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PROBLEMS OF OPTIMIZATION

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1. PROBLEMS OF OPTIMIZATION

1.1 Problems of optimization

We shall be concerned with a real variable \( t \in E_1 \) (time, the independent variable), with a real vector variable \( x = (x^1, \ldots, x^n) \), \( x \in E_n \), \( n \geq 1 \), (state variable), and with a real vector variable \( u = (u^1, \ldots, u^m) \), \( u \in E_m \), \( m \geq 1 \), (control variable).

In a canonical, or Mayer problem we seek the minimum, or the maximum, of a functional

\[
I[x, u] = g(t_1, x(t_1), t_2, x(t_2)) \tag{1.1.1}
\]

in a suitable class \( \Omega \) of pairs of functions \( x(t) = (x^1, \ldots, x^n) \), \( u(t) = (u^1, \ldots, u^m) \), \( t_1 \leq t \leq t_2 \), satisfying a system of ordinary differential equations

\[
\frac{dx}{dt} = f(t, x(t), u(t)), \quad t \in [t_1, t_2] \text{ (a.e.)}, \tag{1.1.2}
\]

constraints

\[
(t, x(t)) \in A, \quad t \in [t_1, t_2] \tag{1.1.3}
\]

\[
u(t) \in U(t, x(t)), \quad t \in [t_1, t_2] \text{ (a.e.)}, \tag{1.1.4}
\]

and boundary conditions

\[
(t_1, x(t_1), t_2, x(t_2)) \in B. \tag{1.1.5}
\]
Here $A$ is a given subset of the tx-space $E_{n+1}$ (which may coincide with the whole space $E_{n+1}$), $B$ a given subset of the $t \times t \times x$-space $E_{n+2}$, $x_1 = (x_1^1, \ldots, x_1^n)$, $x_2 = (x_2^1, \ldots, x_2^n)$, and for every $(t, x) \in A$ a subset $U(t, x)$ of the u-space $E_m$ is given (which may actually be fixed, or coincide with the whole space $E_m$, or depend on $t$ only, or on $x$ only). If $M$ denotes the set of all $(t, x, u)$ with $(t, x) \in A$, $u \in U(t, x)$, then $f(t, x, u) = (f_1, \ldots, f_n)$ is a given function on $M$, while $g$ is a real scalar function on $B$. The differential system (1.1.2) can be written equivalently in the form

$$\frac{dx^i}{dt} = f_i(t, x(t), u(t)), \quad i = 1, \ldots, n.$$  

A pair $x(t) = (x_1^1, \ldots, x_1^n)$, $u(t) = (u_1^1, \ldots, u_m^m)$, $t_1 \leq t \leq t_2$, is said to be admissible provided $x$ is absolutely continuous (AC), $u$ is measurable, and relations (1.1.2-5) hold. Then $\Omega$ is a class of admissible pairs. If $(x, u)$ is an admissible pair, we say that $x$ is an (admissible) trajectory, and that $u$ is an (admissible) strategy, or control function. The point $\eta(x) = (t_1, x(t_1), t_2, x(t_2))$ is denoted as the ends of the trajectory. In Mayer problems, therefore, the functional has the form $I[x] = g(\eta(x))$.

Mayer problems in which we seek the minimum of the functional $I[x, u] = g = t_2 - t_1$ (or $g = t_2$ if $t_1$ is fixed) are said to be problems of minimum time. Mayer problems where $f(x, u)$ is independent of $t$, and $U$ is fixed, or depend on $x$ only, are said to be autonomous.

