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STATEMENT OF PONTRYAGIN'S NECESSARY CONDITION FOR LAGRANGE PROBLEMS, HAMILTON-JACOBI EQUATION, AND FIELDS

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3. STATEMENT OF PONTRYAGIN'S NECESSARY CONDITION FOR LAGRANGE PROBLEMS, HAMILTON-JACOBI EQUATION, AND FIELDS

3.1 The statement

As we have shown in (1.1), Lagrange, Bolza, and Mayer problems can be essentially transformed one into the other very easily by suitable change of space variables. In particular, Lagrange and Bolza problems can be written in an equivalent Mayer form, and then Pontryagin's necessary condition (2.2.1) can be applied to the Mayer problem so obtained. Nevertheless, for Lagrange problems, there are some simplifications, which make it worthwhile to reformulate Pontryagin's necessary condition specifically for problems written in the Lagrange form.

We consider here a Lagrange problem concerning the minimum of the functional

$$I[x,u] = \int_{t_1}^{t_2} f_0(t,x(t),u(t))dt, \quad f_0 \text{ scalar,} \quad (3.1.1)$$

with $x(t) = (x^1, \ldots, x^n)$, $u(t) = (u^1, \ldots, u^m)$, satisfying the differential system

$$\frac{dx}{dt} = f(t,x(t),u(t)), \quad f = (f_1, \ldots, f_n), \quad (3.1.2)$$

boundary conditions $(t_1, x(t_1), t_2, x(t_2)) \in B \subset \mathbb{R}_{2n+2}^2$, and constraints $(t, x(t)) \in A$, $u(t) \in U$.

Here all $n+1$ scalar functions $f_i(t,x,u)$, $i = 0, 1, \ldots, n$, are assumed to be defined and continuous in the set $M$ of all $(t,x,u)$ with $(t,x) \in A$, $u \in U(t)$ (or $M = A \times U$ if $U$ is a fixed set), together with their partial derivatives
The class $\Omega$ of the admissible pairs $x(t), u(t), t_1 \leq t \leq t_2$, is defined as in (1.1), though we must now require explicitly that $f_i(t, x(t), u(t))$ be integrable in $[t_1, t_2]$. As in (2.1) we assume that $x(t), u(t), t_1 \leq t \leq t_2$, is an optimal pair and we need not repeat the assumptions (abcd) of (2.1) for the situation under consideration. In particular, the graph of the optimal trajectory $x$ is in the interior of $A$, $U$ is a fixed set (bounded or not), and the optimal strategy $u$ is bounded. (Alternatively, assumptions corresponding to $c', c''$, etc., can be made.)

We shall need now an $(n+1)$-vector $\tilde{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_n)$, and the Hamiltonian function

$$H(t, x, u, \tilde{\lambda}) = \lambda_0 f_0(t, x, u) + \lambda_1 f_1(t, x, u) + \ldots + \lambda_n f_n(t, x, u). \quad (3.1.3)$$

Also, we shall need the related function

$$M(t, x, \tilde{\lambda}) = \inf_{u \in U} H(t, x, u, \tilde{\lambda}). \quad (3.1.4)$$

Pontryagin's necessary condition for Lagrange problems take now the form below, which we shall deduce explicitly from (2.2.1) at the end of this section.

(Pl') There is a continuous (AC) vector function $\tilde{\lambda}(t) = (\lambda_0, \lambda_1, \ldots, \lambda_n)$, $t_1 \leq t \leq t_2$ (Pontryagin's multipliers) which is never zero in $[t_1, t_2]$, with $\lambda_0$ a constant in $[t_1, t_2]$, $\lambda_0 \geq 0$, such that

$$\frac{d\lambda_i}{dt} = -H_i(t, x(t), u(t), \tilde{\lambda}(t)), \quad i = 1, \ldots, n,$$

for $t$ in $[t_1, t_2]$ (a.e.).
(P2') For every fixed $t$ in $[t_1, t_2]$ (a.e.), the Hamiltonian $H(t, x(t), u(t), \tilde{\lambda}(t))$ as a function of $u$ only with $u$ in $U$ takes its minimum value in $U$ at $u = u(t)$, or

$$M(t, x(t), \tilde{\lambda}(t)) = H(t, x(t), u(t), \tilde{\lambda}(t)), \quad \text{for } t \text{ in } [t_1, t_2] \text{ (a.e.).}$$

(P3') The function $M(t) = M(t, x(t), \tilde{\lambda}(t))$ is continuous (AC) in $[t_1, t_2]$ and

$$\frac{dM}{dt} = \frac{dM(t, x(t), \tilde{\lambda}(t))}{dt} = -H_t(t, x(t), u(t), \tilde{\lambda}(t)),$$

for $t$ in $[t_1, t_2]$ (a.e.).

