T H E   U N I V E R S I T Y   O F   M I C H I G A N

COLLEGE OF LITERATURE, SCIENCE, AND THE ARTS
Department of Mathematics

Technical Report No. 4

# PARAMETRIC PROBLEMS OF OPTIMAL CONTROL

Lamberto Cesari

ORA Project 02416

submitted for:

UNITED STATES AIR FORCE
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
GRANT NO. AFOSR-69-1662
ARLINGTON, VIRGINIA

administered through:

OFFICE OF RESEARCH ADMINISTRATION     ANN ARBOR

February 1969

TABLE OF CONTENTS

TABLE OF CONTENTS (Concluded)

PARAMETRIC PROBLEMS

In the previous chapters we have treated the parameter $t$ as representing "time", the derivatives $dx/dt$ as velocities, and the trajectories $C$: $x = x(t) = (x^1,\ldots,x^n)$, $t_1 \leq t \leq t_2$, as representing laws of motions along some "path curve" $\mathcal{C}$ in the x-space. Though the interpretation of $t$ and $x$, and $dx/dt$, may vary considerably, we have always understood that the actual description $x = x(t)$, $t_1 \leq t \leq t_2$, in terms of the parameter $t$ had some relevance. Nevertheless, there are problems where only the "path curve $\mathcal{C}$" in the x-space $E_n$ is relevant, and not the "law of motion" of the point $x$ along $\mathcal{C}$. In such problems the parameter $t$ is not relevant, the parameter $t$ can be replaced by any other parameter according to standard rules, and all data (constraints, boundary conditions, differential equations, functional) are <u>invariant</u> with respect to such changes of parameter. Such problems will be denoted as <u>parametric problems</u>.

# 18. PARAMETRIC CURVES AND PARAMETRIC INTEGRANDS

## 18.1. SOME REMARKS ON HOMOGENEOUS FUNCTIONS

A real function $f(x,y,\dots,z)$ of m independent variables $x,y,\dots,x$ is said to be homogeneous of degree $\alpha$ in $x,y,\dots,z$, if

$$f(kx,ky,\dots,kz) = k^{\alpha} f(x,y,\dots,z) \text{ for all } k. \qquad (18.1.1)$$

The same function is said to be positively homogeneous if relation (18.1.1) holds for all $k > 0$. For instance $(x^2+y^2+z^2)^{1/2}$ is positively homogeneous of degree 1, $2x+y-z$ is homogeneous of degree 1, $x^2-y^2+xz$ is homogeneous of degree 2, $(x^2+z^2)^{-1/2}$ is positively homogeneous of degree $\alpha = -1$.

If $\alpha > 0$ and f is continuous in its arguments and positively homogeneous of degree $\alpha > 0$, then (18.1.1) holds also for $k = 0$, as it follows by taking $k \to 0$ in (18.1.1). If $\alpha \leq 0$, then a function f positively homogeneous of degree $\alpha \leq 0$ may be not defined, or discontinuous at the point $(0,0,\dots,0)$, and (18.1.1) becomes meaningless for $k = 0$.

(18.1.i) If f is (positively) homogeneous of degree $\alpha$ and of class $C^h$ for all $(x,y,\dots,z) \neq (0,0,\dots,0)$, then its partial derivatives of order h are (positively) homogeneous of degree $\alpha$-h.

It is enough to prove this statement for $h = 1$. From (18.1.1) by differentiation, for instance with respect to x, we have

$$kf_x(kx,\dots,kz) = k^{\alpha} f_x(x,\dots,z)$$

and hence

$$f_x(kx,\ldots,kz) = k^{\alpha-1}f_x(x,\ldots,z) \text{ for all } k \text{ (or all } k > 0). \qquad (18.1.2)$$

(18.1.ii) If f is (positively) homogeneous of degree $\alpha$ and of class $C^1$, then the identity holds

$$xf_x+yf_y+\ldots+zf_z = \alpha f. \qquad (18.1.3)$$

Indeed, from (18.1.1) by differentiation with respect to k, we have

$$xf_x(kx,\ldots,kz)+yf_y(kx,\ldots,kz)+\ldots+zf_z(kx,\ldots,kz) = \alpha k^{\alpha-1}f(x,\ldots,z).$$

and by taking $k = 1$, we obtain

$$xf_x(x,\ldots,z)+yf_y(x,\ldots,z)+\ldots+zf_z(x,\ldots,z) = \alpha f(x,\ldots,z).$$

## 18.2. PARAMETRIC CURVES

We shall now use the equations

$$\mathcal{C}: \quad x = x(t) = (x^1,\ldots,x^n), \quad a \le t \le b, \qquad (18.2.1)$$

to denote oriented parametric curves in $E_n$, or oriented path curves. Here $x(t) = (x^1,\ldots,x^n)$ denotes a continuous vector function in [a,b]. We shall think of t in (1) as an arbitrary parameter for the representation of a path curve $\mathcal{C}$ in $E_n$ or an arbitrary interval [a,b], in other words, we shall allow for an arbitrary "change of parameter". By a change of parameter we shall understand any nondecreasing continuous function

$$t = t(\tau), \quad c \le \tau \le d, \quad t(c) = a, \quad t(d) = b. \qquad (18.2.2)$$

Then the equations

$$\mathcal{C}_1: \quad x = X(\tau) = x(t(\tau)), \qquad c \leq \tau \leq d, \tag{18.2.3}$$

are said to be obtained from (18.2.1) by a change of parameter, we may write $\mathcal{C} \sim \mathcal{C}_1$, and state that the equations (18.2.1) and (18.2.3) are "different representations" of the same oriented path curve, say $\mathcal{C}$. Having assumed $t(\tau)$ nondecreasing and continuous, we see that when $\tau$ describes $[c,d]$ from $c$ to $d$, then $t$ describes $[a,b]$ from $a$ to $b$, and the corresponding point $x$ in $E_n$ sweeps $\mathcal{C}$ in the same sense. We say that the orientation in $\mathcal{C}$ is determined by the increasing direction of the parameter $t$, or $\tau$. We shall say that equations (18.2.1) and equations

$$\mathcal{C}_1: \quad x = X(\tau), \qquad c \leq \tau \leq d, \tag{18.2.4}$$

obtained one from the other by means of one or finitely many successive changes of parameter as above are "equivalent representations" of the same oriented parametric curve, or oriented path curve in $E_n$.

Actually all these "representations" form an equivalent class and we shall see in (App. E.5) how such a concept of equivalence can be formulated. Then an "oriented parametric curve" is simply the equivalent class $\{\mathcal{C}\}$ of all equations like (18.2.1) or (18.2.3) which can be obtained by finitely many arbitrary changes of parameter.

We shall consider only functionals, or integrals, which have the property to take the same value for equivalent representations. Then we shall call them "path curve functionals", or "path curves integrals", since they depend only on the path curve and not on any of its particular representations.

A few examples will illustrate these concepts. For instance

$\mathcal{C}_1$: $x = t$, $y = 0$, $0 \leq t \leq 1$,

$\mathcal{C}_2$: $x = (t+t^2)/2$, $y = 0$, $0 \leq t \leq 1$,

$\mathcal{C}_3$: $x = t^2$, $y = 0$, $0 \leq t \leq 1$,

$\mathcal{C}_4$: $x = \sin t$, $y = 0$, $0 \leq t \leq \pi/2$,

are all representations of the same oriented path curve $\mathcal{C}_1$ in the xy-plane $E_2$, namely, the segment s from (0,0) to 1,0), travelled from (0,0) to (1,0). They can actually be obtained from the first one by a change of parameter as above. Even the equations

$\mathcal{C}_5$: $x = \phi(t)$, $y = 0$, $0 \leq t \leq 1$,

with $\phi(t) = 2t$ for $0 \leq t \leq 1/4$, $\phi(t) = 1/2$ for $1/4 \leq t \leq 3/4$, $\phi(t) = -1+2t$ for $3/4 \leq t \leq 1$ is a representation of the same path curve, or segment s as above.

The curve

$\mathcal{C}_6$: $x = \sin^2(3\pi t/2)$, $y = 0$, $0 \leq t \leq 1$

is a different path curve, namely is the same segment s above travelled three times back and forth. Also, the curve

$\mathcal{C}_7$: $x = 1-t$, $y = 0$, $0 \leq t \leq 1$,

is another path curve, namely the same segment above travelled from (1,0) to (0,0), and $\mathcal{C}_7$ is often said to have been obtained from $\mathcal{C}$ by changing the orientation.

One can convince oneself easily that $\mathcal{C}_6$ and $\mathcal{C}_7$ are different curves than $\mathcal{C}_1$ by observing that some "path curves integrals" may actually take different values on them. For instance, for $I[\mathcal{C}] = \int(x'^2+y'^2)dt$, the length integral, we have $I[\mathcal{C}_1] = 1$, $I[\mathcal{C}_6] = 3$. For $I[\mathcal{C}] = \int[(x'^2+y'^2)^{1/2}+x']dt$ we have $I[\mathcal{C}_1] = 2$, $I[\mathcal{C}_5] = 6$, $I[\mathcal{C}_7] = 0$. As we shall see below, both integrals above are "path curves integrals" that is, they satisfy the conditions which guarantee that they take the same value on equivalent representations. We shall now characterize certain classes of path curves and of their representations which are of interest here.

## 18.3. REGULAR PARAMETRIC CURVES

We shall now consider those path curves possessing representation

$$\mathcal{C}: \quad x = x(t) = (x^1,\dots,x^n), \qquad c \leq t \leq b, \tag{18.3.1}$$

with (a) $x(t)$ is continuous in $[a,b]$ with sectionally continuous derivative $x'(t) = (x^{1'},\dots,x^{n'})$; (b) the vector $x'(t)$ is never zero, that is, $|x'| > 0$, or $x^{1'^2}+\dots+x^{n'^2} > 0$ in $[a,b]$. Note that at the points of discontinuity (jump) of $x'$ we require in (b) that neither the derivative at the left $x'^- = (x^{1'^-},\dots,x^{n'^-})$ nor the derivative at the right $x'^+ = (x^{1'^+},\dots,x^{n'^+})$ are zero. As we know from calculus, these hypotheses assure that $\mathcal{C}$ has a tangent at every point of continuity for $x'$, and a tangent at the left (backward) and one at the right (forward) at the points of discontinuity for $x'$, and the latter points may be corner points of $\mathcal{C}$, or cusps. For $x' = x'(t) = (x^{1'},\dots,x^{n'})$, the direction cosines of the tangent to $\mathcal{C}$ at the point $x = x(t)$ are the n numbers $x^{i'}/|x'|$, $i = 1,\dots,n$. Equations (18.3.1) satisfying (a) and (b) above should be called <u>regular representations</u> of regular path curves $\mathcal{C}$, but the current expression is "regular path curves $\mathcal{C}$", or "regular parametric curves $\mathcal{C}$".

For instance, $\mathcal{C}_1$ and $\mathcal{C}_2$ above are "regular" representations, while $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ are not (why?). $\mathcal{C}_5$ is no regular representation either. We shall consider integrals

$$I[\mathcal{C}] = \int f_o(x,x')dt = \int_a^b f_o(x(t),x'(t))dt, \tag{18.3.2}$$

where $f_o$ is a continuous function of its 2n arguments and is positively homogeneous of degree one in $x'$, or

$$f_o(x,kx') = kf_o(x,x') \text{ for all } k \geq 0, \tag{18.3.3}$$

where $\mathcal{C}$: $x = x(t) = (x^1,\dots,x^n)$, $a \leq t \leq b$, is any path curve in $E_n$ with $x(t)$ continuous with sectionally continuous derivative. Then obviously $f_o(x(t), x'(t))$ is sectionally continuous in $[a,b]$ and (19.3.2) is a Riemann integral.

We shall say that $t = t(\tau)$, $c \leq \tau \leq d$, $t(c) = a$, $t(c) = b$, is a <u>regular</u> change of parameter if $t(\tau)$ is continuous and nondecreasing in $[a,b]$ with sectionally continuous derivative $t'(\tau)$. Then obviously $t'(\tau) \geq 0$ whenever $t'$ exists, and $t'^-(\tau) \geq 0$, $t'^+(\tau) \geq 0$ at the points of discontinuity of $t'$.

(18.3.i). If $f_o(x,x')$ is continuous in $(x,x')$ and positively homogeneous of degree one in $x'$, if $\mathcal{C}_1$: $x = x(t)$, $a \leq t \leq b$, is any path curve in $E_n$ with $x(t)$ continuous in $[a,b]$ with sectionally continuous derivative, if $t = t(\tau)$, $c \leq \tau \leq d$, $t(c) = a$, $t(d) = b$, is any regular change of parameter, and $\mathcal{C}_2$: $x = X(\tau) = x(t(\tau))$, $c \leq \tau \leq d$, then $I[\mathcal{C}_1] = I[\mathcal{C}_2]$.

Indeed, $X(\tau)$ obviously is continuous in $[c,d]$ with sectionally continuous derivative $X'(\tau) = x'(t(\tau))t'(\tau)$, and by force of usual change of variable in Riemann integrals and of relation (18.3.3) we have

$$I[\mathcal{C}_1] = \int_a f_o(x(t),x'(t))dt = \int_c^d f_o(x(t(\tau)),x'(t(\tau)))t'(\tau)d\tau =$$

$$= \int_c^d f_o(x(t(\tau)),x'(t(\tau))t'(\tau))d\tau =$$

$$= \int_c^d f_o(X(\tau),X'(\tau))d\tau = I[\mathcal{C}_2].$$

In particular the Jordan length $L = L(\mathcal{C})$ for a regular representation $\mathcal{C}$: $x = x(t)$, $a \leq t \leq b$, is given by the length integral

$$L(\mathcal{C}) = \int_a^b |x'(t)|dt,$$

and here $f_o(x,x') = |x'|$ obviously satisfies the required conditions on $f_o$, in particular $f_o$ is positively homogeneous of degree one in $x'$. Thus, $L(\mathcal{C})$ does not depend on the representation but only on the "path curve $\mathcal{C}$", as defined above.

For any regular representation the arc length parameter

$$s = s(t) = \int_a^t |x'(\tau)|d\tau, \quad a \leq t \leq b, \quad 0 \leq s \leq L = L(\mathcal{C}),$$

is a continuous strictly increasing function of $t$ in $[a,b]$, hence the inverse function $t = t(s)$, $0 \leq s \leq L$, $t(0) = a$, $t(L) = b$, exists, and

$$\mathcal{C}: \quad x = X(s) = x(t(s)), \quad 0 \leq s \leq L.$$

is then said to be a representation of $\mathcal{C}$ by means of the arc length parameter $s$. Note that for $t = t(s)$, $s = s(t)$, we have $X'(s) = x'(t)t'(s)$, $|X'(s)| = |x'(t)|t'(s)$, where $s'(t) = |x'(t)| > 0$, $t'(s) = |x'(t)|^{-1} > 0$, and hence

$|X'(s)| = 1$ for all s, $0 \leq s \leq L$, and this relation holds even at the points of discontinuity of X' (jumps of X', and corner points or cusps of $\mathcal{C}$), where $|X'(s-0)| = 1$, $|X'(s+0)| = 1$. Another property of X(s) will be stated in a more general context below (relation (18.4.2)).

## 18.4. RECTIFIABLE PARAMETRIC CURVES

We shall often need parametric curves more general than the regular curves but still possessing most of their essential properties (see for an anologous extension for nonparametric curves).

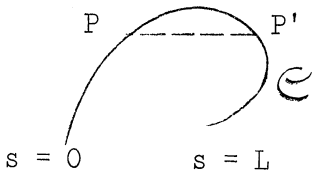
We shall indeed consider the class of those parametric curves $\mathcal{C}$: $x = x(t) = (x^1,\dots,x^n)$, $a \leq t \leq b$, in $E_n$, for which x(t) is absolutely continuous (a,c) in [a,b], that is, each $x^i$ is absolutely continuous in [a,b]. For these curves it is known (see App. E.5) that the Jordan length $L = L(\mathcal{C})$ is (finite) and still given by the integral

$$L(\mathcal{C}) = \int_a^b |x'(t)|\,dt,$$

which is now a Lebesgue integral.

As we shall see in App. E.5, every path curve of finite Jordan length $L(\mathcal{C})$ possesses representations as above, and thus equations $\mathcal{C}$: $x = x(t)$, $a \leq t \leq b$, with $x(t) = (x^1,\dots,x^n)$ continuous and absolutely continuous in [a,b] should be denoted as an AC representation of a path curve of finite Jordan length. We shall use, for the sake of brevity, the simple expression <u>rectifiable path curve</u>.

The existence of such AC representations for curves of finite Jordan length can be immediately understood if we consider the particular representation $\mathcal{C}$: $x = x(s) = (x^1,\ldots,x^n)$, $0 \le s \le L$, of $\mathcal{C}$ by means of the arc length parameter s, so that s describes the interval [0,L] with $L = L(\mathcal{C})$ and the corresponding point x(s) describes the entire curve $\mathcal{C}$. The following property of such a representation is relevant:




$$|x(s)-x(s')| \le |s-s'|, \quad 0 \le s,\ s' \le L. \qquad (18.4.1)$$

Indeed, the first member represents the length of the chord PP' from P = x(s) to P' = x(s'), while the second member represents the length of the arc PP' of $\mathcal{C}$ from P to P' (see App. E.5 for details). Relation (18.4.1) shows that $x(s) = (x^1,\ldots,x^n)$, $0 \le s \le L$, is a Lipschitzian vector function of constant one in [0,L], hence x(s) is certainly continuous and absolutely continuous in [0,L]. The following property of the representation $\mathcal{C}$: $x = x(s)$, $0 \le s \le L$, is also relevant:

$$|x'(s)| = 1, \text{ or } x^{1'^2}+\ldots+x^{n'^2} = 1 \text{ for almost all } s\in[0,L]. \qquad (18.4.2)$$

It is usually said that this property guarantees that any rectifiable curve $\mathcal{C}$ possess a tangent at almost all of its points.

Given any rectifiable path curve $\mathcal{C}$: $x = x(t)$, $a \le t \le b$, (that is, x(t) is AC in [a,b]) we shall allow any change of variable $t = t(\tau)$, $c \le \tau \le d$, $t(c) = a$, $t(d) = b$, with $t(\tau)$ continuous, nondecreasing, absolutely continuous. Then one can show (see App. E.5) for the representation $\mathcal{C}$: $x = X(\tau) = x(t(\tau))$, $c \le \tau \le d$, that $X(\tau)$ is continuous and absolutely continuous in [c,d].

Finally, we shall consider path curves integrals, or parametric integrals for arbitrary rectifiable path curves $\mathcal{C}$: $x = x(t)$, $a \leq t \leq b$, $x = (x^1,\ldots,x^2)$:

$$I[\mathcal{C}] = \int f_o(x,x')dt = \int_a^b f_o(x(t),x'(t))dt, \qquad (18.4.3)$$

where $f_o$ is continuous in its 2n arguments and is positively homogeneous of degree one with respect to x', that is, $f_o$ satisfies (18.3.3), and where $\mathcal{C}$: $x = x(t)$, $a \leq t \leq b$, is any rectifiable path curve. Here (18.4.3) is now a Lebesgue integral.

(18.4.i). For any rectifiable path curve $\mathcal{C}$ the integral (18.4.3) exists (as an L-integral) and its value is independent of the representation of $\mathcal{C}$.

