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LIPSCHITZ FUNCTIONS AND LIPSCHITZ TRANSFORMATIONS

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ADDENDUM VI. LIPSCHITZ FUNCTIONS AND LIPSCHITZ TRANSFORMATIONS

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VI 1. DIFFERENTIABILITY OF LIPSCHITZ FUNCTIONS

If E is any set of the t-space E_ν, $t = (t^1,\ldots,t^\nu)$, and $z(t)$, $t \in E$, any real-valued function defined on E, we say that z is Lipschitzian on E (or uniformly Lipschitzian on E of constant M), if for some constant $M \geq 0$ we have

$$|Z(t) - Z(t')| \leq M|t-t'|$$

for all $t, t' \in E$. A function $z(t)$, $t \in E$, is said to be differentiable at a point $t_o \in \text{int } E$, if there is a linear function

$$L(t) = A_1(t^1-t_o^1)+\ldots+A_\nu(t^\nu-t_o^\nu)$$

such that

$$z(t) = z(t_o) + L(t) + |t-t_o|\, e(t), \qquad t \in N(t_o) \subset E,$$

where $t = (t^1,\ldots,t^\nu)$, $t_o = (t_o^1,\ldots,t_o^\nu)$, $e(t_o) = 0$, and $e(t)$ is continuous in a neighborhood N of t_o. If $z(t)$ is differentiable at t_o, then the usual first order partial derivatives of z at t_o exist, and $z_{t^i}(t_o) = A_i$, $i = 1,..,\nu$.

(VI 1.i) Theorem (H. Rademacher). If z is Lipschitzian in an open set $G \subset E_\nu$, then z is differentiable almost everywhere in G.

<u>Proof</u>. For the sake of simplicity we assume $\nu = 2$, and we write $p = (x,y)$ instead of $t = (t^1,\ldots,t^\nu)$, $p_o = (x_o, y_o)$ instead of $t_o = (t_o^1,\ldots,t_o^\nu)$, and

$A(x_o, y_o)$, $B(x_o, y_o)$ instead of $A_1, \dots, A_\nu$. The proof is divided in a number of parts.

(a) Let us prove that the partial derivatives $A(x,y) = \partial f/\partial x$ and $B(x,y) = \partial f/\partial y$ exist almost everywhere in G. Since f satisfies a uniform Lipschitz condition on G, f(x,y) is AC in each variable x and y, and therefore $\partial f/\partial x$ exists everywhere in G but in a set $E_x \subset G$ whose intersections with each straight line $y = \bar{y}$ has measure zero. An analogous property holds for $\partial f/\partial y$. Let us consider now the Dini's derivate numbers

$$\bar{D}_x f(x,y) = \limsup_{h\to 0} h^{-1}[f(x+h,y) - f(x,y)],$$

$$\underline{D}_x f(x,y) = \liminf_{h\to 0} h^{-1}[f(x+h,y) - f(x,y)].$$

If $R = [A,B]$ is any closed interval $R \subset G$ and $\varepsilon = \text{dist}\,([A,B], \text{bd } G]$, let m, n denote integers with $\varepsilon^{-1} < m < n$, and let

$$H_{mn}(x,y) = \max_{1/n \le |h| \le 1/m} h^{-1}[f(x+h,y) - f(x,y)], \qquad (x,y) \in R$$

$$K_m(x,y) = \lim_{n\to\infty} H_{mn}(x,y), \qquad \bar{D}_x f(x,y) = \lim_{m\to\infty} K_m(z,y).$$

Then $H_{mn}(x,y)$ is continuous in R and monotone nondecreasing as n increases; K_m is monotone nonincreasing as m increases, thus the limits above exist at every $(x,y) \in R$, and $\bar{D}_x f(x,y)$ is certainly Borel measurable in R. Since R is any interval in G, then $\bar{D}_x f(x,y)$ is Borel measurable in G. So is $\underline{D}_x f(x,y)$, and hence the set E_x of the points with $\underline{D}_x f(x,y) < \bar{D}_x f(x,y)$ is certainly measurable. Since E_x has an intersection of measure zero with each straight line $y = \bar{y}$, E_x has 2-measure zero, and thus $\partial f/\partial x$ exist a.e. in G. The same argument holds for $\partial f/\partial y$. Let K denote the set where both $A(x,y) = \partial f/\partial x$ and $B(x,y) = \partial f/\partial y$ exist. Then $G - K = E_x \cup E_y$, and $|K| = |G|$.

(b) Let us prove that given $\lambda > 0$ there is a measurable set $S \subset G$ such that $|S| > |G| - \lambda$, and the convergence of the limits

$$\begin{aligned} &\lambda^{-1}[f(x+h,y) - f(x,y)] \to A(x,y) \qquad \text{as } h \to 0, \\ &\lambda^{-1}[f(x,y+h) - f(x,y)] \to B(x,y) \qquad \text{as } h \to 0, \end{aligned} \tag{VI 1.2}$$

is uniform on S. This is trivial consequence of Egoroff's theorem for the convergence of limits

$$\begin{aligned} &\pm n[f(x\pm 1/n,y) - f(x,y)] \to A(x,y) \text{ as } n \to \ , \\ &\pm n[f(x,y\pm 1/n) - f(x,y)] \to B(x,y) \text{ as } n \to \ , \end{aligned} \tag{VI 1.3}$$

where $n = 1,2,\ldots$, and we denote by S a corresponding set of uniform convergence of (VI 1.3) with $|S| > |G| - \lambda$. Now, given ε, $0 < \varepsilon \le 1$, there is an N such that $(x,y) \in S$, $n \ge N$, implies

$$|\pm n[f(x\pm 1/n,y) - f(x,y)] - A(x,y)| \le \varepsilon, \quad |\pm n[f(x,y\pm 1/n) - f(x,y)] - B(x,y)| \le \varepsilon,$$

and we may well assume $N \ge M/\varepsilon$. If h_o is any number, $0 < h_o < (N+1)^{-1}$, then for any h, $0 < h \le h_o$, there is an integer n such that $(n+1)^{-1} < h \le n^{-1}$, hence $n+1 > h^{-1} \ge h_o^{-1} > N > N+1$, or $n > N \ge M/\varepsilon$, as well as $1 < (nh)^{-1} < 1+1/n \le 2$, and $M/n \le \varepsilon$. Thus, for $(x,y) \in S$, $0 < h \le h_o$, we have

$$|h[f(x+h,y) - f(x,y)] - A(x,y)|$$
$$\leq |h^{-1}[f(x+1/n,y) - f(x,y)] - A(x,y)| + |h^{-1}[f(x+h,y) - f(x+1/n,y)]|$$
$$\leq (nh)^{-1}|n[f(x+1/n,y) - f(x,y)] - A(x,y)| + Mh^{-1}|h-1/n|$$
$$\leq 2\varepsilon + M/n \leq 2\varepsilon + \varepsilon = 3\varepsilon.$$

Analogous reasoning holds for $-h_o \leq h < 0$, as well as for the expression $h^{-1}[f(x,y+h) - f(x,y)] - B(x,y)$. This proves the uniform convergence of the limits (VI 1.2) on S.

