THE UNIVERSITY OF MICHIGAN

COLLEGE OF LITERATURE, SCIENCE, AND THE ARTS
Department of Mathematics

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ONE-DIMENSIONAL PROBLEMS OF OPTIMIZATION

Lamberto Cesari

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1.1. THE CANONIC PROBLEM OF OPTIMIZATION

We shall deal here with a real variable $t \in E_1$, with a real vector variable $x = (x^1,\ldots,x^n) \in E_n$, $n \geq 1$, and a real vector variable $u = (u^1,\ldots,u^m) \in E_m$, $m \geq 1$. Here t is the independent variable and is usually called <u>time</u>, and then x is called the <u>space variable</u> (or state, or phase variable), and u the <u>control variable</u>. Accordingly, x^i, $i = 1,\ldots,n$, are the space, or state, or phase coordinates, or variables, and u^j, $j = 1,\ldots,m$, the control coordinates, or variables.

We shall denote by A a given subset of the tx-space $E_1 \times E_n$, and for each $(t,x) \in A$ we shall denote by $U(t,x)$ a given subset of the u-space E_m. We do not exclude that A coincides with the whole tx-space, and that $U(t,x)$ coincides with the whole u-space. Often U is a fixed subset of E_m, or U may depend on t only, or on x only. In any case A is called the time-state space, and $U(t,x)$ the control space at time t and state x. We shall denote by M the set of all (t,x,u) with $(t,x) \in A$, $u \in U(t,x)$, thus $M \subset E_1 \times E_n \times E_m$. Note that A is the projection of M on the tx-space, while, for each $(\bar{t},\bar{x}) \in A$, $U(\bar{t},\bar{x})$ is the intersection of M with the linear subspace $(t = \bar{t}, x = \bar{x})$ of $E_1 \times E_n \times E_m$. We shall denote by $f(t,x,u) = (f_1,\ldots,f_n)$ a given vector function defined (at least) on M. Let B be a given subset of the $t_1x_1t_2x_1$- space E_{2N+2}, $x_1 = (x_1^1,\ldots,x_1^n)$, $x_2 = (x_2^1,\ldots,x_2^n)$, and let $g(t_1,x_1,t_2,x_2)$ be a given real-valued function defined on B. We shall consider pairs of functions $x(t) = (x^1,\ldots,x^n)$, $u(t) = (u^1,\ldots,u^m)$, $t_1 \leq t \leq t_2$, satisfying

(a) $x(t)$ is absolutely continuous (AC) in $[t_1,t_2]$, that is, each component $x^i(t)$ is AC in $[t_1,t_2]$;

(b) $u(t)$ is measurable in $[t_1,t_2]$, that is, each component $u^i(t)$ is measurable in $[t_1,t_2]$;

(c) $(t,x(t)) \in A$ for every $t \in [t_1,t_2]$;

(d) $u(t) \in U(t,x(t))$ almost everywhere (a.e.) in $[t_1,t_2]$;

(e) $dx/dt = f(t,x(t),u(t))$ a.e. in $[t_1,t_2]$, that is,

$$dx^i/dt = f_i(t,x(t),u(t)), \quad i = 1,\ldots,n, \text{ a.e. in } [t_1,t_2]; \qquad (1.1.1)$$

(f) $(t_1,x(t_1),t_2,x(t_2)) \in B$.

Since $x(t)$ is AC, that is, each $x^i(t)$ is AC in $[t_1,t_2]$, we conclude that all dx^i/dt, and hence all $f_i(t,x(t),u(t))$, $i = 1,\ldots,n$, are L-integrable in $[t_1,t_2]$.

A pair $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, as above is said to be <u>admissible</u>, and for such a pair $x(t)$ is called a <u>trajectory</u>, and $u(t)$ a <u>stategy</u>, or <u>control</u>, or <u>steering</u> function. It is then said that $u(t)$ transfers, or steers the state variable x from the value $x(t_1)$ at time t_1 to the value $x(t_2)$ at time t_2. Also we may say that the trajectory $x(t)$ is generated by the strategy $u(t)$. If x,u is any admissible pair, then we may well say that x is an <u>admissible</u> <u>trajectory</u>, and that u is an <u>admissible</u> <u>strategy</u>.

In many applications U depends on t only, the point $(t_1,(x(t_1))$ is given and is an interior point of A, and the differential system $dx/dt = f(t,x,u(t))$ with $u(t)$ measurable and values $u(t) \in U(t)$ satisfies local conditions for existence and uniqueness of the solution $x(t)$, at least in a right neighborhood of t_1. In this situation (and analogous ones) $u(t)$ determines $x(t)$

uniquely, at least in a right neighborhood of t_1. In any case we shall use the terminology above (u transfers $x(t_1)$ to $x(t_2)$, x is generated by u) even if conditions for uniqueness theorems do not hold.

Often $U(t,x)$ is defined for every $(t,x) \in A$ by inequalities, as in the examples: (1) $m = 1$, $0 \leq u \leq 1$; (2) $m = 2$, $a_s u^1 + b_s u^2 + c_s \leq 0$, $s = 1,2,3$, with a_s, b_s, c_s, $s = 1,2,3$, real given numbers, or real functions of t and x in A. More generally, $U(t,x)$ may be defined by means of inequalities $F_s(t,x,u) \leq 0$, $s = 1,\ldots,N$. Because of these and analogous examples, condition (d) is often called a unilateral condition on the control variable $u = (u^1,\ldots,u^m)$.

Often A itself is defined by means of inequalities, as in the examples: (4) $0 \leq t \leq 1$, $x \in E_n$; (5) $n = 1$, $a \leq t \leq b$, $c \leq x \leq d$; (6) $n = 2$, $e_s t + f_s x^1 + g_s x^2 + h_s \leq 0$, $s = 1,2,3$, with e_s, f_s, g_s, h_s given constants. More generally A may be defined by equalities of the form $G_s(t,x) \leq 0$, $s = 1,\ldots,P$. Because of these and analogous examples, condition (c) is often denoted as a unilateral constraint on the time-state variables t,x.

Condition (f) concerns the <u>end</u> values $(t_1,x(t_1),t_2,x(t_2))$ of the trajectories, in particular the initial values $t_1,x(t_1)$, and the terminal values t_2, $x(t_2)$. We shall often denote by $\eta(x) = (t_1,x(t_1),t_2,x(t_2))$ the <u>ends</u> of the trajectory x. Thus, condition (f) is an abstract form of the boundary conditions for the trajectories.

Often B is of the form $B_1 \times B_2$, and then requirement (f) separates into requirements $(t_1,x(t_1)) \in B_1$, $(t_2,x(t_2)) \in B_2$ concerning initial and terminal values. Often $(t_1,x(t_1))$ is a fixed point, that is, B_1 is a single point of the t_1x_1-space. Also, B_2 may be called a target. In particular, if t_2 is fixed, say $t_2 = t_{20}$, then $B_2 = \{t_2\} \times B_{20}$, where B_{20} is a subset of the x_2-space E_n, and we may perfer to call B_{20} a target, to be reached at a fixed time $t_2 = t_{20}$. If t_2 is not fixed, assume that B_2 can be thought of as the set of all (t,x) with $t_2' \leq t \leq t_2''$, $x \in B(t) \subset E_n$, $B(t)$ a given subset of E_n depending on t. Then $B_2(t)$ is a "moving target" to be reached in the time interval $[t_2', t_2'']$. If $B_2(t)$ is a single point, say $x_2(t) \in E_n$, then we have the problem of joining a given fixed initial point $(t_1 = t_{10}, x_1 = x_{10})$ to a given "curve" $x = x_2(t)$, $t_2' \leq t \leq t_2''$.

Often, boundary conditions are assigned by means of a set of inequalities and equalities concerning the $(2n+2)$-vector $\eta(x) = (t_1, x(t_1), t_2, x(t_2))$, say

$$b_e(t_1, x(t_1), t_2, x(t_2)) = 0, \quad e = 1, \ldots, k_1,$$

$$b_e(t_1, x(t_1), t_2, x(t_2)) \leq 0, \quad e = k_1 + 1, \ldots, k_1 + k_2.$$

Then B is the set of all $(2n+2)$-vectors (t_1, x_1, t_2, x_2) satisfying the same set of $k_1 + k_2$ relations.