Proof. First let S denote the unit sphere in $E_n$, and K the compact graph of the path curve $\mathcal{C}$, say $K = [x \epsilon E_n | x = x(t),\ a \leq t \leq b]$. Then $f_o$ is continuous on the compact set K x S, hence bounded there, $|f_o(x,x')| \leq M$ on K x S. Now if $X = X(s)$, $0 \leq s \leq L$, is the canonic representation of $\mathcal{C}$ by means of the arc-length parameter s, then $X(s) \in K$ for $0 \leq s \leq L$, and also $X'(s) = 1$ a.e. in [0,L] by force of (E.4.iii). Then $|f_o(X(s),X'(s)| \leq M$ for s in [0,L] (a.e.), and $f_o(X(s),X'(s))$, which is certainly measurable (App. F, remark in F1), is also L-integrable. Now $s = s(t)$, $a \leq t \leq b$, $0 \leq s \leq L = L(\mathcal{C})$, and we know that s(t) is AC in [a,b]. By force of (E.3.vi) we have then

$$\int_0^L f_o(X(s),X'(s)ds = \int_a^b f_o(X(s(t)),X'(s(t)))s'(t)dt =$$

$$= \int_a^b f_o(X(s(t)),X'(s(t))s'(t))dt,$$

since s (t) $\geq 0$ and $f_o$ is positively homogeneous of degree one. For every t in $[a,b]$ we have now $X(s(t)) = x(t)$, and for almost every $t$ in $[a,b]$, namely whenever $s'(t)$ exists and is finite, we have $X'(s(t))s'(t) = x'(t)$. Thus $f_o(x(t),x'(t))$ is L-integrable in $[a,b]$ and

$$\int_0^L f_o(X(s),X'(s))ds = \int_a^b f_o(x(t),x'(t))dt = I[\mathcal{C}].$$

## 18.5. LAGRANGE, MAYER, AND BOLZA PARAMETRIC PROBLEMS IN THEIR MODERN FORM

Let $A$ be any closed set of the x-space $E_n$ and, for every $x \in A$, let $U(x)$ be a closed set of the u-space $E_m$ with the property that $u \in U(x)$, $k \geq 0$ implies $k\,u \in U(x)$; in other words, $U(x)$ is made up of rays issued from the origin of the u-space $E_m$, or $U(x)$ is a closed cone in $E_m$ with vertex at the origin. Let $M$ denote the set of all $(x,u)$ with $x \in A$, $u \in U(x)$, and let $f(x,u) = (f_1,\dots,f_n)$ be any continuous vector function on $M$ such that $f(x,ku) = kf(x,u)$ for all $k \geq 0$, $x \in A$, $u \in U(x)$, or $f_i(x,ku) = kf_i(x,u)$, $i = 1,\dots,n$. We shall say that $f(x,u) = (f_1,\dots,f_n)$ is a parametric vector function. Let $B$ be a closed subset of the $x_1x_2$-space $E_{2n}$, $x_1 = (x_1^1,\dots,x_1^n)$, $x_2 = (x_2^1,\dots,x_2^n)$.

We shall say that $x(t) = (x^1,\dots,x^n)$, $u(t) = (u^1,\dots,u^m)$, $a \leq t \leq b$, is an admissible parametric pair provided (a) $x(t)$ is AC in $[a,b]$; (b) $u(t)$ is measurable in $[a,b]$; $x(t)\in A$ for every $t\in[a,b]$; (d) $u(t)\in U(x(t))$ for $t\in[a,b]$ (a.e.); $(x(a), x(b)) \in B$; (f)

$$dx/dt = f(x(t),u(t)), \quad t\in[a,b] \quad (a.e.). \tag{18.5.1}$$

In most applications it is enough to assume $u(t)$ sectionally continuous in $[a,b]$ and $x(t)$ continuous with sectionally continuous derivative, so that (18.5.1) is satisfied at every point of continuity of $u(t)$, or $x'(t) = f(x(t),u(t))$, and even at the points of discontinuity (jumps) of $u$, were $x'^+(t) = f(x(t),u^+(t+0)$, $x'^-(t) = f(x(t), u(t-0))$. Nevertheless, we shall not exclude the case where $u(t)$ is only measurable in [a,b], and x(t) is continuous and absolutely continuous in [a,b], so that $x'(t)$ exists [a,b] and (18.5.1) is satisfied a.e. in [a,b].

To define functionals, we may consider a given continuous function $g(x_1,x_2)$ defined on B, and take as functional

$$I[x,u] = g(x(t_1),x(t_2)). \tag{18.5.2}$$

Alternatively, we may consider a (real-valued) function $f_o(x,u)$ continuous in M and positively homogeneous of degree one with respect to $x'$, or $f_o(x,ku) = kf_o(x,u)$ for all $k \geq 0$, and take as functional

$$I[x,u] = \int_a^b f_o(x(t),\ u(t))|\,dt. \tag{18.5.3}$$

Finally, we may consider a functional of the form

$$I[x,u] = g(x(t_1),x(t_2))+\int_a^b f_o(x(t),u(t))dt. \tag{18.5.4}$$

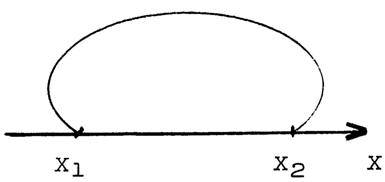
The problems of the maxima or minima of I[x,u] are called parametric problems of optimal control (parametric Mayer, Lagrange, or Bolza problem respectively).

As in (1.8), parametric isoperimetric problems are those for which we require in addition that certain integrals, analogous to the one above, or

$$\int_a^b h_j(x(t),\ u(t))dt = C_j, \quad j = 1,\dots,\mu,$$

have assigned values $C_j$, where here each $h_j$ is as usual a continuous function of $(x,x')$ which is positively homogeneous of degree one in $x'$. The transformation of isoperimetric problems in Lagrange, Mayer, Bolza problems as above, and of each of these into any other is the same as in (1.7), (1.8) and is left as an exercise for the reader. We leave also as an exercise to verify that the problems above are <u>invariant</u> with respect to the allowed changes of variables. Finally, we note that whenever $f(x,u) = u$, or $f_i(x,u) = u^i$, $i = 1,\dots,n$, $m = n$, then the differential system (18.5.1) reduces to $dx/dt = u$, or $dx^i/dt = u^i$, $i = 1,\dots,n$. If, in addition, we have $U = E_n$, then the problems above reduce to free problems of the calculus of variations.

<u>An example</u>. The isoperimetric problem of joining two points $1 = (x_1,0)$, $2 = (x_2,0)$, $x_1 < x_2$, of the x-axis by means of a path curve $C: x = x(t)$, $y = y(t)$, $0 \leq t \leq 1$, lying in the half plane $y \geq 0$ (of the xy-plane $E_2$), of given length L, and such that the area of the region between C and the segment 12 of the x-axis is maximum (Cfr. (1.8)) has now the following parametric formulation: determine the maximum of the parametric integral




$$I = 2^{-1}\int_0^1 (xy'-yx')dt \qquad (18.5.5)$$

with the constraints

$$I = \int_0^1 (x'^2+y'^2)^{1/2}\, dt = L, \qquad (18.5.6)$$

$$x(0) = x_1, \quad y(0) = 0, \quad x(1) = x_2, \quad y(0) = 0, \quad x_1 < x_2, \quad y(t) \geq 0.$$

Here we expect the optimal solution to be the unique arc of circumference of length L and chord 12. For $L < |x_2-x_1|$ the problem has no solution, of course, and for $L > (\pi/2)|x_2-x_1|$ the optimal curve is clearly parametric, and thus cannot be obtained by the nonparametric approach of (1.8). With the introduction of an auxiliary space variable z, and control variables u, v, and control space $U = E_2$, this problem is reduced to the parametric Lagrange problem of the minimum of the functional

$$I[x,y,z,u,v] = 2^{-1}\int_0^1 (xv-yu)dt \qquad (18.5.7)$$

with differential equations and constraints

$$dx/dt = u, \quad dy/dt = v, \quad dz/dt = (u^2+v^2)^{1/2}, \qquad (18.5.8)$$

$x(0) = x_1$, $y(0) = 0$, $x(1) = x_2$, $y(1) = 0$, $z(0) = 0$, $z(1) = L$, $y(t) \geq 0$,

and no constraint on $u$, $v$, that is, $(u,v) \in U = E_2$.

A variant of this problem is as follows: determine the closed curve $\mathcal{C}$: $x = x(t)$, $y = y(t)$, $0 \leq t \leq 1$, in $E_2$ of given length $L$ enclosing the maximum area. It can be formulated as the search for the parametric curves $\mathcal{C}$ for which the integral (18.5.5) takes its maximum value and the integral (18.5.6) the value $L$ with the constraints $x(0) = x(1)$, $y(0) = y(1)$. With the same changes as above we have again the problem of the maximum of (18.5.7) with differential equations (18.5.8) and constraints $x(0) = x(1)$, $y(0) = y(1)$, $z(0) = 0$, $z(1) = L$, and again $(u,v) \in U = E_2$.

## 19. PONTRYAGIN'S TYPE NECESSARY CONDITION FOR PARAMETRIC OPTIMAL SOLUTIONS

### 19.1. GENERAL CONSIDERATIONS CONCERNING NECESSARY CONDITIONS FOR PARAMETRIC PROBLEMS

From the considerations above and examples, it is clear, that parametric problems can be thought of as particular cases of the nonparametric ones, and thus all necessary conditions for optimal solutions for nonparametric problems in a sense, hold as well for parametric ones (free problems, Lagrange, Mayer, Bolza, isoperimetric problems, etc.). Nevertheless, it is well evident that for parametric problems the necessary conditions cannot determine, in general, the optimal solution $\mathcal{C}$: $x = x(t)$, $a \leq t \leq b$, as functions of an arbitrary parameter t, since any other representation, say $\mathcal{C}$: $x = X(\tau)$, $c \leq \tau \leq d$, of the same path curve $\mathcal{C}$, must also satisfy the same necessary conditions. On the other hand the special properties of integrals, differential equations, boundary conditions, and constraints, make it often possible to write the necessary conditions in suitable forms from which the optimal "path curve $\mathcal{C}$" can be determined.

One method to avoid this undetermination is to choose the arc length s as a parameter. This can be done by always requiring that $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, satisfy the relations $t_1 = 0$, $|dx/dt| = 1$ for t in $[t_1,t_2]$ (a.e.). Then $t = s$ is the arc length, $t_2 = L$ is the length of the path curve, and thus a unique representation for $\mathcal{C}$ has been characterized. Note that with this convention the arc length integral becomes

$$\int_{t_1}^{t_2} |dx/dt|\, dt = \int_0^L ds = L,$$

and then the problems of minimum length of the path (in the parametric form) are the anologues of the problems of minimum time (in the nonparametric forms). Nevertheless, particularly for free problems, suitable forms of the necessary conditions have been devised which do not need the strict parametrization above.

## 19.2. PONTRYAGIN'S NECESSARY CONDITION FOR PARAMETRIC MAYER PROBLEMS

Let A be any closed set of the x-space $E_n$, and for every $x \in A$ at $U(x)$ be a closed cone in $E_m$ with vertex at the origin. Let M be the set of all $(x,u)$ with $x \in A$, $u \in U(x)$ ($M = A \times U$ if U does not depend on x). Let $f(x,u) = (f_1,\ldots,f_n)$ be a continuous vector function in M such that $f(x,ku) = kf(x,u)$ for all $x \in A$, $u \in U(x)$, $k \geq 0$, and assume that each $f_i$ has first order partial derivatives $f_{ix^j}$ $(i,j = 1,\ldots,n)$ all continuous in M. Let B be a closed subset of the $x_1x_2$-space $E_{2n}$, and let $g(x_1,x_2)$ be a given continuous function on B.

Let $\lambda = (\lambda_1,\ldots,\lambda_n)$ denote an n-vector and

$$H(x,u,\lambda) = \lambda_1 f_1(x,u)+\ldots+\lambda_n f_n(x,u)$$

the Hamiltonian. For every $x \in A$, $\lambda \in E_n$, let

$$M(x,\lambda) = \text{Inf } H(x,u,\lambda)$$

where Inf is taken for $u \in U$ (or $u \in U(x)$).

Let $\Omega$ be the class of all parametric admissible pairs $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$ (see (18.5)), and let $I[x,u]$ be the functional $I = g(x(t_1),$

$x(t_2))$, or $I = g(\eta(x))$, where $\eta(x)$ denotes the end points $(x(t_1),x(t_2))$ of the path curve $\mathcal{C}$: $x = x(t)$, $t_1 \leq t \leq t_2$.

Let $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, be a given optimal pair in $\Omega$, for which, that is, $I[x,u] \leq I[\tilde{x},\tilde{u}]$ for all pairs $\tilde{x}$, $\tilde{u}$ in $\Omega$. Let us assume that each point $x(t)$ of the optimal path $\mathcal{C}$: $x = x(t)$, $t_1 \leq t \leq t_2$, be interior to A. Let us assume that U does not depend on x, that is, U is a fixed closed subset of $E_m$. (An alternative hypothesis is that each point $u(t)$ is interior to U).

Also, let us assume that B possesses at the point $\eta(x) = (x(t_1), x(t_2))$ a tangent space B', and that $g(x_1,x_2)$ is differentiable at the same point $\eta(x)$ of B. Then the differential dg of g at $\eta(x)$ is

$$dg = \sum_i (g_{x_1^i}\, dx_1^i + g_{x_2^i}\, dx_2^i),$$

where $h = (dx_1,dx_2) = (dx_1^i,dx_2^i, i = 1,\ldots,n)$ is any element h of B'.

Under these hypotheses:

$(\Pi_1)$ there is a continuous (AC) vector function $\lambda(t) = (\lambda_1,\ldots,\lambda_n)$ (Pontryagin's multipliers) which is never zero in $[t_1,t_2]$ such that

$$d\lambda_i/dt = -H_{x^i}(x(t),u(t),\lambda(t)), \quad i = 1,\ldots,n, \quad t \in [t_1,t_2] \text{ (a.e.)},$$

$(\Pi_2)$ For every $t \in [t_1,t_2]$ the Hamiltonian $H(x(t), u, \lambda(t))$ as a function of u only in U takes its minimum value at $u = u(t)$, or

$$M(x(t),\lambda(t)) = H(x(t),u(t),\lambda(t)), \quad t \in [t_1,t_2] \text{ (a.e.)};$$

$(\Pi_3)$ the function $M(t) = M(x(t),\lambda(t))$ is the zero constant in $[t_1,t_2]$.

$(\Pi_4)$ (transversality relation). There is a constant $\lambda_o \geq 0$ such that

$$\lambda_o dg+[\Sigma_{j=1}^n \lambda_j(t)dx^j]_1^2 = 0$$

for every vector $h \in B'$.

Here we have $dx^i/dt = \partial H/\partial\lambda_i$, $d\lambda_i/dt = -\partial H/\partial x^i$, $i = 1,\ldots,n$, or

$$dx^i/dt = f_i(x(t),u(t)), \quad i = 1,\ldots,n,$$

$$d\lambda_i/dt = -\Sigma_j f_{jx^i}\lambda_j, \quad i = 1,\ldots,n,$$

for $t \in [t_1,t_2]$, (a.e.).

We can always require $|dx/dt| = 1$ a.e. in $[t_1,t_2]$, or $f_1^2+\ldots+f_n^2 = 1$, $t_1 = 0$, and then $t$ coincides with the arc length parameter $s$, and $t_2 = L$ is the Jordan length of $\mathcal{C}$. Note that $H$ has the property $H(x,ku,\lambda) = kH(x,u,\lambda)$ for every $k \geq 0$. Hence, either $H$ has negative values and then $M = -\infty$, or $H \geq 0$, and then $u = 0$ is certainly a minimum for $H$. Since $u = 0$ yields $dx/dt = f = 0$ we see that the requirement $|dx/dt| = 1$ (a.e.) guarantees that $u(t) \neq 0$ a.e. in $[t_1,t_2]$, and that the minimum zero of $H(x(t),u,\lambda(t))$ is taken along the entire line $ku(t)$, $k \geq 0$.

Note that the properties above are necessary not only for an absolute minimum as stated, but even for a strong relative minimum. In other words we need only to know that $I[x,u] \leq I[\tilde{x},\tilde{u}]$ for all parametric pairs $\tilde{x},\tilde{u}$ whose parametric curve $\tilde{\mathcal{C}}$: $x = \tilde{x}(t)$, $t_1 \leq t \leq t_2$, satisfy the same boundary conditions as $\tilde{\mathcal{C}}$, and lie in any given small neighborhood of $\mathcal{C}$.

Example. Problem of the line of minimum length between two given points

$1 = x_1 = (x_1^1,\dots,x_1^n)$, $2 = x_2 = (x_2^1,\dots,x_2^n)$ as a parametric Mayer problem. Here we have an (n+1)-vector $x = (x^1,\dots,x^n,x^{n+1})$ and an n-vector $u = (u^1,\dots,u^n)$, with differential equations, boundary conditions and functional

$$dx^i/dt = u^i, \quad i = 1,\dots,n, \quad dx^{n+1}/dt = |u| = ((u^1)^2+\dots+(u^n)^2)^{1/2},$$

$$x^i(t_1) = x_1^i, \quad x^i(t_2) = x_2^i, \quad i = 1,\dots,n, \quad x^{n+1}(t_1) = 0,$$

$$I[x,u] = g = x^{n+1}(t_2) = \text{minimum}.$$

Here $U = E_n$, and

$$H(u,\lambda) = \lambda_1 u^1+\dots+\lambda_n u^n+\lambda_{n+1}|u|.$$

Thus, either $H(u,\lambda)$ has negative values, and then $M(\lambda) = \text{Inf}_u H(u,\lambda) = -\infty$, or $H(u,\lambda) \geq 0$, and then $u = 0$ is always a minimum for $H(u,\lambda)$ in U with $H(0,\lambda) = 0$. If $H(u,\lambda) \geq 0$ and H has its minimum also at some point $u \neq 0$, then this minimum is zero, and $H(u,\lambda)$ takes the value zero along the entire line $ku$, $k \geq 0$. In addition, we must have, by differentiation of H, $\lambda_i+\lambda_{n+1}u^i|u|^{-1} = 0$, and $u^1: u^2:\dots:u^n:|u| = \lambda_1:\lambda_2:\dots:\lambda_n:-\lambda_{n+1}$. On the other hand, the equations for the multipliers are $d\lambda_i/dt = 0$, $i = 1,\dots,n, n+1$, hence $\lambda_i$ = constant in $[t_1,t_2]$, hence $u_i$ = constant in $[t_1,t_2]$, and all $x^i$ are linear functions of t. This shows that the optimal path from 1 to 2 in $E_n$ (if any) is the segment $\mathcal{C} = 12$. The conditions of existence theorem 1 of section 20 below are satisfied, hence the minimum exists, and is given by the segment $\mathcal{C} = 12$.

## 19.3. PONTRYAGIN'S NECESSARY CONDITION FOR PARAMETRIC LAGRANGE PROBLEMS

Let A be any closed set of the x-space $E_n$, and for every $x \in A$ let $U(x)$

be a closed cone in $E_n$ with vertex at the origin. Let M be the set of all $(x,u)$ with $x \in A$, $u \in U(x)$ ($M = AxU$ if U does not depend on x). Let $f(x,u) = (f_o, f) = (f_1,\dots,f_n)$ be any continuous vector function on M such that $f(x,ku) = kf(x,u)$ for all $x \in A$, $u \in U(x)$, $k \geq 0$, and assume that each $f_i$ has first order partial derivatives $f_{ix^j}$ ($i = 0,1,\dots,n$, $j = 1,\dots,n$), all continuous in M. Let B be a closed subset of the $x_1x_2$-space $E_{2n}$. Let $\Omega$ be the class of all parametric admissible pairs $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$ (defined as in (18.5) with n differential equations $dx^i/dt = f_i(t,x,u)$, $i = 1,\dots,n$, and the additional requirement that $f_o\ (x(t),u(t))$ be integrable in $[t_1,t_2]$. Let $I[x,u]$ be the functional $I = \int_{t_1}^{t_2} f_o(x(t),u(t))dt$, defined for all pairs x, u in $\Omega$.

Let $\lambda = (\lambda_o,\lambda_1,\dots,\lambda_n)$ denote an (n+1)-vector and

$$H(x,u,\lambda) = \lambda_o f_o(x,u)+\dots+\lambda_n f_n(x,u)$$

the Hamiltonian. For every $x \in A$, $\lambda \in E_{n+1}$, let

$$M(x,\lambda) = \operatorname{Inf} H(x,u,\lambda),$$

where Inf is taken for every $u \in U$ (or $U = U(x)$).