In a Lagrange problem we seek the minimum, or the maximum, of a functional

$$I[x, u] = \int_{t_1}^{t_2} f_0(t, x(t), u(t))dt$$  

(1.1.6)
in a class $\Omega$ of pairs $x(t) = (x^1, \ldots, x^n)$, $u(t) = (u^1, \ldots, u^m)$, $t_1 \leq t \leq t_2$, 

satisfying a system of ordinary differential equations (1.1.2) constraints

(1.1.3), (1.1.4), and boundary conditions (1.1.5). Here $f_0(t, x, u)$ is a scalar function defined on $M$, and $A$, $B$, $U$, $M$ are as before. A pair $x(t)$, $u(t)$, 

$t_1 \leq t \leq t_2$, is said to be admissible under the same requirements as before, 

and the additional one that $f_0(t, x(t), u(t))$ be $L$-integrable in $[t_1, t_2]$. 

For $f_0 = 1$, we have $I = t_2 - t_1$, and we deal again with a problem of minimum 

time. A Lagrange problem is said to be autonomous if $f_0(x, u)$, $f(x, u) = 

(f_1, \ldots, f_n)$ do not depend on $t$, and $U$ is fixed, or depend on $x$ only. 

In a \textit{Bolza} problem we seek the minimum or the maximum of a functional 

\begin{equation} 
I[x, u] = \int_{t_1}^{t_2} f_0(t, x(t), u(t)) dt + g(t_1, x(t_1), t_2, x(t_2)) \tag{1.1.7} 
\end{equation} 

in a class $\Omega$ of pairs $x(t) = (x^1, \ldots, x^n)$, $u(t) = (u^1, \ldots, u^m)$, $t_1 \leq t \leq t_2$, 

satisfying the same requirements as for a Lagrange problem. 

In each of the problems above isoperimetric constraints may be required 

of the form 

\begin{equation} 
J_s = \int_{t_1}^{t_2} h_s(t, x(t), u(t)) dt = C_s, \ s = 1, \ldots, N, \tag{1.1.8} 
\end{equation} 

for given functions $h_s(t, x, u)$ defined on $M$, and given constants $C_s$, $s = 1, \ldots, N$. 

We shall see below that these constraints can be rewritten so as to be included 

in the general requirements (1.1.2-5). 

In a \textit{free} problem we seek the minimum or the maximum of a functional 

\begin{equation} 
I[x] = \int_{t_1}^{t_2} f_0(t, x(t), x'(t)) dt \tag{1.1.9} 
\end{equation}
in a class $\Omega$ of function $x(t) = (x^1, \ldots, x^n)$, $t_1 \leq t \leq t_2$, satisfying constraints and boundary conditions.

\[(t, x(t)) \in A, \ t \in [t_1, t_2], \quad (1.1.10)\]

\[(t_1, x(t_1), t_2, x(t_2)) \in B. \quad (1.1.11)\]

As before, $A$ and $B$ are given subsets of $E_{n+1}$ and of $E_{2n+2}$, respectively, and $f_0(t, x, u)$ is a given scalar function on $M = A \times E_n$. A function \(x(t) = (x^1, \ldots, x^n), t_1 \leq t \leq t_2\), is said to be admissible provided $x$ is absolutely continuous (AC), relations (1.1.10), (1.1.11) are satisfied, and $f_0(t, x(t), x'(t))$ is $L$-integrable in $[t_1, t_2]$, where $x' = dx/dt$. We shall consider classes $\Omega$ of admissible functions $x$. Here, too, isoperimetric constraints may be required of the form

\[\int_s = \int_{t_1}^{t_2} h_s(t, x(t), x'(t))dt = C_s, \quad s = 1, \ldots, N, \quad (1.1.12)\]

for given functions $h_s(t, x, u)$ defined on $M = A \times E_n$, and given constants $C_s, s = 1, \ldots, N$. We shall see below that free problems, with or without isoperimetric constraints, can be written as particular Lagrange and Mayer problems.