The question as to whether we can satisfy the boundary conditions and all the constraints is sometimes a very difficult one. Often it means whether a certain task can be performed at all with the given constraints, that is, in applications, within the limitation inherent to a certain physical plant. The question is often referred to as to whether a given system is <u>controllable</u>. Given all the constraints, say A, B, and U above, the question of controllability is equivalent to the question as to whether the corresponding class of all admissible pairs is not empty. Questions of controllability will be discussed in [ ]. In existence theorems we shall often assume that the class of admissible pairs is not empty, and we shall consider nonempty subclasses Ω of admissible pairs.

Once a nonempty class Ω of admissible pairs $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, is assigned, we may consider the values taken by the functional

$$I[x,u] = g(t_1,x(t_1),t_2,x(t_2)) \qquad (1.1.2)$$

in the class Ω. Clearly the value of this functional depends only upon the ends $\eta(x)$ of the trajectory, or

$$I[x,u] = g(\eta(x)).$$

The functional $I[x,u]$ is often called <u>cost functional</u>, or <u>cost</u>, or <u>performance index</u>.

For instance, we may assume t_1 and $x_1 = x(t_1)$ as fixed, we may leave t_2 undetermined, and we may try to minimize the distance of the end point $x(t_2)$ from a fixed point x_o of E_n. Then $g = |x(t_2) - x_o|$ is of the form (1.1.2).

We shall say that a given pair x,u in Ω is optimal, or that x,u gives an absolute minimum for the cost functional $I[x,u]$ in Ω, if $I[x,u] \leq I[\tilde{x},\tilde{u}]$ for all pairs $\tilde{x},\tilde{u}$ in Ω. An equivalent way to describe this property is as follows. Let i denote the infimum of $I[x,u]$ in the class Ω, or

$$i = \operatorname*{Inf}_{\Omega} I[x,u] .$$

Then an element x,u of the class Ω is optimal if and only if, (a) i is finite, and (b) $I[x,u] = i$. There may be more than one element (x,u) in Ω which is optimal (for $I[x,u]$ in Ω) but the value taken by $I[x,u]$ is the same, namely i.

Analogous definitions hold for absolute maxima, but problems of maximum can be reduced to problems of minimum by replacing g by -g. Thus, for the sake of simplicity, we shall always refer to problems of minimum.

We have described above the canonic problem of optimization, often referred to as a Mayer problem of optimal control.

1.2. MAYER, LAGRANGE, BOLZA PROBLEMS

First, a particular case of the canonic problem above is of interest, that is, the case in which $g = t_2 - t_1$ or, t_1 being fixed, simply $g = t_2$. Then the problem of the minimum of $I[x,u] = g = t_2$, is called a problem of minimum time, or a brachistochrone problem.

Often the cost functional $I[x,u]$ is given in the form

$$I[x,u] = \int_{t_1}^{t_2} f_o(t,x(t),u(t))dt \ , \qquad (1.2.1)$$

where $f_o(t,x,u)$ is a real-valued single-valued function defined in M, and Ω is then restricted to only those admissible pairs for which $f_o(t,x(t),u(t))$ is L-integrable. This is said to be a Lagrange problem of optimization. These problems are readily reduced to Mayer problems by introducing a new state variable x^o, the new state vector $\tilde{x} = (x^o,x) = (x^o,x^1,\dots,x^n)$, an additional differential equation

$$dx^o/dt = f_o(t,x(t),u(t)) \ ,$$

and an additional initial condition $x^o(t_1) = 0$. Then the functional above becomes $I[x,u] = x^o(t_2)$, and we have now a Mayer problem with $g = x^o(t_2)$, and the (n+1)-vector $\tilde{x}$ replacing n-vector x. Again, if we take $f_o = 1$ in (1.2.1), then the functional (1.2.1) reduces to $I = \int_{t_1}^{t_2} dt = t_2 - t_1$, and we have a problem of minimum time.

Often the cost functional $I[x,u]$ is given in the form

$$I[x,u] = \int_{t_1}^{t_2} f_o(t,x(t),u(t)) + g(t_1,x(t_1),t_2,x(t_2))$$

with f_o and g as above. This is said to be a Bolza problem of optimization. Again the same change of variables as above yields $I[x,u] = x^o(t_2) + g(t_1,x(t_1),t_2,x(t_2))$, and we have a Mayer problem concerning the (n+1)-vector $\tilde{x}$ and a new function $\tilde{g} = x^o(t_2) + g(t_1,x(t_1),t_2,x(t_2))$.

Actually, Mayer, Lagrange, and Bolza problems are equivalent. Indeed, Mayer and Lagrange problems are obviously particular cases of the Bolza problems above, since they can be obtained by taking $f_o = 0$, or $g = 0$, respectively, in (1.1.4). We have already shown that both Lagrange and Bolza problems can be reduced to Mayer problems. It remains only to prove that Mayer problems can be reduced to Lagrange problems. This can be obtained by introducing an auxiliary variable x^o, an additional differential equation $dx^o/dt = 0$, the boundary condition

$$x^o(t_1) = (t_2 - t_1)^{-1} g(t_1,x(t_1),t_2,x(t_2)) , \qquad (1.2.3)$$

and the functional

$$J = \int_{t_1}^{t_2} x^o(\tau)d\tau.$$

Since $x^o(\tau)$ is constant in $[t_1,t_2]$ with value (1.2.3), then $J = g(t_1,x(t_1),t_2,x(t_2))$. The canonic (Mayer) control variable u, and functional (1.1.2), is now reduced to a Lagrange problem with state variable $\tilde{x} = (x^o,x) = (x^o,x^1,\ldots,x^n)$, control variable u, functional J, and differential system (1.1.1.) plus the equation $dx^o/dt = 0$.

1.3. FREE PROBLEMS, ISOPERIMETRIC PROBLEMS, AND FURTHER VARIANTS OF THE CANONIC PROBLEM

Often problems of optimization are written in terms of the trajectory x only (assumed always to be an AC vector function), and its derivative x', the functional being of the form

$$I[x] = \int_{t_1}^{t_2} f_o(t,x(t),x'(t))dt \ . \tag{1.3.1}$$

In this situation we shall retain possible unilateral constraints on the time-state variables, or $(t,x(t)) \in A$, where A is a given subset of the tx-space E_{n+1}, but we do not require constraints on the values of the derivatives $x'(t)$. Such a problem is said to be a <u>free</u> problem. For instance, if we take $f_o = (1 + |u|^2)^{1/2}$, $u = (u^1,\dots,u^n)$, then

$$I[x] = \int_{t_1}^{t_2} (1 + |x'(t)|^2)^{1/2} \, dt \ ,$$

and we are looking for a path of minimum length (if any) of the type C: $x = x(t)$, $t_1 \leq t \leq t_2$, x AC in $[t_1,t_2]$, satisfying the constraints $(t,x(t)) \in A$, and $(t_1,x(t_1),t_2,x(t_2)) \in B$.

These problems are readily reduced to Lagrange problems by introducing the control variable $u = (u^1,\dots,u^n)$, (thus $m = n$), the control space $U = E_n$, and the differential system $dx^i/dt = u^i$, $i = 1,\dots,n$, or $dx/dt = u$, in other words, $f(t,x,u) = u$, or $f_i(t,x,u) = u^i$, $i = 1,\dots,n$. Then we have a Lagrange problem with

$$I[x,u] = \int_{t_1}^{t_2} f_o(t,x(t),u(t))dt$$

and differential system and constraints

$$(t,x(t)) \in A, \ u(t) \in U = E_n, \ dx/dt = u \ .$$

This problem is finally reduced to a Mayer problem as mentioned above. See below for more details and examples.

