove that T is a convex set. Indeed, if $b_1, b_2 \in T$ and r_1, r_2 are the corresponding numbers, then, for $o \leq \alpha \leq 1$, we have

$$F(v) - [\alpha r_1 + (1-\alpha) r_2 - (\alpha b_1 + (1-\alpha)\, b_2) \cdot v]$$

$$= \alpha[F(v) - (r_1 + b_1 \cdot v)] + (1-\alpha)\, [F(v) - (r_2 + b_2 \cdot v)] \geq o$$

for all $v \in Q$. Hence, $\alpha b_1 + (1-\alpha)\, b_2 \in T$. Moreover, T contains the origin since $F(u) \geq o$ for all $u \in Q$.

Let us prove that T is not contained in any (n-1)-dimensional subspace of E_n. If it were, there would be a unit vector $e = \zeta_1 \bar{e}_1 + \dots + \zeta_n \bar{e}_n$ such that $e \cdot b = o$ for all $b \in T$. Note that for $b = \bar{e}_i$, $i = s + \sigma + 1, \dots, n$, and $v = v_1 \bar{e}_1 + \dots + v_n \bar{e}_n$, $v \in Q$, we have $v_{s+\sigma+1} = \dots = v_n = o$, and

$$F(v) - b \cdot v \geq o - \bar{e}_i \cdot (v_1 \bar{e}_1 + \dots + v_{s+\sigma} \bar{e}_{s+\sigma}) = o;$$

hence, $b = \bar{e}_i \in T$, and $o = b \cdot e = \bar{e}_i \cdot (\zeta_1 \bar{e}_1 + \dots + \zeta_n \bar{e}_n) = \zeta_i$.

We have proved that $\zeta_i = o$ for all $i = s + \sigma + 1, \ldots, n$.

For $b = -\bar{e}_i$, $i = 1, \ldots, s$, and $v = v_1\bar{e}_1 + \ldots + v_n\bar{e}_n$, $v \in Q$, we have $v_1 \geq o, \ldots, v_s \geq o$, and

$$F(z) - b \cdot v \geq o + \bar{e}_i \cdot (v_1\bar{e}_1 + \ldots + v_n\bar{e}_n) = v_i \geq o;$$

hence $b = -\bar{e}_i \in T$, and $o = b \cdot e = -\bar{e}_i (\zeta_1\bar{e}_1 + \ldots + \zeta_n\bar{e}_n) = -\zeta_i$. We have proved that $\zeta_i = o$ for all $i = 1, \ldots, s$. Thus,

$$e = \zeta_{r+1}\, \bar{e}_{r+1} + \ldots + \zeta_{r+\sigma}\, \bar{e}_{r+\sigma},$$

or $e \in E_\sigma$. Then, λe belongs to Q for all λ real, and $F(\lambda e) + F(-\lambda e) > o$ for some $\lambda \neq o$ by force of 2. Hence, either $F(\lambda e) > o$, or $F(-\lambda e) \neq o$, or both. It is not restrictive to assume $F(\lambda e) > o$. Since $\lambda e \in \text{Rint}Q$, then F has a supporting plane $z(v) = F(\lambda e) + b \cdot (v - \lambda e)$ at the point λe, by force of (5.iii) of [1]. Then $F(v) \geq z(v)$ for all v, so $b \in t$, $e \cdot b = o$, and $z(\gamma e) = F(\lambda e) + b \cdot (\gamma - \lambda e) = F(\lambda e)$ for all γ real. Thus, in the directions $\pm e$, the function $z(u)$ is constant and positive. But $z(o) \leq F(o) = o$, a contradiction. We have proved that T is n-dimensional.

We know that a convex set in E_n contained in no (n-1)-dimensional manifold has an interior point. Therefore, let $\bar{b}$ be an interior point of T, and let $\varepsilon > o$ be so chosen that $|b - \bar{b}| \leq \varepsilon$ implies $b \in T$. Let $\bar{r}$ be a constant such that $F(v) \geq w(v) = \bar{r} + b \cdot v$ for all $v \in Q$. Suppose that $\liminf\, [F(v) - w(v)] \neq +\infty$, where lim inf is taken as $|v| \to +\infty$ with $v \in Q$. Then, there is a constant $a > o$ and a sequence $[v_k]$ such that $|v_k| \to +\infty$, $v_k \in Q$, $F(v_k) - w(v_k) < a$ for all k. Without loss of generality we can assume that $v_k/|v_k|$ converges to a unit vector $\bar{v}$ as $k \to \infty$. Then, $\bar{b} + \varepsilon\,\bar{v} \in T$, and there is a constant r_1 such that $z(v) = r_1 + (\bar{b} + \varepsilon\,\bar{v})\cdot v \le F(v)$ for all v. Thus,