(P4') (Transversality relation):

$$-M(t_1)dt_1 + \sum_{j=1}^{n} \lambda_j(t_1)dx_j^{1} + M(t_2)dt_2 - \sum_{j=1}^{n} \lambda_j(t_2)dx_j^{2} = 0$$

for every vector $h \in B'$, or (2.1.3), that is,

$$\left[ M(t)dt - \sum_{j=1}^{n} \lambda_j(t)dx_j \right]^2_1 = 0. \quad (3.1.5)$$

The transversality relation is identically satisfied if $t_1, x_1, t_2, x_2$ are fixed, that is, for the boundary conditions which correspond to the case that "both end points and times are fixed" $(dt_1 = dx_1^{i} = dt_2 = dx_2^{i} = 0, i = 1, \ldots, n)$.

Here $x, u$ is an admissible pair itself so that the differential equations

$$\frac{dx^i}{dt} = f^i_1(t, x(t), u(t)), \quad i = 1, \ldots, n, \quad (3.1.6)$$

hold and these equations, together with (P1), yield the canonic equations.
\[
\frac{dx_i}{dt} = \frac{\partial H}{\partial x_i}, \quad \frac{d\lambda_i}{dt} = -\frac{\partial H}{\partial x_i}, \quad i = 1, \ldots, n. \tag{3.1.7}
\]

These same equations as well as (P3) can also be written in the explicit forms

\[
dx_i/\ dt = f_i(t,x(t),u(t)), \quad i = 1, \ldots, r,
\]
\[
d\lambda_i/\ dt = -\sum_{j=0}^{n} \lambda_j(t)f_j(t,x(t),u(t)), \quad i = 1, \ldots, n,
\]
\[
dM/\ dt = -\sum_{j=0}^{n} \lambda_j(t)f_j(t,x(t),u(t)), \quad t_1 \leq t \leq t_2.
\]

If we denote by \(x^0\) an auxiliary variable satisfying the differential equation \(dx^0/\ dt = f_0(t,x,u)\) and initial condition \(x^0(t_1) = 0\), then

\[
I[x,u] = \int_{t_1}^{t_2} f_0 dt = x^0(t_2) - x^0(t_1) = x^0(t_2),
\]

and the two equations

\[
\frac{dx^0}{dt} = \frac{\partial H}{\partial x_0}, \quad \frac{d\lambda^0}{dt} = -\frac{\partial H}{\partial x^0},
\]

can be added to the canonic equations (3.1.7), since \(\partial H/\partial x_0 = f_0\), and \(x_0^0 = 0\), and hence \(\lambda^0\) is a constant, as stated.

Finally, let us note that for autonomous problems, that is, for problems where \(f_0, f_1, \ldots, f_n\) do not depend on \(t\), but only on \(x\) and \(u\), we have

\[
f_{it} = \partial f_i/\partial t = 0. \quad \text{Hence, (P3') yields dM/\ dt = 0, and the function M(t) is a constant in [t_1, t_2].}
\]
In particular, for autonomous problems between two fixed points \(x_1 = (x^i_1, i = 1, \ldots, n)\) and \(x_2 = (x^i_2, i = 1, \ldots, n)\), in an undeterminate time, then we can take \(t_1 = 0\), \(dx^i_1 = dx^i_2 = 0, \ i = 1, \ldots, n\), \(dt_2\) arbitrary.
Hence, (P1') yields \( M(t_2)dt_2 = 0 \), or \( M(t_2) = 0 \), and \( M(t) \) is the constant zero in \([t_1, t_2]\). This occurs, for instance, in a problem of minimum time (between two fixed points), where \( f_0 = 1 \), and then \( I = t_2 \).

3.2 Derivation of properties (P1'-4) from (P1-4)

We shall now deduce (P1'-4) from the analogous relations (P1-4) of (2.2). To this purpose let us first transform the Lagrange problem above into a Mayer problem. As mentioned in (3.1), we introduce an auxiliary variable \( x^{n+1} \), the extra differential equation \( dx^{n+1}/dt = f_0(t, x(t), u(t)) \), and the extra boundary condition \( x^{n+1}(t_1) = 0 \). Thus

\[
I[x, u] = \int_{t_1}^{t_2} f_0 \, dt = x^{n+1}(t_2) - x^{n+1}(t_1) = x^{n+1}(t_2).
\]