Let $x\ (t)$, $u\ (t)$, $t_1 \leq t \leq t_2$, be an optimal pair in $\Omega$, for which that is $I[x\ ,u\ ] \leq I[\tilde{x},\tilde{u}]$ for every pair $\tilde{x}$, $\tilde{u}$ in $\Omega$. Let us assume that B possesses at the point $\eta(x\ ) = (x\ (t_1),x\ (t_2))$ a tangent space B' of vectors $h = (dx_1,dx_2) = (dx_1^i,dx_2^i,\ i = 1,\dots,n)$. Let us assume that each point $x\ (t)$ of the optimal path : $x = x\ (t)$, $t_1 \leq t \leq t_2$, is interior to A. Let us assume that U does not depend on x, that is, U is a fixed closed subset of $E_m$ (an alternative hypothesis is that each point $u(t)$ is interior to $U(x(t))$.

Under these hypotheses:

$(M_1')$ There is a continuous (AC) vector function $\lambda(t) = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ (Pontryagin's multipliers), which is never zero in $[t_1, t_2]$, with $\lambda_0$ a constant in $[t_1, t_2]$, $\lambda_0 \geq 0$, such that

$$d\lambda_i/dt = -H_{x^i}(x\ (t), u\ (t), \lambda(t)), \quad i = 1, \ldots, n, \quad t \in [t_1, t_2]\ (\text{a.e.});$$

$(M_2')$ For every $t \in [t_1, t_2]$ the Hamiltonian $H(x\ (t), u, \lambda(t))$ as a function of $u$ in $U$ takes its minimum value at $u = u\ (t)$, or

$$M(x\ (t), \lambda(t)) = H(x\ (t), u\ (t), \lambda(t)), \quad t \in [t_1, t_2]\ (\text{a.e.});$$

$(M_3')$ The function $M(t) = M(x\ (t), \lambda(t))$ is the constant zero in $[t_1, t_2]$.

$(M_4')$ (transversality relation). We have

$$[\Sigma_{j=1}^{n} \lambda_j(t) dx^j]_1^2 = 0$$

for every vector $h \in B'$.

Here we have $dx^i/dt = \partial H/\partial \lambda_i$, $d\lambda_i/dt = -\partial H/\partial x^i$, $i = 1, \ldots, n$, or

$$dx^i/dt = f_i(x\ (t), u\ (t))$$

$$d\lambda_i/dt = -\Sigma_{j=0}^{n} \lambda_i f_{jx^i}(x\ (t), u\ (t))$$

for $t \in [t_1, t_2]$ (a.e.).

We can always require $|dx/(dt)| = 1$ a.e. in $[t_1, t_2]$, or $f_1^2 + \ldots + f_n^2 = 1$, and $t_1 = 0$, and then $t$ coincides with the arc length parameter $s$, and $t_2 = L$ is the Jordan length of $\mathcal{C}$.

# 20. EXISTENCE THEOREMS FOR OPTIMAL PARAMETRIC SOLUTIONS

## 20.1. EXISTENCE THEOREMS FOR AN OPTIMAL SOLUTION

As in Section 4 we are interested here in existence theorems for optimal path curves in parametric control problems. We give here only two such theorems, for Mayer and Lagrange parametric problems respectively. Free problems and parametric problems can be reduced to these problems. We refer to Part II for more existence theorems for the optimum of parametric problems.

(20.1.i). <u>Existence Theorem</u> (for parametric Mayer problems). Let A be a compact subset of the x-spece $E_n$, and for any $x \in A$ let $U(x)$ be a closed subset of the u-space $E_m$ such that $u \in U(x)$ implies $ku \in U(x)$ for all $k \geq 0$. Let M be the set of all $(x,u)$ with $x \in A$, $u \in U(x)$, and let $f(x,u) = (f_1,\dots,f_n)$ be any vector valued function continuous on M such that $f(x,ku) = kf(x,u)$ for all $k \geq 0$ and $(x,u) \in M$. Assume that $|f(x,u)| \geq c|u|$ for some constant $c > 0$ and all $(x,u) \in M$. Assume that the compact set $U_1(x) = U(x) \cap S$ be upper semicontinuous on A, where S is the unit sphere in $E_m$. Assume that the set $Q(x) = f(x,U(x))$ be a convex subset of $E_n$ for every $x \in A$. Let B be a closed subset of the $x_1x_2$-space $E_{2n}$ and $g(x_1,x_2)$ a continuous scalar function on B. Let $\Omega$ be the class of all parametric admissible pairs $x(t),u(t),t_1 \leq t \leq t_2$, whose path curves (trajectories $\mathcal{C}$: $x = x(t)$, $t_1 \leq t \leq t_2$) lie in A, have equibounded lengths $L(\mathcal{C})$, and end points $(x(t_1),x(t_2)) \in B$. Then, the cost functional $I[x,u] = g(x(t_1),x(t_2))$ has an absolute minimum in $\Omega$.

If A is not compact but closed, then existence theorem (20.1.i) still holds provided the following additional condition is satisfied: (b) every trajectory $\mathcal{C}$ of $\Omega$ possesses at least a point $x^*$ on a compact subset P of $E_n$.

Condition (b) is certainly satisfied if the first end point $x(t_1)$ of the trajectories $\mathcal{C}$ of $\Omega$ are assumed to belong to a compact subset $B_1$ of $E_n$, in particular if the first end point is fixed. The same is true if the second end point $x(t_2)$ satisfies an anologous condition.

The condition that the path curves $\mathcal{C}$ of $\Omega$ have equibounded lengths $L(\mathcal{C})$ can be removed provided we assume the following hypothesis which actually concerns A and $\Omega$:($\alpha$). If A is compact, then there exists a number $M > 0$ such that for every two points $x_1, x_2 \in A$ with $(x_1,x_2) \in B$ there is at least one admissible parametric pair $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, in $\Omega$ whose trajectory $\mathcal{C}$: $x = x(t)$, $t_1 \leq t \leq t_2$, satisfies $x(t_1) = x_1$, $x(t_2) = x_2$, $L(\mathcal{C}) \leq M$. If A is not compact but closed and property (b) holds, then A is the countable union of compact subsets $A_s$ of A, each $A_s$ having the property above for some number $M_s$ which may depend on s.

(20.1.ii). <u>Existence Theorem 2</u> (for parametric Lagrange problems). Let A be a compact subset of the x-space $E_n$, and for any $x \in A$ let $U(x)$ be a closed subset of the u-space $E_m$ such that $u \in U(x)$ implies $ku \in U(x)$ for all $k \geq 0$. Let M be the set of all $(x,u)$ with $x \in A$, $u \in U(x)$, and let $\tilde{f} = (f_o,f) = (f_o,f_1,\ldots,f_n)$ be any vector valued continuous function on M such that $\tilde{f}(x,ku) = k\tilde{f}(x,u)$ for all $k \geq 0$ and $(x,u) \in M$. Assume that $f_o(x,u) \geq c|u|$ for some constant $c > 0$ and all $(x,u) \in M$. Assume that the compact set $U_1(x) = U(x) \cap S$ be upper semicontinuous on A, where S is the unit sphere in $E_m$. Assume that the set $Q(x)$ of all $z \in E_{n+1}$, $z = (z^o,z)$, with $z^o \geq f_o(x,u)$, $z = f(x,u)$, $u \in U(x)$, be convex for every $x \in A$. Let B be a closed subset of the $x_1x_2$-space $E_{2n}$. Let $\Omega$ be the class of all parametric admissible pairs $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, whose path curves (trajectories) $\mathcal{C}$: $x = x(t)$, $t_1 \leq t \leq t_2$, lie in A, and end points $(x(t_1), x(t_2)) \in B$. Then the cost functional $I[x,u] = \int_{t_1}^{t_2} f_o(x, u)dt$ has an absolute minimum in $\Omega$.

If A is not compact but closed, then existence theorem (20.1.ii) still holds under the additional condition (b) listed after theorem (20.1.i).

Remark. In both theorems (20.1.i) and (20.1.ii) the class $\Omega$ of all parametric admissible pairs $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, satisfying the requested properties, can be replaced by any closed subclass $\Omega_o \subset \Omega$. The class $\Omega_o$ is said to be closed if, roughly speaking, limits of elements of $\Omega_o$ belong to $\Omega_o$. Precisely, we require that $(x_k, u_k) \in \Omega_o$, $k = 1,2,\ldots$, $x_k \to x$, $(x_k, u_k) \in \Omega$ implies $(x, u) \in \Omega_o$. By convergence $x_k \to x$ it is meant that, by a suitable parametrization $x_k \to x$ uniformly (or at least in the metric $\mathcal{C}$ (App. F.).

## 21. DEDUCTION OF CLASSICAL NECESSARY CONDITIONS FOR FREE PARAMETRIC PROBLEMS

### 21.1. EULER-TYPE NECESSARY CONDITIONS FOR FREE PARAMETRIC PROBLEMS IN $E_n$

We consider here the integral

$$I[\mathcal{C}] = \int_{t_1}^{t_2} f_o(x(t),x'(t))dt\,, \qquad (21.1.1)$$

where $f_o = f_o(x,x') = f_o(x^1,\dots,x^n,x^{1'},\dots,x^{n'})$ is a continuous function of $(x,x')$ which is positively homogeneous of degree one in $x'$. As usual we shall determine necessary conditions for a rectifiable parametric curve $\mathcal{C}$: $x = x(t) = (x^1,\dots,n^n)$, $t_1 \leq t \leq t_2$ ( precisely, for a representation of such a curve) to give the minimum (or maximum) of $I[\mathcal{C}]$ in the class $\Omega$ of all such curves satisfying usual boundary conditions and constraints

$$(x(a),x(b)) \in B, \quad x(t) \in A \text{ for } t_1 \leq t \leq t_2, \qquad (21.1.2)$$

where B is a given subset of the $x_1x_2$-space $E_{2n}$ and A is a given subset of the x-space $E_n$. As we have seen in section 18.4 for Lagrange problem, the integral (21.1.1) is defined for any such curve $\mathcal{C}$ and is independent of the representation of $\mathcal{C}$.

To obtain results analogous to the ones in section 7, we shall assume also that $f_o$ has continuous first order partial derivatives $f_{ox^i}$, $f_{ox^{i'}}$, $i = 1,\dots,n$, for all $(x,x') \in A \times E_n$ with $x' \neq 0$. (This exception is well justified if we note that already for the usual length integral we have $f_o = |x'| = (x^{1'^2}+\dots+x^{n'^2})^{1/2}$ and the partial derivatives $f_{ox^{i'}}$ do not exist at $x' = 0$).

Nevertheless, the necessary conditions for nonparametric problems of section 7 still hold unchanged for the minimum of a parametric integral (21.1.1) at a regular parametric curve $\mathcal{C}$: $x = x(t) = (x^1,\dots,x^n)$, $t_1 \leq t \leq t_2$. Thus, Euler equations (E) of (7.1) become

$$f_{ox^{i\prime}}(x(t),x'(t)) - \int_{t_1}^{t} f_{ox^i}(x(\tau),x'(\tau))d\tau = c_i, \quad i = 1,\dots,n, \qquad (21.1.3)$$

for some constants $c_i$, and in this form they hold for all $t_1 \leq t \leq t_2$. Alternatively, we can write them in the form

$$(d/dt)f_{ox^{i\prime}}(x(t),x'(t)) = f_{ox^i}(x(t),x'(t)) = 0, \quad i = 1,\dots,n \qquad (21.1.4)$$

at all points which are not points of discontinuity for $x'$, and even at these points with the conventions mentioned in (7.1).

Since $f_o$ does not depend on $t$, thus the problem is autonomous, equation (7.1.1) can now be written in the form (7.1.6)

$$f_o - \sum_{i=1}^{n} x^{i\prime} f_{ox^{i\prime}} = c, \quad t_1 \leq t \leq t_2, \qquad (21.1.5)$$

for some constant $c$. But now we see that this relation is trivial and identically satisfied as a consequence of Euler relations for homogeneous functions (18.1.3). Weierstrass and Legendre necessary conditions (7.1.iv), (7.1.v) also hold with no change.

The remark shall be made that here $x = x(t)$, $t_1 \leq t \leq t_2$, is assumed to be a regular curve, hence $|x'(t)| > 0$ for all $t$, where $x'$ is sectionally continuous, and we have $|x'^+| > 0$, $|x'^-| > 0$ even at the points of (jump) discontinuity for $x'$. Thus, there is some constant $m > 0$ such that $|x'(t)| \geq m > 0$

for all t in $[t_1,t_2]$. The critical point $x' = 0$ is thus avoided. Of course in the proofs, we have also to take as comparison curves only those curves $\mathcal{C}$: $x = x(t)$, $t_1 \leq t \leq t_2$, with say $|x'(t)| \geq m/2$, and thus $f_o$ is continuous in $(x,x')$ with all its first order partial derivatives in the entire set $A \times U_o$ where now $U_o = [u \in E_n, |u| \geq m/2]$.

Finally, as mentioned in (19.1), let us remind that the necessary conditions for parametric problems cannot determine any of the $\infty$-many representations of the optimal path curve, provided we do not explicitely require by other means one or another of these representations. For instance, if we require $t_1 = 0$, $|dx/dt| = 1$ in $[t_1,t_2]$, then $t = s$ is the arc length parameter and $t_2 = L = L(\mathcal{C})$ the Jordan length of $\mathcal{C}$.

## 21.2. FURTHER NECESSARY CONDITIONS FOR FREE PARAMETRIC PROBLEMS IN $E_2$

There is a number of interesting forms for the necessary conditions which are specific for free parametric problems. We shall limit outselves to the case $n = 2$. We shall write $x,y$ instead of $x^1$, $x^2$ below. We shall write $\mathcal{C}$ in the form $\mathcal{C}$: $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, and $I[\mathcal{C}]$ takes the form

$$I[\mathcal{C}] = \int f_o(x,y,x',y')dt = \int_{t_1}^{t_2} f_o(x(t),y(t),x'(t),y'(t))dt, \quad (21.2.1)$$

where

$$f_o(x,y,kx',ky') = kf_o(x,y,x',y') \text{ for all } k \geq 0.$$

Also, we shall limit ourselves to the case where $\mathcal{C}$ joins two fixed given points $1 = (x_1,y_1)$, $2 = (x_2,y_2)$ in the xy-plane $E_2$. We shall denote by K the class of all regular parametric curves $\mathcal{C}$ joining $1 = (x_1,y_1)$ and $2 = (x_2,y_2)$. First, let us repeat in the present notations the necessary condition mentioned above for any n.

(Euler necessary condition). If C: $x = x(t), y = y(t), t_1 \leq t \leq t_2$ (21.2.i) gives a minimum or a maximum of I in K (absolute, or strong relative, or weak relative), then the Euler equations are satisfied

$$f_{ox'} - \int_{t_1}^{t} f_{ox} dt = C_1, \quad f_{y'} - \int_{t_1}^{t} f_{oy} dy = C_2, \tag{21.2.2}$$

$C_1$, $C_2$ constants, for all $t_1 \leq t \leq t_2$. Thus

$$f_{ox} - \frac{d}{dt} f_{ox'} = 0, \quad f_{oy} - \frac{d}{dt} f_{oy'} = 0 \tag{21.2.3}$$

at all points which are not of discontinuity for $x'(t)$, $y'(t)$, and, if $f_o$ is of class $C^2$ and $x(t)$, $y(t)$ are of class $C^2$, also

$$\begin{aligned} &f_{ox'x'}x'+f_{ox'y}y'+f_{ox'x'}x''+f_{ox'y'}y''-f_{ox} = 0, \\ &f_{oy'x}x'+f_{oy'y}y'+f_{oy'x}x''+f_{oy'y'}y''-f_{oy} = 0. \end{aligned} \tag{21.2.4}$$

As mentioned above, equation (7.1.6) now reduces to the trivial Euler relation for homogeneous function of degree one:

$$x'g_{ox'}+y'f_{oy'} = f_o. \tag{21.2.5}$$

as a consequence of the (positive) homogeneity of f in x', y', of degree one. If f is of class $C^2$, then by differentiating (21.2.5) with respect to x' we have $f_{ox'}+x'f_{ox'x'}+y'f_{ox'y'} = f_{ox'}$ and hence

$$x'f_{ox'x'}+y'f_{ox'y'} = 0, \tag{21.2.6}$$

and analogously, by differentiation with respect to y', we have

$$x'f_{ox'y'}+y'f_{oy'y'} = 0. \tag{21.2.7}$$

These equations have a very simple algebraic consequence:

$$f^* = \frac{f_{ox'y'}}{y'^2} = -\frac{f_{ox'y'}}{x'y'} = \frac{f_{oy'y'}}{x'^2} \tag{21.2.8}$$

It may be pointed out that for all $(x'y') \neq (0,0)$ at least one of these three fractions has a meaning and thus the function $f^*(x,y,x',y')$ is defined by (21.2.8) for all $(x',y') \neq (0,0)$, and is continuous in $x,y,x',y'$. Since the second derivatives of f are positively homogeneous of degree -1 in $x',y'$, we conclude that $f^*$ is positively homogeneous of degree -3 in $x',y'$. For instance:

$$f_o = (x'^2+y'^2)^{1/2}, \qquad f^* = (x'^2+y'^2)^{-3/2},$$

$$f_o = A(x,y)x'+B(x,y)y', \qquad f^* = 0.$$

(21.2.ii) (Weierstrass necessary condition). If $f_o$ is of class $C^2$ and $x(t)$, $y(t)$ are also of class $C^2$, then a necessary condition for an extremum of I in K is that

$$f_{oy'x}-f_{ox'y}+f^*(x'y''-x''y') = 0, \tag{21.2.9}$$

for all $t_1 \leq t \leq t_2$. Thus, the curvature $1/r$ of C is given by

$$1/r = \frac{x'y''-x''y'}{(x'^2+y'^2)^{3/2}} = \frac{f_{ox'y}-f_{oxy'}}{(x'^2+y'^2)^{3/2}f^*} \tag{21.2.10}$$

If the parameter s (arc length) is used, then $x'^2 + y'^2 = 1$ and this relation becomes

$$1/r = \frac{f_{ox'y} - f_{oxy'}}{f^*} \tag{21.2.11}$$

Proof. First of all, $f_o$ is positively homogeneous of degree 1 in $x',y'$ (for all $x,y$), that is

$$f_o(x,y,kx',ky') = kf_o(x,y,x',y') \text{ for all } k \geq 0$$

and all $x,y,x',y'$. By differentiation with respect to x and y we conclude that both $f_{ox}$ and $f_{oy}$ are positively homogeneous of degree 1 in $x',y'$. As a consequence of the identity (21.2.5) we have, by differentiation,

$$x'f_{oxx'} + y'f_{oxy'} = f_{ox}, \quad x'f_{oyx'} + y'f_{oyy'} = f_{oy}. \tag{21.2.12}$$

By introducing these expressions of $f_{ox}$ and $f_{oy}$ into the Euler equations (21.2.4) we have

$$f_{ox'x}x' + f_{ox'y}y' + f_{ox'x'}x'' + f_{ox'y'}y'' - x'f_{oxx'} - y'f_{oxy'} = 0,$$

$$f_{oy'x}x' + f_{oyy'}y' + f_{oy'x'}x'' + f_{oy'y'}y'' - x'f_{oyx'} - y'f_{oyy'} = 0,$$

and finally

$$y'f_{ox'y} - y'f_{oxy'} + f_{ox'x'}x'' + f_{ox'y'}y'' = 0,$$

$$x'f_{oy'x} - x'f_{oyx'} + f_{oy'x'}x'' + f_{oy'y'}y'' = 0.$$

By using the expression of the function f* we have also

$$y'[f_{ox'y}-f_{oxy'}+f^*y'x''-f^*x'y''] = 0$$

$$x'[f_{oy'x}-f_{oyx'}-f^*y'x''+f^*x'y''] = 0,$$

where the two brackets are equal and opposite in sign, and $x',y'$ are never both zero. Thus

$$f_{oy'x}-f_{oyx'}+f^*(x'y''-x''y') = 0$$

for all $t_1 \leq t \leq t_2$. Theorem II is thereby proved.