$$F(v_k)-w(v_k) \ge r_1 + (\bar{b} + \varepsilon\bar{v}) \cdot v_k - r - \bar{b} \cdot v_k$$

$$= r_1 - \bar{r} + \varepsilon\bar{v} \cdot v_k$$

$$= r_1 - \bar{r} + \varepsilon\,|v_k|\;\bar{v} \cdot (v_k/|v_k|) \to +\infty$$

as $k \to \infty$, a contradiction. We have proved that $F(v)-w(v) \to +\infty$ as $|v| \to +\infty$ with $v \in Q$.

We have proved that 2 implies 1 for the case where $o \in \text{Rint } Q$, $F(u) \ge o$ and $F(o) = o$. In general, let R be as usual the linear manifold of minimum dimension r containing Q, $o \le r \le n$. If $r = o$, then Q is reduced to a single point u_o, we can take $w(u) = F(u_o)$, and 1. holds trivially in force of the chosen conventions. If $r \ge 1$, then Rint $Q \ne o$ and we can take a point $u_o \in \text{Rint } Q$. Then $F(u)$ has a supporting plane at u_o by force of (5.iii) of [1], say $z(u) = F(u_o) + b_1\cdot(u-u_o)$. Let $G(u) = F(u)-z(u)$. Then $G(u) \ge o$ for all $u \in Q$ and $G(u_o) = o$. Finally, we can transfer u_o to the origin by a translation in E_n. We can now apply to G the results proved above, and translate them in terms of F. Theorem (2.i) is thereby proved.

3. Properties of Seminormality of Convex Functions

Let A be a closed subset of the x-space E_n, and for every $x \in Q$ let $Q(x)$ be a convex subset of the u-space E_m. Let M be the set of all (x,u) with $x \in A$, $u \in Q(x)$, and let $f_o(x,u)$ be a real-valued function on M.

(3.i) If for some $\bar{x} \in A$ and $\bar{u} \in Q(\bar{x})$ there are numbers r, $b = (b_1, \ldots, b_m)$ real, $\nu > o$ such that

$$f_o(\bar{x},u) > r + b \cdot u + \nu|u-\bar{u}| \text{ for all } u \in Q(\bar{x}),$$

then for no $u_1 \in E_m$, $u_1 \neq o$, it may occur that

$$\bar{u} + \lambda u_1 \in Q(\bar{x}) \ , \tag{3}$$

$$f_o(\bar{x},\bar{u}) \ = \ 2^{-1}[f_o(\bar{x},\bar{u} + \lambda u_1) + f_o(\bar{x},\bar{u} - \lambda u_1)] \tag{4}$$

for all λ real.

<u>Proof</u>. Suppose that there is $u_1 \in E_m$, $u_1 \neq o$ such that both (3) and (4) hold for all λ real. Then

$$f_o(\bar{x},\bar{u}) \ = \ 2^{-1}[f_o(\bar{x},\bar{u} + \lambda u_1) + f_o(\bar{x},\bar{u} - \lambda u_1)]$$

$$\geq 2^{-1} \{[r + b\cdot(\bar{u} + \lambda u_1) + \nu|\lambda u_1|] + [r + b\cdot(\bar{u} - \lambda u_1) + \nu|-\lambda u_1|]\}$$

$$= r + b\cdot\bar{u} + 2\nu|\lambda| \ |u_1| .$$

This is impossible since λ can be arbitrarily large. Statement (3.i) is thereby proved.

Again, let A be a closed subset of the x-space E_n, and for every $x \in A$ let $Q(x)$ be a convex subset of the u-space E_m. Let M be the set of all (x,u) with $x \in A$, $u \in Q(x)$, and let $f_o(x,u)$ be a real-valued function defined on M. We shall assume that, for every $x \in A$, $f_o(x,u)$ is convex in u in the convex set $Q(x)$.