1.2 Some relations among the problems above

Mayer, Lagrange, and Bolza problems are equivalent. On one hand, we see that Mayer and Lagrange problems can be thought of as particular Bolza problems with $f_0' = 0$ and $g = 0$, respectively. On the other hand, Lagrange and Bolza problems can be reduced to Mayer problems by the introduction of an additional variable $x^0$, the $(n+1)$-vector $\tilde{x} = (x^0, x) = (x^0, x^1, \ldots, x^n)$, and auxiliary
differential equation \( \frac{dx^0}{dt} = f^0 \) and boundary condition \( x^0(t_1) = 0 \). Then, Lagrange problems are reduced to the Mayer problem

\[
I[\tilde{x}, u] = x^0(t_2),
\]

\[
\frac{dx}{dt} = f(t, x(t), u(t)), \quad \frac{dx^0}{dt} = f^0(t, x(t), u(t)),
\]

\[
(t, \tilde{x}(t)) \in \tilde{A} = A \times E_1, \quad u(t) \in U(t, x(t)),
\]

\[
(t_1, u(t_1), t_2, x(t_2)) \in A, \quad x^0(t_1) = 0.
\]

Analogously, Bolza problems are reduced to the same Mayer problem with functional

\[
I[\tilde{x}, u] = x^0(t_2) + g(t_1, x(t_1), t_2, x(t_2)).
\]

Finally, Mayer problems can be reduced to Lagrange problems in the state variable \( \tilde{x} = (x^0, x^1, \ldots, x^n) \),

\[
I[x, u] = \int_{t_1}^{t_2} x^0(t) dt
\]

\[
\frac{dx}{dt} = f(t, x(t), u(t)), \quad \frac{dx^0}{dt} = 0,
\]

\[
(t, \tilde{x}(t)) \in \tilde{A} = A \times E_1, \quad u(t) \in U(t, x(t)),
\]

\[
(t_1, x(t_1), t_2, x(t_2)) \in B, \quad (t_2 - t_1)x^0(t_1) - g(t_1, x(t_1), t_2, x(t_2)) = 0.
\]

The equivalence of Mayer, Lagrange, and Bolza problems is thus proved.

Isoperimetric constraints (1.1.9) can be included in the Mayer, Lagrange, Bolza problems above by simply introducing additional variables \( x^{n+s} \), \( s = 1, \ldots, N \),
auxiliary differential equations and boundary conditions

$$\frac{dx^{n+s}}{dt} = h_s(t, x(t), u(t)),$$

$$x^{n+s}(t_1) = 0, x^{n+s}(t_2) = c_s, s = 1, \ldots, N.$$

Free problems can be written as Lagrange problems

$$I[x,u] = \int_{t_1}^{t_2} f_0(t, x(t), u(t)) dt,$$
$$dx/dt = u(t),$$
$$(t, x(t)) \in A, u(t) \in U = E_n,$$

$$(t_1, x(t_1), t_2, x(t_2)) \in B.$$

Thus, free problems are Lagrange problems with $m = n, f = u$, $U = E_n$ (thus, $f_i = u_i$, $i = 1, \ldots, n$). Possible isoperimetric constraints (1.1.10) can be written as above in the form (1.1.12) by the use of auxiliary variables $x^{n+s}, s = 1, \ldots, N$.

1.3 Examples

1. For instance, the simple problem of the path of minimum length between two sets $B_1$ and $B_2$ in $E_{2+n}$ is a free problem of the form

$$I[x] = \int_{t_1}^{t_2} (1 + |x'(t)|^2)^{1/2} dt, x(t) = (x^1, \ldots, x^n), t_1 \leq t \leq t_2,$$

$$(t_1, x(t_1)) \in B_1, (t_2, x(t_2)) \in B_2$$

with $B_1, B_2 \subset E_{n+1}$. As a Lagrange problem it becomes
\[ I[x, u] = \int_{t_1}^{t_2} (1 + |u(t)|^2)^{1/2} dt, \]
\[ \frac{dx}{dt} = u(t), \]
\[ (t_1, x(t_1)) \in B_1, (t_2, x(t_2)) \in B_2, u(t) \in U = E_n, \]
\[ x(t) = (x^1, \ldots, x^n), u(t) = (u^1, \ldots, u^n), t_1 \leq t \leq t_2. \]