Often, constraints are assigned in the form

$$I_j[x,u] = \int_{t_1}^{t_2} g_j(t,x(t),u(t))dt = c_j, \; j = 1,\dots,N,$$

that is, by requiring that certain additional functionals take on given fixed values c_j, $j = 1,\dots,N$. These are called <u>isoperimetric</u> problems (or isoperimetric requirements). Again these problems can be reduced to Mayer form by introducing N additional state variables y^j, $j = 1,\dots,N$, additional differential equations

$$dy^j/dt = g_j(t,x(t),u(t)), \; j = 1,\dots,N,$$

and additional boundary conditions

$$y^j(t_1) = 0, \; y^j(t_2) = c_j, \; j = 1,\dots,N.$$

Then we have again problems as above concerning the (n+N)-vector $\tilde{x} = (x^1,\dots,x^n,y^1,\dots,y^N)$.

Finally, let us mention that classes Ω of admissible pairs are often defined by many kinds of restrictions besides boundary conditions. For instance, we may require that certain components $x^j(t)$ take given values at certain fixed points $\tau_1,\tau_2,\dots,\tau_M$ other than end points, or that the trajectory $x(t)$ passes through given points $x_1,\dots,x_M$ at certain fixed points $\tau_1,\tau_2,\dots,\tau_M$ as before. More generally we may assume that the trajectory $x(t)$ passes through given sets $B_1,\dots,B_M$ at certain fixed points $\tau_1,\dots,\tau_M$ as before, or

$$t_1 < \tau_1 < \tau_2 < \dots < \tau_M < t_2, \; x(\tau_s) \in B_s, \; s = 1,\dots,M .$$

Finally, the points $\tau_1,\ldots,\tau_M$ may be variable and required to satisfy certain requirements. It happens that these requirements can be often expressed in the abstract form

$$(t_1,\tau_1,\ldots,\tau_M,t_2,x(t_1),x(\tau_1),\ldots,x(\tau_M),x(t_2)) \in \tilde{B} ,$$

where $\tilde{B}$ is a given subset of a Euclidean space of suitable dimension.

The differential system (1.1.1) relates trajectories and strategies to each other. In applications, a given physical system is said to be monitored, or regulated, by the differential system (1.1.1), and in some cases (1.1.1) is said to represent the physical plant, or briefly the plant.

Note that the differential system (1.1.1) may be linear in u,

$$dx/dt = B(t,x)\ u + C(t,x) ,$$

or linear in x,

$$dx/dt = A(t,u)\ x + C(t,u)$$

or may take more particular forms, as

$$dx/dt = A(t)\ x + B(t)\ u + C(t) ,$$

or

$$dx/dt = Ax + Bu + C(t) ,$$

where A,B,C here represent matrices of the types $n \times n$, $n \times m$, $n \times 1$. respectively, whose entries may be constant or depending upon the variables just indicated.

For free problems, as mentioned, the differential system is reduced to $dx/dt = u$ (or $f = u$ with $m = n$, $U = E_n$).

1.4. EXAMPLES

The problem of the (nonparametric) curve of minimum length of the tx-plane E_2 joining two given points $1 = (t_1,x_1)$, $2 = (t_2,x_2)$, $t_1 < t_2$, is usually written as a free problem with functional

$$I = \int_{t_1}^{t_2} (1 + x'^2)^{1/2}\, dt$$

and boundary conditions $x(t_1) = x_1$, $x(t_2) = x_2$ t_1,t_2, x_1,x_2 fixed, $t_1 < t_2$. The same problem can be written as a Lagrange problem with $n = 1$, $m = 1$, functional

$$I = \int_{t_1}^{t_2} (1 + u^2)^{1/2} dt ,$$

and differential equation and boundary condition

$$dx/dt = u,\ x(t_1) = x_1,\ x(t_2) = x_2 .$$

The same problem can be written as a Mayer problem with $n = 2$. $m = 1$, functional $I = y(t_2)$, and differential equations and boundary conditions

$$dx/dt = u,\ dy/dt = (1 + u^2)^{1/2} ,\ x(t_1) = x_1,\ x(t_2) = x_2,\ y(t_1) = 0.$$

Any Mayer problem of minimum time ($I = t_2 - t_1$, or $I = t_2$ if t_1 is fixed) is equivalent to a corresponding Lagrange problem with $f_0(t,x,u) = 1$.

The isoperimetric problem with $n = N = 1$, $x(t_1) = 0$, $x(t_2) = 0$, $t_1 < t_2$, $A = [t_1 \leq t \leq t_2,\ x \geq 0]$, $I = \int_{t_1}^{t_2} x\, dt = \text{maximum}$, $J = \int_{t_1}^{t_2} (1 + x'^2)^{1/2}\, dt = L$, with $t_2 - t_1 \leq L \leq \pi(t_2 - t_1)/2$, is equivalent to the Mayer problem with $n = 3$, $m = 1$, $x(t_1) = 0$, $x(t_2) = 0$, $y(t_1) = 0$, $z(t_2) = 0$, $A = [t_1 \leq t \leq t_2$,

$x \geq 0$, $y \geq 0$, $z \geq 0]$, $U = [-\infty < u < +\infty]$, $dx/dt = u$, $dy/dt = (1 + u^2)^{1/2}$, $dz/dt = x$, $I = z(t_2) = \text{maximum}$.

1.5. SYSTEMS WHERE THE STRATEGY IS UNIQUELY DETERMINED BY THE TRAJECTORY

Let us note first that, given an admissible pair $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, the trajectory x may not determine uniquely the strategy u. This occurs, for instance, for $x' = u^2$, with $n = m = 1$, and control space $U = [-1 \leq u \leq 1]$. The trajectory $x(t) = t$, $0 \leq 1 \leq 1$, is generated by the strategy $u(t) = 1$, $0 \leq t \leq 1$, but also by $u(t) = -1$, $0 \leq t \leq 1$, as well as by any other measurable function $u(t)$, $0 \leq t \leq 1$, with values $u(t) = \pm 1$.

There are important situations, however, where the trajectory x of any admissible pair $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, determines uniquely the strategy u (a.e. in $[t_1,t_2]$). In other words, an <u>admissible</u> trajectory x determines uniquely the corresponding strategy u. These systems, or problems, may be denoted sometime as systems TDS (trajectory determining the strategy).

This certainly occurs for (usual solutions) of free problems ($m = n$, $f = u$, $U = E_n$) since then (1.1.1) reduces to $u = dx/dt$, and the strategy $u = x'$ is uniquely determined by the trajectory (a.e.). This does not mean that the trajectory x is arbitrary, even in free problems! For instance, in the free problem of the minimum of $I = \int_0^1 x'^2 \, dt$ with $n = 1$, $A = E_2$, $x(0) = 0$, $x(1) = 1$, the AC function $x = t^{1/2}$, $0 \leq t \leq 1$, is not an admissible trajectory since x'^2 is not L-integrable in $[0,1]$.

Another case of TDS systems is represented by problems monitored by a differential equation of order n, say linear and of the form,

$$dy^n/dt^n + a_1(t)\, d^{n-1} y/dt^{n-1} + \dots + a_n(t)\, y = u$$

in the scalar functions y, and where u is a scalar strategy. By the canonic substitution $x' = y$, $x^2 = y'$, $\dots$, $x^n = y^{(n-1)}$, we obtain a system (1.1.1) of the form

$$dx^1/dt = x^2, \dots,\; dx^{n-1}/dt = x^n,\; dx^n/dt = -a_n(t)x^1 - \dots - a_1(t)x^n + u\ ,$$

and again the strategy u is uniquely determined by the trajectory x,

$$u(t) = dx^n/dt + a_n(t)x^1 + \dots + a_1(t)x^n,\ \text{(a.e.)}\ .$$

An analogous case of systems TDS is represented by problems depending on higher derivatives, or the problem of the minimum of

$$I = \int_{t_1}^{t_2} f_o(t,\, y(t),\, y'(t), \dots, y^{(n)}(t))dt$$

for scalar functions y which are AC together with $y', \dots, y^{(n-1)}$, satisfying constraints of the form $(t,\, y(t), \dots, y^{(n-1)}(t)) \in A$, and $(t_1, y(t_1), \dots, y^{(n-1)}(t_1),\, t_2,\, y(t_2), \dots, y^{(n-1)}(t_2)) \in B$ where $A \subset E_n$, $B \subset E_{2n+2}$ are fixed sets, and there are no constraints on $y^{(n)}$.