Remark. We have actually proved that, if C satisfies equations (21.2.2) then it satisfies also the equation (21.2.9). The converse is also true. We have only to reverse the argument. Thus, the system of two equations (21.2.2) and the only equation (21.2.9) are equivalent. If $f_o$ is of class $C^2$, then any curve C: $x = x(t)$, $y = y(t)$, with $x(t), y(t) \in C^2$, $x',y'$ never zero, satisfying equations (21.2.2) or (21.2.9) is said to be an extremal for (21.2.1).

Concerning Weierstrass and Legendre necessary conditions, let us observe that here we have

$$x'f_{ox'}(x,y,x',y')+y'f_{oy'}(x,y,x',y') = f(x,y,x',y')$$

hence

$$\begin{aligned} E(x,y,x',y',X',Y') &= \\ &= f_o(x,y,X',Y')-f_o(x,y,x',y')-(X'-x')f_{ox'}(x,y,x',y')-(Y'-y')f_{oy'}(x,y,x',y') \\ &= f_o(x,y,X',Y')-X'f_{ox'}(x,y,x',y')-Y'f_{oy'}(x,y,x',y') \\ &= X'[f_{ox'}(x,y,X',Y')-f_{ox'}(x,y,x',y')]+Y'[f_{oy'}(x,y,X',Y')-f_{oy'}(x,y,x',y')]. \end{aligned}$$

If we take

$$p = \frac{x'}{(x'^2+y'^2)^{1/2}} = \cos \theta, \qquad q = \frac{y'}{(x'^2+y'^2)^{1/2}} = \sin \theta ,$$

$$P = \frac{X'}{(X'^2+Y'^2)^{1/2}} = \cos \bar{\theta}, \qquad Q = \frac{Y'}{(X'^2+Y'^2)^{1/2}} = \sin \bar{\theta} , \tag{21.2.13}$$

then, since $f_{ox'}, f_{oy'}$ are positively homogeneous of degree 0, we have also

$$E(x,y,x',y',X',Y') = (X'^2+Y'^2)^{1/2} E(x,y,p,q,P,Q) ,$$

$$E(x,y,p,q,P,Q) = P[f_{ox'}(x,y,P,Q)-f_{ox'}(x,y,p,q)]+Q[f_{oy'}(x,y,P,Q)-f_{ox}(x,y,p,q)]. \tag{21.2.14}$$

Also note that with the notations above the quadratic form Q defined in (7.1.18) becomes

$$Q = f_{ox'x'}\zeta^2+2f_{ox'y'}\zeta\eta+f_{oy'y'}\eta^2 =$$

$$= f^*(y'\zeta^2-2x'y'\zeta\eta+x'^2\eta^2) = f^*(y'\zeta-x'\eta)^2. \tag{21.2.15}$$

(21.2.iii). If a given curve $C$ of K gives a minimum [maximum] of I in K, then for every element $(x,y,x',y')$ of $C$ and any pair of real numbers $X',Y'$ not both zero we have

$$E(x,y, \cos \theta, \sin \theta, \cos \bar{\theta}, \sin \bar{\theta}) \geq 0$$

for every $\bar{\theta}$, and where $\theta$ is defined by (21.2.13).

(21.2.iv) (Legendre necessary condition). Let $f_o$ be a class $C^2$, and the curve $C$ of K gives a minimum [maximum] of I in K, then along $C$ we have

$f^* \geq 0$ [$f^* \leq 0$].

These statements are immediate consequences of (7.1.iv), (7.1.v) when one uses the expressions (21.2.14) and (21.2.15) for E and Q.

We mention that all the conditions above (Euler, Weierstrass, Legendre, etc.) are necessary not only for an absolute minimum as stated but even for a strong relative minimum. In other words it is enough we know that $I[\mathcal{C}] \leq I[\tilde{\mathcal{C}}]$ for all parametric curves $\tilde{\mathcal{C}}$ satisfying the same boundary conditions as $\mathcal{C}$ and lying in any given neighborhood of $\mathcal{C}$.

We mention here the following formula due to Schwarz, which gives a direct relation between E and $f^*$:

$$E(x,y,\cos\theta_0,\sin\theta_0,\cos\theta,\sin\theta) =$$

$$= f_0(x,y,\cos\theta,\sin\theta)-\cos\theta f_{0x'}(x,y,\cos\theta_0,\sin\theta_0)-\sin\theta f_{0y'}(x,y,\cos\theta_0,\sin\theta_0)$$

$$= \int_{\theta_0}^{\theta} f^*(x,y,\cos\alpha,\sin\alpha)\sin(\theta-\alpha)\,d\alpha. \qquad (21.2.16)$$

The proof is based on the usual Euler relation $x'f_{0x'}+y'f_{0y'} = f_0$ and the fundamental theorem of calculus:

$$E = f_0(x,y,\cos\theta,\sin\theta)-[\cos\theta f_{0x'}(x,y,\cos\theta_0,\sin\theta_0)+\sin\theta f_{0y'}(x,y,\cos\theta_0,\sin\theta_0)]$$

$$= \cos\theta[f_{0x'}(x,y,\cos\theta,\sin\theta)-f_{0x'}(x,y,\cos\theta_0,\sin\theta_0)]+$$

$$+ \sin\theta[f_{0y'}(x,y,\cos\theta,\sin\theta)-f_{0y'}(x,y,\cos\theta_0,\sin\theta_0)]$$

$$= \cos\theta\int_{\theta_0}^{\theta} (d/d\alpha)f_{0x'}(x,y,\cos\alpha,\sin\alpha)d\alpha+\sin\theta\int_{\theta_0}^{\theta} (d/d\alpha)f_{0y'}(x,y,\cos\alpha,\sin\alpha)d\alpha$$

By performing the differentiations and using (21.2.8), we have now

$$E = \cos\theta\int_{\theta_o}^{\theta}[-f^*\sin^2\alpha\sin\alpha d\alpha - f^*\sin\alpha\cos\alpha]d\alpha + \sin\theta\int_{\theta_o}^{\theta}[f^*\sin\alpha\cos\alpha\sin\alpha + f^*\cos^2\alpha\cos\alpha]d\alpha$$

$$= \int_{\theta_o}^{\theta} f^*(-\cos\theta\sin\alpha + \sin\theta\cos\alpha)d\alpha = \int_{\theta_o}^{\theta} f^*(x,y,\cos\alpha,\sin\alpha)\sin(\theta-\alpha)d\alpha.$$

From Schwarz formula we can now prove directly that $(\alpha) f^*(x,y,x',y') \geq 0$ for all $x',y'$ (not both zero) implies $E(x,y,x',y',X',Y') \geq 0$ for all $x',y',X',Y'$; $(\beta) E(x,y,x',y',X',Y') \geq \theta$ for all $x',y',X',Y'$ implies $f^*(x,y,x',y') \geq \theta$ for all $x',y'$.

To prove $(\alpha)$ we can always assume that the angles $\theta$, $\theta_o$ are such that $\theta_o-\pi \leq \theta \leq \theta_o+\pi$. If $\theta_o \leq \theta \leq \theta_o+\pi$, then in Schwarz formula we have $\theta_o \leq \alpha \leq \theta \leq \theta_o+\pi$, $\sin(\theta-\alpha) \geq 0$ and thus $f^* \geq 0$ implies $E \geq 0$. Analogous reasoning holds in the other case. Note that, if $f^* \geq 0$ and $f^* = 0$ in no interval, then $E > 0$ for all $\theta \neq \theta_o$.

To prove $(\beta)$ let us assume that $E \geq 0$. By Schwarz formula and mean value theorem we have

$$E(x,y,\cos\theta_o,\sin\theta_o,\cos\theta,\sin\theta) = f^*(x,y,\cos\tilde{\theta},\sin\tilde{\theta})[1-\cos(\theta-\theta_o)]$$

where $\tilde{\theta}$ is a convenient number between $\theta$ and $\theta_o$, $\theta_o-\pi \leq \theta \leq \theta_o+\pi$. By dividing by $1-\cos(\theta-\theta_o) > 0$ and taking the limit as $\theta \to \theta_o$, we obtain $f^*(x,y,\cos\tilde{\theta},\sin\tilde{\theta}) \to f^*(x,y,\cos\theta_o,\sin\theta_o) \geq 0$. The reasoning for $(\alpha)$ has already shown that if $E > 0$ for $\theta \neq \theta_o$, we cannot have $f^*(x,y,\cos\alpha,\sin\alpha) = 0$ in an interval.

Remark. As in (7.1; last remark ) both Weierstrass and Legendre necessary conditions above deal with convexity properties of $f_o(x,x')$ with respect to $x'$. Both conditions state that $f_o(x,x')$ is convex with respect to $x'$ at every $(x(t),x'(t))$, $t_1 \leq t \leq t_2$. We shall encounter other convexity properties for existence theorems (section 22) and for sufficiency theorems (section 23). As for nonparametric problems, an interpretation of these conditions in terms of functional analysis is due to L. Tonelli.

## 21.3. JACOBI TYPE NECESSARY CONDITION

We need now convenient forms for the second variation (10.1.4), the accessory problem, and the Jacobi equations (10.1.7), or (10.1.8). By force of (21.2.15) the expression for $\omega$ is

$$\begin{aligned} 2\omega = {} & f^*(y'\xi'-x'\eta')^2+f_{oxx}\xi^2+2f_{oxy}\xi\eta+f_{oyy}\eta^2+ \\ & +2f_{oxx'}\xi\xi'+2f_{oxy'}\xi\eta'+2f_{ox'y}\xi'\eta+2f_{oyy'}\eta\eta'. \end{aligned} \tag{21.3.1}$$

Let u denote $u = y'\xi-x'\eta$. Then

$$u' = (y'\xi'-x'\eta')+(y''\xi-x''\eta).$$

$$\begin{aligned} u'^2 = {} & (y'\xi'-x'\eta')^2+(y''\xi^2-2x''y''\xi\eta+x''\eta^2)+ \\ & + 2(y'y''\xi\xi'-x''y'\xi'\eta-x'y''\xi\eta'+x'x''\eta\eta'), \end{aligned}$$

and $2\omega$ takes the form

$$\begin{aligned} 2\omega = {} & f^*u'^2+(f_{oxx}-y''^2f^*)\xi^2+2(f_{oxy}+x''y''f^*)\xi\eta+(f_{oyy}-x''^2f^*)\eta^2 \\ & + 2(f_{oxx'}-y'y''f^*)\xi\xi'+2(f_{oxy'}+x'y''f^*)\xi\eta'+ \\ & + 2(f_{ox'y}+x''y'f^*)\xi'\eta+2(f_{oyy'}-x'x''f^*)\eta\eta'. \end{aligned}$$

If L, M, N denote the expressions

$$L = f_{oxx'}-y'y''f^*, \quad N = f_{oyy'}-x'x''f^*$$
$$M = f_{oxy'}+x'y''f^* = f_{ox'y}+x''y'f^*, \tag{21.3.2}$$

then we have

$$2\omega = f^*u'^2+2L\xi\xi'+2M(\xi\eta'+\xi'\eta)+2N\eta\eta'$$
$$+ (f_{oxx}-y''^2f^*)\xi^2+2(f_{oxy}+x''y''f^*)\xi\eta+(f_{oyy}-x''^2f^*)\eta^2.$$

On the other hand, the relation holds

$$(d/dt)(l\xi^2+2M\xi\eta+N\eta^2) = 2L\xi\xi'+2M(\xi\eta'+\xi'\eta)+2N\eta\eta'$$
$$+(L'\xi^2+2M'\xi\eta+N'\eta^2).$$

If $L_1$, $M_1$, $N_1$ denote the expressions

$$L_1 = f_{oxx}-y''^2f^*-L', \quad N_1 = f_{oyy}-x''^2f^*-N'$$
$$M_1 = f_{oxy}+x''y''f^*-M', \tag{21.3.3}$$

we have also

$$2\omega = f^*u'^2+(d/dt)(L\xi^2+2M\xi\eta+N\eta^2)+L_1\xi^2+2M_1\xi\eta+N_1\eta^2. \tag{21.3.4}$$

Now we shall prove that $L_1$, $M_1$, $N_1$ are proportional to $y'^2$, $-x'y'$, $x'^2$. First, from (21.3.2) and (21.2.12) we have

$$Lx'+My' = x'f_{oxx'}+y'f_{ox'y} = f_{ox},$$

$$Mx'+Ny' = x'f_{oyx'}+y'f_{oyy'} = f_{oy}.$$

$$Lx''+My'' = x''f_{oxx'}+y''f_{ox'y},$$

$$Mx''=Ny'' = x''f_{oyx'}+y''f_{oyy'}.$$

By differentiation of the first relation we have

$$L'x'+M'y'+Lx''+My'' = f_{oxx}x'+f_{oxy}y'+f_{oxx'}x''+f_{oxy'}y''$$

and, by comparison and using (21.2.9), also

$$L'x'+M'y' = f_{oxx}x'+f_{oxy}y'+(f_{oxy'}-f_{ox'y})y'' ,$$

$$Lx'+My' = f_{oxx}x'+f_{oxy}y'-f^*y''(x'y''-x''y') ,$$

and finally, by (21.3.3), we obtain

$$L_1x'+M_1y' = 0.$$

Analogously, we have

$$M_1x'+N_1y' = 0$$

and hence, $L_1:M_1:N_1 = y'^2:-x'y':x'^2$. Ifg* denotes the proportionality factor, we have

$$g^* = \frac{L_1}{y'^2} = -\frac{M_1}{x'y'} = \frac{N_1}{y'^2}$$

and then (21.3.4) yields

$$2\omega = f^*u'^2+g^*u^2+(d/dt)(L\xi^2+2M\xi\eta+N\eta^2)$$

Finally, the second variation (10.1.3) becomes

$$\phi''(0) = \int_{t_1}^{t_2}(f^*u'^2+g^*u^2)dt. \qquad (21.3.5)$$

From (10.1) we know that, if E gives a minimum for I, then the integral (21.3.5) has also a minimum. The Euler equation for (21.3.5) is now given by (7.1) in the form

$$g^*u - \frac{d}{dt}\left(f^* \frac{du}{dt}\right) = 0. \qquad (21.3.6)$$

This is Jacobi's accessory equation.

The concept of conjugate points for parametric curves can now be defined in terms of Jacobi's accessory equation (21.3.6). A point $3 = [x(t_3),y(t_3)]$ is said to be a focus, or a conjugate point to $1 = [x(t_1),y(t_1)]$ on the extremal arc E: $x = x(t)$, $y = y(t)$, $t_1 \leq t \leq t_2$, if there is a nonidentically zero solution $u(t)$, $t_1 \leq t \leq t_2$, of equation (21.3.6) with $u(t_1) = u(t_3) = 0$. Then Jacobi's necessary condition has the usual formulation:

(21.3.i). If f is of class $C^4$, $E_{12}$ is an arc of extremal E which gives an extremum of I[y] in K, and $\Delta = f_{ox'x'}f_{oy'y'}-f^2_{ox'y'} \neq 0$ along $E_{12}$, then there is no conjugate point to 1 between 1 and 2 on $E_{12}$.

For applications it is important to consider the case where a two-parameter family of extremals $E_{\alpha\beta}$: $x = \phi(t,\alpha,\beta)$, $y = \psi(t,\alpha,\beta)$ is known, the extremal arc $E_{12}$ is contained on the extremal $E_o = E_{\alpha_o\beta_o}$: $x = \phi(t,\alpha_o,\beta_o)$, $y = \psi(t_o,\alpha_o,\beta_o)$, and the functional determinant $\Lambda = \partial(\phi,\psi)/\partial(\alpha,\beta) \neq 0$ along

$E_{12}$. Then the functions

$$u_1(t) = \psi_t(t,\alpha_o,\beta_o)\phi_\alpha(t,\alpha_o,\beta_o)-\phi_t(t,\alpha_o,\beta_o)\psi_\alpha(t,\alpha_o,\beta_o),$$

$$u_2(t) = \psi_t(t,\alpha_o,\beta_o)\phi_\beta(t,\alpha\ ,\beta\ )-\phi_t(t,\alpha_o,\beta_o)\psi_\beta(t,\alpha_o,\beta_o),$$

are two particular solutions of Jacobi's accessory equation (21.3.6). Also, the conjugate point $3 = [x(t_3),y(t_3)]$ to 1 on $E_o$ correspond to the first zero $t > t_1$ of the function $D(t,t_1) = u_1(t)u_2(t_1)-u_2(t)u_1(t_1)$. In other words, the necessary condition (21.3.i) can be expressed by saying that $D(t,t_1) \neq 0$ for all $t_1 < t < t_2$. We omit the proof of this statement, which is also similar to the one we have given for the nonparametric case in (10.2).

As in (10.2) we may consider a one-parameter family of extremals $E_a$, $a' \leq a \leq a''$, of which one $E_{a_o}$ with $a' < a_o < a''$, contains $E = E_{12}$. If $E_a$ has an envelope G then no point of contact of $E_{a_o}$ with G can fall between 1 and 2 on $E_{12}$.

## 22. EXISTENCE THEOREMS FOR FREE PARAMETRIC PROBLEMS

### 22.1. TONELLI'S EXISTENCE THEOREMS FOR FREE PARAMETRIC PROBLEMS

Free parametric problems as formulated in (21.1) are particular cases of Lagrange parametric problems with $m = n$, $U = E_n$, and differential equations $dx^i/dt = u^i$, $i = 1,\dots,n$, or $f(x,u) = u$. Thus, existence theorem (20.1.ii) applies, and we shall reword it explicitely in the present situation, as we have done in section 11 for free nonparametric problems.

As usual, A is a closed subset of the x-space $E_n$, $f_o(x,x')$ a real-valued continuous function on $A \times E_n$, and the boundary conditions are assigned by means of a subset B of the $x_1x_2$-space $E_{2n}$. We denote by $\Omega$ the class of all continuous path curves $\mathcal{C}$: $x = x(t)$, $t_1 \leq t \leq t_2$, with $x(t) \in A$ for all $t \in [t_1,t_2]$, and $(x(t_1),x(t_2)) \in B$. As in 11.1 we shall need the property of convexity of $f_o(x,x')$ with respect to x', that is, we shall require that

$$f_o(x,\alpha u+(1-\alpha)v) \leq \alpha f_o(x,u)+(1-\alpha)f_o(x,v)$$

for all $u,v \in E_n$, $\alpha$ real, $0 \leq \alpha \leq 1$, and every $x \in A$. We shall consider the integral

$$I[\mathcal{C}] = \int f_o(x,x')dt = \int_{t_1}^{t_2} f_o(x(t),x'(t))dt$$

which is defined on every element $\mathcal{C}$ of $\Omega$ and is independent on the representation of $\mathcal{C}$.

(22.1.i) <u>Existence Theorem</u> (for free parametric problems) (Tonelli 1914). If A is compact, if $f_o(x,x')$ is continuous in $A \times E_n$, if $f_o(x,x') \in A \times E_n$

with $x' \neq 0$, if B is closed, and $\Omega$ is not empty, then I has an absolute minimum in $\Omega$.

If A is not compact but closed then existence theorem (22.1.i) still holds provided the following additional conditions are satisfied: (a) each curve $\mathcal{C}$ of the class $\Omega$ possesses at least one point $x^*$ on a compact subset P of A; (b) $f_o(x,x') \geq \mu|x'|^{-1}$ for all $(x,x') \in A \times E_n$ with $|x| \geq R$, and suitable constants $\mu > 0$, $R \geq 0$.

Condition (a) is certainly satisfied if the initial point $x_1 = x(t_1)$ is fixed, or the terminal point $x_2 = x(t_2)$ is satisfied.

Note that for A compact and $f_o(x,x') > 0$ for all $(x,x') \in A \times E_n$ with $x' \neq 0$, there is a constant $c > 0$ such that $f_o(x,x') \geq c|x'|$ for all $(x,x') \in A \times E_n$. Indeed $f_o(x,x') > 0$ on the compact set $A \times S$, S the unit sphere in $E_n$, hence $f_o$ has a positive minimum c on $A \times S$, and finally $f_o(x,x') \geq c|x'|$ on $A \times E_n$.