For every $\bar{x} \in A$ and $\delta > o$ let $N_\delta(\bar{x})$ be the set of all $x \in A$ with $|x-\bar{x}| \leq \delta$.

For every $x \in A$ let $\tilde{\tilde{Q}}(x)$ be the set of all $(z^o, u) \in E_{m+1}$ with $z^o \geq f_o(x,u)$, $u \in Q(x)$. Then $\tilde{\tilde{Q}}(x)$ is a convex subset of E_{m+1} whose projection on the u-space E_m is $Q(x)$.

For every $\bar{x} \in A$ and $\delta > o$ let

$$Q(\bar{x};\delta) = \bigcup_{x \in N_\delta(\bar{x})} Q(x) \subset E_m ,$$

$$\tilde{\tilde{Q}}(\bar{x};\delta) = \bigcup_{x \in N_\delta(\bar{x})} \tilde{\tilde{Q}}(x) \subset E_{m+1} .$$

These sets may not be convex, but the sets

$$\text{co } Q(\bar{x};\delta), \qquad \text{co } \tilde{\tilde{Q}}(\bar{x};\delta)$$

certainly are convex subsets of E_m and E_{m+1} respectively, and the projection of co $\tilde{\tilde{Q}}(x;\delta)$ on the u-space E_m is co $Q(\bar{x};\delta)$.

Let

$$\Phi(\bar{x},u,\delta) = \text{Inf } [z^o | (z^o,u) \in \text{co } \tilde{\tilde{Q}}(\bar{x};\delta)] , \quad u \in \text{co } Q(\bar{x};\delta) .$$

(3.ii) Theorem. Suppose that for a given $\bar{x} \in A$ and $\bar{u} \in$ Rint $Q(\bar{x})$ it occurs that for no $u_1 \in E_m$, $u_1 \neq o$, we have $\bar{u} + \lambda u_1 \in Q(\bar{x})$ and $f_o(\bar{x},\bar{u}) = 2^{-1}[f_o(\bar{x},\bar{u} + \lambda u_1) + f_o(\bar{x},\bar{u}-\lambda u_1)]$ for all λ real. Suppose that for any system of numbers r, $b = (b_1,\dots,b_m)$ real such that $f_o(\bar{x},u) \geq r + b\cdot u$ for all $u \in Q(\bar{x})$, any compact subset K of E_m, and $\varepsilon > o$, there is some $\delta_o = \delta_o(\varepsilon,K) > o$ such that $u \in K \cap \text{co } Q(\bar{x};\delta_o)$ implies $\Phi(\bar{x},u,\delta) \geq r + b\cdot u-\varepsilon$. Then, there are numbers $\bar{r}$, $\bar{b} = (\bar{b}_1,\dots,\bar{b}_m)$ real and $\nu > o$, $\delta > o$ such that $f_o(\bar{x},\bar{u}) \leq \bar{r} + \bar{b}\cdot\bar{u} + \varepsilon$, and

$f_o(x,u) \geq \bar{r} + \bar{b} \cdot u + \nu\ |u-\bar{u}|$ for all $x \in A$, $|x-\bar{x}_o| \leq \delta$, and all $u \in Q(x)$.

Proof. Let $v(u) = r_1 + b_1 \cdot u$, $u \in E_m$, be the supporting plane of $f_o(\bar{x},u)$ at $u = \bar{u}$. This supporting plane exists since $\bar{u} \in Q(\bar{x})$ and by force of (5.iii) of [1]. Let $w(u) = \bar{r} + \bar{b} \cdot u$, $u \in E_m$, be the linear function satisfying the requirements of (2.i), part 1, for $f_o(\bar{x},u)$ thought of as a convex function of u alone on the convex set $Q(\bar{x})$. Then, for $o < \alpha \leq 1$ and all $u \in Q(\bar{x})$ we have

$$f_o(\bar{x},u) - [\alpha w(u) + (1-\alpha)v(u)] =$$

$$= \alpha[f_o(\bar{x},u) - w(u)] + (1-\alpha)\ [f_o(\bar{x},u) - v(u)] \geq o\ .$$