As a Mayer problem it becomes

\[ I[x, y, u] = y(t_2), \]
\[ \frac{dx}{dt} = u(t), \frac{dy}{dt} = (1 + |u(t)|^2)^{1/2}, \]
\[ (t_1, x(t_1)) \in B_1, (t_2, x(t_2)) \in B_2, u(t) \in U = E_n, \]
\[ x(t) = (x^1, \ldots, x^n), y(t), u(t) = (u^1, \ldots, u^n), t_1 \leq t \leq t_2. \]

2. As another example, let us consider the Lagrange problem in \( E_n \)

\[ I[x, u] = \int_{t_1}^{t_2} (|x(t)|^2 + |u(t)|^2) dt \]
\[ \frac{dx}{dt} = Ax + Bu, u \in U, \]
\[ (t_1, x(t_1)) \in B_1, (t_2, x(t_2)) \in B_2, \]
\[ x(t) = (x^1, \ldots, x^n), u(t) = (u^1, \ldots, u^m), t_1 \leq l \leq t_2, \]

where \( A, B \) are constant matrices of the types \( n \times n, n \times m, U \) a fixed subset of \( E_m \), and \( B_1, B_2 \) subsets of \( E_{n+1} \). In Mayer form it becomes
\[ I[x, y, u] = y(t_2), \]

\[ \frac{dx}{dt} = Ax + Bu, \frac{dy}{dt} = |x(t)|^2 + |u(t)|^2, \]

\((t_1, x(t_1)) \in B_1, (t_2, x(t_2)) \in B_2, y(t_1) = 0,\]

\[ x(t) = (x^1, \ldots, x^n), y(t), u(t) = (u^1, \ldots, u^m), t_1 \leq t \leq t_2. \]

3. As a further example we consider the problem of the maximum of the integral

\[ I[x] = \int_{t_1}^{t_2} x(t)dt, \]

with constraints and boundary conditions

\[ x(t) \geq 0, \int_{t_1}^{t_2} (1 + x'^2(t))^{1/2} dt = L \]

\[ x(t_1) = 0, x(t_2) = 0, t_1 < t_2, \text{ fixed.} \]

The same problem can be written in the Mayer form

\[ I[x, y, t, u] = y(t_2), \]

\[ \frac{dx}{dt} = u, \frac{dy}{dt} = x, \frac{dz}{dt} = (1 + u^2(t))^{1/2}, \]

\[ x(t_1) = 0, x(t_2) = 0, y(t_1) = 0, z(t_1) = 0, z(t_2) = L, \]

\( t_1 < t_2 \text{ fixed, } u(t) \in U = E_1, \)

\[ (t, x(t)) \in A = [(t, x)| t_1 \leq t \leq t_2, x \geq 0]. \]
4. Let us consider finally the Bolza problem

\[ I[x,u] = x^2(t_1) + x^2(t_2) + \int_{t_1}^{t_2} x^2(t) \, dt \]

\[ \frac{dx}{dt} = u, \; u(t) \in U = [1 \leq u \leq 2], \; n = m = 1. \]

Its equivalent Mayer form is

\[ I[x,y,u] = x^2(t_1) + x^2(t_2) + y(t_2) \]

\[ \frac{dx}{dt} = u, \; \frac{dy}{dt} = x^2, \]

\[ y(t_1) = 0, \; u(t) \in U = [1 \leq u \leq 2], \; n = 2, \; m = 1. \]

An equivalent Lagrange problem is as follows:

\[ I[x,y,u] = \int_{t_1}^{t_2} (x^2(t) + y(t)) \, dt, \]

\[ \frac{dx}{dt} = u, \; \frac{dy}{dt} = 0 \]

\[ (t_2-t_1) \gamma(t_1) = x^2(t_1) + x^2(t_2), \; u(t) \in U = [1 \leq u \leq 2], \]

\[ n = 2, \; m = 1. \]