Thus, given ε, $0 < \varepsilon \leq 1$, there is some $h_o(\varepsilon) > 0$ such that $(x,y) \in S$, $0 < |h| \leq h_o(\varepsilon)$, implies

$$|h^{-1}[f(x+h,y) - f(x,y)] - A(x,y)| < \varepsilon, \quad |h^{-1}[f(x,y+h) - f(x,y)] - B(x,y)| < \varepsilon.$$

(c) We may well assume above that S is closed, $S \subset G$, where G is bounded, hence S is closed and bounded, that is, compact. Now $A(x,y)$, $B(x,y)$, as uniform limits of continuous functions, are continuous on the compact set S, hence uniformly continuous on S. Thus, given $\varepsilon > 0$, there is some $h_1(\varepsilon) > 0$ such that $p = (x,y) \in S$, $p' = (x',y') \in S$, $|p-p'| \leq h_1(\varepsilon)$, implies $|A(p) - A(p')|$, $|B(p) - B(p')| \leq \varepsilon$.

(d) From (VI 5.iv) we know that S has density one at almost all of its points. Thus, there is a measurable subset S_o of S, $S_o \subset S \subset G$, $|S_o| = |S| = |G| - \lambda$, such that, if $p_o = (x_o,y_o) \in S_o$, if $K(p_o,r)$ is the circle of center p_o and radius r, and if $S_o(p_o,r)$ denotes the set $S_o(p_o,r) = S \cap K(p_o,r)$, then $|S_o(p_o,r)/|K(p_o,r)| \to 1$ as $r \to 0$. Thus, given $\varepsilon > 0$, there is an $h_2(\varepsilon;p_o) > 0$ such that $0 < r \leq h_2(\varepsilon;p_o)$ implies

$|S_o(p_o,r)|/|K(p_o,r)| > 1-\varepsilon^2(1+\varepsilon)^{-2}$.

(e) Let $p_o = (x_o,y_o)$ be a point of S_o, let ε be any number, $0 < \varepsilon \leq 1$, let $r > 0$ be any number $r < h_o(\varepsilon)$, $r \leq h_1(\varepsilon)$, $r \leq h_2(\varepsilon;p_o)$, and let $p = (x,y)$ be any point of $K(p_o,r_o)$, $r_o = r(1+\varepsilon)^{-1}$. If $h = |p-p_o|$, $0 \leq h \leq r_o = r(1+\varepsilon)^{-1}$, then the measure of the set complementary to S_o in the circle $K(p_o,h(1+\varepsilon)$ is less than $\pi h^2(1+\varepsilon)^2 \cdot \varepsilon^2(1+\varepsilon)^{-2} = \pi h\varepsilon^2$. On the other hand, the circle $K(p,h\varepsilon)$ of center p and radius $h\varepsilon$ lies in $K(p_o,r)$ since $h+h\varepsilon = h(1+\varepsilon) \leq r$, and $K(p,h\varepsilon)$ has area $\pi h^2\varepsilon^2$. Hence, there is some point $\bar{p} = (\bar{x},\bar{y}) \in S_o \cap K(p,\varepsilon)$, and $|\bar{p}-p_o| \leq |\bar{p}-p| + |p-p_o| \leq h\varepsilon + h = h(1+\varepsilon) \leq r$.

We have now

$$f(x,y) - f(x_o,y_o) = [f(x,y) - f(\bar{x},\bar{y})]$$
$$+ [f(\bar{x},\bar{y}) - f(x_o,\bar{y})] + [f(x,\bar{y}) - f(x_o,y_o)] = \Delta_1 + \Delta_2 + \Delta_3. \tag{VI 1.4}$$

Note that $|p-\bar{p}| \leq h\varepsilon$, and hence

$$|\Delta_1| = |f(p) - f(\bar{p})| \leq M|p-\bar{p}| \leq Mh\varepsilon. \tag{VI 1.5}$$

Note that $\bar{p} = (\bar{x},\bar{y}) \in S$, $p_o = (x_o,y_o) \in S$, with $|\bar{p}-p_o| \leq r \leq h_o(\varepsilon)$, and hence

$$|(\bar{x}-x_o)^{-1} [f(\bar{x},\bar{y}) - f(x_o,\bar{y})] - A(\bar{x},\bar{y})| \leq \varepsilon,$$
$$|(\bar{y}-y_o)^{-1} [f(x_o,\bar{y}) - f(x_o,y_o)] - B(x_o,y_o)| \leq \varepsilon.$$

Finally, since p_o, $\bar{p} \in S$, $|\bar{p}-p_o| \leq r \leq h_1(\varepsilon)$, we have

$$|A(\bar{x},\bar{y}) - A(x_o, y_o)| \leq \varepsilon,$$

and thus

$$(\bar{x}-x_o)^{-1}[f(\bar{x},\bar{y}) - f(x_o,\bar{y})] = A(x_o,y_o) + \eta_1,$$

$$(\bar{y}-y_o)^{-1}[f(x_o,\bar{y}) - f(x_o,y_o)] = B(x_o,y_o) + \eta_2,$$

with $|\eta_1| \leq 2\varepsilon$, $|\eta_2| \leq \varepsilon$. Now

$$|\Delta_2 - (x-x_o)A(x_o,y_o)| = |f(\bar{x},\bar{y}) - f(x_o,\bar{y}) - (x-x_o)A(x_o,y_o)|$$

$$= |(\bar{x}-x_o)[A(x_o,y_o) + \eta_1] - (x-x_o)A(x_o,y_o)|$$

$$\leq |x-\bar{x}|\,|A(x_o,y_o)| + |\bar{x}-x_o|\,|\eta_1|$$

$$\leq M|p-\bar{p}| + 2\varepsilon|\bar{p}-p_o|$$

$$\leq Mh\varepsilon + 2\varepsilon(h+h\varepsilon) = (M + 2 + 2\varepsilon)h\varepsilon \leq (M+4)h\varepsilon. \qquad \text{(VI 1.6)}$$