Let α_o, $o < \alpha_o \leq 1$, be so small that $\alpha_o|w(\bar{u})-v(\bar{u})| < \varepsilon/4$, and let

$$z(u) = \alpha_o w(u) + (1-\alpha_o)v(u) - \varepsilon/4.$$

Then

$$f_o(\bar{x},u)-z(u) = \alpha_o[f_o(\bar{x},u)-w(u)] + (1-\alpha_o)[f_o(\bar{x},u)-v(u)] + \varepsilon/4 \geq \varepsilon/4;$$

$$\lim\ [f_o(\bar{x},u)-z(u)] = +\infty \quad \text{as } |u| \to +\infty \quad \text{with } u \in Q(\bar{x}); \tag{6}$$

$$f_o(\bar{x},\bar{u})-z(\bar{u}) = v(\bar{u})-z(\bar{u}) = \alpha_o[v(\bar{u})-w(\bar{u})] + \varepsilon/4 \leq \varepsilon/2. \tag{7}$$

From (6) we conclude that, for some $m > o$ we have

$$f_o(\bar{x},u) - z(u) \geq 2\varepsilon \text{ for all } u \in Q(\bar{x}) \text{ with } |u-\bar{u}| = m.$$

By force of the hypotheses in (3.ii) we conclude that there is some $\delta > o$ such that

$$\Phi(\bar{x},u,\delta)-z(u) \geq \varepsilon/8 \qquad \text{for all } u \in Q(\bar{x};\delta),\ |u-\bar{u}| \leq m, \tag{8}$$

$$(\bar{x},u,\delta)-z(u) \geq 9\varepsilon/8 \qquad \text{for all } u \in Q(\bar{x};\delta),\ |u-\bar{u}| = m. \tag{9}$$

If $\nu = \varepsilon/8m$, then (8) implies

$$\Phi(\bar{x},u,\delta)-z(u)-\nu|u-\bar{u}| \geq (\varepsilon/8)-\nu|u-\bar{u}| \geq (\varepsilon/8)-(\varepsilon/8) = o$$

for all $u \in Q(\bar{x};\delta)$, $|u-\bar{u}| \leq m$. For $|u-\bar{u}| > m$, let $\alpha = m/|u-\bar{u}|$ so $o < \alpha < 1$, and let us take $u_o = \alpha(u-\bar{u}) + \bar{u}$, or $u_o = \alpha u + (1-\alpha)\bar{u}$. Then $u_o \in \text{co } Q(\bar{x};\delta)$, $|u_o-\bar{u}| = (m/|u-\bar{u}|)|u-\bar{u}| = m$, and, by the convexity of Φ, also

$$\Phi(\bar{x},u_o,\delta)-z(u_o) \leq \alpha[\Phi(\bar{x},u,\delta)-z(u)] + (1-\alpha)[\Phi_o(\bar{x},\bar{u},\delta)-z(\bar{u})].$$

From here we deduce that

$$\Phi(\bar{x},u_o,\delta)-z(u_o) \geq \Phi(\bar{x},\bar{u},\delta)-z(\bar{u}) +$$

$$+(1/\alpha)\ \{[\Phi(x,u_o,\delta)-z(u_o)] - [\Phi(\bar{x},\bar{u},\delta)-z(\bar{u})]\}$$

$$\geq (1/\alpha)\ ((9\varepsilon/8)-\varepsilon) = \varepsilon/8\alpha\ .$$

Since $\alpha\nu|u-\bar{u}| = \varepsilon/8$ we have

$$\Phi(\bar{x},u,\delta)-z(u) - \nu|u-\bar{u}| \geq (\varepsilon/8\alpha) - (\varepsilon/8\alpha) = o.$$

We have proved that $\Phi(\bar{x},u,\delta) \geq z(u) + \nu|u-\bar{u}|$ for all $u \in Q(\bar{x};\delta)$. This implies that $f_o(x,u) \geq z(u) + \nu|u-\bar{u}|$ for all $x \in A$, $|x-\bar{x}| \leq \delta$, $u \in Q(x)$. Statement (3.ii) is thereby proved.