We denote now by \( \tilde{x} \) the \((n+1)\)-vector \( \tilde{x} = (x^1, \ldots, x^n, x^{n+1}) \), and by \( \tilde{f}(t, x, u) \) the \((n+1)\)-vector function \( \tilde{f}(t, x, u) = (f_1, \ldots, f_n, f_0) \). We have now the Mayer problem of the minimum of the functional

\[
J[\tilde{x}, u] = g = x^{n+1}(t_2) = I[x, u],
\]

with differential equations

\[
\frac{dx^i}{dt} = f_i(t, x, u), \quad i = 1, \ldots, n, \quad \frac{dx^{n+1}}{dt} = f_0(t, x, u),
\]

boundary conditions \((t_1, x(t_1), t_2, x(t_2)) \in B, x^{n+1}(t_1) = 0\), and the same constraints as before \((t, x(t)) \in A, u(t) \in U(t, x(t))\), where \( \tilde{A} = A \times E_1 \). The boundary conditions can be expressed in the equivalent form \((t_1, \tilde{x}(t_1), t_2, \tilde{x}(t_2)) \in \tilde{B}\) where \( \tilde{B} \) is the closed subset of the space \( E_{2n+4} \) defined by
\[ B = B \times (x_1^{n+1} = 0) \times (x_2^{n+1} \in E_1) \], since we have assigned the fixed value 
\[ x_1^{n+1} = 0 \] for \( x^{n+1}(t_1) \), and we have left undetermined the value \( x_1^{n+1} = x_1^{n+1}(t_2) \).

Thus, in any case, we have \( dx_1^{n+1} = 0 \), and \( dx_2^{n+1} \) is arbitrary. In other words, the new set \( B' \) is the set \( B' = B \times (0) \times (E_1) \), since \( dx_2^{n+1} \) is arbitrary. According to (2.2), we need now an \((n+1)\)-vector \( \tilde{\lambda} = (\lambda_1, \ldots, \lambda_n, \lambda_{n+1}) \), the Hamiltonian function

\[ \tilde{H}(t, x, u, \tilde{\lambda}) = \lambda_1 f_1 + \ldots + \lambda_n f_n + \lambda_{n+1} f_0, \]

and the related function

\[ \tilde{M}(t, x, \tilde{\lambda}) = \inf_{u \in U} \tilde{H}(t, x, u, \tilde{\lambda}), \]

where we have written \( x \) instead of \( \tilde{x} \) since these functions do not depend on \( x^{n+1} \). Also, we shall need nonnegative constant \( \lambda_0 \geq 0 \).

We are now in a position to write explicitly statements \((P1-4)\) of (2.2.i), for the so obtained Mayer problem, where \( n+1 \) replaces \( n \) of (2.2.i). The following remarks are relevant. First, from \((P1)\) of (2.2.i), we obtain for the multiplier \( \lambda_{n+1} \) the equation \( d\lambda_{n+1}/dt = -H_{x^{n+1}} \), where now \( H_{x^{n+1}} = 0 \), hence \( d\lambda_{n+1}/dt = 0 \), and \( \lambda_{n+1}(t) \) is a constant in \([t_1, t_2]\). Secondly, let us write \((P4)\) for a very particular case of the new problem, namely, when

\[ dt_1 = dx_1^i = dt_2 = dx_2^i = 0, \quad i = 1, \ldots, n, \quad dx_1^{n+1} = 0, \quad \text{and} \quad dx_2^{n+1} \text{ is arbitrary}. \]

Here \( g \) is the function \( g(t_1, \tilde{x}, t_2, \tilde{x}) = x_2^{n+1} \), and hence \( g_{x_2^{n+1}} = \partial g/\partial x_2^{n+1} = 1 \), while all the other partial derivatives of \( g \) are zero. Then \((P4)\) reduces to the simple relation \([\lambda_0 - \lambda_{n+1}(t_2)]dx_2^{n+1} = 0 \) with \( dx_2^{n+1} \) arbitrary; hence \( \lambda_{n+1}(t_2) = \lambda_0 \), and since \( \lambda_{n+1}(t) \) is a constant we have \( \lambda_{n+1}(t) = \lambda_0 \), a
nonnegative constant in \([t_1, t_2]\). We can now identify the multiplier \(\lambda_{n+1}\) with the constant \(\lambda_0\). Then the \((n+1)\)-vector \(\lambda\) can be written in the form
\[
\tilde{\lambda} = (\lambda_1, \ldots, \lambda_n, \lambda_0),
\]
and \(\tilde{H}\) becomes \(\tilde{H} = \lambda_0 f_0 + \lambda_1 f_1 + \ldots + \lambda_n f_n\), that is, the Hamiltonian \(\tilde{H}\) takes the form (3.1.3). For this new problem, with the above notation, (P1-4) now become the statements (P'1-4) listed above. The auxiliary variable \(x^{n+1}\) should now be denoted by \(x^0\), as mentioned at the end of statements (P'1-4).