Analogously,

$$|\Delta_3 - (y-y_o)B(x_o,y_o)| = |f(x_o,\bar{y}) - f(x_o,y_o) - (y-y_o)B(x_o,y_o)|$$

$$= |(\bar{y}-y_o)[B(x_o,y_o) + \eta_2] - (y-y_o)B(x_o,y_o)|$$

$$\leq |y-y_o|\,|B(x_o,y_o)| + |\bar{y}-y_o|\,|\eta_2|$$

$$\leq M|p-\bar{p}| + \varepsilon|\bar{p}-p_o|$$

$$\leq Mh\varepsilon + \varepsilon(h+h\varepsilon) = (M+1+\varepsilon)h\varepsilon \leq (M+2)h\varepsilon. \qquad \text{(VI 1.7)}$$

By (VI 1.4), (VI 1.5), (VI 1.6), (VI 1.7) we have finally

$$|f(x,y) - f(x_o,y_o) - (x-x_o)A(x_o,y_o) - (y-y_o)B(x_o,y_o)|$$

$$\leq |\Delta_1| + |\Delta_2 - (x-x_o)A(x_o,y_o)| + |\Delta_3 - (y-y_o)B(x_o,y_o)|$$

$$\leq Mh\varepsilon + (M+4)h\varepsilon + (M+2)h\varepsilon = (3M+b)h\varepsilon,$$

for all $p = (x,y) \in K(p_o,r_o)$, and where $h = |p-p_o| = ((x-x_o)^2 + (y-y_o)^2)^{1/2}$. This proves that f is differentiable at every point $(x_o,y_o) \in S_o$, where $S_o \subset G$, $|S_o| > |G|-\lambda$, and λ is arbitrary.

If now we take $\lambda = 1/n$ and we denote by S_{on} the corresponding set S_o, $n = 1,2,\ldots$, and we set $\Sigma = \bigcup_n S_{on}$, we see that $S_{on} \subset \Sigma \subset G$, $|G-\Sigma| \leq |G-S_{on}| \leq 1/n$; hence $|\Sigma| = |G|$. On the other hand, f is differentiable at every point $(x_o,y_o) \in S_{on}$, and hence at every point of Σ. Theorem (VI 1.i) is thereby proved.

VI 2. TRANSFORMATIONS OF CLASS K

A one-to-one transformation $q = Tp$, or $T\colon G \to U$, with inverse $p = T^{-1}q$ or $T^{-1}\colon U \to G$, of an open set G in E_ν into an open set U of E_ν such that both T and T^{-1} are uniformly Lipschitzian in G and U respectively is said to be a transformation of class K. Thus, if T is such a transformation, there is a constant $M > 0$ such that, for all points, $p, p' \in G$ and $q, q' \in U$ we have

$$|Tp-Tp'| \leq M|p-p'|, \qquad |T^{-1}q-T^{-1}q'| \leq M|q-q'|. \qquad \text{(VI 2.1)}$$

If $x = (x^1,\ldots,x^\nu) \in G$, $u = (u^1,\ldots,u^\nu) \in U$, then T and T^{-1} can be represented by real-valued functions

$$T: \quad u^i = u^i(x), \quad i = 1,\dots,\nu, \quad x \in G,$$

$$T^{-1}: \quad x = x^i(u), \quad i = 1,\dots,\nu, \quad u \in U, \qquad \text{(VI 2.2)}$$

and $u^i(x)$, $x^i(u)$ are all Lipschitzian functions of constant M in G and U respectively. Conversely, if T and T^{-1} are given by relations (VI 2.2) with $u^i(x)$, $x^i(u)$ uniformly Lipschitzian in G and I respectively with constant M, then certainly T and T^{-1} are of class K with constant νM.

Note that, as proved in (VI 1), all functions $u^i(x)$, $x^i(u)$ are differentiable almost everywhere in G and U respectively, hence they possess first order partial derivatives $\partial u^i/\partial x^s$, $\partial u^s/\partial u^i$, $i,s = 1,\dots,\nu$, almost everywhere in G and U respectively. We shall denote by [du/dx] and [dx/du] the nxn matrices of these partial derivatives, and then the usual Jacobians

$$du/dx = \det[du/dx], \qquad dx/du = \det[dx/du]$$

are defined a.e. in G and U respectively.

We shall prove below (VI 2.iii) that, under transformations of class K, sets of measure zero in G and U correspond. We shall need a few lemmas.

(VI 2.i) (lemma). If W is any measurable set in G, and $W^* = T(W)$ is the image of W under a transformation $T: G \to U$ satisfying $|Tu-Tv| \le M\,|u-v|$ for all $u,v \in G \subset E_\nu$, then $|W^*| \le 2^\nu M^\nu |W|$, where $|W^*|$ denotes the outer measure of W^*.

<u>Proof</u>. For the sake of simplicity let us assume $\nu = 2$. Given $\varepsilon > 0$ we may cover W with countably many circles K_i, $i = 1,2,\dots$, such that

$\Sigma_{i=1}^{\infty} K_i \leq W + \varepsilon$. Let ρ_i denote the radius of K_i, and let K_i' be the image of K_i under T. If q_i, q_i' denote any two points of K_i', $q_i = Tp_i$, $q_i' = Tp_i'$, p_i, $p_i' \in K_i$, then $|p_i-p_i'| \leq 2\rho_i$, $|q_i-q_i'| \leq 2M\rho_i$; hence K_i' is contained in the circle, say K_i'', of center any point q_i of K_i and radius $2M\rho_i$. Then

$$|W^*| \leq \Sigma_{i=1} |K_i''| = \Sigma_{i=1} 4\pi M^2 \rho_i^{\ 2} = 4M^2 \Sigma_{i=1} |K_i| \leq 4M^2 \, [|W|+\varepsilon].$$

Since $\varepsilon > 0$ is arbitrary, the lemma follows.