4. The Sets $\approx{Q}(x)$ and Their Seminormality Properties

Let A be a given closed subset of the x-space E_n and for every $x \in A$ let $U(x)$ be a given subset of the u-space E_m. Let M be the set of all (x,u) with

$x \in A$, $u \in U(x)$, that we shall suppose closed. Let $f_o(x,u)$, $f(x,u) = (f_1,\dots,f_n)$ be real-valued functions defined on M. For every $x \in A$ let $Q(x)$, $\tilde{\tilde{Q}}(x)$ be the sets

$$Q(x) = [z \in E | z = f(x,u) \text{ for some } u \in U(x)] \subset E_n ,$$

$$\tilde{\tilde{Q}}(x) = [(z^o,z) | z^o \geq f_o(x,u),\ z = f(x,u) \text{ for some } u \in U(x)] \subset E_{n+1} .$$

Then, the projection of $\tilde{\tilde{Q}}(x)$ on the z-space E_n is $Q(x)$.

We say that the sets $\tilde{\tilde{Q}}(x)$ satisfy property (Q) at $\bar{x} \in A$ provided

$$\tilde{\tilde{Q}}(\bar{x}) = \cap_\delta \text{ cl co } \tilde{\tilde{Q}}(\bar{x};\delta) = \cap_\delta \text{ cl co } \bigcup_{x \in N_\delta(\bar{x})} Q(x) .$$

We say that the sets $\tilde{\tilde{Q}}(x)$ satisfy property (Q) in A if they have the above mentioned property at every $\bar{x} \in A$. Sets satisfying property (Q) are necessarily convex and closed.

For any fixed $\bar{x} \in A$ let us consider the following scalar function defined on $Q(\bar{x})$:

$$T(z;\bar{x}) = \text{Inf } [f_o(\bar{x},u) | z = f(\bar{x},u),\ u \in U(\bar{x})]$$

$$= \text{Inf } [z^o | (z^o,z) \in \tilde{\tilde{Q}}(\bar{x})],\ z \in Q(\bar{x}) .$$

Then, for $\bar{x} \in A$, we have $-\infty \leq T(z;\bar{x}) < +\infty$ for all $z \in Q(\bar{x})$. We consider $T(z;\bar{x})$ as a function of z in $Q(\bar{x})$.

Note that the convexity of $\tilde{\tilde{Q}}(x) \subset E_{n+1}$ implies the convexity of $Q(\bar{x}) \subset E_n$, but $Q(\bar{x})$ may be not closed even if $Q(x)$ is closed. As usual we denote by $R = R(x)$ the linear manifold of minimum dimension r containing $Q(x)$, so that Rint $Q(x) \subset Q(x) \subset R(x) \in E_n$.

We say that condition (α) holds at a point $\bar{x} \in A$ provided

$$(\alpha) \text{ If } (z^o, z) \in \cap_\delta \text{ cl co } \tilde{\tilde{Q}}(\bar{x},\delta), \text{ then } z \in Q(\bar{x}).$$

We showed in ([1], no. 4) that (α) is a necessary condition for property (Q).

We say that condition (X) holds at a point $\bar{x} \in A$ provided (X) for every $\bar{z} \in Q(\bar{x})$ there is at least one $\bar{u} \in U(\bar{x})$ with $\bar{z} = f(\bar{x},\bar{u})$ such that the following holds: given $\varepsilon > o$ there are numbers $\delta > o$ and r, $b = (b_1,\dots,b_n)$ real such that

$$(X') \quad f_o(x,u) \geq r + \Sigma_j b_j f_j(x,u) \text{ for all } x \in N_\delta(\bar{x}) \text{ and } u \in U(x),$$

$$(X'') \quad f_o(\bar{x},\bar{u}) \leq r + \Sigma_j b_j f_j(\bar{x},\bar{u}) + \varepsilon.$$

(4.i) If conditions (α) and (X) hold at the point $\bar{x} \in A$, then $\tilde{\tilde{Q}}(\bar{x})$ is closed and convex, and the sets $\tilde{\tilde{Q}}(x)$ satisfy property (Q) at the point $\bar{x}$.

This statement was proved in ([1], (4.i)).

(4.ii) If $\tilde{\tilde{Q}}(x)$ is convex, then either $T(z;\bar{x}) = -\infty$ for all $z \in \text{Rint } Q(\bar{x})$, or $T(z;\bar{x}) > -\infty$ for all $z \in Q(\bar{x})$. In the latter case, $T(z;\bar{x})$ is finite everywhere and a convex function on the convex set $Q(\bar{x})$, $T(z;\bar{x})$ is bounded below on every bounded subset of $Q(\bar{x})$, and $T(z;\bar{x})$ is continuous in the convex set Rint $Q(\bar{x})$ open with respect to $R(\bar{x})$. Finally, if $\tilde{\tilde{Q}}(x)$ is convex and closed, and $T(z;\bar{x}) > -\infty$ for all $z \in Q(\bar{x})$, then $T(z;\bar{x})$ is (continuous on Rint $Q(\bar{x})$ and) lower semicontinuous at every point $z \in Q(\bar{x}) - \text{Rint } Q(\bar{x})$.