For instance, for $n = 2$, hence $m = n = 2$, $U = E_2$, the functions $f_o(x,y,x',y')$ below satisfy the conditions of existence theorem (22.1.i), where we write $(x,y)$ instead of $(x^1,x^2)$, and $(x',y')$ instead of $(u^1,u^2)$:

$$f_o = (x'^2+y'^2)^{1/2}, \qquad f_o = (1+x^2+y^2)(2x'^2+3y'^2)^{1/2},$$

$$f_o = (1+|x|+|y|)[(x'^2+y'^2)^{1/2}-2^{-1}x'-3^{-1}y']$$

$$f_o = (1+x^2+y^2)[(x'^2+y'^2)^{1/2}-2^{-1}(1+x^2+y^2)^{-1}(x'+y')].$$

<u>Remark</u>. In theorem (22.1.i) the class $\Omega$ of all parametric continuous curves $\mathcal{C}$ satisfying the requested properties, can be replaced by any <u>closed</u> subclass

$\Omega_o \subset \Omega$. The class $\Omega_o$ is said to be closed if, roughly speaking, limits of elements of $\Omega_o$ belong to $\Omega_o$. Precisely, we require that $C_k \in \Omega_o$, $k = 1,2,\ldots$, $C_k \to C_o$, $C_o \in \Omega$ implies $C_o \in \Omega_o$. By convergence $C_k \to C_o$ it is meant that by suitable parametrizations $C_k$: $x = x_k(t)$, $C$: $x = x_o(t)$, and $x_k \to x_o$ uniformly (or at least in the metric $\rho$ (App. F. )). For instance, the class of all such curves $C$ whose lengths $L(C)$ are all $\leq M$ for some constant M, is closed. In theorem (22.1.i) the condition $f_o(x,x') > 0$ can be omitted provided $\Omega$ is a nonempty complete class of curves $C$ whose lengths are all $\leq M$ for some constant M.

We shall denote as usual by $\Sigma$ the unit sphere $|u| = 1$ in $E_{2n}$, and by $N_\delta(x_o)$ the set of all $x \in A$ with $|x-x_o| \leq \delta$.

We shall say that a parametric integral $f_o(x,u)$, $(x,u) \in M = A \times E_m$, is <u>quasi normally convex</u> in u at the point $(x_o,u_o) \in A \times \Sigma$ provided, given $\epsilon > 0$, there are a number $\delta = \delta(x_o,u_o,\epsilon) > 0$ and a linear function $z(u) = b \cdot u$, $b = (b_1,\ldots,b_m)$, $u = (u^1,\ldots,u^m)$ (which may also depend on $x_o$, $u_o,\epsilon$) such that

(a) $f_o(x,u) \geq z(u)$ for all $x \in N_\delta(x_o)$, $u \in U(x)$,

(b) $f_o(x,u) \leq z(u)+\epsilon$ for all $x \in N_\delta(x_o)$, $u \in U(x)$, $|u-u_o| \leq \delta$.

The same $f_o$ is said to be <u>normally convex</u> in u at the point $(x_o,u_o) \in A \times \Sigma$ provided, given $\epsilon > 0$, there are numbers $\delta = \delta(x_o,u_o,\epsilon) > 0$, $\gamma = \gamma(x_o,u_o,\epsilon) > 0$, and a linear function $z(u) = b \cdot u$ as above (which may also depend on $x_o$, $u_o,\epsilon$) such that (b) holds, and (a') $f_o(x,u) \geq z(u)+\nu|u-u_o|$ for all $x \in N_\delta(x_o)$, $u \in U(x)$.

The same $f_o$ is said to be quasi normally convex in $\omega$, or normally convex in u, if it has these properties at every point $(x_o,u_o) \in A \times \Sigma$.

A parametric integrand $f_o$ can be convex in u without being quasi normally convex, nor normally convex. This situation is shown by the function $f_o(x,u) = xu$ (with $n = 1$, $m = 1$, $A = E_1$, $U = E_1$, x, u scalars), at the points $x_o = 0$, $u_o = 0$.

The following statement gives a useful characterization of the parametric integrands which are normally convex in u:

(i) If A is closed, and $f_o(x,u)$, $(x,u) \in A \times E_m$, is a parametric integrand, then $f_o$ is normally convex in u if and only if $f_o$ is convex in u at every $x_o \in A$, and for every $x_o \in A$, the graph of $z^o = f_o(x_o;u)$, $u \in E_m$, contains no whole straight line through the origin. A proof of (i) is given in App. F.

A point $x_o \in A$ is said to be a <u>zero</u> for the parametric integral $f_o(x,u)$, $(x,u) \in A \times E_m$, if $f_o(x_o,u) = 0$ for at least one $u \in E_m$, $u \neq 0$. Then $f_o(x_o,ku) = 0$ for all $k \geq 0$, that is, for all u of a ray from the origin in the u-space $E_m$.

## 22.2. MORE TONELLI EXISTENCE THEOREMS FOR FREE PARAMETRIC PROBLEMS IN $E_2$

We state here a few of the existence theorems for the absolute minimum for parametric free problems proved by Tonelli for n = 2. To simplify notations we shall assume throughout this number that A is a closed subset of the xy-plane $E_2$, that U is the fixed set $U = E_2$, (or the uv-plane $E_2$, or the x'y'-plane $E_2$), and that $f_o(x,y,x',y')$ is a continuous function in $A \times E_2$ with

$f_o(x,y,kx',ky') = kf_o(x,y,x',y')$ for all $(x,y) \in A$, $(x',y') \in E_2$, $k \geq 0$, (thus $f_o(x,y,0,0) = 0$ for all $(x,y) \in A$). The cost functional

$$I[C] = \int_a^b f_o(x(t),y(t),x'(t),y'(t))dt \qquad (22.2.1)$$

is then defined (as a Lebsegne integral, 18.4) for every AC representation $C$: $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, of a rectifiable path curve $C$ lying in $A$. A function $f_o(x,y,x',y')$ as above will be said a parametric integrand.

Existence Theorem PFT1. If $A_o$ is compact, if $f_o(x,y,x',y')$ is convex in $(x',y')$ for every $(x,y) \in A$, and normally convex in $(x',y')$, if $f_o(x,y,x',y') \geq 0$ in $A \times E_2$, and the zeros of $f_o$ are on finitely many continuous open curves of class $C^2$ with no multiple points, any two of them having at most one point in common and forming no closed curve in the xy-plane $E_2$, then the cost functional (22.2.1) has an absolute minimum in any nonempty complete class $\Omega$ of rectifiable path curves $\mathcal{C}$ lying in $A$.

If $A$ is not compact, this theorem still holds under the additional hypotheses (a), (b) listed under theorem P3.

For instance the conditions of theorem PFT1 relative to $f_o$ are satisfied by

$$f_o = (1+x^2y^2)(x'^2+y'^2)^{1/2}-x',$$

$$f_o = (1+(y-x^2)^2)(x'^2+2y'^2)^{1/2}-x',$$

$$f_o = (x'^2+y'^2)^{1/2}-(1+x^2y^2)^{-1}y' .$$

Existence Theorem PFT2. If $A_o$ is compact, if $f_o(x,y,x',y')$ is convex in

$(x',y')$ for every $(x,y) \in A$, and normally convex in $(x',y')$, if $f_o(x,y,x',y') \geq 0$ in $A \times E_2$, and assume that there is a fixed arc $\gamma$ of the circumference $x'^2+y'^2 = 1$ of length less than $\pi$, and such that $f_o(x,y,x',y') > 0$ for all $(x,y) \in A$, $x'^2+y'^2 = 1$, $(x',y') \notin \gamma$. Then the cost functional (22.2.1) has an absolute minimum in any nonempty complete class $\Omega$ of rectifiable path curves $\mathcal{C}$ lying in A.

If A is not compact but closed, then this statement is still valid provided the conditions above hold in every compact part $A_o$ of A and the additional hypotheses (a), (b) hold listed at the end of theorem P3.

In other words, we assume in PFT2 that the zeros of $f_o$ can be arbitrary points of A, even all points of A, provided $f_o = 0$ at most at directions $(x',y')$ belonging to a fixed arc $\gamma$ of length $< \pi$ of the unit circumference. For instance, the conditions above concerning $f_o$ are satisfied by

$$f_o = (x'^2+y'^2)^{1/2}-x', \quad f_o = (1+x^2+y^2)((x'^2+y'^2)^{1/2}-x').$$

<u>Existence Theorem PFT3</u>. Let A be compact, let $f_o(x,y,x',y')$ be convex in $(x',y')$ for every $(x,y) \in A$, and normally convex, and let $f_o(x,y,x',y') \geq 0$ in $A \times E_2$. Assume that A can be divided into finitely many parts by means of finitely many parallel straight lines of the xy-plane, and that to each part $\Gamma_s$ there corresponds a fixed arc $\gamma_s$ of the circumference $x'^2+y'^2 = 1$ of length $< \pi$ such that $f_o(x,y,x',y') > 0$ for $(x,y) \in \Gamma_s$, $x'^2+y'^2 = 1$, $(x',y') \notin \gamma_s$. Then the cost functional (22.2.1) has an absolute minimum in any nonempty complete class $\Omega$ of rectifiable path curves $\mathcal{C}$ lying in A.

If A is not compact but closed, then this statement is still valid provided the conditions above hold in the compact parts $A_m = A \times [(x,y), x^2+y^2 \leq m^2]$, $m = 1,2,\ldots$, and in addition conditions (a) and (b) are satisfied.

For instance, the conditions of theorem PFT3 relative to $f_o$ are satisfied by

$$f_o = (1+x^2+y^2)^{-1}[(x'^2+y'^2)^{1/2}-x'\cos x-y'\sin x].$$

For every pair $(x',y')$ with $x'^2+y'^2 = 1$ let us write $x' = \cos\theta$, $y' = \sin\theta$. Then $f_o = (1+x^2+y^2)^{-1}(1-\cos(\theta-x)) \geq 0$, and $f_o = 0$ only for $x = \theta$, that is $x' = \cos x$, $y' = \sin x$. If we draw the parallels to the y-axis, $x = k\pi/2$, $k = 0, \pm 1, \pm 2, \ldots$, then in each strip between any two consecutive of these straight lines there exists an arc $\gamma$ of the circumference $x'^2+y'^2 = 1$ of length exactly $\pi/2$ on which lie all pairs $(x',y')$ with $x'^2+y'^2 = 1$ and $f_o(x,y,x',y') = 0$ for some $(x,y)$ of the strip. The conditions of theorem PFT3 are satisfied.

Remark. The conditions of theorem PFT3 hold if $f_o$ does not depend on x, or does not depend on y, or does not depend on (x,y).

Existence Theorem PFT4. Let A be compact, and let $f_o$ be convex in $(x',y')$ for every $(x,y) \in A$, and normally convex in $(x',y')$. Assume that there are two continuous functions $P(x,y)$, $Q(x,y)$, $(x,y) \in A$, such that $Pdx + Qdy$ is the total differential of a function $\Phi(x,y)$ of class C in A, and such that $f_o(x,y,x',y')-x'P(x,y)-y'Q(x,y) > 0$ for all $(x,y) \in A$ and all $(x',y') \neq (0,0)$. Then the cost functional (22.2.1) has an absolute minimum in any nonempty complete class $\Omega$ of rectifiable path curves $\mathcal{C}$ lying in A. If A is not compact but closed, the same remarks hold as in the previous theorems.

For instance the conditions of theorem PFT4 relative to $f_o$ are satisfied by

$$f_o = (x'^2+y'^2)^{1/2}+yx'+xy'.$$

Indeed, by taking $P = y$, $Q = x$, $\Phi = xy$, then $d\Phi = Pdx+Qdy$, and

$$f_o-(yx'+xy') = (x'^2+y'^2)^{1/2} > 0 \text{ for all } (x',y') \neq (0,0).$$

Remark. Theorem PFT4 holds even if the function $F(x,y,x',y') = f_o(x,y,x',y') -x'P(x,y)-y'Q(x,y)$ satisfies in A any one of the conditions stated for $f_o$ in theorems PFT4, PFT1, 2, 3. In particular, theorem PFT4 holds if $f_o(x,y,x',y')$ is convex in $(x',y')$, normally convex in $(x',y')$, of class $C^1$ in $(x',y')$, and if there is some $(x'_o,y'_o)$ fixed with $x'^2_o+y'^2_o = 1$ such that $f_{ox'}(x,y,x'_o,y'_o)dx +f_{oy'}(x,y,x'_o,y'_o)dy$ is the exact differential in A of a function $\Phi(x,y)$ of class $C^1$ in A. Finally, the conclusion of PFT4 holds if $f_o(x',y')$ is independent of $(x,y)$, $f_o(x',y')$ is of class $C^1$, and $f_o(x',y')$ is convex and quasi normally convex.

As an example, the function

$$f_o = (1+(x+y)^2)((x'^2+y'^2)^{1/2}-2x'-y')$$

is obviously convex and normally convex in $(x',y')$, and $f_{ox'}(x,y,1,0)dx +f_{oy'}(x,y,1,0)dy = [-1-(x+y)^2](dx+dy)$ is the exact differential in $E_2$ of $\Phi = -(x+y)-3^{-1}(x+y)^2$. Thus, $f_o$ satisfies the condition of the second paragraph of the remark above.

## 23. SUFFICIENT CONDITIONS FOR PARAMETRIC FREE PROBLEMS

### 23.1. SUFFICIENT CONDITIONS FOR A RELATIVE MINIMUM

Let R be a simply connected region of the xy-plane, and assume that (a) certain functions $p(x,y)$, $q(x,y)$, $(x,y) \in R$, are given of class $C^1$ in R and never both zero; (b) the line integral

$$I^*[C] = \int_C A dx + B dy = \int f_{ox'}(x,y,p,q)dx + f_{oy'}(x,y,p,q)dy \quad .$$

depends only on the end points of any parametric regular curve in R. Then we say that R is a field for the integral $I[C] = \int f_o(x,y,x',y')dt$. Note that the integral $I^*[C]$ is still the integral (12.1.1) if we observe that the coefficient $F = f_o - pf_{ox'} - qf_{oy'}$ of dt in (12.1.1) is zero here by force of (21.2.5). We shall assume also that f is of class $C^2$ hence A and B are of class $C^1$, and, by force of (b).

$$0 = A_y - B_x = f_{ox'y} + f_{ox'x'}p_y + f_{ox'y'}q_y - f_{oy'x} - f_{oy'x'}p_x - f_{oy'x'}q_x$$

Hence, the solutions of the equations in R

$$dx/dt = p(x,y) \quad , \quad dy/dt = q(x,y) \quad . \tag{23.1}$$

are of class $C^2$, fill R and one and only one passes through each point of R, satisfy the identities $x' = p$, $y' = q$, $x'' = p_x x' + p_y y'$, $y'' = q_x x' + q_y y'$, and hence by (21.2.8), successively

$$f_{oy'x} - f_{ox'y} + f^*(-x'y'p_x + x'^2 q_x - y'^2 p_y + x'y'q_y) = 0,$$

$$f_{oy'x} - f_{ox'y} + f^*(x'y'' - x''y') = 0 ,$$

which is the Weierstrass equation. Thus, the solutions of the equations (23.1) are the extremals of the integral I. They are the extremals of the field. Note, that both p and q could be replaced by kp, kq, where $k = k(x,y) > 0$ is any function of class $C^1$ in R. The equations (23.1) change, but the solutions are still extremals of I.

Conversely, assume that a simply covered region R of the x,y-plane is simply covered by a one-parameter family of extremals, say $x = x(t,c)$, $y = y(t,c)$. In other words, a certain region $R_o$ of the auxiliary tc-plane is mapped one-one onto the region R by these equations. Assume that $x(t,c), y(t,c)$, together with the inverse functions $t(x,y), c(x,y)$ are of class $C^2$. Then the function

$$p(x,y) = x'[t(x,y),c(x,y)] \quad , \quad q(x,y) = y'[t(x,y),c(x,y)] \quad ,$$

are of class $C^1$ in R, the given extremals satisfy the equations $dx/dt = p$, $dy/dt = q$ in R, and the condition (b) is automatically satisfied.

Given a field, let us observe that on any arc E of an extremal of the field we have $I^*[E] = I[E]$, since $f_{x'}p + f_{y'}q = f$ by force of (21.2.5). Now, for any parametric regular curve $C_{12}$: $x = X(t)$, $y = Y(t)$, $t_1 \leq t \leq t_2$, lying in R and joining any two points 12 on an extremal E of the field, we have

$$I[C_{12}] - I[E_{12}] = I[C_{12}] - I^*[E_{12}] = I[C_{12}] - I^*[C_{12}]$$

$$= \int_{t_1}^{t_2} [f(X,Y,X',Y') - f_{x'}(X,Y,p,q)X' - f_{y'}(X,Y,p,q)Y']dt$$

$$= \int_{t_1}^{t_2} E(X,Y;p,q,X',Y')dt \quad .$$

As for (12.1.iii) in (12.6), this formula yields sufficient conditions for a minimum.

(23.1.i) (sufficient condition for a strong relative minimum). If the extremal $E_{12}$ is contained in a field R with slope functions $p(x,y), q(x,y)$, if $E(x,y,p,q,X',Y') \geq 0$ for all $X',Y'$ and all points $(x,y) \in R$, then for every parametric regular curve $C_{12}$ of R having the same end points as $E_{12}$ we have $I[C_{12}] \geq I[E_{12}]$. If $E > 0$ for all pairs $(X',Y')$ of numbers not both zero and not proportional to $(p,q)$, then $I[C_{12}] > I[E_{12}]$ for all curves $C_{12}$ in R not identical with $E_{12}$.

This statement is the analogue of (12.6.i). By similar argument we prove the analogue of (12.6.ii):

(23.1.ii) (sufficient condition for a weak relative minimum). Suppose that an extremal $E_{12}$: $x = x(t)$, $y = y(t)$, $t_1 \leq t \leq t_2$, is given with $f^* > 0$ along $E_{12}$, and that no point 3 between 1 and 2 on $E_{12}$, nor 2 is conjugate to 1. Then there is an $\epsilon > 0$ such that for any curve $C_{12}$: $x = X(t)$, $y = Y(t)$, $t_1 \leq t \leq t_2$, having the same points 1 and 2 of $E_{12}$, distinct from $E_{12}$, and such that

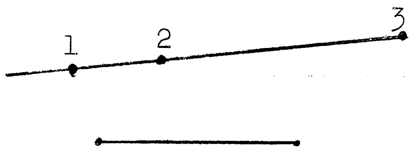
$$|X(t)-x(t)| \leq \epsilon, \quad |Y(t)-y(t)| \leq \epsilon, \quad |X''(t)-x'(t)| \leq \epsilon, \quad |Y'(t)-y'(t)| \leq \epsilon ,$$

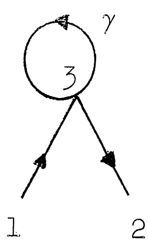
we have $I[C_{12}] > I[E_{12}]$.