(VI 2.ii) (lemma). Any measurable set W in E_ν is the countable union of compact sets and of a set of measure zero.

<u>Proof</u>. As usual assume $\nu = 2$ and divide E_2 into the congruent squares $Q_{ij} = [i \leq x < i+1,\ j \leq y < j+1]$, $i,j = 0, \pm 1, \pm 2, \ldots$. It is enough to prove the lemma for each of the sets $W \cap Q_{ij}$. Let us denote by W any of these sets. From [53] we know that there are two sets S and P, $S \subset E \subset P$, with $|P-S| = 0$, where S is the countable union of closed sets, say $S = \cup_j F_j$, $F_j \subset W \subset Q$, and P is the countable intersection of open sets say $P = \cap_j P_j$, $W \subset P_j$. Each set F_j is certainly bounded hence compact, and now

$$W = (\cup_j F_j) \cup (W - \cup_j F_j)$$

where $W-\cup_j F_j = W-S \subset P-S$, and $|W-\cup_j F_j| = |P-S| = 0$.

(VI 2.iii) <u>Theorem</u>. If T: $G \to U$ is any transformation satisfying $|Tu-Tv| \leq M|u-v|$ for all $u,v \in G$, then the image W^* of any measurable set $W \subset G$ is measurable, and if $|W| = 0$ then also $|W^*| = 0$. If T is of class K, then measurable subsets of U and G correspond, and the sets of measure zero also correspond.

Proof. That $|W| = 0$ implies $|W^*| = 0$ is a consequence of (VI 2.i). Now, if W is a measurable subset of G, then $W = (\cup_j F_j) \cup Z$, where the sets F_j are compact, and Z has measure zero. If $F^* = T(F_j)$, $j = 1,2,\ldots$, and $Z^* = T(Z)$, then $|Z^*| = 0$, and all sets F_j^* are compact as the continuous images of compact sets. Finally $W^* = T(W) = (\cup_j F_j^*) \cup Z^*$, and W^* is certainly measurable.

VI 3. THE MAGNIFICATION RATIO AND THE JACOBIAN

Let $T: G \to U$ be a given transformation of class K. For every point $p_o \in G$ let us consider the hypercube $Q = [p | x^i - x_o^i| \leq \ell,\ i = 1,\ldots,\nu]$ of center p_o and sidelength 2ℓ, where we suppose ℓ sufficiently small so that $Q \subset G$, and $p_o = (x_o^1,\ldots,x_o^\nu)$, $p = (x^1,\ldots,x^\nu)$. Let $Q' = T(Q)$ be the image of Q under T. As we know from (VI 2.iii), Q' is a measurable set of measure $|Q'|$. If the limit exists

$$\Delta(p_o) = \Delta(x_o^1,\ldots,x_o^\nu) = \lim |T(Q)|/|Q| \text{ as } \ell \to 0, \qquad \text{(VI 3.1)}$$

then we say that $\Delta(p_o)$ is the magnification ratio of T at the point p_o.

(VI 3.i) Theorem. If $T: G \to U$, or $T: u^i = u^i(x)$, $i = 1,\ldots,\nu$, $x \in G$, is a transformation of class K, then the magnification ratio exists a.e. in G, and $\Delta = |du/dx|$ a.e. in G.

Proof. For the sake of simplicity we assume $\nu = 2$, so as T is given by the real functions $u = u(x,y)$, $v = v(x,y)$, $(x,y) \in G$, and we know already that the functions $u(x,y)$, $v(x,y)$ are differentiable at almost every point $p = (x,y) \in G$. It is enough to prove that the statement holds at every point $p_o = (x_o,y_o) \in G$ where u and v are both differentiable. Let $p_o = (x_o,y_o)$ be such a point. Let $A = u_x$, $B = u_y$, $C = v_x$, $D = v_y$, where all these derivatives are computed at (x_o,y_o), and note that

$$u(x_o+h,y_o+k) = u(x_o,y_o) + Ah + Bk + (h^2+k^2)^{1/2}R(h,k),$$

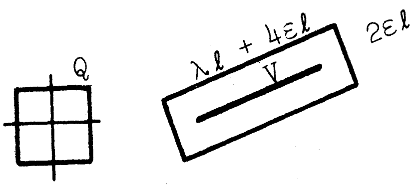
$$v(x_o+h,y_o+k) = v(x_o,y_o) + Ch + Dk + (h^2+k^2)^{1/2}R'(h,k),$$

where R, $R' \to 0$ as h, $k \to 0$. Let $u_o = u(x_o,y_o)$, $v_o = v(x_o,y_o)$ and let $\bar{T}$ be the affine transformation

$$\bar{T}: \quad u = u_o + Ah + Bk, \qquad v = v_o + Ch = Dk. \qquad \text{(VI 3.2)}$$

Then $\bar{T}$ maps Q into a parallelogram V of center (u_o,v_o), $u_o = u(x_o,y_o)$, $v_o = v(x_o,y_o)$, which may be degenerated into a segment or a point.

Suppose first $L = AD-BC = 0$. Then V is degenerated to a segment (or a point) of length $\lambda\ell$, where $\lambda \geq 0$ is a fixed number which depends on A, B, C, D. Let ε be an arbitrary number, $0 < \varepsilon \leq 1$. Let $\delta > 0$ be sufficiently small so as for $h^2+k^2 \leq 2\delta^2$ we have $|R|$, $|R'| \leq \varepsilon$. Let us assume $\ell \leq \delta$ and so small that $Q \subset G$. As $p = (x,y)$ describes Q, the point $\bar{T}p$ describes the segment V. On the other hand, $|h|$, $|k| \leq \ell$, $h^2+k^2 \leq 2\ell^2 \leq 2\delta^2$, and