This statement was proved in ([1], (8.i)).

(4.iii) Theorem. If $T(z;\bar{x}) > -\infty$ in $Q(\bar{x})$, then the sets $\tilde{\tilde{Q}}(x)$ have property (Q) at $\bar{x}$ if and only if properties (α) and (X) hold at the point $\bar{x}$.

This statement was proved in ([1], (9.i)).

(4.iv) If $\tilde{\tilde{Q}}(\bar{x})$ is convex, and $T(z;\bar{x}) > -\infty$ in $Q(\bar{x})$, then the set $\tilde{\tilde{Q}}(\bar{x})$ contains a straight line if, and only if, there are $z_o \in Q(\bar{x})$ and $z_1 \in E_n$, $z_1 \neq o$, such that $z_o + \lambda z_1 \in Q(\bar{x})$ and

$$T(z_o;\bar{x}) = 2^{-1} [T(z_o + \lambda z_1; \bar{x}) + T(z_o - \lambda z_1;\bar{x})] \tag{10}$$

for all λ real.

Proof. Note that for any $\bar{z} \in Q(\bar{x})$ the points $(z^o,\bar{z})$ with $z^o > T(\bar{z};\bar{x})$ belong to Rint $\tilde{\tilde{Q}}(\bar{x})$. Note that $\tilde{\tilde{Q}}(\bar{x})$ contains a straight line if and only if its closure cl $\tilde{\tilde{Q}}(\bar{x})$ contains a straight line by force of (2.ii). Finally, the graph of $T(z;\bar{x})$, $z \in Q(\bar{x})$, certainly belongs to cl $\tilde{\tilde{Q}}(\bar{x})$. Thus, if (10) holds for some z_o, z_1 and all λ real, then certainly cl $\tilde{\tilde{Q}}(\bar{x})$ and $\tilde{\tilde{Q}}(\bar{x})$ contain straight lines. Conversely, if $\tilde{\tilde{Q}}(\bar{x})$ contains a straight line ℓ, then ℓ cannot be vertical, and thus $\ell = [z^o + \lambda z_1^o, \bar{z} + \lambda z_1, \lambda \text{ real}]$ for some $z_1 \in E_n$, $z_1 \neq o$, z_1^o real. Then $z^o \geq T(\bar{z};\bar{x})$, and for every $\varepsilon > o$ the point $(T(\bar{z};\bar{x}) + \varepsilon,\bar{z})$ belongs to Rint $\tilde{\tilde{Q}}(\bar{x})$, hence the straight line $[T(\bar{z};\bar{x}) + \varepsilon + \lambda z_1^o, \bar{z} + \lambda z_1, \lambda \text{ real}]$ belongs certainly to $\tilde{\tilde{Q}}(\bar{x})$, hence to cl $\tilde{\tilde{Q}}(\bar{x})$, and finally

$$T(\bar{z};\bar{x}) + \varepsilon + \lambda z_1^o \geq T(\bar{z} + \lambda z_1;\bar{x})$$

for all λ real. Since $\varepsilon > o$ is arbitrary, we have also

$$T(\bar{z};\bar{x}) + \lambda z_1^o \geq T(\bar{z} + \lambda z_1;\bar{x}) ,$$

and by exchanging λ with $-\lambda$ also

$$T(\bar{z};\bar{x}) - \lambda z_1^o \geq T(\bar{z} - \lambda z_1;\bar{x}) .$$

By addition then

$$T(\bar{z};\bar{x}) \geq 2^{-1}[T(\bar{z} + \lambda z_1;\bar{x}) + T(\bar{z} - \lambda z_1;\bar{x})] ,$$

where T is convex in its first argument, and hence the opposite relation is also true. We conclude that

$$T(\bar{z};\bar{x}) = 2^{-1}[T(\bar{z} + \lambda z_1;\bar{x}) + T(\bar{z} - \lambda z_1;\bar{x})]$$

for all λ real. Statement (4.iv) is thereby proved.