3.3 Examples of application of Pontryagin's necessary condition for Lagrange problems

(a) Exercise 1 of (2.3) can be written as the Lagrange problem of the minimum of the integral
\[
I[x, y, u] = \int_{t_1}^{t_2} dt = t_2 - t_1
\]
(where we can take \(t_1 = 0\)), with differential equations, constraints, and boundary conditions
\[
\frac{dx}{dt} = y, \quad \frac{dy}{dt} = u, \quad u \in U = [-1 \leq u \leq 1],
\]
\[
x(t_1) = a, \quad y(t_1) = b, \quad x(t_2) = 0, \quad y(t_2) = c.
\]
Then \(f_0 = 1, f_1 = y, f_2 = u\), and the Hamiltonian is \(H = \lambda_0 + \lambda_1 y + \lambda_2 u\), with \(\lambda_0\) a constant. The analysis is now the same as for exercise 1 of (2.3), and the optimal time \(t_2\) is the same.

Note that here the Hamiltonian is \(H = \lambda_0 + \lambda_1 y + \lambda_2 u\), with \(\lambda_0 \geq 0\) constant, and \(\lambda_1, \lambda_2\) satisfying \(d\lambda_1/dt = 0, d\lambda_2/dt = -\lambda_1\), hence \(\lambda_1 = \lambda_2\).
3.8

\[ \lambda_2 = -c_1 t + c_2, \quad c_1, c_2 \] constant, as for exercise 1 of (2.3). Again here \( H_t = 0, \)
\( \frac{dM}{dt} = 0, \) and \( M \) is a constant. On the other hand, \( dt_1 = 0, \)
\( dx_1 = dy_1 = dy_2 = 0, \) arbitrary, and \( (P4') \) yields \( M(t_2)dt_2 = 0, \) or
\( M(t_2) = 0. \) Thus, \( M(t) \) is the constant zero in \([t_1, t_2], \) as stated in general
at the end of (3.1). We can take \( \lambda_0 = 1, \) and \( c_1, c_2 \) as stated in the Remark
after exercise 1 of (2.3).

(b) The problem of the curve of minimum length between two points \( 1 = (t_1, x_1), \)
\( 2 = (t_2, x_2), \) \( t_1 < t_2, \) in the tx-plane, was written in exercise 5 of (2.3) as
the Lagrange problem of the minimum of the functional

\[ I[x, u] = \int_{t_1}^{t_2} (1 + u^2(t))^{1/2} dt \]

with differential equation and boundary condition

\[ \frac{dx}{dt} = u, \quad x(t_1) = x_1, \quad x(t_2) = x_2. \]

Then \( f_0 = (1 + u^2)^{1/2}, \) \( f_1 = u, \) and the Hamiltonian is \( H(t, x, u, \lambda) = \)
\( \lambda_0 (1 + u^2)^{1/2} + \lambda_1 u, \) with \( \lambda_0 \geq 0 \) a constant, and \( \lambda_1 \) satisfying \( \frac{d\lambda_1}{dt} = 0, \)
hence \( \lambda_1 = c_1, \) a constant. Here \( H \) has a minimum as a function of \( u \) in
\( -\infty < u < +\infty, \) if \( \lambda_0 > |\lambda_1|, \) and for the minimum we must have
\( \lambda_0 u(1 + u^2)^{-1/2} + \lambda_1 = 0, \) where both \( \lambda_0 \) and \( \lambda_1 \) are constants. Thus, \( u \) is a
constant \( c \) in \([t_1, t_2], \) \( x \) is linear, and we obtain the usual segment \( s = 12, \) as
in exercise 5 of (2.3). Concerning the remaining consideration, again we have
\( H_t = 0, \) \( \frac{dM}{dt} = 0, \) and \( M(t) \) is a constant. On the other hand, \( t_1, x_1, t_2, x_2 \) are
fixed, and \( (P4') \) is identically satisfied. We can take \( \lambda_0 = 1 \) and
\( c_1 = -c(1 + c^2)^{-1/2} \) as in exercise 5 of (2.3).
3.4 Examples of transversality relations for Lagrange problems