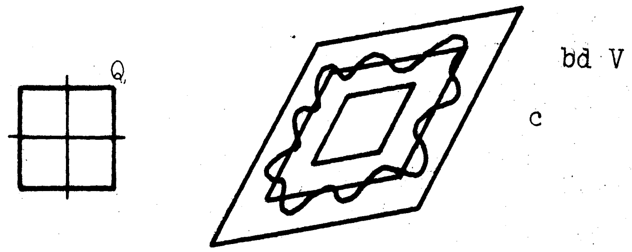


$$|T_p - \overline{T}_p| \leq (R^2+R'^2)^{1/2}(h^2+k^2)^{1/2} \leq (2\varepsilon^2)^{1/2}(2\ell^2)^{1/2}) = 2\varepsilon\ell.$$

Hence, $T(Q)$ is completely contained in an interval of center (u_o,v_o) and side-lengths $\lambda\ell + 4\varepsilon\ell$ and $4\varepsilon\ell$. Thus $|T(Q)| \leq (\lambda\ell + 4\varepsilon\ell)(4\varepsilon\ell)$, $|Q| = \ell^2$, and $|T(Q)|/|Q \leq (\lambda + 4\varepsilon)4\varepsilon$. Thus $|T(Q)|/|Q \to 0 = L$ as $\ell \to 0$, and $\Delta(p_o) = L$.

Let us suppose now $L = |AD-BC| \neq 0$. Then V is a nondegenerated parallelogram of center (u_o,v_o) and area $|V| = L|Q|_o$. The minimum distance of bd V from (u_o,v_o) will be $\mu\ell$ where $\mu > 0$ is a fixed number which depends on A,B,C,D. Let V_1, V_2 be the parallelograms of center (u_o,v_o), affine to V, such that bd V_1 and bd V_2 have minimum distance from (u_o,v_o) equal to $(1-\varepsilon)\mu\ell$, $(1+\varepsilon)\mu\ell$, respectively. Let ε be an arbitrary number, $0 < \varepsilon \leq 1$. Let $\delta > 0$ be sufficiently small so as for $h^2+k^2 \leq 2\delta^2$ we have $|R|$, $|R'| \leq \mu\varepsilon/4$. Let us assume $\ell \leq \delta$ and so small so that $Q \subset G$. As $p = (x,y)$ describes bd Q, then $|h|$, $|k| \leq \ell$, $h^2+k^2 \leq 2\ell^2 \leq 2\delta^2$, and

$$|Tp-\overline{T}p| = (R^2+R'^2)^{1/2}(h^2+h^2)^{1/2} \leq (2\mu^2\varepsilon^2/16)^{1/2}(2\ell^2)^{1/2} = \mu\ell\varepsilon/2.$$



Thus, the image of bd Q under T is an oriented curve C which lies between the parallelograms V_1 and V_2.

Actually, as p describes once bd Q, then $\overline{T}p$ describes once the boundary of V, while Tp describes the curve C with $|Tp-\overline{T}p| \leq \mu\ell\varepsilon/2$. This shows that all points of the interior parallelogram V_1 are linked by bd V as well as by C and that $V_1 \subset T(Q) \subset V_2$; hence

$$(1-\varepsilon)^2|V| \le |V_1| \le |T(Q)| \le |V_2| = (1+\varepsilon)^2|V|$$

with $|V| = L|Q|$, and finally

$$(1-\varepsilon)^2 L \le |T(Q)|/|Q| \le (1+\varepsilon)^2 L,$$

for every ℓ with $0 < \ell \le \delta$ and $Q \subset G$. This proves that $|T(Q)|/|Q| \to L$ as $\ell \to 0$, or $\Delta(p_0) = L$.

We have proved that $\Delta(p_0) = L = |AC-BD|$ at every point $p_0 = (x_0, y_0) \in G$ where both $u(x,y)$, $v(x,y)$ are differentiable, that is, almost everywhere in G.

(VI 3.ii) Let $T: G \to U$, or $T: u^i = u^i(x)$, $i = 1,\dots,\nu$, $x \in G$, be a transformation of class K. Let A be a measurable subset of G where $D(x) \le \beta$ for some constant β, and $A' = T(A)$. Then A' is measurable and $|A'| \le \beta|A|$.

Proof. As usual we may assume $\nu = 2$, and $|A| < \infty$. Let A_0 be the set of all points $p \in A$ of density one and at which $\Delta(p)$ exists. Then $|A_0| = |A|$. Given $\varepsilon > 0$, for every point $p \in A_0$ there are squares K of diameter as small as we want with

$$|A\cap K|/|K| = |A_0 \cap K|/|K| \ge 1-\varepsilon, \qquad |T(K)| \le (\beta+\varepsilon)|K|.$$

By Vitali's covering theorem there is a countable system of points $p_s \in A_0$ and corresponding disjoint squares K_s, $s = 1,2,\dots$, such that

$$p_s \in K_s,\ |A\cap K_s|/|K_s| = |A_0\cap K_s|/|K_s| \ge 1-\varepsilon,\ |T(K_s)| \le (\beta+\varepsilon)|K_s|,$$

$$s = 1,2,\dots, \qquad |A - \cup_s K_s| = 0.$$

Then $|T(A-\bigcup_s K_s)| = 0$, and

$$|A'| = |T(A)| = T[(\bigcup_s A K_s) \cup (A-\bigcup_s K_s)]$$

$$= \Sigma_s |T(A\cap K_s)| + |T(A-\bigcup_s K_s)|$$

$$\leq \Sigma_s T(K_s) \leq (\beta+\varepsilon)|K_s| \leq (\beta+\varepsilon)(1-\varepsilon)^{-1} \Sigma_s |A\cap K_s|$$

$$\leq (\beta+\varepsilon)(1-\varepsilon)^{-1}|A|.$$

Since $\varepsilon > 0$ is arbitrary, we have proved that $|A'| \leq \beta|A|$.

(VI 3.iii) Let $T: G \to U$, or $T: u^i = u^i(x)$, $i = 1,\dots,\nu$, $x \in G$, be a transformation of class K. Let A be a measurable subset of G where $D(x) \geq \alpha$ for some constant $\alpha \geq 0$, and $A' = T(A)$. Then A' is measurable, and $|A'| \geq \alpha |A|$.