Note that for $\bar{x} \in A$, the set $\tilde{\tilde{Q}}(\bar{x})$ convex, and $T(z;\bar{x}) > -\infty$ in $Q(\bar{x})$, then certainly the set $Q(\bar{x})$ is also convex, the real valued function $T(z;\bar{x})$ is convex on the convex set $Q(\bar{x})$, and hence $T(z;\bar{x})$ possesses a supporting plant π: $z^o = r + b\cdot z$, $z \in E_n$, at every point $\bar{z} \in$ Rint $Q(\bar{x})$, by force of statement (5.iii) of [1]. That is,

$$T(z;\bar{x}) \geq r + b\cdot z \text{ for all } z \in Q(\bar{x}),$$

$$T(\bar{z};\bar{x}) = r + b\cdot \bar{z},$$

and π is actually the supporting plane of the convex set $\tilde{\tilde{Q}}(\bar{x})$ at the point $(z^o,\bar{z})$, $z^o = T(\bar{z};\bar{x})$.

At points $\bar{z} \in Q(\bar{x})$- Rint $Q(\bar{x})$ (if any) the situation may be different. Indeed, $T(z;\bar{x})$ may be not continuous at $\bar{z}$, not even lower semicontinuous, T may have no supporting plane at $\bar{z}$, in the sense that the convex set $\tilde{\tilde{Q}}(\bar{x})$ may have vertical supporting plane at $(z^o,\bar{z})$, $z^o = T(\bar{z};\bar{x})$. Actual examples of these occurrences have been given in ([1], Section 8). Even if $Q(\bar{x})$ is closed, and hence $T(z,\bar{x})$ is lower semicontinuous at the points $\bar{z} \in Q(\bar{x})$ - Rint $Q(\bar{x})$, yet the supporting plane of $Q(\bar{x})$ at $(z^o,\bar{z})$ may be vertical. Nevertheless, the following statement holds:

(4.v) If $\bar{x} \in A$, if $T(z;\bar{x}) > -\infty$ in $Q(\bar{x})$, if the sets $\tilde{\tilde{Q}}(x)$ have property (Q) at $\bar{x}$, then for every $\bar{z} \in Q(\bar{x})$ and $\varepsilon > o$ there are numbers r, $b = (b_1,\ldots,b_n)$ real and $\delta > o$ such that

$$T(z;\bar{x}) \geq r + b\cdot z \quad \text{for all } z \in Q(\bar{x}),$$

$$T(\bar{z};\bar{x}) < r + b\cdot\bar{z} + \varepsilon.$$

For $\bar{z} \in$ Rint $Q(\bar{x})$ this statement is only a corollary of (4.ii) and of (5.iii) of [1], as mentioned above. For $\bar{z} \in Q(\bar{x})$ - Rint $Q(\bar{x})$ statement (4.iv) has been actually proved in ([1], proof of (9.i)).

(4.vi) If $\bar{x} \in A$, if $T(z;\bar{x}) > -\infty$ in $Q(\bar{x})$, if the sets $\tilde{\tilde{Q}}(x)$ have property (Q) at $\bar{x}$, if r, $b = (b_1,\ldots,b_n)$ are real numbers such that $T(z;\bar{x}) \geq r + b\cdot z$ for all $z \in Q(\bar{x})$, then given any compact subset K of E_n and numbers $\varepsilon > o$ there is some other number $\delta_o = \delta_o(\varepsilon,K) > o$ such that $T(z;\bar{x},\delta) \geq r + b\cdot z - \varepsilon$ for all $z \in$ co $Q(\bar{x};\delta_o)$.