## 24. EXAMPLES AND APPLICATIONS

### 24.1. THE INTEGRAL $I[\mathcal{C}] = \int [a(x'^2 + y'^2)^{1/2} + \frac{1}{2}(x'y - xy')dt, \ a > 0$

Let us consider this integral in the class $\Omega$ of all oriented parametric rectifiable path curves of the x,y-plane joining the point $1 = (x_1, y_1)$ to the point $2 = (x_2, y_2)$. First of all, let us observe that $I[C] = aJ[C] + H[C]$, where $J[C]$ denotes the length of C, and $H[C]$ can be interpreted as the signed area "linked" by the oriented closed curve $C' = C + (21)$ made up of the curve C travelled from 1 to 2 and of the segment (21) from 2 to 1. To see this, one has only to take a system of polar coordinates of center 1, since then $H[C] = 2^{-1} \int_{C'} r^2 d\theta$. Let us prove now that $I[C]$ has neither absolute maximum, nor absolute minimum in $\Omega$. We shall prove precisely, that the supremum j of $I[C]$ in K is $+\infty$, and the infimum i is $-\infty$. Indeed, if $2c = |12|$ is the distance from 1 to 2, and we take a point 3 at an arbitrary distance d from 2 on the straight line from 1 containing (12), then the curve $C = (13) + (32)$, made up of the two oriented segments (13) and (32), belongs to $\Omega$, and $H[C] = 0$, $I[C] = J[C] = 2c + 2d > 0$, and this number can be made as large as we want. On the other hand, if we take a circle $\gamma$ of radius r, counterclockwise oriented, say $\gamma$: $x = \alpha + r \cos t$, $y = \beta + r \sin t$, $0 \leq t \leq 2\pi$, then we have $H[C] = -\pi r^2 < 0$, $I[C] = 2\pi ra - \pi r^2 = 2(2a - r) < 0$ for $r > 2a$. If we choose $(\alpha, \beta)$ in such a way that $\gamma$ passes through a fixed point 3, we take any two fixed curves $C_{13}$ and $C_{32}$, then



$C = C_{13} + \gamma + C_{32}$ belongs to $\Omega$, and $I[C] = I[C_{13}] + I[C_{32}] + \pi r(2a - r)$ can be made negative and in absolute value as large as we want. Thus $j = +\infty$, $i = -\infty$, and $I[C]$ has neither absolute maximum, nor absolute minimum in $\Omega$. Note that instead of taking r large, we could fix any $r > 2a$, and take $C = C_{13} + n\gamma + C_{32}$, that is, we could travel $\gamma$ any number n of times. By taking n sufficiently large, we see that I[C] may have, nevertheless, relative extrema.

We have here

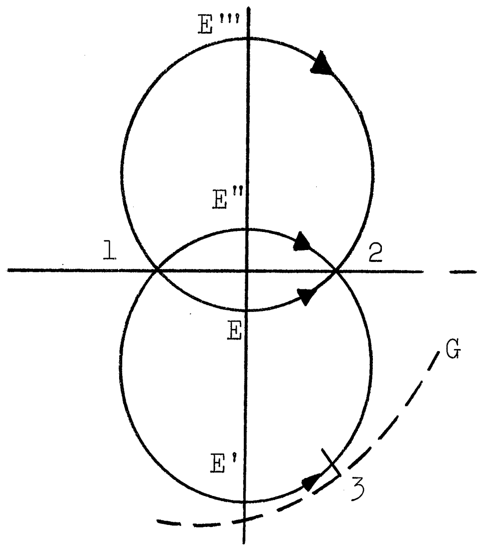
$$f_o = a(x'^2 + y'^2)^{1/2} + \frac{1}{2}(x'y - xy') ,$$

$$f_{ox'} = ax'(x'^2 + y'^2)^{-1/2} + \frac{1}{2}y , \quad f_{ot'} = ay'(x'^2 + y'^2)^{-1/2} - \frac{1}{2}x ,$$

$$f_{ox'x'} = ay'^2(x'^2 + y'^2)^{-3/2} , \quad f^* = a(x'^2 + y'^2)^{-3/2} . \qquad (24.1.1)$$

Since $f^* > 0$, we conclude that I[C] may have in $\Omega$ at most (relative) minima. For every fixed point (x,y), let us use formulas (21.2.13), so that we express $f_{ox'}$ and $f_{oy'}$ as functions $\alpha(\theta)$, $\beta(\theta)$ of $\theta$ only. We have $\alpha(\theta) = \frac{1}{2}y + \alpha \cos \theta$, $\beta(\theta) = -\frac{1}{2}x + a \sin \theta$, and thus we can have $\alpha(\theta_1) = \alpha(\theta_2)$, $\beta(\theta_1) = \beta(\theta_2)$ only if $\theta_1 - \theta_2$ is a multiple of $2\pi$. This reasoning shows that, if a curve C of $\Omega$ gives a minimum for I[C] in $\Omega$, then C has no corner point. On the other hand, from (24.1.1) we deduce $f_{ox'y} = \frac{1}{2}$, $f_{oxy'} = -\frac{1}{2}$, and from (37.11), we have $1/r = 1/a$, that is, $r = a$. Thus, if C gives a minimum for I[C] in $\Omega$, then C is an arc of a circle of radius a. Thus, a necessary condition in order that I[C] has a minimum in $\Omega$ is that the two points 1 and 2 have a distance $12 \leq 2a$. Thus for $c > a$ there are not even relative minima in $\Omega$. Assume

$c \leq a$. First let us study the case $c < a$. It is not restrictive to assume $1 = (-c,0)$, $2 = (c,0)$. There are four arcs of circle of radius a joining 1 and 2, say E', E", E''', $E_4$ given by the equations




E: $x = a\sin t$, $y = b - a\cos t$, $-t_o \leq t \leq t_o$,

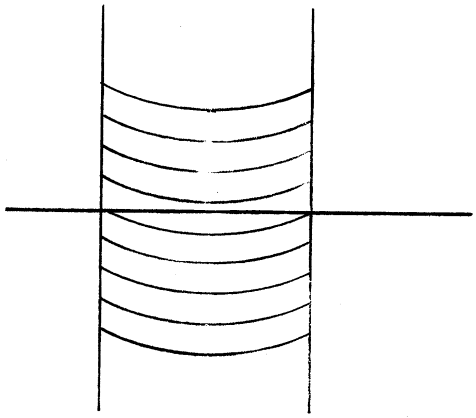
E': $x = a\sin t$, $y = -b - a\cos t$, $-t_1 \leq t \leq t_1$,

E": $x = a\sin t$, $y = -b + a\cos t$, $-t_o \leq t \leq t_o$,

E''': $x = a\sin t$, $y = b + a\cos t$, $-t_1 \leq t \leq t_1$,

where $0 < t_o < \pi/2 < t_1 < \pi$, $b = (a^2 - c^2)^{1/2} > 0$, $\sin t_o = \sin t_1 = c/a$, $\cos t_o = b/a$, $\cos t_1 = -b/a$. Now we have here $f_{oy'x} = -1/2$, $f_{oxy'} = -1/2$, $f^* = a(x'^2 + y'^2)^{-3/2} = a^{-2}$, hence Weierstrass' equation (21.2.9) yields $-1 + a^{-2}(x'y'' - x''y') = 0$. We verify immediately that we have $x'y'' - x''y' = a^2$ on E and E', and $x'y'' - x''y' = -a^2$ on E" and E'''. Thus E and E' satisfy Weierstrass' equation and E" and E''' do not satisfy this equation. Hence E" and E''' do not give any extremum of I[C]. Let us discuss E'. The family of circles of radius a through the point 1 has an envelope G which is the circle of center 1 and radius 2a. Thus the point 3 opposite to 1 on E' is point of contact of E' with the envelope G, hence 3 is a conjugate point to 1 on E' and is between 1 and 2. Thus E' is an extremal arc, but does not satisfy Jacobi's necessary condition. Thus E' does not give any extremum of I[C].

Let us now discuss $E = E_{12}$. The family of extremal arcs $C_\alpha$: $x = a\sin t$, $y = \alpha - a\cos t$, $-t_o \leq t \leq t_o$, $-\infty < \alpha < +\infty$, fills simply the region $R = [-c \leq x \leq c, -\infty < y < +\infty]$ and $t = t(x,y) = \arcsin(x/a)$, $\beta = \beta(x,y) = y + (a^2 - x^2)^{1/2}$ and both $t(x,y)$, $\beta(x,y)$ are certainly of class $C^2$ in R. Also, we have



$p = p(x,y) = (a^2-x^2)^{1/2}$, $q = q(x,y) = x$, and both p, q are certainly of class $C^1$. Finally, we have $E_b = E = E_{12}$, and $p^2+q^2 = a^2$ in R. The corresponding Hilbert integral is obviously the integral of an exact differential since $A_y = B_x = 1/2$.

$$I^* = \int f_{ox'}dx+f_{oy'}dy = \int (p+ \frac{1}{2} y)dx+(q- \frac{1}{2} x)dy$$

$$= \int [(a^2-x^2)^{1/2}+ \frac{1}{2} y]dx+ \frac{1}{2} xdy = \int Adx+Bdy.$$

We have here a field R. For any pair of numbers $(\bar{p},\bar{q})$ with $\bar{p}^2+\bar{q}^2 = a^2$, we have now, by using (21.2.14)

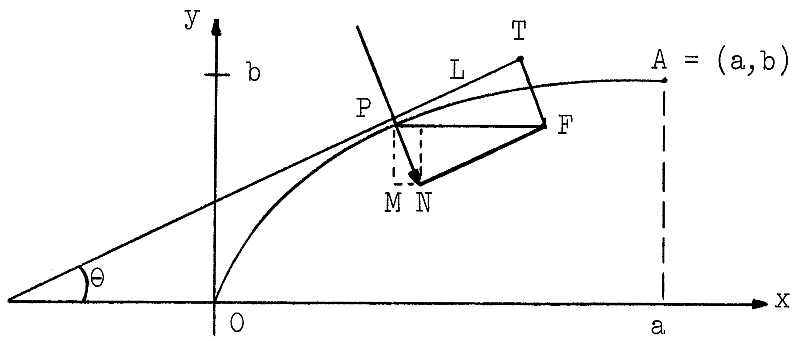
$$E(x,y,p,q,\bar{p},\bar{q}) = \bar{q}(\bar{p}-p) \in \bar{p}(\bar{q}-q) = a^2-(p\bar{p}+q\bar{q}) \geq 0,$$

and = sign holds only if $\bar{p},\bar{q} \div p,q$. By (23.1.i) we conclude that $I[C_{12}] \geq I[E_{12}]$ for every curve $C_{12}$ of $\Omega$ lying in R, and = sign holds only if $C_{12}$ coincides with $E = E_{12}$. Thus E gives a strong relative minimum of I[C] in $\Omega$.

Remark. The curves E", E''' above do not satisfy the Weierstrass equation and thus are not extremals. Nevertheless -E", -E''', the curves we obtain by changing their orientation do satisfy the Weierstrass equation, and are extremals.

## 24.2. SOLID OF REVOLUTION IN MOTION IN A FLUID OFFERING THE LEAST RESISTANCE (Newton's Problem, 1636)

(a). Let Ox be the x-axis parallel and in the direction opposite to fluid motion. Let C = OA be a section of the solid in the half plane $y \geq 0$. Let us assume that, for a given velocity, the resistance is taken as proportional to the area normal to the direction of motion. If PF = r measures the resistance due to fluid motion on unit area perpendicular to the direction of motion, then the component of resistance normal to the surface will be PN = r sin θ, (see illustration). We assume that the fluid resistance is frictionless, that is, that the tangential component of the resistance is zero. In other words, the solid offers no resistance to the tangential component PT of r. We should now resolve PN into two components, one $PL = PN \sin\theta = r \sin^2\theta$ in the direction of motion, and one PM normal to the x-axis. Since PM is equal and opposite




to P'M' due to fluid impact on the opposite side of the axis of symmetry and the effect of these two forces cancel each other, we need only consider PL. By taking $2\pi y\,dy$ as area element normal to the direction of motion, we obtain

$$2\pi\int_0^b y\ r\ \sin^2\theta\ dy = 2\pi r \int_{t_1}^{t_2} \frac{yy'^3}{x'^2+y'^2}\ dt.$$

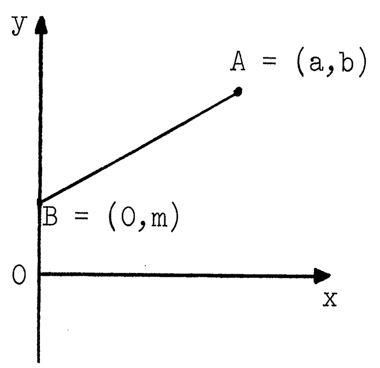
where A = (a,b), $\sin\theta = y'(x'^2+y'^2)^{-1/2}$. Thus we have to consider the line

$$I[C] = \int_{t_1}^{t_2} \frac{yy'^3 dt}{x'^2+y'^2} \qquad (24.2.1)$$

in the class of parametric regular curves joining (0,0) to (a,b).

Note that we consider only curves C of the first quadrant in the x,y-plane with the property that for each arc P'P" of C taken in the sense from O to A, say $P' = (x',y')$, $P'' = (x'',y'')$, we have $x' \leq x''$, $y' \leq y''$. It is clear that for curves not having this property, the presence of vertices would certainly increase the resistance, and, in any case, the formula above could not be valid. The hypothesis of a frictionless resistance is already not in agreement with experiments; nevertheless, the present analysis is indicative as a limit case.

Note that a gives the elongation of the solid, and 2b its diameter. Any cylindrical continuation of the solid has no bearing in the resistance, because of the frictionless hypothesis, and no attempt is made to discuss here the effect of the vortex at the tail of the solid.



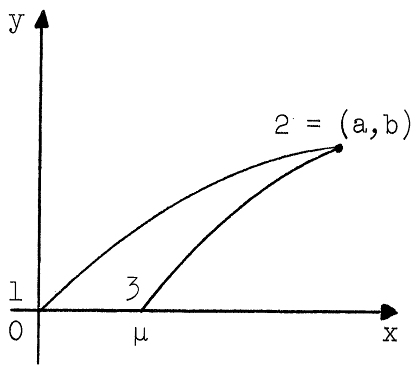

For a polygonal line P = OBA, $B = (0,m)$, $0 \leq m \leq b$, the solid of revolution is a truncated cone (a cone for m = 0, a cylinder for m = b). If $\theta$ is the angle of BA with the x-axis, the formula above gives a resistance $\phi(m) = 2^{-1}[m^2+(b^2-m)^2\sin\theta]$, where $a \tan\theta = b-m$. Thus

$$\phi(m) = 2^{-1}[a^2+(b-m)^2]^{-1}[a^2m^2+(b-m)^2b^2].$$

An analysis of this function of m shows that the minimum in [0,b] is taken in the interior, hence $\phi'_{min} < \phi(0) < \phi(b)$, and thus the cone is not the solid of least resistance. The problem under discussion can now be formulated as follows: Determine the minimum of (24.2.1) in the collection $\Omega$ of all parametric curves C: $x = x(t)$, $y = y(t)$, $t_1 \le t \le t_2$, with $x(t_1) = 0$, $y(t_1) = 0$, $x(t_2) = a$, $y(t_2) = b$, $x(t), y(t)$ AC in $[t_1, t_2]$, $x'(t)$, $y'(t) \ge 0$ in $[t_1, t_2]$.

We shall first discuss this problem in the subclass K of all curves C of $\Omega$ with $x'(t)$, $y'(t)$ sectionally continuous. In this situation we can well assume $x'$, $y'$ never both zero. By direct argument, Euler equation, and theory of fields, we shall completely determine the unique solution in K. In (c) we shall apply general existence theorems and other statements, so as to prove directly that the minimizing solution in $\Omega$ exists, and is smooth enough to belong to K.

(b) We may well exclude that a curve C which gives the minimum of I in K contains a segment of the x-axis. Indeed, if $C_{12}$ of the collection K contains such a segment, then $C_{12}$ has the segment 13 as initial segment. Let




$C_{32}$: $x = x(t)$, $y = y(t)$, $t_3 \le t \le t_2$, $3 = (\mu, 0)$, $0 < \mu < a$, and denote by $D_{12}$ the curve $D_{32}$: $x = X(t) = x(t) - (\mu/b)[b - y(t)]$, $y = y(t)$. Then $X' = x' + (\mu/b)y' \ge x'$ and certainly $X' > x'$ in some subinterval. Also, $X' \ge x' \ge 0$, $Y' = y' \ge 0$, and $D_{12}$ is in K. Finally

$$I[D_{12}] = \int_{t_3}^{t_2} yy'^3(X'^2+y'^2)^{-1}dt < \int_{t_3}^{t_2} yy'^3(x'^2+y'^2)dt = I[C_{12}].$$

Thus $C_{12}$ does not give the absolute minimum of I[C] in K. Because of this remark we may well assume that, for the curves C of K, we have $x'(t) \geq 0$, $y(t) \geq 0$ for $t_1 \leq t \leq t_3$, $y(t) > 0$ for $t_1 < t \leq t_2$.

We now study the integral (40.1) in K. We have

$$f_o = \frac{yy'^3}{x'^2+y'^2}, \quad f_{ox'} = \frac{-2yx'y'^3}{(x'^2+y'^2)^2}, \quad f_{oy'} = \frac{yy'^2(3x'^2+y'^2)}{(x'^2+y'^2)^2} \qquad (24.2.2)$$

If a curve C of K gives a minimum of I in K, let us first consider any arc $\alpha$ of C (if any), say $\alpha$: $x = x(t)$, $y = y(t)$, $t' \leq t \leq t''$, with $x(t)$, $y(t) \in C^1$ and $y,x',y' > 0$ in $[t',t'']$. Euler equations must hold along $\alpha$. Indeed, $y,x',y' \geq m > o$ in $[t',t'']$ for some $m > 0$, and then, for any given $\eta_1$, $\eta_2$ zero outside $(t',t'')$, we certainly have $y+a\eta_2$, $x'+a\eta_1'$, $y'+a\eta_2' > 0$ in $[t',t'']$ for all $|a|$ sufficiently small. Thus, the curves $X = x+a\eta_1$, $y = y+a\eta_2$ belong to K for all $|a|$ sufficiently small, and the usual reasoning holds. The Euler equation $f_x-(d\ dt)f_{x'} = 0$ for any such arc $\alpha$ yields $f_{x'}$ = constant, or

$$\frac{yy'^3x'}{(x'^2+y'^2)^2} = c \ , \qquad (24.2.3)$$

where c is a positive constant, since $y$, $y'$, $x' > 0$ along $\alpha$. By introducing the parameter $q = x'/y' = \text{cotg}\theta$, we deduce from (24.2.3)

$$y = cq^{-1}(1+q^2)^2 = c(q^{-1}+2q+q^3).$$

and hence, successively, using (24.2.3) again

$$y' = c(-q^{-2}+2+3q^2)q',$$

$$x' = c(-q^{-1}+2q+3q^3)q',$$

$$x = c(q^2+\frac{3}{4}q^4-\log q)+d\ ,$$

where d is another constant of integration. Then

$$\begin{aligned} dx/dq &= c(-q^{-1}+2q+3q^3) = cq^{-1}(3q^2-1)(q^2+1), \\ dy/dq &= c(-q^{-2}+2+3q^2) = cq^{-2}(3q^2-1)(q^2+1), \end{aligned} \qquad (24.2.4)$$

hence $dy/dx = q^{-1} = \tan\theta$, and

$$d^2y/dx^2 = (d/dx)(q^{-1}) = -q^{-2}(dq/dx) = -c^{-1}q^{-1}(3q^2-1)^{-1}(q^2+1)^{-1},$$

where $c > 0$, $q > 0$. As q increases from 0 to $3^{-1/2}$, both x,y decrease from $+\infty$ to some $(x_o,y_o)$ with $dy/dx > 0$, $d^2y/dx^2 > 0$. As q increases from $3^{-1/2}$ to $+\infty$, both x,y increase from $(x_o,y_o)$ to $+\infty$, with $dy/dx > 0$, $d^2y/dx^2 < 0$. The point $(x_o,y_o)$ is a cusp for $\Gamma$ with tangent of slope $3^{1/2}$ (forming an angle of 60° with the x-axis, and $(x_o,y_o)$ divides $\Gamma$ into two parts $\Gamma_1$, $\Gamma_2$, the first concave upwards, the second concave downwards. By (24.2.2), differentiating, and using (21.2.8), we have

$$f^* = 2y(3q^2-1)y'^{-3}(1+q^2)^{-3},$$

and Legendre's necessary condition (21.2.iv) yields $f^* \geq 0$, and since $y > 0$, $y' > 0$, $q > 0$, we deduce $q \geq 3^{-1/2}$. Therefore, any such arc as $\alpha$ on C is a subarc of the curve $\Gamma_1$ concave downwards, with $q \geq 3^{-1/2}$, hence, decreasing slope $\tan\theta = 1/q$, and $\pi/3 \geq \theta > 0$.