Proof. As usual we assume $\nu = 2$. Since the statement is trivial for $\alpha = 0$, we may well assume $\alpha > 0$. As in the proof of (VI 3.ii) we assume $|A| < +\infty$. Let A_o be the set of all points $p \in A$ of density one and at which $\Delta(p)$ exists. Then $|A_o| = |A|$. Given ε, $0 < \varepsilon \leq \alpha$, for every point $p \in A_o$ there are squares K of diameter as small as we want with

$$p \in K, \quad |A\cap K|/|K| = |A_o\cap K|/|K| \geq 1-\varepsilon, \quad |T(K)| \geq (\alpha-\varepsilon)|K|.$$

Then

$$|K-A| = |K| - |A\cap K| \leq |K| - (1-\varepsilon)|K| = \varepsilon|K|,$$

and by (VII 2.i) also

$$T(K-A) \leq 4M^2\varepsilon^2|K|.$$

We shall assume $\varepsilon > 0$ sufficiently small so that $\varepsilon+4M^2\varepsilon^2 < \alpha$. By Vitali's covering theorem there is a countable system of points $p_s \in A_o$ and corresponding disjoint squares K_s, $s = 1,2,\ldots,$ such that

$$p_s \in K_s, \quad |A\cap K_s|/|K_s| = |A_o\cap K_s|/|K_s| \geq 1-\varepsilon,$$

$$T(K_s) \geq (\alpha-\varepsilon)|K_s|, \qquad T(K_s-A) \leq 4M^2\varepsilon^2|K_s|, \; s = 1,2,\ldots,$$

$$||A-\cup_s K_s| = 0.$$

Then $|T(A-\cup_s K_s)| = 0$ and

$$\begin{aligned} |A'| = |T(A)| &= T[(\cup_s A\cap K_s) \cup (A-\cup_s K_s)] \\ &= \Sigma_s|T(A\cap K_s)| + |T(A-\cup_s K_s)| \\ &= \Sigma_s T(K_s) - \Sigma_s T(K_s-A) \\ &\geq (\alpha-\varepsilon)\,\Sigma_s|K_s| - 4M^2\varepsilon^2\,\Sigma_s|K_s| \\ &\geq (\alpha-\varepsilon-4M^2\varepsilon^2)\,\Sigma_s|A\cap K_s| = (\alpha-\varepsilon-4M^2\varepsilon^2)|A|. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have proved that $|A'| \geq \alpha|A|$.

VI 4. THE RULE OF MULTIPLICATION OF JACOBIANS

(VI 4.i) Let $T: U \to G$, or $T: x = x(u)$, $u \in U$, $x = (x^1,\ldots,x^\nu)$, $u = (u^1,\ldots,u^\nu)$, be a transformation of class K. Let $z^i = z^i(x)$, $i = 1,\ldots,\nu$, $x \in G$, be Lipschitzian functions in G, write $z = z(x)$, $x \in G$, $z = (z^1,\ldots,z^\nu)$, and let $Z(u) = z(x(u)) = (Z^1,\ldots,Z^\nu)$, $u \in U$. Then the functions Z^i are differentiable a.e. in T, (VI 4.1) $\partial Z^i/\partial u^j = \Sigma_{r=1}^{\nu} (\partial z^i/\partial x^r)(\partial x^r/\partial u^j)$, $i,j = 1,\ldots,\nu$, and the rule of multiplication of Jacobians hold:

$$dZ/du = (dz/dx)(dx/du), \qquad \text{a.e. in } U. \tag{VI 4.2}$$

<u>Proof</u>. Each function $Z^i(u) = z^i(x(u))$ is uniformly Lipschitzian in U as the composition of Lipschitzian functions. Hence, each function $Z^i(u)$ is certainly differentiable a.e. in U by (VI 1.i). Actually, if $H_o \subset G$ is the set of points where some z^i are not differentiable in G, then meas $H_o = 0$, and we know from (VI 2.iii) that the set $H_1 = T^{-1}(H_o) \subset U$ also has measure zero in U. Let H_2 be the set of points in U where some of the functions $x^j(u)$ are not differentiable. Then meas $H_2 = 0$, and if $H = H_1 \cup H_2$, also meas $H = 0$. If $u_o \in U - H$, and $x_o = Tu_o$, then all z^i are differentiable at x_o, and all x^j are differentiable at u_o. Now the proof that all $Z^i(u) = z^i(x(u))$ are differentiable at u_o and that formulas (VI 4.1) hold is the same as in calculus. Formula (VI 4.2) follows by the composition rule of matrices.

(VI 4.ii) If $T: U \to G$ is a transformation of class K, say $T: x = x(u)$, $u \in U$, and $T^{-1}: u = u(x)$, $x \in G$, then

$$[dx/du][du/dx]_{x=x(u)} = I,$$

$$[du/dx]_{x=x(u)}\,[dx/du] = I,$$

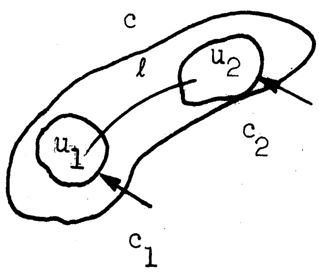
a.e. in U, where I is the unitary matrix, and hence the products of the relative determinants (jacobians) is equal to one a.e. in U. An analogous statement holds by exchanging x with u.

A consequence of (VI 4.i)

(VI 4.iii) If T: $U \to G$ is a transformation of class K, and U is connected, say T: $x = x(u)$, $u \in U$, T^{-1}: $u = u(x)$, $x \in G$, then either $dx/du > 0$ a.e. in U, or $dx/du < 0$ a.e. in U. If $dx/du > 0$ a.e. in U, then $du/dx > 0$ a.e. in G, and T is said to be <u>positive</u>.