By the canonic substitution $x^1 = y$, $x^2 = y^1, \dots, x^n = y^{(n-1)}$, $u = y^{(n)}(t)$, we see that the problem above is transformed into a Lagrange problem concerning the minimum of the functional

$$I = \int_{t_1}^{t_2} f_o(t,\, x(t),\, u(t))dt\ ,$$

with state variable $x = (x^1, \dots, x^n)$, control variable u, $m = 1$, $U = E_1$, differential system $dx^1/dt = x^2, \dots, dx^{n-1}/dt = x^n$, $dx^n/dt = u$, constraints $(t,\, x(t)) \in A$, $(t_1, x(t_1), t_2, x(t_2)) \in B$. Thus any admissible trajectory determines the corresponding strategy uniquely $u(t) = dx^n/dt$.

An example of an analogous situation is the problem of the minimum of

$$I = \int_{t_1}^{t_2} f_o(t, y(t), (Lg)(t))dt ,$$

with constraints analogous to those of the previous case, and where L is a differential operator of order n. Then the cost functional can be written in the usual form

$$I = \int_{t_1}^{t_2} f_o(t, y(t), u(t))dt,$$

with the same constraints and differential equation $(Lx)(t) = u(t)$. Again an admissible strategy certainly determines the strategy u uniquely.

Another case of TDS systems occurs whenever $m \leq n$, systems (1.1.1) has the linear form

$$dx/dt = B(t,x)\, u + C(t,x), \quad t_1 \leq t \leq t_2 , \tag{1.5.1}$$

and the n x m matrix $B(t,x)$ has always maximal rank m. Then, for t in $[t_1,t_2]$ (a.e.), $u(t) = (u^1,\dots,u^m)$ can be recovered by linear operations from the n-vector $dx/dt - C(t, x(t))$ and the n x m matrix $B(t, x(t))$. In harmony with the previous considerations, we assume here $x(t)$ AC and the matrices B and C with continuous entries.

1.B. THE CANONIC PROBLEM IN TERMS OF ORIENTOR FIELDS

1.6. ORIENTOR FIELDS AND THE IMPLICIT FUNCTION THEOREM

Let us consider again the canonic problem of optimization of (1.1).

For every $(t,x) \in A$ let us denote by $Q(t,x)$ the set of all $z = (z^1,\ldots,z^n) \in E_n$ such that $z = f(t,x,u)$ for some $u \in U(t,x)$, or $z^i = f_i(t,x,u)$, $u \in U(t,x)$, in symbols

$$Q(t,x) = [z \in E_n | z = f(t,x,u),\ u \in U(t,x)] = f(t,x,U(t,x)) \ .$$

The last notation is most common, since it states that $Q(t,x)$ is the image in E_n of the set $U(t,x)$ of E_m, image obtained by means of the vector function $f(t,x,u) = (f_1,\ldots,f_n)$.

For every admissible pair $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, we have

$$dx/dt = f(t,x(t),u(t)),\ u(t) \in U(t,x(t)) \text{ a.e. in } [t_1,t_2] \ ,$$

$$(t,x(t)) \in A \text{ for all } t \in [t_1,t_2] \ ,$$

and these relations obviously imply

$$dx/dt \in Q(t,x(t)) \text{ a.e. in } [t_1,t_2] \ ,$$

$$(t,x(t)) \in A \text{ for all } t \in [t_1,t_2] \ .$$

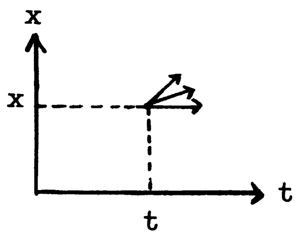
This remark obviously suggests that instead of the differential system and constraints

$$dx/dt = f(t,x,u),\ (t,x) \in A,\ u \in U(t,x) \ , \tag{1.6.1}$$

we simply write

$$dx/dt \in Q(t,x), \ (t,x) \in A.$$

Relations (1.6.1) and (1.6.2) mean that at every point $(t,x) \in A$ we can choose any direction dx/dt of the set $Q(t,x)$. In other words,



at every point $(t,x) \in A$ we have a possible set of directions $dx/dt \in Q(t,x)$ instead of only one direction, say $dx/dt = F(t,x)$ as for usual differential systems. We say that (1.6.2) defines an <u>orientor field</u> in A. This term is the analogue of "direction field" as defined by usual differential systems (in normal form).

We have shown above that for every admissible pair $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, the trajectory $x(t)$, $t_1 \leq t \leq t_2$, can be interpreted as an (AC) solution of the orientor field (1.6.2) (satisfying boundary conditions (f)). Conversely, every AC solution of the orientor field (1.6.2) (satisfying (f)) can be interpreted as a solution of the differential system (1.6.1), precisely, there exists a measurable vector function $u(t)$, $t_1 \leq t \leq t_2$, with $u(t) \in U(t,x(t))$, $dx/dt = f(t,x(t),u(t))$ a.e. in $[t_1,t_2]$. In other words, to every solution x of the orientor field (1.6.2) (satisfying (f)) we can always associate some measurable vector function u such that x, u is an admissible pair. This statement can be proved under very mild hypotheses, indeed it suffices to know that both A and M are closed sets, $(A \subset E_{n+1}, M \subset E_{n+1+m})$. Precisely, we shall prove below the simple statement:

(1.6.i) <u>Implicit Function Theorem for Orientor Fields</u>. If A is a closed subset of the tx-space E_{n+1}, if $U(t,x)$ is a subset of E_m for every $(t,x) \in A$, and the set M of all $(t,x,u) \in E_{n+1+m}$ with $(t,x) \in A$, $u \in U(t,x)$ is closed, if

$x(t)$, $t_1 \leq t \leq t_2$, is an AC vector function such that $(t,x(t)) \in A$ for all $t \in [t_1,t_2]$ and $x'(t) \in Q(t,x(t))$ for almost all $t \in [t_1,t_2]$, then there exists a measurable function $u(t)$, $t_1 \leq t \leq t_2$, such that $u(t) \in U(t,x(t))$ and $x'(t) = f(t,x(t),u(t))$ for almost all $t \in [t_1,t_2]$.

Note that A is the projection of M on the tx-space, and that for every $(t,x) \in A$ the set $U(\bar{t},\bar{x})$ is the intersection of M with the subspace $[t = \bar{t}, x = \bar{x}]$ in E_{n+1+m}. Thus, the assumption that M is closed certainly implies that $U(\bar{t},\bar{x})$ is a closed subset of E_m. Thus, for every $(t,x) \in A$, the set $U(t,x)$ is necessarily closed.

The above shows that, under very mild hypothesis (f continuous on M and M closed) the usual requirements (1.6.1) can be written in the equivalent form of an orientor field (1.6.2), and vice versa. Statement (1.6.i) will be proved in (1.7) below.

1.7. GENERAL FORMS OF THE IMPLICIT FUNCTION THEOREM

A real single-valued function $f(x)$ defined on a metric space x is said to be lower [upper] semicontinuous at the point $x_o \in X$ provided $f(x_o) \leq \underline{\lim}\, f(x)$ as $x \to x_o$ in $X[f(x_o) \geq \overline{\lim}\, f(x)$ as $x \to x_o$ in X]; in other words, provided, given $\varepsilon > 0$, there is some $\delta = \delta(\varepsilon, x_o) > 0$ such that $f(x) > f(x_o) - \varepsilon$ [$f(x) < f(x_o) + \varepsilon$] for all $x \in X$ whose distance from x_o is $\leq \delta$. The same function f is said to be lower [upper] semicontinuous in X if f is lower [upper] semicontinuous at every point of X.

(1.7.i) Every real single-valued function $f(x)$ lower [or upper] semicontinuous in the metric space X is B-measurable.