By (24.2.2) and (21.2.14), and algebraic manipulations, we see that

$$E(x,y,p,q,P,Q) = y(P^2+Q^2)^{-1}(p^2+q^2)^{-2}(pQ-Pq)^2[(p^2-q^2)Q+2pqQ] \quad (24.2.5)$$

and, by using relations (21.2.13), also

$$E(x,y, \cos \theta, \sin \theta, \cos \bar{\theta}, \sin \bar{\theta}) = y \sin (\theta-\bar{\theta})^{-2}\sin(2\theta+\bar{\theta}) \quad (24.2.6)$$

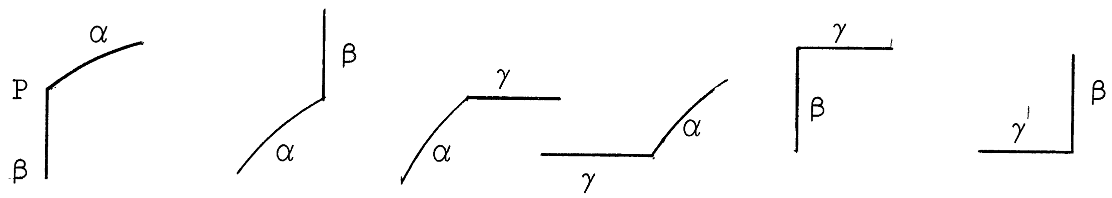
The reasoning with which we prove Weierstrass' necessary condition involves an arbitrary direction $\bar{\theta}$. Here $\bar{\theta}$ must be an arbitrary admissible direction, hence $0 \leq \theta \leq \pi/2$, and then the usual reasoning can be repeated without change. Therefore, Weierstrass' necessary condition for a minimum implies that we must have $\sin(2\theta+\bar{\theta}) \geq 0$ all along $\alpha$ for all $0 \leq \bar{\theta} \leq \pi/2$. Since along $\alpha$ we have $2\pi/3 \geq 2\theta > 0$ the requirement above yields $-\bar{\theta} \leq 2\theta \leq \pi-\theta$, and we conclude that along $\alpha$ we must have $0 \leq 2\theta \leq \pi/2$. Thus we conclude that $\alpha$ must be a subarc of $\Gamma_1$ with decreasing slope $\theta$, $\pi/4 \geq \theta > 0$. Precisely such an arc $\alpha$ has a representation

$$x = c(q^2+ \frac{3}{4} q^4-\log q)+b, \qquad y = c(q^{-1}+2q+q^3), \quad (24.2.7)$$

for some constants b and $a > 0$, and $q_1 \leq q \leq q_2$, with $1 \leq q_1 < q_2 < +\infty$. Since y increases with q for $q \geq 1$, we see that $y \geq 4c > 0$. Thus such an arc $\alpha$ cannot have an end point at the origin.

If we denote as exceptional those elements $(x,y,x',y')$ of the curve C with $y > 0$, and $x',y'$ not both positive, then, for each exceptional element, we have $x \geq 0$, $y > 0$, $x'y' = 0$, $x' \geq 0$, $y' \geq 0$, $x'+y' > 0$, hence either $x' = 0$, $y' > 0$, $\theta = \pi/2$, or $x' > 0$, $y' = 0$, $\theta = 0$. Assume that $x = x(t_o)$, $y = y(t_o)$, and that $\theta_o$ is the slope of the tangent to C at $(x,y)$ [or of the tangent to the right, or to left, to C at $(x,y)$]. Then there cannot be nonexceptional elements in some neighborhood of $t_o$, since, otherwise $\theta_o = \pi/2$, or $\theta_o = 0$, should be the limit of numbers $\theta$ which are in some interval $[\mu_1, \mu_2]$ with $\pi/4 \geq \mu_1 > \mu_2 > 0$. Thus, the exceptional elements of C form subarcs of C and these are segments $\beta$ parallel to the y-axis, or segments $\gamma$ parallel to the x-axis.

We conclude that a parametric (regular) curve C of the collection K which gives the minimum of I[C] in K is made up of finitely many arcs of the types $\alpha$, $\beta$, $\gamma$. There arises the possibility of adjacent arcs of the types $\alpha$, $\beta$, $\gamma$, hence of the six types $\beta\alpha$, $\alpha\beta$, $\alpha\gamma$, $\gamma\alpha$, $\beta\gamma$, $\gamma\beta$, shown in the illustration.




As we shall see, the combination $\beta\alpha$ is the only possible one. Let us consider a pair of adjacent arcs $\beta\alpha$. Together they form a subarc of C, say $\beta\alpha$: $x = x(t)$, $y = y(t)$, $t' \leq t \leq t''$. We may keep $x(t)$ fixed, and replace $y(t)$ by a function $Y(t) = y(t) + a\eta_2(t)$, $\eta_2(t) = 0$ outside $(t', t'')$. Then for a sufficiently small, the curve $x = x(t)$, $y = Y(t)$, $t_1 \leq t \leq t_2$, belongs to K, and the usual reasoning applies, with which we prove the Euler equation $f_{y'} - \int_{t_1}^{t} f_y \, dt =$ constant. From here we deduce the Erdman corner relation in y, say $f_{y'}|^- = f_{y'}|^+$ at the corner point $P = (x,y)$. If we consider $f_{y'}$ as a function of $\theta$ by means of relations (21.2.13), we have

$$\alpha(\theta) = f_{oy'}(x,y,x',y') = f_{oy'}(x,y,\ \cos\theta,\ \sin\theta) =$$

$$= y \sin^2\theta(3 \cos^2\theta + \sin^2\theta) = y(3 \sin^2\theta - 2 \sin^4\theta).$$

In particular we have $\alpha(0) = 0$, $\alpha(\pi/4) = y$, $\alpha(\pi/2) = y$, and also $\alpha'(\theta) = 2y \sin\theta \cos\theta(3-4 \sin^2\theta)$. Hence $\alpha'(\theta) > 0$ for $0 < \theta \leq \pi\ 4$, and finally, $0 < \alpha(\theta) \leq y$ for $0 < \theta \leq \pi/4$. Thus $f^-_{y'} = \alpha(\pi/2) = y > 0$, $f^+_{y'} = \alpha(\theta) \leq y$, since we know that on $\alpha$ we have $\pi/4 \geq \theta > 0$. Thus, the Erdman condition is satisfied only if $\theta = \pi/4$.

The combination $\alpha\beta$ can be treated in an analogous way. We have here $f^-_{y'} = \alpha(\theta)$ where $\pi/4 > \theta > 0$, and hence $1 > \alpha(\theta) > 0$, while $f^+_{y'} = \alpha(\pi/2) = 1$. Thus the Erdman corner condition cannot be satisfied, and the combination $\alpha\beta$ is excluded.

The combination $\alpha\gamma$ can be discussed by noting the Erdman corner condition $f^-_{x'} = f^+_{x'}$ must hold at the corner point. We have

$$\beta(\theta) = f_{x'}(x,y,x',y') = f_{x'}(x,y,\ \cos\theta,\ \sin\theta) = -2y \cos\theta \sin^3\theta\ .$$

In particular $f^-_{x'} = \beta(\theta)$, and $\beta(\theta) < 0$ for $0 < \theta \leq \pi/4$ and $y > 0$, while

$f^-_{x'} = \beta(0) = 0$. Thus the Erdman condition cannot be satisfied, and the combination $\alpha\gamma$ is excluded.

The combination $\gamma\alpha$ with $P = (x,y)$ and $y = 0$ is already excluded by the initial argument. For $y > 0$, we have $f^-_{x'} = 0$, $f^+_{x'} = \beta(\theta)\quad 0$, and the Erdman condition is not satisfied.

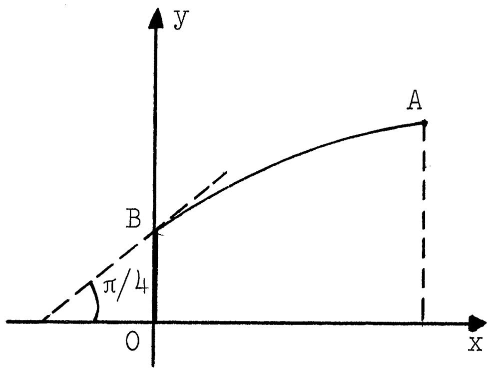
The combination $\gamma\beta$ can be eliminated by observing that, if $\gamma = 12$, $\beta = 23$,

$1 = (x_1,y_1)$, $2 = (x_2,y_2)$, $3 = (x_3,y_3)$, then $x_1 < x_2 \leq x_3$, $y_1 = y_2 < y_3$, and if $K = (y_3-y_2)/(y_2-y_1)$, $K = \tan\theta$, and $\delta$ is the segment 13 of slope k, then

$$I[\delta] = 2^{-1}(y_3^2-y_1^2)\sin^2\theta < I(\gamma\beta) = 2^{-1}(y_3^2-y_1^2).$$

Analogous reasoning holds for the combination $\beta\gamma$.

We conclude that, if a curve C of the collection K gives the minimum of $I[C]$ in K, then C is made up of exactly one segment $\beta = OB$, of the y-axis issued from the origin, and of exactly an arc $\alpha = BA$ whose tangent at B form an angle $\theta = \pi/4$ with the x-axis. Thus such an arc $\alpha = BA$ has a representation




(40.5) with $1 \leq q \leq q_2$. We have now to show that the parameters c, d can be determined so as $\alpha$ joins A to the y-axis and has $\theta = \pi/4$ at B. The last requirement implies $x(1) = 0$, hence by (40.4), $d = -7c/4$, and

$$x = x(t,c) = c(t^2 + \frac{3}{4}t^4 - \log t - \frac{7}{4})$$
$$y = y(t,c) = c(t^{-1}+2t+t^3) = ct^{-1}(1+t^2)^2 \qquad (24.2.6)$$

where $t \geq 1$, $c > 0$, hence $x(1,c) = 0, y(1,c) = 4c > 0$.

We must determine now c ant $t = t_2 > 1$ in such a way that $x(c,q) = a$, $y(c,q) = b$. If we consider the function

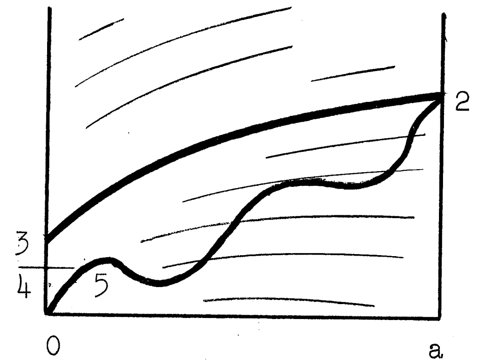
$$\mu(t) = \frac{x(t,c)}{y(t,c)} = (t^2+1)^{-1}(t^3 + \frac{3}{4}t^5 - \frac{7}{4}t - t\log t), \quad 1 \leq t < +\infty,$$

we have $\mu(1) = 0$, $\mu(+\infty) = +\infty$, and by using (24.2.4)

$$\mu'(t) = y^{-2}(x'y-xy') = (t^2+1)^{-2}(3t^2-1)(\frac{1}{4}\ t^4+t^2+\log\ t+ \frac{11}{4})$$

Hence $\mu'(t) > 0$ for $t \geq 1$, $\mu(t)$ is an increasing function of $t$, and there is one and only one value $t = t_2$ for which $\mu(t) = a/b$. Since $y(t)$ is an increasing function of $t$, we can now determine $c > 0$ in such a way that $y(c,t_2) = b$, and hence $x(c,t_2) = a$.

If $E_o$ denotes the curve we have so determined, $E_o \in K$, we have now to prove that $I[L] \leq I[C_{12}]$ for all curves $C_{12}$ in $K$ joining $1 = (0,0)$ to $2 = (a,b)$. Let us consider the one-parameter family of extremals (24.2.6) for $0 \leq x \leq a$,




$t \geq t$, $c > 0$. They simply cover the strip $R = [0 \leq x \leq a,\ y > 0]$, and we shall denote by $c = c(x,y)$, $t = t(x,y)$, the corresponding functions. We have as usual $p(x,y) = x'[c(x,y),q(x,y)]$, $q(x,y) = y'[c(x,y),q(x,y)]$. Note that $t(0,y) = 1$, $c(0,y) = y/4$, $p(0,y) = = q(0,y) = 4c = y$. By using relations (24.2.4) and (24.2.6) we have for the functional determinant,

$$\Delta = x_t y_c - x_c y_t = ct^{-2}(3t^2-1)(t^2+1)(ty_c - x_c) =$$

$$= ct^{-2}(3t^2-1)(t^2+1)(t^2+1)(1+t^2+ \frac{1}{4}\ t^4+\log\ t+ \frac{7}{4}) > o$$

for all $t \geq 1$. Since the functions $x(t,c)$, $y(t,c)$ are certainly of class $C^2$, the same holds for $t(x,y)$, $c(x,y)$. The Hilbert integral is here

$$I* = \int f_{ox'}dx+f_{oy'}dy = \int A(x,y)dx+B(x,y)dy,$$

where, by force of (24.2.6), we have

$$A = f_{ox'} = \frac{-2x'yy'^3}{(x'^2+y'^2)^2} = -2c,$$

$$B = f_{oy'} = \frac{yy'^2(3x'^2+y'^2)}{(x'^2+y'^2)^2} = \frac{3x'^2+y'^2}{x'y'} = (3t+t^{-1})c,$$

$$A_y = -2c_y, \quad B_x = (3-t^{-2})ct_x+(3t+t^{-1})c_x.$$

If we write relations (24.2.6) in the form $x = cP(t)$, $y = cQ(t)$, we have by differentiation

$$c_xP+ct_xP' = 1, \quad c_xQ+ct_xQ' = 0,$$

$$c_yP+ct_yP' = 0, \quad c_yQ+ct_yQ' = 1,$$

with $\Delta = x_ty_c-x_cy_t = c(P'Q-PQ') > 0$. If $\Delta_o = P'Q-PQ'$, we have

$$\Delta_oc_x = -Q', \quad \Delta_oct_x = Q, \quad \Delta_oc_y = P', \quad \Delta_oct_y = -P.$$

By algebraic substitutions we have now

$$A_y = -2c_y = -2\Delta_o^{-1}P' = -\Delta_o^{-1}(4t+6t^3-2t^{-1}),$$

$$B_x = \Delta_o^{-1}[(3-t^{-2})Q-(3t+t^{-1})Q'] = -\Delta_o^{-1}(4t+6t^3-2t^{-1}).$$

Thus $A_y = B_x$, and I* is the integral of an exact differential. This proves that R is a field. Also, E is given by (24.2.5) and hence $E > 0$ in R for all (PQ) not both zero and not proportional to (p,q). If the curve C joining 0 to 2 has an arc 04 in common with the y-axis, then we have

$$I[C_{02}]-I[E_{02}] = I[C_{42}]-I[E_{42}] \geq 0,$$

and = sign holds only if $C_{42}$ coincides with $E_{42}$. If $C_{02}$ has no segment in common with the y-axis, then take a point $4 = (0,\epsilon)$, and the corresponding point $5 = (x_5,\epsilon)$ on C. Then, If $C_{45}$ is the segment 45, we have

$$I[C_{45}+C_{52}]-I[E_{45}] = \int_0^{x_5} E(x,\epsilon,p,q,0,1)dx$$

$$+ \int_{C_{52}} E(\bar{x},\bar{y},p,q,P,Q)dt.$$

As $\epsilon \to 0$, $x_5 \to 0$, and the first integral approaches zero, while the second has a positive limit (for any curve $C_{02}$ distinct from $E_{02}$). Thus $I[C_{02}] \geq I[E_{02}]$ in all cases. It is proved, thereby, that $E_{02}$ is the absolute minimum of I[C] in K.

(c) First let us prove that the class $\Omega$ defined in (a) is complete. Let $C_k$: $x = x_k(t)$, $y = y_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k = 1,2,\ldots$, be a sequence of curves in $\Omega$, approaching in the metric $\rho$ a given curve C: $x = x(t)$, $y = y(t)$, $t_1 \leq t \leq t_2$. Then $t_{1k} \to t_1$, $t_{2k} \to t_2$, and $C_k \in \Omega$ implies $x_k(t) \leq x_k(t')$, $y_k(t) \leq y_k(t')$ for all t,t' with $t_{1k} \leq t \leq t' \leq t_{2k}$. For t,t' with $t_1 < t < t' < t_2$, we have also $t_{1k} \leq t < t' \leq t_{2k}$ for all k sufficiently large and hence as $k \to \infty$ the inequalities above for $x_k, y_k$ imply $x(t) \leq x(t')$, $y(t) \leq y(t')$. Since x(t), y(t) are continuous in $[t_1,t_2]$, these relations hold for every $t_1 \leq t \leq t \leq t_2$. Thus $C \in \Omega$, and $\Omega$ is complete.

As usual we write $x' = \sqrt{x'^2+y'^2}\cos\theta$, $y' = \sqrt{x'^2+y'^2}\sin\theta$, $0 \leq \theta \leq 2\pi$, for every pair $(x',y') \neq (0,0)$. Let us prove that the curves C of $\Omega$ have equibounded length. Indeed,

$$L(C) = \int_{t_1}^{t_2}(x'^2+y'^2)^{1/2}dt \leq \int_{t_1}^{t_2}(|x'|+|y'|)dt = \int_{t_1}^{t_2}(x'+y')dt =$$

$$= (x(t_2)-x(t_1))+(y(t_2)) = a+b.$$

In (b) we determined the Weierstrass function E of the integrand $f_o$. From the expression of E given in (24.2.6) we can see that E can be negative, for instance, for $y > \theta$, $\theta = \pi/3$, $\bar{\theta} = \pi/2$. Thus, $f_o(x,y,x',y') = (x'^2+y'^2)yy'^3$ is not always convex in y'. Nevertheless if $0 \leq \theta \leq \pi/4$, $\pi-2\theta \leq \bar{\theta} \leq \pi/2$, $y \geq 0$, then $E \geq 0$.

We now introduce an auxiliary function $\bar{f}_o(x,y,x',y')$. To do this, we first rewrite f*:

$$f^* = f_{ox'x'}/y'^2 = -f_{ox'y'}/x'y' = f_{oy'y'}/y'^2 = (x'^2+y'^2)^{-3}2yy'(3x'^2-2y'^2),$$

or

$$f^*(x,y,x',y') = (x'^2+y'^2)^{-3/2}\cdot 2y(3\cos^2\theta-2\sin^2\theta).$$

We now define $\bar{f}^*$ by taking

$$\bar{f}^*(x,y,x',y') = (x'^2+y'^2)^{-3/2}\cdot 2y\varphi(\theta),$$

where $\varphi(\theta)$ is the periodic function of period $\theta$ defined by $\varphi(\theta) = 3\cos^2\theta-2\sin^2\theta$ for $o \leq \theta \leq \pi/4$, $\varphi(\theta) = 0$ for $\pi/4 \leq \theta \leq \pi/2$, $\varphi(\theta) = \varphi(\pi-\theta)$ for $\pi/2 \leq \theta \leq \pi$, $\varphi(\theta) = \varphi(2\pi-\theta)$ for $\pi \leq \theta \leq 2\pi$. Here $\varphi(\theta)$ is continuous in $(-\infty,+\infty)$ but the points $(2k+1)(\pi/4)$, $k = o,\pm 1,\pm 2,\ldots$ . Also, because of the symmetries $\varphi(\theta) = \varphi(\pi-\theta)$, $\varphi(\theta) = \varphi(2\pi-\theta)$, we have $\int_0^{2\pi}\varphi(\theta)\cos\theta\, d\theta = 0$. Now we can define $\bar{f}_o$ as a positively homogeneous function of degree one in x',y', by taking

$$\bar{f}_o(x,y,\cos\Theta,\sin\Theta) = \cos\Theta\, f_{ox'}(x,y,1,0) + \sin\Theta\, f_{oy'}(x,y,1,0)$$

$$+ \int_0^{\Theta} \bar{f}_{ox}(x,y,\cos\omega,\sin\omega)\sin(\Theta-\omega)\,d\omega.$$

By Schwarz formula (21.2.16) we conclude that for $0 \le \Theta \le \pi/4$, $\bar{f}_o$ and $f_o$ coincide, that is,

$$\bar{f}_o(x,y,x',y') = f_o(x,y,x',y') \text{ for all } x' \ge 0, \quad y' \ge 0, \quad 0 \le y' \le x'.$$

Now for $\pi/4 \le \Theta \le \pi/2$, we have

$$\bar{f}_o(x,y,\cos\Theta,\sin\Theta) = \cos\Theta\, f_{ox'}(x,y,\cos\tfrac{\pi}{4},\sin\tfrac{\pi}{4}) + \sin\Theta\, f_{oy'}(x,y,\cos\tfrac{\pi}{4},\sin\tfrac{\pi}{4})$$

$$= -\frac{1}{2}\, y\cos\Theta + y\sin\Theta\ ,$$

and hence

$$\bar{f}_o(x,y,x',y') = y(-\tfrac{1}{2}x' + y') \text{ for all } x' \ge 0, \quad y' \ge 0, \quad x' \le y'.$$

This function $\bar{f}_o(x,y,x',y')$ is continuous with $f_{ox'}$, $f_{oy'}$ for all $x,y,x',y'$, with $(x',y') \ne (0,0)$. Also, $\bar{f}_o$ possesses continuous partial derivatives of all orders for $x'^2 - y'^2 \ne 0$. Finally one can verify that $\bar{f}_o(x,y,\cos\frac{\pi}{2},\sin\frac{\pi}{2}) = f_o(x,y,\cos\frac{\pi}{2},\sin\frac{\pi}{2})$.