<u>Proof</u>. By (VI 4.i) and (VI 4.ii) we see that $(dx/du)(du/dx)_{x=x(u)} = 1$ a.e. in U, hence $dx/du \neq 0$ a.e. in U, and dx/du and du/dx have the same sign in corresponding points (a.e. in U). It remains to prove that dx/du has the same sign wherever it is different from zero in U. To prove this we need the concept of topological degree [3] and a few of its properties. If U is an point of U, c any cell in U, (c closed), T any continuous map from U into E_ν, and x any point $x \in E_\nu - T(\text{bd } c)$, then the topological degree $O(x;c,T)$ is an integer ($\gtreqless 0$), and (α) $O(x;c,T)$, as a function of x in E_ν, is constant on each component of the open set $E_\nu - T(\text{bd } c)$, and (β) $O(x,c,T) \neq 0$ implies that $x = Tu$ for some $u \in \text{int } c$. Also, (φ) if T is 1-1 then $O(x;T,c) = \pm 1$ for every point $x \in T(\text{int } c)$. Thus, if T is 1-1, since T(int c) is a single component of $E_\nu - T(\text{bd } c)$, we deduce from (α) that either $O(x;c,T) = 1$ for all $x \in T(\text{int } c)$, or $O(x;c,T) = -1$ for all $x \in T(\text{int } c)$. Now, if $u \in U$, and c,c' are any two cells in U, with $u \in \text{int } c$, $c \subset \text{int } c'$, $c \subset c' \subset U$, then we can

prove that (δ) $O(x;c,T) = O(x;c',T)$. Indeed, if we divide $\delta(c'-c)$ into finitely many not overlapping cells $c_1,\dots,c_N$, then by an additive property of the topological degree we have $O(x;c',T) = O(x,c,T) + \Sigma_{j=1}^{N} O(x;c_j,T)$. Then $O(x;c,T) \neq O(x;c',T)$ would imply $O(x;c_j,T) \neq 0$ for at least one j, hence by (β), there would be a counter image $u' \in T^{-1}x$ in the interior of c_j, hence $T(u) = T(u') = x$, $u \neq u'$, a contradiction, since T is one-one. Thus $O(x;c,T) = O(x;c',T)$.



Finally, if u_1, u_2 are any two points of U and U is connected, then we can take a line ℓ in U joining u_1 and u_2, and a cell c containing ℓ and hence u_1 and u_2 in its interior. If c_1, c_2 are two cells with $u_i \in \text{int } c_i$, $c_i \in \text{int } c$, $i = 1,2$, then by (δ), $O(x_i,c_1,T) = O(x_i;c,T)$, $i = 1,2$, and by (α), $O(x_1,c,T) = O(x_2,c,T)$. Hence (ε) $O(x_1;c_1,T) = O(x_1,c,T) = O(x_2,c,T) = O(x_2,c_2,T)$. Let us assume now that T be as in the statement. We have seen in the proof of (VI 3.i) that for any $u \in U$ with $dx/du \neq 0$, we can take a cube c (cell) as small as we want, with $u \in \text{int } c$, $c \subset U$, and $O(x,c,T) = \text{sgn}(dx/dt)$. Thus, if u_1, u_2 are any two points of U where the Jacobian is $\neq 0$, and c_1, c_2 corresponding cubes, and $x_i = Tu_i$, $i = 1,2$, then, by (ε),

$$\text{sgn}(dx/dt)_{u=u_1} = O(x_1;c_1,T) = O(x_2,c_2,T) = \text{sgn}(dx/dt)_{u=u_2}.$$

Statement (VI 4.iii) is thereby proved.

VI 5. TRANSFORMATION OF MULTIPLE INTEGRALS

(VI 5.i) Theorem. Let $T: G \to U$ be a transformation of class K. Thus, T is one to one, and $T: u = u(x)$, $x \in G$, $T^{-1}: x = x(u)$, $u \in U$, are uniformly Lipschitzian. Let $f(x)$, $x \in G$, be any measurable function in G, and let $F(u) = f(x(u))$, $u \in U$. Then $F(u)$ is measurable in U, and

$$\int_U F(u)du = \int_G f(x)|du/dx|dx, \qquad \text{(VI 5.1)}$$

whenever either $F(u)$ or $f(x)|du/dx|$ is known to be L-integralbe, and then both $F(u)$ and $f(x)|du/dx|$ are L-integrability and (VI 5.1) holds.

Proof. As usual we assume $\nu = 2$ and we use the notation $T: u = u(x,y)$, $v = v(x,y)$, $(x,y) \in G$, for T, and we write $f(x,y)$, $F(u,v) = f(x(u,v), y(u,v))$, and $L(x,y) = |u_x v_y - u_y v_x|$.

(a) Let us assume first that A is any measurable subset of G with $|A| < +\infty$, and that $f(x,y)$, $(x,y) \in A$, is any bounded measurable nonnegative function. Then $A' = T(A)$ is also measurable with $|A'| < \infty$ by force of (VI 2.i), F and fL are bounded and measurable, as we shall prove that

$$\int_{A'} F(q)dq = \int_A f(p)L(p)dp. \qquad \text{(VI 5.2)}$$

For every $\delta > 0$ and $k = 0,1,2,\dots$, let $e_k = [p|k\delta \leq f(p) < k+1]$. Then each set $e_k \subset G$ is measurable, and so is the set $e'_k = T(e_k)$. Then

$$\Sigma_{k=0}^{\infty} k\delta|e'_k| \leq \int_A F(q)dq \leq \Sigma_{k=0}^{\infty} (k+1)\delta|e'_k|, \qquad \text{(VI 5.3)}$$

where the left sum converges as $\delta \to 0$ since $\int_{A'} F(q)dq < +\infty$. Further

$\Sigma_k |e'_k| = |A'| < +\infty$, and hence the right sum also converges as $\delta \to 0$, and the two limits coincide with $\int_{A'} F(q)dq$.

For every $j = 0,1,2,\ldots$, let $s_j = [p \in A \mid j\delta \leq \Delta(p) < (j+1)\delta]$. Then s_j is measurable, the sets $s'_j = Ts_j$ are measurable, and by force of (VI 3.ii), (VI 3.iii) we have

$$j\delta |e_k s_j| \leq |e'_k s_j| \leq (j+1)\delta |e_k s_j|. \tag{VI 5.4}$$

By summation on j we have

$$\Sigma_j\, j\delta |e_k s_j| \leq |e'_k| \leq \Sigma_j\, (j+1)\delta |e_k s_j|,$$

where the sums converge as $\delta \to 0$ since $|e'_k| < +\infty$, and $\Sigma_j |e_k s_j| = |e_k| < +\infty$. From (VI 5.3) and (VI 5.4) we have

$$\delta^2 \Sigma_k\, k\, \Sigma_j j |e_k s_j| \leq \int_{A'} F(q)dq \leq \delta^2 \Sigma_k (k+1)\, \Sigma_j (j+1) |e_k s_j|. \tag{VI 5.5}$$