Indeed, for every real number a the set of points $x \in X$ where $f(x) > a$

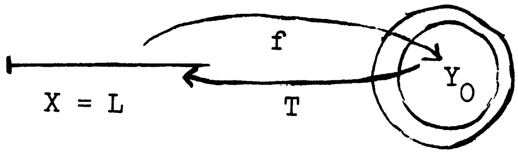
$[f(x) < a]$ is open in X (cfr. Kelley [ ], p. 119 and p. 40).

(1.7.ii) Every function $f(x)$ lower [upper] semicontinuous in the compact metric space X has an absolute minimum [absolute maximum] in X.

Proof. Suppose $f(x)$ is lower semicontinuous in the compact metric space X, let $m = \text{Inf } f(x)$ for all $x \in X$ and let $[x_k]$ by any sequence of points $x_k \in X$ with $f(x_k) \to m$ as $k \to \infty$. Since X is compact, there is at least one point $x_o \in X$ and a subsequence $[x_{k_s}]$ such that $x_{k_s} \to x_o$ as $s \to \infty$. Since $f(x)$ is lower semicontinuous at x_o, given $\varepsilon > 0$ there is some $\delta > 0$ such that $f(x) > f(x_o) - \varepsilon$ for all $x \in X$ at a distance $< \delta$ from x_o. Thus $f(x_{k_s}) > f(x_o) - \varepsilon$ for all s sufficiently large, and $f(x_{k_s}) \to m$ implies $m \geq f(x_o) - \varepsilon$, and thus m is finite. Since this relation holds for all $\varepsilon > 0$, we have $m \geq f(x_o)$. On the other hand, $x_o \in X$, and hence $m \leq f(x_o)$. By comparison, we conclude that $m = f(x_o)$, and thus f has an absolute minimum in X. Analogous reasoning holds for upper semicontinuity and the existence of an absolute maximum.

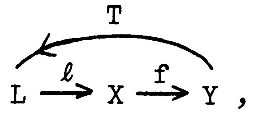
(1.7.iii) Given any two metric spaces X, Y of which X is the countable union of compact subspaces of X, let $f : X \to Y$ be any continuous mapping, and let $Y_o = f(X)$ be the image of X in Y. Then, there is a B-measurable map $\varphi : Y_o \to X$ such that $f\varphi$ is the identity map on Y_o, that is, $f[\varphi(y)] = y$ for every $y \in Y_o$.

Proof. (a) Let us suppose that X is replaced by a closed set $L \subset [0,+\infty]$. In this situation, let $T(y) = \text{Inf } f^{-1}(y)$ for $y \in Y_o$. Here $f^{-1}(y)$ is a nonempty set of real nonnegative numbers, and the operator Inf applies. Actually, f is a continuous map, hence $f^{-1}(y)$ is a closed nonempty subset of $[0,+\infty]$, and hence $f^{-1}(y)$ has a minimum, or $T(y) = \text{Min } f^{-1}(y)$, hence $T(y) \in f^{-1}(y)$, and



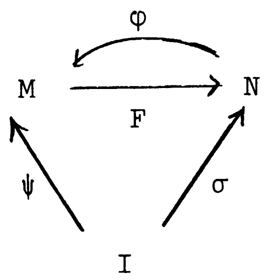
$f[Ty)] = y$ for every $y \in Y_o$. Let us prove that $T : Y_o \to L$ is a lower semicontinuous (real single-valued) function. Suppose this is not the case. Then, there is a point $y_o \in Y_o$, a number $\varepsilon > 0$, and a sequence $[y_k]$ such that $y_k \in Y_o$, $y_k \to y_o$, $T(y_k) < T(y_o) - \varepsilon$, or $0 \leq x_k < x_o - \varepsilon$, with $x_k = T(y_k)$, $f(x_k) = y_k$, $x_o = T(y_o)$, $f(x_o) = y_o$. Here $[x_k]$ is a bounded sequence, hence there is a subsequence, say $[x_{k_s}]$ with $x_{k_s} \to \bar{x}$ with $0 \leq \bar{x} \leq x_o - \varepsilon$. Here $x_{k_s} = T(y_{k_s}) \in L$, and thus $\bar{x} \in L$ since L is closed. Also, $f(x_{k_s}) = f(T(y_{k_s})) = y_{k_s} \to y_o$, $x_{k_s} \to \bar{x}$, hence $f(\bar{x}) = y_o$ since f is continuous on L. Thus, $\bar{x} \in f^{-1}(y_o) \leq x_o - \varepsilon$, a contradiction, since $x_o = \text{Min}\, f^{-1}(y_o)$. We have proved that $T : Y_o \to L$ is lower semicontinuous in Y_o, hence B-measurable because of (1.4.i), and $f(T(y)) = y$ for every $y \in Y_o$. Statement (1.4.iii) is proved for X replaced by any closed subset L of $[0,+\infty]$.

(b) Let us consider now the general case. By hypothesis $X = \cup X_\alpha$, where X_α, $\alpha = 1,2,\ldots$, is a sequence of compact subsets of X. By general topology (Hocking and Young [ ], Th. 3.28) we know that each X_α is the continuous image $\ell_\alpha : L_\alpha \to X_\alpha$, $\alpha = 1,2,\ldots$, of some closed subset L_α of $[0,1]$, say $L_\alpha \subset [\varepsilon, 1 - \varepsilon]$ for some $\varepsilon > 0$. Let us denote by L the set which coincides with $L'_\alpha = \alpha + L_\alpha$ in $[\alpha, \alpha + 1]$, that is, the displacement L'_α of L_α in $[\alpha, \alpha + 1]$. Then L is a closed subset of $[0,+\infty)$ and we shall denote by $\ell : L \to X$ the map which coincides with ℓ_α on L'_α. Then ℓ is a continuous map of L onto X. We have the situation



and, by force of (a), there is a B-measurable map $T : Y_o \to L$ such that $(f\ell)(T(y)) = y$ for every $y \in Y_o$, or $(f\ell T)(y) = y$, since $f\ell$ is a continuous map, and $Y_o = f(X) = f(\ell(X)) = (f\ell)(X)$. If we take $\varphi = \ell T$, $\varphi : Y_o \to X$, we have $f(\varphi(y)) = y$ for every $y \in Y_o$. Obviously, φ is a B-measurable map as the superposition of a continuous map ℓ on the B-measurable map T. Statement (1.7.iii) is now completely proved.

(c) We are now in a position to prove the implicit function theorem (1.6.i). As usual we denote by M the set of all (t,x,u) with $(t,x) \in A$ and $u \in U(t,x)$, hence $M \subset E_{1+n+m}$. Also, we denote by N the set of all (t,x,z) with $(t,x) \in A$, $z = f(t,x,u)$, $u \in U(t,x)$, hence $N \subset E_{1+2n}$. Let $F : M \to N$ denote the continuous map defined by $(t,x,u) \to (t,x,z)$ with $z = f(t,x,u)$. Here M and N are metric spaces since $M \subset E_{1+n+m}$, $N \subset E_{1+2n}$; also, M is closed by hypothesis, and M is the countable union of compact subsets, say $M_\alpha = [(t,x,u) \in M \,|\, |t| + |x| + |u| \le \alpha]$, $\alpha = 1,2,\dots$. Finally, $N = F(M)$. By force of (1.7.iii), there is a B-measurable map $\varphi : N \to M$ such that $F\varphi$ is the identity map, that is, $F\varphi(t,x,z) = (t,x,z)$ for every $(t,x,z) \in N$. Now, let us consider the map $\sigma : I \to N$ on



$I = [t_1,t_2]$, defined by $t \to (t,x(t),x'(t))$, where $x(t)$, $t_1 \le t \le t_2$, is an AC solution of the orientor field $x' \in Q(t,x) = f(t,x,U(t,x))$. Then $\psi = \varphi\sigma$: $I \to M$, and ψ maps t into some $(t,x(t),u(t)) \in M$, with $F(t,x(t),u(t)) = (t,x(t),x'(t))$, or $x'(t) = f(t,x(t),u(t))$. Actually, σ is defined not on all of I, but in the subset I_o of I where $x'(t)$ exists, and meas I_o = meas $I = t_2 - t_1$, and thus the concluding relation holds in I_o, that is, $x'(t) = f(t,x(t),u(t))$ a.e. in $[t_1,t_2]$. On the other hand, $x'(t)$ is measurable, that is, σ is measurable, φ is B-measurable, hence $u(t)$ is measurable. The implicit function theorem (1.6.i) is thereby proved.