<u>Proof</u>. If the statement were not true, then there would be some $\varepsilon > o$ and a sequence of numbers δ_k and points z_k with $\delta_k > o$, $\delta_k \to o$, $z_k \in K \cap$ co $Q(\bar{x};\delta_k)$,

such that $T(z_k;\bar{z},\delta_k) \leq r + b\cdot z_k - \varepsilon$. Hence there would be also real numbers z_k^o with

$$(z_k^o, z_k) \in \text{co}\ \tilde{\tilde{Q}}(\bar{x};\delta_k) \ , \tag{11}$$

$$r + b\cdot z_k - \varepsilon \leq z_k^o \leq r + b\cdot z_k - \varepsilon/2, \quad k = 1,2,\ldots \tag{12}$$

Since the points z_k belong to the compact set K, $[z_k]$ is bounded, and, by force of (12), $[z_k^o]$ also is bounded. Thus, there is a subsequence, say still [k], such that (z_k^o, z_k) converges as $k \to \infty$ to some (z^o, z). Now, for every $\delta > o$ and k sufficiently large so that $\delta_k \leq \delta$, we deduce from (11) that $(z_k^o, z_k) \in \text{co}\ \tilde{\tilde{Q}}(\bar{x};\delta)$ and hence $(z^o, z) \in \text{cl co}\ \tilde{\tilde{Q}}(\bar{x};\delta)$. Since this holds for every $\delta > o$ we have also $(z^o, z) \in \bigcap_{\delta>o} \text{cl co}\ \tilde{\tilde{Q}}(\bar{x};\delta) = \tilde{\tilde{Q}}(\bar{x})$ by force of property (Q) at $\bar{x}$. On the other hand, from (12) we deduce $z^o \leq r + b\cdot z - \varepsilon/2$, a contradiction, since $z^o \geq T(z;\bar{x}) \geq r + b\cdot z$. Statement (4.v) is thereby proved.

(4.vii) Theorem. If $T(z;\bar{x}) > -\infty$ in $Q(\bar{x})$, if the set $\tilde{\tilde{Q}}(\bar{x})$ contains no straight line, then the sets $\tilde{\tilde{Q}}(x)$ have property (Q) at $\bar{x}$ if, and only if, the following properties hold: (α) If $(z^o, z) \in \bigcap_\delta \text{cl co}\ \tilde{\tilde{Q}}(\bar{x};\delta)$, then $z \in Q(\bar{x})$; (X^*) for every $\bar{z} \in Q(\bar{x})$ there is at least one point $\bar{u} \in U(\bar{x})$ with $\bar{z} = f(\bar{x},\bar{u})$ such that given $\varepsilon > o$, there are numbers $\delta > o$, $\nu > o$, and r, $b = (b_1,\ldots,b_n)$ real with

(X'^*) $f_o(x,u) > r + \sum_j b_j f_j(x,u) + \nu|f(x,u)|$ for all $x \in N_\delta(\bar{x})$ and $u \in U(x)$,

(X'') $f_o(\bar{x},\bar{u}) \leq r + \sum_j b_j f_j(\bar{x},\bar{u}) + \varepsilon$.

Proof. Let us prove the sufficiency part. First conditions (α) and (X*) hold, hence condition (X) holds, and by force of (4.iii) the sets $\tilde{\tilde{Q}}(x)$ have property (Q) at $\bar{x}$. On the other hand, the set $\tilde{\tilde{Q}}(\bar{x})$ contains no straight line by force of (3.i). Let us prove the necessity part. First, the sets $\tilde{\tilde{Q}}(x)$ have property (Q) at $\bar{x}$; hence conditions (α) and (X) hold by force of (4.iii). Also, the real-valued function $T(z;\bar{x})$ is convex over the convex set $Q(\bar{x})$, and statement (3.ii) holds. In addition, the set $\tilde{\tilde{Q}}(\bar{x})$ contains no straight line by hypothesis, and hence the real function $T(z;\bar{x})$ has the property that for no $z_o \in Q(\bar{x})$, $z_1 \in E_n$, $z_1 \neq o$, it occurs that $z_o + \lambda z_1 \in Q(\bar{x})$ for all λ real and $T(z_o;\bar{x}) = 2^{-1}[T(z_o + \lambda z_1;\bar{x}) + T(z_o - \lambda z_1;\bar{x})]$ for all λ real. Finally, by force of (4.ii), there are numbers $\bar{r}$, $\bar{b} = (\bar{b}_1,\dots,\bar{b}_n)$ real and $\nu > o$, $\delta > o$, such that $T(z;x) \geq \bar{r} + \bar{b}\cdot z + \nu|z-\bar{z}|$ for all $x \in A$, $|x-\bar{x}| \leq \delta$, $z \in Q(x)$.

Reference

[1] Cesari, L., Seminormality and upper semincontinuity in optimal control. Journal of Optimization Theory and Applications, Vol. 6, 1970, pp. 114-137.