Again, the left side converge as $\delta \to 0$ since $\int_{A'} F(q)dq < +\infty$, and the right side also converges if we note that

$$\delta\, \Sigma_k \Sigma_j\, k |e_k s_j| = \delta\, \Sigma_k\, k |e_k| \leq \int_A f(p)dp,$$

$$\delta\, \Sigma_k \Sigma_j\, j |e_k s_j| = \Sigma_j j |e_j| \leq |A'|$$

$$\Sigma_k \Sigma_j |e_k s_j| = |A|,$$

and we can rearrange the sums since all terms are nonnegative. Consequently,

$$\delta^2 \Sigma_k \Sigma_j\, (k+1)(j+1) |e_k s_j| - \delta^2 \Sigma_k \Sigma_j kj |e_k s_j| \leq \eta(\delta), \tag{VI 5.6}$$

where

$$\eta(\delta) \leq \delta \int_A f(p)dp + \delta|A'| + \delta^2|A|,$$

and $\eta(\delta) \to 0$ as $\delta \to 0$. On the other hand, for all $p \in e_k s_j$ we have

$$kj\, \delta^2|e_k s_j| \leq f(p)\Delta(p) \leq (k+1)(j+1)\delta^2|e_k s_j|,$$

hence

$$\delta^2 \Sigma_k \Sigma_j kj|e_k s_j| \leq \int_A f(p)\Delta(p)dp \leq \delta^2 \Sigma_k \Sigma_j (k+1)(j+1)|e_k s_j|$$

and from (VI 5.5) and (VI 5.6) also

$$\left|\int_{A'} F(q)dq - \int_A f(p)\Delta(p)dp\right| \leq \eta(\delta).$$

Since $\eta(\delta) \to 0$ as $\delta \to 0$, we have proved that

$$\int_{A'} F(q)dq = \int_A f(p)\Delta(p)dp$$

and (VI 5.2) is thereby proved.

(b) Now let A be any measurable subset of G, and $f(x,y)$, $(x,y) \in A$, any nonnegative measurable function on A. Then there are compact sets C_n, $n = 1,2,\ldots$, such that $C_n \subset A$, $C_n \subset C_{n+1}$, $|C_n| < +\infty$, $n = 1,2,\ldots$, and $|A-A_o| = 0$ where $A_o = \cup_n C_n$. If $C'_n = T(C_n)$, $A'_o = T(A_o)$, then $|A'-A'_o| = 0$ where $A'_o = \cup_n C'_n$. Now let us define $f_n(p)$, $p \in A$, by taking $f_n(p) = 0$ in $A-C_n$; $f_n(p) = f(p)$ if $f(p) \leq n$ and $p \in C_n$; and $f_n(p) = n$ if $f(p) \geq n$ and $p \in C_n$. Then $f_n(p) \to f(p)$ as $n \to \infty$ for all $p \in A_o$, and if $F_n(q) = f_n(T_q^{-1})$,

$q \in A'$, then $F_n(q) \to F(q)$ as $n \to \infty$ for all $q \in A'_0$. Finally, both sequences $[f_n(p),\ p \in A,\ n=1,2,..]$, $[F_n(q),\ q \in A',\ n=1,2,...]$ are nonnegative monotone nondecreasing sequences. By part (a) above we have

$$\int_{C'_n} F_n(q)dq = \int_{C_n} f_n(p)\Delta(p)dp, \quad n=1,2,...,$$

and also

$$\int_{A'} F_n(q)dq = \int_A f_n(p)\Delta(p)dp, \quad n = 1,2,....$$

By Lebesgue monotone convergence theorem, F is L-integrable if and only if $\int_{A'} F_n(q)dq$ as a finite limit as $n \to \infty$. Analogously, $f\,\Delta$ is L-integrable if and only if $\int_A f_n(p)\Delta(p)dp$ has a finite limit as $n \to \infty$. Therefore,

$$\int_{A'} F(q)dq = \int_A f(p)\Delta(p)dp$$

whenever one of these integrals exists, and then the other one also exists, and equality holds.

(c) Now, let $f(x,y)$, $(x,y) \in G$, be any measurable function. Let G^+, G^- be the subsets of G where $f \geq 0$, of $f < 0$ respectively. Then, if $U^+ = T(G^+)$, $U^- = T(G^-)$, by force of part (b) above we have

$$\int_{U^+} F(q)dq = \int_{G^+} f(p)\Delta(p)dp, \qquad \int_{U^-} F(q)dq = \int_{G^-} f(p)\Delta(p)dp. \tag{VI 5.7}$$

The first relation in (VI 5.7) holds if we know that either F is L-integrable in U^+, or that $f\Delta$ is L-integrable in G^+, and an analogous statement holds for the second relation in (VI 5.7).

By combining the two relations in (VI 5.7) we have

$$\int_U F(q)dq = \int_G f(p)\Delta(p)dp, \qquad \text{(VI 5.8)}$$

and this relation certainly holds if we know that F is L-integrable in U, or that $f\Delta$ is L-integrable in G. Finally, by (VI 5.8) and statement (VI 3.iii), we deduce statement (VI 5.i).

Statement (VI 5.i) can be extended in the following way:

(VI 5.ii) Let T be a one to one transformation from G to U which preserves measurable sets. Assume that F is integrable on U and there are countably many sets U_n, n=1,2,..., such that $U_n \subset U_{n+1}$; $\bigcup_n U_n$ = U-B with $|B| = 0$; and T is of class K on each U_n. Then relation (VII 5.1) holds.

This theorem follows from (VI 5.i) by application of Lebesgue monotone convergence theorem.

Statement (VI 5.ii) applies for instance in the transformation of cartesian into polar coordinates.

Bibliographical notes. The property of differentiability almost everywhere of Lipschitz functions was proved by H. Rademaker [94]. The concept of transformation of class K (or class K_ℓ, $\ell \geq 1$) was studied by C. B. Morrey [82b]. For the concept of topological degree mentioned in the proof of (VI 4.iii) the reader may see P. Alexandroff and H. Hopf [3], or T. Rado and P. Reichelderfer [96]. Another proof of (VI 4.iii) is given in the last mentioned work [96, p. 132].