Statement (1.6.i) we have just proved is the form we shall use in this exposition of a type of statements which have been given the name of implicit function theorems. Statements (1.7.iv) and (1.7.v) below are two more examples of such statements.

(1.7.iv) Let M be a topological space which is the countable union of compact metrizable subsets, let N be a Hausdorff space, and I a measure space. Let $f : M \to N$ be a continuous map, and $\sigma : I \to N$ a measurable map such that $\sigma(I) \subset f(M)$. Then there exists a measurable map $\psi : I \to M$ such that $f(\psi(x)) = \sigma(x)$ for all $x \in M$.

(1.7.v) Same as (1.7.iv) when M is a separable metric space.

For the proof of these statements we refer to E. J. McShave and R. B. Warfield [ ]. For other inplicit function theorems see C. Castaing [ ].

1.C. GENERALIZED SOLUTIONS FOR THE CANONIC PROBLEM OF OPTIMAL CONTROL

1.8. GENERALIZED SOLUTIONS

Often a given problem has no optimal solution, but the mathematical problem and the corresponding set of solutions can be modified in such a way that an optimal solution exists and yet neither the system of trajectories, nor the corresponding values of the cost functional are essentially modified. The modified (or generalized) problem and its solutions are of interest in themselves, and have often relevant physical interpretations. Besides, generalized solutions can be thought of as limits of usual solutions in a suitable topology.

Here we introduce generalized solutions as usual problems involving a finite number of ordinary strategies which are thought of as being used at the same time according to some probability distribution (Gamkrelidze's chattering states).

Briefly, instead of considering the usual cost functional, differential system, boundary conditions, and constraints

$$I[x,u] = g(t_1,x(t_1),t_2,x(t_2)) ,$$

$$dx/dt = f(t,x(t),u(t)), \quad f = (f_1,\dots,f_n) , \qquad (1.8.1)$$

$$(t_1,x(t_1),t_2,x(t_2)) \in B, \quad (t,x(t)) \in A, \quad u(t) \in U(t,x(t)) ,$$

we shall consider new cost functional differential system, boundary conditions, and constraints

$$I[x,p,v] = g(t_1,x(t_1),t_2,x(t_2)) ,$$

$$dx/dt = h(t,x(t),p(t),v(t)), \quad h = (h_1,\dots,h_n) \ , \tag{1.8.2}$$

$$(t_1,x(t_1),t_2,x(t_2)) \in B, \ (t,x(t)) \in A, \ v(t) \in V(t,x(t)), \ p(t) \in \Gamma \ .$$

Precisely, $v(t) = (u^{(1)},\dots,u^{(\gamma)})$ represents a system of some γ ordinary strategies $u^{(1)}(t),\dots,u^{(\gamma)}(t)$, each $u^{(j)}(t)$ having its values in $U(t,x(t)) \subset E_m$. Thus, we think of $v = (u^{(1)},\dots,u^{(\gamma)})$ are themselves vectors u with values in $U(t,x)$. In other words,

$$v = (u^{(1)},\dots,u^{(\gamma)}), \ u^{(j)} \in U(t,x), \ j = 1,\dots,\gamma \ ,$$

or

$$v \in V(t,x) = [U(t,x)]^{\gamma} = U \times \cdots \times U \subset E_{m\gamma} \ , \tag{1.8.3}$$

where the last term is the product of U by itself taken γ times, and thus V is a subset of the Euclidean space $E_{m\gamma}$. In (1.8.2) $p = (p_1,\dots,p_\gamma)$ represents a probability distribution. Hence, p is a point of the simplex Γ of the Euclidean space E_γ defined by $p_j \geq 0$, $p_1 + \cdots + p_\gamma = 1$. Finally, in (1.8.2) the new control variable is (p,v) with values $(p,v) \in \Gamma \times V(t,x) \subset E_{\gamma+m\gamma}$. In (1.8.2) $h = (h_1,\dots,h_m)$ and all g_i are defined by

$$h_i(t,x,p,v) = \sum_{j=1}^{\gamma} p_j f_i(t,x,u^{(j)}), \ i = 1,2,\dots,n.$$

or

$$h(t,x,p,v) = \sum_{j=1}^{\gamma} p_j f(t,x,u^{(j)}) \ . \tag{1.8.4}$$

The new control variable (p,v) is thus an $(m\gamma+\gamma)$-vector. Note that h in (1.8.4) can be thought of as a convex combination of $f(t,x,u^{(j)})$, $j = 1,\dots,\gamma$, with coefficients p_j. On the other hand, $p_1,\dots,p_\gamma$ can be thought of also as probabilities.

A <u>generalized solution</u> $x(t)$, $p(t)$, $v(t)$, $t_1 \leq t \leq t_2$, or an <u>admissible system</u>, is now a system of functions x, p, v satisfying

(a) $x(t)$ is AC in $[t_1,t_2]$;

(b) $p(t)$, $v(t)$ are measurable in $[t_1,t_2]$;

(c) $(t,x(t)) \in A$ for every $t \in [t_1,t_2]$;

(d) $p(t) \in \Gamma$, $u^{(j)}(t) \in U(t,x(t))$ for almost all $t \in [t_1,t_2]$;

(e) $dx/dt = h(t,x(t),p(t),v(t))$ for almost all $t \in [t_1,t_2]$;

(f) $(t_1,x(t_1),t_2,x(t_2)) \in B$.

Then $x(t)$ is said to be a <u>generalized trajectory</u> generated by the <u>generalized strategy</u> $p(t)$, $v(t)$, that is, by the γ usual strategies $u^{(j)}(t)$ with probability distribution $p(t)$, $t_1 \leq t \leq t_2$.

It is essential to show that every usual solution can be interpreted as a generalized solution. Indeed, if we take $p_1 = 1$, $p_2 = \cdots = p_\gamma = 0$, $u = u^{(1)}$, then obviously all relations (1.8.2), (1.8.3) reduce to relations (1.8.1). If we denote by Ω the class of all usual admissible pairs $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, and by Ω^* the class of all admissible generalized systems $x(t)$, $p(t)$, $v(t)$, $t_1 \leq t \leq t_2$, then we have just proved that $\Omega \subset \Omega^*$. If j denotes the infimum of $I[x,p,v] = g(\eta(x))$ in Ω), then $j \leq i$.

As we shall see in the future we have $j = i$ under very weak assumptions (usually satisfied in most applications). Under the same hypotheses we shall prove also that generalized trajectories can be uniformly approximated by usual trajectories, and that the family of generalized solutions represents the closure in a suitable topology of the family of usual solutions. Under suitable hypotheses, the set of generalized solutions will be proved to be compact the same topology [(2.7) and App. F].

We shall denote by $R(t,x)$ the set of all $z = (z^1,\dots,z^n) \in E_n$ with $z = h(t,x,p,v)$ for $p \in \Gamma$, $v \in V(t,x)$. In view of (1.8.4) $R(t,x)$ is made up of convex combinations of γ points of $Q(t,x)$. We shall choose for γ the minimum integer such that $R(t,x)$ is the "convex hull" of $Q(t,x)$, or

$$R(t,x) = \text{co } Q(t,x) \text{ for all } (t,x) \in A \ .$$

Such number γ is always $\leq n + 1$. Any higher values of γ will give rise to the same set $R(t,x)$ and, as we shall see below, will not produce essentially any new trajectory.