Let $C$ be any curve of the class $\Omega$ and let $\bar{I}(C)$ be the integral $\bar{I}(C) = \int_C \bar{f}_o\, ds$. Let us prove that $I[C] \ge \bar{I}[C]$ for all $C$ of the class $\Omega$. Along $C$ we consider the angle $\Theta$ defined as usual by $x' = \cos\Theta\sqrt{x'^2+y'^2}$, $y' = \sin\Theta \sqrt{x'^2+y'^2}$. Since $x \ge 0$, $y \ge 0$ along $C$ (where the derivatives exist), we have

$0 \leq \Theta \leq \pi/2$. It is clear that for all t for which $0 \leq \Theta \leq \pi/4$, we have $\bar{f}_o(x,y,x',y') = f_o(x,y,x',y')$. For all t with $\pi/4 \leq \Theta \leq \pi/2$, we have

$$f_o(x,y,\cos\Theta,\sin\Theta)-\bar{f}_o(x,y,\cos\Theta,\sin\Theta) =$$

$$= f_o(x,y,\cos\Theta,\sin\Theta)-[\cos\Theta\, f_{ox'}(x,y,\cos\tfrac{\pi}{4},\sin\tfrac{\pi}{4})+\sin\Theta\, f_{oy'}(x,y,\cos\tfrac{\pi}{4},\sin\tfrac{\pi}{4})]$$

$$= E(x,y,\cos\tfrac{\pi}{4},\sin\tfrac{\pi}{4},\cos\Theta,\sin\Theta) \geq 0$$

since E is defined by (21.2.4) and $y \geq 0$, and $2(\pi/4)+\Theta = \pi/2+\Theta$ lies between $3\pi/4$ and $\pi$, and hence $E \geq 0$.

Note that, by the very definition of $\bar{f}_o$, and remarks above we have

$$E(x,y,\ \cos\Theta,\ \sin\Theta,\ \cos\bar{\Theta},\ \sin\bar{\Theta}) \geq 0$$

for all $\Theta,\bar{\Theta}$, that is, $\bar{f}_o(x,y,x',y')$ is a convex function of $(x',y')$.

By the remark after (22.1.i), the integral $\bar{I}[C]$ admits an absolute minimum in $\Omega$ given by at least one curve $C_o$: $x = x(t)$, $y = y(t)$, $t_1 \leq t \leq t_2$, with $x' \geq 0$, $y' \geq 0$ a.e. in $[t_1,t_2]$, $x(t),y(t)$ AC, $x(t_1) = y(t_1) = 0$, $x(t_2) = a$, $y(t_2) = b$. We shall prove that along $C_o$ we have either $0 \leq \Theta \leq \pi/4$, or $\Theta = \pi/2$, and hence $\bar{I}[C_o] = I[C_o]$, and hence $\bar{I}[C] \geq \bar{I}[C_o]$ for all $C \in \Omega$ implies $I[C] \geq \bar{I}[C] \geq \bar{I}[C_o] = I[C_o]$, that is, $C_o$ gives the minimum of $I[C]$ in $\Omega$.

We shall begin with an analysis of the extremal arcs $\gamma$: $x = x(t)$, $y = y(t)$, $\alpha \leq t \leq \beta$, of the integral $\bar{I}[C]$. Along such an arc $\gamma$ Euler equations are satisfied. Since $\bar{f}_{ox} \equiv 0$, $\bar{f}_{oy} \equiv 0$, the Euler equation, say relative to x, gives $\bar{f}_{ox'}$ = constant. If on such an arc $\gamma$ we have $0 \leq \Theta \leq \pi/4$, then $\bar{f}_o = f_o$, and the equation becomes

$$y \sin^3\Theta \cos\Theta = C \qquad (24.2.7)$$

If on such an arc $\gamma$ we have $\pi/4 \leq \Theta \leq \pi/2$, then, by $\bar{f}_o = \frac{1}{2} yx'+yy'$, we obtain $\bar{f}_{oy'} = y$, and hence the Euler equation yields

$$y = C\ , \qquad (24.2.8)$$

and then $\Theta = 0$. We conclude that on any extremal arc $\gamma$ relative to $\bar{I}[C]$ with $0 \leq \Theta \leq \pi/2$ we actually have $0 \leq \Theta \leq \pi/4$, and relation (24.2.8) holds. If such an extremal arc contains a point on the x-axis then $C = 0$ and then $y \equiv 0$, $\Theta \equiv 0$. If an extremal arc $\gamma$ satisfying (24.2.7) is known to have $C = 0$, there again $\sin\Theta \equiv 0$, $\Theta \equiv 0$, $y =$ constant.

Let $\gamma$ an extremal arc relative to $\bar{I}[C]$, satisfying $0 \leq \Theta \leq \pi/4$, and then (24.2.7). If it happens that $\Theta = 0$ at the first end point of $\gamma$, then $C = 0$, and then $\Theta \equiv 0$ along $\gamma$, and $\gamma$ is a segment parallel to the x-axis. If it happens that $\Theta > 0$ at the first end point of $\gamma$, then y must be positive and must increase, and hence $\sin^3\Theta \cos\Theta$ decreases. But the derivative of $\varphi(\Theta) = \sin^3\Theta \cos\Theta$ is $\varphi'(\Theta) = \sin^2\Theta(3\cos^2\Theta - \sin^2\Theta) > 0$ for $0 < \Theta \leq \pi/4$, and thus $\Theta$ must decrease, but $\Theta$ cannot reach the value zero since the product $y \sin^3\Theta \cos\Theta$ preserves a constant positive value.

Assume that at some point $\bar{M}$ of the minimizing curve $C_o$ we have $\bar{y} > 0$ and $0 < \bar{\Theta} < \pi/4$. Let $y_o$, $\omega_1$, $\omega_2$, be numbers with $0 < y_o < \bar{y}$, $0 < \omega_1 < \bar{\Theta} < \omega_2 < \pi/4$. Then there is a neighborhood $V(M,\rho)$ where for every $\omega$, $\omega_1 \leq \omega \leq \omega_2$, and every $\omega'$, $0 \leq \omega' < 2\pi$, we have

$$\bar{f}_o(x,y, \cos\omega, \sin\omega) = f_o(x,y, \cos\omega, \sin\omega)$$

$$\bar{f}_{o*}(x,y, \cos\omega, \sin\omega) = f_{o*}(x,y, \cos\omega, \sin\omega) > 0$$

$$\bar{E}(x,y, \cos\omega, \sin\omega, \cos\omega', \sin\omega') > 0$$

The first two relations are obvious, the last one is due to Schwarz formula and the fact that $\bar{f}_{ox} \geq 0$ in all of the interval $(\omega,\omega')$, and $\bar{f}_{ox} > 0$ in at least a neighborhood of $\omega$. Thus, if we choose any two points $Q_1$, $Q_2$ on $C_o$ sufficiently close to $\bar{M}$, one preceeding and one following $\bar{M}$ on $C_o$, and such that the chord $Q_1Q_2$ forms an angle sufficiently close to $\theta$, then there is an extremal arc $\gamma$ joining $Q_1Q_2$ on which $\bar{I}$ has a value which is less then or equal to $I[C_{Q_1Q_2}]$ and equality holds if and only if $C_{Q_1Q_2}$ coincides with $\gamma$, that is, itself an extremal arc. This must be the case, since $C_o$ is a minimizing curve for $\bar{I}$ in $\Omega$. Thus, each point $\bar{M}$ of $C_o$ with $y > 0$, $0 < \theta < \pi/4$, is interior point of an extremal arc $\gamma$ contained in $C_o$. By repeating this argument to the right and to the left of $\bar{M}$ we see that $C_o$ contains an arc $P_1'' P_2$ whose second end point is $P_2$ and whose first end point is $P_1''$ either on the x-axis, or on the y-axis.

We have already seen in (b) that $C_o$ cannot contain any segment of the x-axis. Thus $P_1''$ is on the y-axis. Note $P_1'' = (0,y_1'')$ cannot coincide with 0, since no extremal arc can join 0 to $P_2$.

The arc $OP_1''$ of $C_o$ cannot contain points $\bar{M} = (x,y)$ with $x > 0$ $0 < y < y_1''$, since then, by repeating the same argument we would obtain a new extremal arc from a new point $P_1''$ to a point $\bar{P}_2$ which should be below $P_2$, a contradiction. Thus the arc $OP_1''$ of $C_o$ is a segment of the y-axis. We have now proved that on $C_0$ we have always either $0 < \theta \leq \pi/4$, or $\theta = \pi/2$, and thus as observed, $C_0$ gives to $\bar{I}[C]$ its minimum value and consequently $C_0$ gives to $I[C]$ too its minimum value. The solution now found coincides obviously with the one obtained by the elementary method.

24.3. CURVES OF MINIMUM LENGTH ENCLOSING A GIVEN AREA

(a) Let $\Omega$ be the class of all closed oriented rectifiable path curves C of the xy-plane for which the integral

$$J(C) = \frac{1}{2} \int_C (xy'-x'y)dt = \alpha > 0$$

has a given fixed positive value $\alpha$. Let us determine a curve C in $\Omega$ for which the integral

$$I(C) = \int_C (x'^2+y'^2)^{1/2}dt$$

has its minimum value. In other words, we attempt to determine, among the curves C which "enclose" a given area $\alpha$, one which has minimum length.

Note that in $\Omega$ we have all possible curves, as above, namely even those which intersect themselves, and link points of the plane as many times as we want in one or the other sense. On the other hand, we assume for the curves C only representations C: $x = x(t)$, $y = y(t)$, $t_1 \leq t \leq t_2$, with $x(t)$ and $y(t)$ both AC in $[t_1,t_2]$. We have seen in (18.4) that it is not restrictive to assume both $x'(t)$, $y'(t)$ essentially bounded.

Certainly $\Omega$ is not empty, and for the minimum of $I[C]$ we may well consider only the subclass $\Omega_0$ of those curves C of $\Omega$ which have a length $I(C)$ less than or equal to a given number $L > 0$. Since neither J nor I are modified by a rigid displacement of C in the xy-plane it is not restrictive to assume that all curves C of $\Omega_o$ pass through a given point, say (0,0). Then the curves C of $\Omega_o$ are all interior to the full circle $A_o$ of center (0,0) and radius L. We can assume for each curve C of $\Omega_o$ a representation

$$C: \quad x = x(t), \quad y = y(t), \quad 0 \leq t \leq L,$$

with x(t), y(t) AC in [0,L], $x'^2+y'^2 = 1$, and $x(0) = y(0) = x(L) = y(L) = 0$. From Hilbert theorem, the lower semicontinuity of Jordan length, and properties of continuity of the integral I, we conclude that $\Omega$ is a complete class.

Now we have

$$f_o(x,y,x',y') = (x'^2+y'^2)^{1/2}$$

and hence

$$f* = (x'^2+y'^2)^{-3/2} > 0$$

$f_o > 0$, for all $(x,y) \in A_o$ and $(x',y') = (0,0)$. We have also

$$f_1(x,y,x',y') = \frac{1}{2}(xy'-x'y)$$

obviously of the form $Px'+Qy'$ with $P = -y/2$, $Q = x/2$, both continuous in $A_o$.

By existence theorem (20.1.ii) we conclude that there exists a curve $C_o$ in $\Omega$ for which I[C] has its minimum value. By (16.1.i) we know that this curve $C_o$ must satisfy the multiplier rule. In other words, there are two constants $\lambda_o$, $\lambda_1$ not both zero, such that, if

$$F(x,y,x',y') = \lambda_o(x'^2+y'^2)^{1/2}+(\lambda_1/2)(xy'-x'y),$$

then the Euler equations relative to F are satisfied with all their consequences. In particular, the Weierstrass form (21.2.ii) holds:

$$1/2 = (F_{x'y} - F_{xy'})/F_* ,$$

$$F_{x'} = \lambda_o x'(x'^2+y'^2)^{-1/2} - (\lambda_1 y)/2 \qquad F_{y'} = \lambda_o y'(x'^2+y'^2)^{-1/2} + (\lambda_1 x)/2,$$

$$F_{x'y} = -\lambda_1/2, \qquad F_{y'x} = \lambda_1/2, \qquad F_{x'y} - F_{y'x} = -\lambda_1,$$

$$F_* = \lambda_o f_{o*} = \lambda_o (x'^2+y'^2)^{-3/2},$$

and finally

$$1/2 = -\lambda_o/\lambda_1.$$

Thus $r = a$, and $C_o$ is a circle of radius a, and $J(C_o) = \pi a^2$. Thus $a = (\alpha/\pi)^{1/2}$, and $I[C_o] = 2\pi a = 2\pi(\alpha/\pi)^{1/2} = 2(\pi\alpha)^{1/2}$. We have also

$$I[C] \geq I[C_o]$$

and hence

$$(I(C))^2 \geq (I(C_o))^2 \geq 4\pi\alpha = 4\pi J[C]$$

for every curve C in $\Omega$, or

$$J(C) \leq \frac{1}{4\pi} (I(C))^2.$$

This is the isoperimetric inequality. We have proved above that = sign holds only for circles.

(b) As an exercise, we can obtain the same solution by the formalism for the classical Mayer problems. We attempt to determine two AC functions

$x(t)$, $y(t)$, $o \leq t \leq L$, satisfying the differential equation

$$x'^2+y'^2 = 1,$$

the boundary conditions $x(0) = x(L)$, $y(0) = y(L)$, the isoperimetric condition

$$J[C] = \frac{1}{2} \int_0^L (xy'-x'y)dt = \alpha$$

and L minimum.

If z and w denote auxiliary variables, we have a Mayer problem for the four variables z, x, y, w, the differential equations

$$z' = 1, \quad x'^2+y'^2-1 = 0, \quad w' - \frac{1}{2} (xy'-x'y) = 0,$$

the boundary conditions

$$z(0) = 0, \quad x(0) = x(L), \quad y(0) = y(L), \quad w(0) = 0, \quad w(L) = \alpha,$$

and so that $z(L)$ = minimum.

If $\lambda_o$, $\lambda_1$, $\lambda_2$ are three multipliers (functions) and

$$F = \lambda_o(z'-1)+\lambda_1(x'^2+y'^2-1)+\lambda_2(w'- \frac{1}{2} (xy'-x'y))-,$$

then we must satisfy the Euler equations

$$F_z = (d/dt)F_{z'}, \quad F_x = (d/dt)F_{x'}, \quad F_y = (d/dt)F_{y'}, \quad F_w = (d/dt)F_{w'}.$$

First and last equations yield $0 = (d/dt)\lambda_o$, $0 = (d/dt)\lambda_2$, and hence $\lambda_o = C_o$, $\lambda_2 = C_2$, constants. Second and third equations yield

$$\lambda_2(-y'/2) = (d/dt)(2\lambda_1 x'+\lambda_2 y/2),$$

$$\lambda_2(x'/2) = (d/dt)(2\lambda_1 y'-\lambda_2 x/2),$$

and by integration

$$-\lambda_2 y/2 = 2\lambda_1 x'+\lambda_2 y/2 + C_2' ,$$

$$\lambda_2 x/2 = 2\lambda_1 y'-\lambda_2 x/2 + C_2' ,$$

and successively

$$2\lambda_1 x' = -\lambda_2(y-C_2) \qquad 2\lambda_1 y' = \lambda_2(x-C_1)$$

or

$$2\lambda x' = -(y-C_2) \qquad 2\lambda y' = x-C_1$$

$$(x-C_1)dx+(y-C_2)dy = 0$$

$$(x-C_1)^2+(y-C_2)^2 = C^2$$

This formalism has yielded the same result we had obtained above rigorously.

(c) A nonparametric problem related to the previous one is the following: Determine an AC function $y(x)$, $x_1 \leq x \leq x_2$, satisfying the boundary conditions $y(x_1) = y_1$, $y(x_2) = y_2$, $(x_1,y_1,x_2,y_2$ fixed), the isoperimetric condition

$$J[x] = \int_{x_1}^{x_2} y dx = \alpha ,$$

and for which the integral

$$I[x] = \int_{x_1}^{x_2} (1+y'^2)^{1/2} dx$$

has its minimum value.

By multiplier rule we shall determine numbers $\lambda_o$, $\lambda_1$ so that, if

$$F = \lambda_o(1+y'^2)^{1/2}+\lambda_1 y,$$

then

$$F_y = (d/dt)F_{y'}$$

or

$$\lambda_1 = (d/dx)[\lambda_o y'(1+y'^2)^{-1/2}].$$

From here we obtain successively

$$(d/dx)[y'(1+y'^2)^{-1/2}] = 1/\lambda$$

$$y'(1+y'^2)^{-1/2} = (1/\lambda)/(x-C)$$

$$[\lambda^2-(x-C)^2]y'^2 = (x-C)^2$$

$$dy/dx = (x-C)[\lambda^2-(x-C)^2]^{-1/2}$$

$$y-d = [\lambda^2-(x-C)^2]^{1/2}$$

$$(x-C)^2+(y-d)^2 = \lambda^2.$$

We shall suppose that $\lambda$ is larger than the distance 12 of the points 1 and 2. Then the solution is one of the two arcs of circle through 1 and 2 of length L. The only requirement is that L is larger than 12.

(d) We discuss now the same problem as in (c) as a nonparametric Mayer problem. Let $w = w(t)$ be the area between the curve, the x-axis, and the straight lines parallel to the y-axis of abscissas $x_1$ and x. Then $dw = ydx$, or $w' = y$. We have now three variables z, y, w, three differential equations

$$z' - \sqrt{4+y'^2} = 0, \quad w'-y = 0,$$

boundary conditions

$$z(0) = 0, \quad y(x_1) = y_1, \quad y(x_2) = y_2, \quad w(x_1) = 0, \quad w(x_2) = \alpha,$$

and we attempt to determine the functions x, y, w so that $z(x_2)$ = minimum. If $\lambda_o$ is a constant, $\lambda_1$, $\lambda_2$ functions of t, and

$$F = \lambda_o(z' - \sqrt{1+y'^2}) + \lambda_1(w'-y),$$

then

$$F_z = (d/dx)F_{z'}, \quad F_y = (d/dx)F_{y'}, \quad F_w = (d/dx)F_{w'}.$$

First and last equations yield $0 = (d/dx)(\lambda_o)$, $0 = (d/dx)\lambda_1$, hence $\lambda_o = C_o$, $\lambda_1 = C_2$, constants. The second equation yields

$$-\lambda_1 = (d/dx)(-\lambda_o(1+y'^2)^{-1/2}y')$$

hence

$$y'(1+y'^2)^{-1/2} = 1/\lambda(x-C_1)$$

and, as before, $(x-C_1)^2+(y-C_2)^2 = \lambda^2$.