An Example. We give here an example of a problem which has no optimal usual solution but one well determined optimal generalized solution. Let $m = 1$, $n = 2$, $A = E_2$, U made up of the two points $u = +1$ and $u = -1$. Let us consider the differential equations $dx/dt = y^2$, $dy/dt = u$, boundary conditions $x(0) = 0$, $y(0) = y(1) = 0$, and functional $I[x,y,u] = g = x(1)$. Since $x(0) = 0$, $dx/dt = y^2 \geq 0$, we have $x(1) \geq 0$. If i denotes the infimum of $I[x,y,u]$ in the class Ω of all admissible pairs $x(t)$, $y(t)$, $u(t)$, $0 \leq t \leq 1$, then $i \geq 0$. On the other hand, let $x_k(t)$, $y_k(t)$, $u_k(t)$, $0 \leq t \leq 1$, $k = 1,2,\dots$, be the sequence of admissible pairs defined by $u_k(t) = +1$ for $j/k \leq t \leq j/k + 1/2k$, $u_k(t) = -1$ for $j/k + 1/2k \leq t \leq (j + 1)/k$, $j = 0,1,\dots,k - 1$, hence $y_k(t) = t - j/k$ and $y_k(t) = (j + 1)/k - t$ for t in the corresponding intervals, then $0 \leq y_k(t) \leq 1/2k$, $0 \leq y_k^2(t) \leq 1/4k^2$ for all $0 \leq t \leq 1$, and $x_k(1) \leq 1/4k^2$. Thus $0 \leq I[x_k, y_k, u_k] \leq 1/4k^2$ and $I[x_k, y_k, u_k] \to 0$ as $k \to +\infty$. This proves that $i = 0$. For no admissible pair $x(t)$, $y(t)$, $u(t)$, $0 \leq t \leq 1$, we can have $I[x,y,u] = 0$, since this would imply $x(1) = 0$, $x(t) = 0$ for all $0 \leq t \leq 1$,

$y(t) = 0$ for all $0 \leq t \leq 1$, and finally $u(t) = 0$ a.e. in $[t_1, t_2]$, a contradiction, since $u = \pm 1$. We have proved that the problem above has no optimal usual solution. The corresponding generalized problem with $\gamma = 2$ has now differential equations $dx/dt = y^2$, $dy/dt = p_1 u^{(1)} + p_2 u^{(2)}$, boundary conditions $x(0) = 0$, $y(0) = y(1) = 0$, functional $J = g = x(1)$, and here $u^{(1)}, u^{(2)} = \pm 1$, $p_1 \geq 0$, $p_2 \geq 0$, $p_1 + p_2 = 1$. Since $w = p_1 u^{(1)} + p_2 u^{(2)}$ takes on all possible values between -1 and +1, we can replace the second equation by $dy/dt = w$ with $w = W = [-1 \leq w \leq 1]$. The obvious optimal solution is here $x(t) = 0$, $y(t) = 0$, $w(t) = 0$ for all $0 \leq t \leq 1$, or $u^{(1)} = -1$, $u^{(2)} = +1$, $p_1 = 1/2$, $p_2 = 1/2$ for all $0 \leq t \leq 1$. Here the infimum of J is still zero, and $J_{min} = j = i = 0$.

1.9 MORE ON GENERALIZED SOLUTIONS

We have given above the definitions concerning generalized solutions for the canonic, or Mayer, problem. For solutions written in the Lagrange form

$$I[x,u] = \int_{t_1}^{t_2} f_o(t,x,u)dt ,$$

$$dx/dt = f(t,x,u),\ (t,x) \in A,\ u \in U(t,x),\ \eta(x) \in B ,$$

$$x = (x^1,\dots,x^n),\ f = (f_1,\dots,f_n),\ u = (u^1,\dots,u^m) ,$$

the corresponding generalized problem is

$$I[x,p,v] = \int_{t_1}^{t_2} h_o(t,x,p,v)dt , \tag{1.9.1}$$

$$dx/dt = h(t,x,p,v),\ (t,x) \in A,\ (p,v) \in \Gamma \times V(t,x),\ \eta(x) \in B , \tag{1.9.2}$$

where

$$x = (x^1,\dots,x^n),\ h = (h_1,\dots,h_n)$$

$$v = (u^{(1)}, \dots, u^{(\gamma)}),\ u^{(j)} = (u^{j1}, \dots, u^{jm}) \in U(t,x),\ j = 1, \dots, \gamma ,$$

$$p = (p_1, \dots, p_\gamma) \in \Gamma \equiv [p \mid p_j \geq 0,\ j = 1, \dots, \gamma,\ p_1 + \cdots + p_\gamma = 1]$$

$$h_o(t,x,u) = \sum_{j=1}^{\gamma} p_j f_o(t,x,u^{(j)}) ,$$

$$h(t,x,u) = \sum_{j=1}^{\gamma} p_j f(t,x,u^{(j)}) .$$

By reducing this problem to the Mayer form by the device mentioned in (1.2) we obtain a generalized Mayer problem as in (1.5) above.

Analogously, for free problems

$$I[x] = \int_{t_1}^{t_2} f_o(t,x,u)dt,\ x' = u \in U = E_n ,$$

$$m = n,\ x = (x^1, \dots, x^n),\ \eta(x) \in B ,$$

we obtain the corresponding generalized free problem by taking

$$I[x,p,v] = \int_{t_1}^{t_2} h_o(t,x,p,v)dt , \tag{1.9.3}$$

$$dx/dt = p_1 u^{(1)} + \cdots + p_\gamma u^{(\gamma)} , \tag{1.9.4}$$

where

$$x = (x^1, \dots, x^n),\ (p,v) \in \Gamma \times E_{n\gamma},\ \eta(x) \in B ,$$

$$p = (p_1, \dots, p_\gamma) \in \Gamma = [p \mid p_j \geq 0,\ j = 1, \dots, \gamma,\ p_1 + \cdots + p_\gamma = 1] ,$$

$$v = (u^{(1)}, \dots, u^{(\gamma)}),\ u^{(j)} \in U = E_n,\ j = 1, \dots, \gamma,$$

$$h_o(t,x,p,v) = \sum_{j=1}^{\gamma} p_j f_o(t,x,u^{(j)}) .$$

Note that, while for usual solutions of free problems the trajectory determines the strategy since $dx/dt = u$ (that is, they form a TDS system (1.5)), this is not the case for generalized solutions of free problems, where the strategy $p(t) = (p_1,\dots,p_\gamma)$, $v(t) = (u^{(j)}(t), j = 1,\dots,\gamma)$, $t_1 \le t \le t_2$, is certainly not determined by the relation $dx/dt = p_1u^{(1)} + \cdots + p_\gamma u^{(\gamma)}$, with $p \in \Gamma$, $u^{(j)} \in E_n$. For instance, again for the free problem of the minimum of $I = \int x'^2\, dt$, the trajectory $x = 0$ can be generated by any generalized strategy $p(t) = (p_1,p_2)$, $v(t) = (u^{(1)},u^{(2)})$ with $0 = p_1 u^{(1)} + p_2 u^{(2)}$, say $p_1 = p_2 = 1/2$, $u^{(1)} = 1$, $u^{(2)} = -1$, as well as say by $p_1 = 1/3$, $p_2 = 2/3$, $u^{(1)} = 2$, $u^{(2)} = -1$.

An Example. We consider here the free problem of minimum of

$$I = \int_0^1 (x^2 + |x'^2 - 1|)dt$$

with $n = 1$, $x(0) = 0$, $x(1) = 0$. Here $x^2 + |x'^2 - 1| \ge 0$, hence $I \ge 0$, and, if i denotes the infimum of I in the class Ω of all AC functions $x(t)$, $0 \le t \le 1$, with $x(0) = 0$, $x(1) = 0$, and such that I is finite, then certainly $i \ge 0$. Let us consider now the sequence $[x_k]$ of trajectories defined by $x_k(t) = t - j/k$ for $j/k \le t \le j/k + 1/2k$, $x_k(t) = (j + 1)/k - t$ for $j/k + 1/2k \le t \le (j + 1)/k$, $j = 0,1,\dots,k - 1$, then $0 \le x_k(t) \le 1/2k$, $x'(t) = \pm 1$ a.e. in $[0,1]$, and $0 < I[x_k] \le 1/4k^2$. Thus $I[x_k] \to 0$ as $k \to \infty$, and hence $i = 0$. Obviously there is no AC function $x(t)$, $0 \le t \le 1$, with $x(0) = x(1) = 0$ and $I[x] = 0$, since this would imply $x(t) = 0$ for $0 \le t \le 1$, $x'(t) = \pm 1$ a.e. in $[0,1]$ a contradiction. This proves that the problem above has no absolute minimum in Ω.

The corresponding generalized problem concerns the minimum of

$$J = \int_0^1 (x^2 + p_1|(u^{(1)})^2 - 1| + p_2|(u^{(2)})^2 - 1|)dt$$

with $x(0) = 0$, $x(1) = 0$, $p_1 \geq 0$, $p_2 \geq 0$, $p_1 + p_2 = 1$, and differential equation

$$dx/dt = p_1 u^{(1)} + p_2 u^{(2)}, \quad u^{(1)}, u^{(2)} \in E_1 .$$

Obviously $w = p_1 u^{(1)} + p_2 u^{(2)}$ can take any possible value in E_1 as $u = x'$ in the original free problem. If j denotes the infimum of the new functional J then obviously $j \geq 0$ since the integrand is still nonnegative. On the other hand, $j \leq i = 0$ since the class Ω of generalized solutions contains the class Ω of usual solutions. Thus, $j = i = 0$. An optimal solution is obviously given by $u^{(1)}(t) = 1$, $u^{(2)}(t) = -1$, $p_1(t) = 1/2$, $p_2(t) = 1/2$, $0 \leq t \leq 1$, hence $dx/dt = 0$, $x(t) = 0$ in $[0,1]$, and $J_{min} = j = i = 0$.

1.D. THE GENERALIZED SOLUTIONS IN TERMS OF ORIENTOR FIELDS

1.10. CONVEX HULL OF THE SET OF DIRECTIONS

We consider now generalized solutions for the canonic problem as defined in (1.8), where we have already introduced the sets $R(t,x) = \text{co } Q(t,x)$, $(t,x) \in A$, with $R(t,x) = h(t,x,\Gamma,V(t,x))$.

For every admissible generalized system $x(t)$, $p(t)$, $v(t)$, $t_1 \leq t \leq t_2$, we have

$$dx/dt = h(t,x(t),p(t),v(t)),\ p(t) \in \Gamma,\ v(t) \in [U(t,x(t))]^{\gamma} \text{ a.e. in } [t_1,t_2],$$

$$(t,x(t)) \in A \text{ for all } t \in [t_1,t_2] ,$$

and these relations obviously imply

$$dx/dt \in \text{co } Q(t,x(t)) \text{ a.e. in } [t_1,t_2] ,$$

$$(t,x(t)) \in A \text{ for all } t \in [t_1,t_2] .$$

In other words, instead of the differential equations and constraints

$$dx/dt = h(t,x,p,v),\ (t,x) \in A,\ v \in [U(t,x)]^{\gamma},\ p \in \Gamma , \qquad (1.10.1)$$

we simply write the orientor field

$$dx/dt \in \text{co } Q(t,x),\ (t,x) \in A . \qquad (1.10.2)$$

We have shown that for any admissible generalized system $x(t)$, $p(t)$, $v(t)$, $t_1 \leq t \leq t_2$, the generalized trajectory $x(t)$, $t_1 \leq t \leq t_2$, can be interpreted as an (AC) solution of the orientor field (1.10.2) (satisfying boundary conditions (f)). Conversely, every AC solution of the orientor field (1.10.2) (satisfying (f)) can be interpreted as a solution of the differential system (1.10.1), precisely, there exist measurable vector functions p(t), v(t), $t_1 \leq t \leq t_2$, with $p(t) \in \Gamma$, $v(t) \in U(t,x(t)) = [U(t,x(t))]^{\gamma}$, $dx/dt = h(t,x(t), p(t),v(t))$ a.e. in $[t_1,t_2]$, or x(t), p(t), v(t), $t_1 \leq t \leq t_2$, is an admissible generalized system. This can be proved under very mild hypotheses. Precisely, the following statement holds:

(1.10.i) Implicit function theorem for the orientor fields of generalized solutions (1.10.2). If A is a closed subset of the tx-space E_{n+1}, if U(t,x) is a subset of E_m for every $(t,x) \in A$, and the set N of all $(t,x,p,v) \in E_{n+1+m+m\gamma}$ with $(t,x) \in A$, $p \in \Gamma$, $v \in V(t,x) = (U(t,x))^{\gamma}$, is closed, if x(t), $t_1 \leq t \leq t_2$, is an AC vector function such that $(t,x(t)) \in A$ for all $t \in [t_1,t_2]$ and $dx/dt \in \text{co}\, Q(t,x(t))$ for almost all $t \in [t_1,t_2]$, then there exist measurable functions p(t), v(t), $t_1 \leq t \leq t_2$, such that $p(t) \in \Gamma$, $v(t) \in V(t,x(t)) = (U(t,x(t)))^{\gamma}$, and $dx/dt = h(t,x(t),p(t),v(t))$ for almost all $t \in [t_1,t_2]$.

This statement is simply a corollary of (1.6.1) and is obtained by applying (1.6.1) to the generalized problem (1.10.1).

We have shown that, under mild hypotheses (A closed, N closed) the generalized solutions can be written in the equivalent form of the orientor field (1.10.2), and vice versa.

1.11. QUASI SOLUTIONS

Relations (1.6.2) and (1.10.2) lead to the following further extension of the concept of trajectory. Let us denote by S(t,x) the closed convex hull of Q(t,x), or

$$S(t,x) = \text{cl } R(t,x) = \text{cl co } Q(t,x) ,$$

so that obviously $S(t,x) \supset R(t,x) \supset Q(t,x)$. Whenever $R(t,x)$ is known to be closed, then $S = R$. In particular, if $Q(t,x)$ is compact, then $R(t,x) = \text{co } Q(t,x)$ is known to be compact, hence closed, and $S = R$. In general, $S(t,x)$ may well be larger than $R(t,x)$.

We shall say that $x(t)$, $t_1 \leq t \leq t_2$, is a quasi solution provided (a) $x(t)$ is AC in $[t_1,t_2]$, (b) $(t,x(t)) \in A$ for all $t \in [t_1,t_2]$, and (c)

$$dx/dt \in \text{cl co } Q(t,x(t)) \text{ for almost all } t \in [t_1,t_2] .$$

It is clear that a generalized solution is a quasi solution. Whenever $R(t,x)$ is closed, hence $S(t,x) = R(t,x)$, then any quasi solution is a generalized solution.

In general it may well occur that a quasi trajectory is not a generalized trajectory. For instance, if $S(t,x(t))$ is actually larger than $R(t,x(t))$ for all t of a set $H \subset [t_1,t_2]$ of positive measure and $dx/dt \in S(t,x(t)) - R(t,x(t))$ for all $t \in H$, then clearly $x(t)$ cannot be a generalized trajectory. A simple example of this occurrence is as follows. Let $n = 1$, $m = 1$, and consider differential equations, boundary conditions, and constraints:

$$dx/dt = (u + 1)^{-1} ,$$

$$x(0) = 0, \; u \in U = [0 \leq u < +\infty] ,$$

and let us seek the minimum of the functional $I[x,u] = x(1)$. Here $Q(t,x)$ is the half closed half open segment $Q(t,x) = [z|0 < z \leq 1]$, hence Q is convex but not closed. Hence $R(t,x) = Q(t,x)$ is also convex and not closed. Finally $S(t,x) = [z|0 \leq z \leq 1]$ is the corresponding closed segment. The problem has no usual optimal solution since I is positive, can be made as small as we want, but not zero. The problem clearly has no optimal generalized solution either, but there is an optimal quasi solution, namely $x(t) = 0$, $0 \leq t \leq 1$, for which $I = 0$.