We shall now apply transversality relation (P4') of Pontryagin's necessary condition for Lagrange problems (3.1) to a number of particular but rather typical cases. We restate here that $x(t), u(t), t_1 \leq t \leq t_2$, is an optimal pair, and that, therefore, by (P2'), (F3'),

$$M(t) = M(t, x(t), \tilde{\lambda}(t)) = H(t, x(t), u(t), \tilde{\lambda}(t))$$

$$= \lambda_0 f_0(t, x(t), u(t)) + \sum_{i=1}^{n} \lambda_i(t) f_i(t, x(t), u(t))$$

for $t$ in $[t_1, t_2]$ (a.e.). (3.4.1)

Also we mention here that, for autonomous problems, that is, $f_0(x, u), \ldots, f_n(x, u)$ depending on $x$ and $u$ only, $M(t)$ is constant in $[t_1, t_2]$.

(a) As mentioned in 3.1, in the case of both end points and times fixed, or $l = (t_1, x_1), 2 = (t_2, x_2)$ both fixed, transversality relation (P4') is trivial, $dt_1 = dx_1^i = dt_2 = dx_2^i = 0, i = 1, \ldots, n$.

(b) Let us consider the case concerning first end point $l = (t_1, x_1)$ fixed, second end point $2 = (t_2, x(t_2))$ on a given curve $\Gamma$: $x = b(t), t' \leq t \leq t''$, hence $x(t_2) = b(t_2)$, where $b(t) = (b_1, \ldots, b_n)$ (case (b) of 1.6). Let us assume that 2 is not an end point of $\Gamma$, that is, $t' < t_2 < t''$, and that $b$ is differentiable at $t_2$. By the same argument as in (2.4) we obtain the (finite) transversality relation.
\[ M(t_2) = \sum_{i=1}^{n} \lambda_i(t_2) b_i(t_2) = 0. \]  \hspace{1cm} (3.4.2)

(c) Let us consider the case concerning first end point \( l = (t_1, x_1) \) fixed, second end point \( x_2 \) fixed in \( E_n \) (target), to be reached at some undetermined time \( t_2 \geq t_1 \) (case (c) of (1.6)). By the same argument of (2.4) and the use of \( (P_4') \) (instead of \( (P_4) \)), we obtain the unique (finite) transversality relation

\[ M(t_2) = 0. \] \hspace{1cm} (3.4.3)

Thus, for autonomous problems, \( M(t) \) is the constant zero in \([t_1, t_2]\).

(d) Let us consider the case concerning first end point \( l = (t_1, x_1) \) fixed, second end point \( x_2 \) on a given set \( S \) in \( E_n \) (target), with \( S \) to be reached at some undetermined time \( t_2 \geq t_1 \) (case (d) of (1.6)). Let us consider first the case, say (d1), where \( S = E_n \) is the whole \( x \)-space, and assume \( t_2 > t_1 \). (In the illustration, we have taken \( n = 2 \).) The same argument as in (2.4), and the use of \( (P_4') \) (instead of \( (P_4) \)), yields the \( n+1 \) (finite) equations

\[ M(t_2) = 0, \quad \lambda_i(t_2) = 0, \quad i = 1, \ldots, n. \] \hspace{1cm} (3.4.4)

Let us now consider the case, say (d2), where \( S \) is a given curve in \( E_n \), say \( S: x = b(\tau), \tau' \leq \tau \leq \tau'', \tau \) a parameter, \( b(\tau) = (b_1(\tau), \ldots, b_n(\tau)) \), and
assume that \( x \) hits the target \( S \) at a
time \( t_2 > t_1 \) (as above), and at a point
\( x_2 \) which is not an end point of \( S \), pre-
cisely \( x_2 = x(t_2) = b(\tau_0) \) for some \( \tau_0 \),
\( \tau' < \tau_0 < \tau'' \). The same argument as in (2.4) yields the two (finite) equations

\[
M(t_2) = 0, \quad \sum_{j=1}^{n} \lambda_j(t_2) b'_j(\tau) = 0. \tag{3.4.5}
\]

The second relation states that the vector \( \lambda_i(t_2) \), \( i = 1, \ldots, n \), is orthogonal
at 2 to the line \( l \) which is tangent to the curve \( S \) at the same point.

Let us consider now the case, say (d3), where \( S \) is a given \( k \)-dimensional
manifold in \( F_n \). If we assume that \( S \) is a given parametrically as before, say
\( S: \quad x = b(\tau), \quad b(\tau) = (b_1, \ldots, b_n), \quad \tau = (\tau_1, \ldots, \tau_k), \) then \( x_2 = x(t_2) = b(\tau_0) \) for
some \( \tau_0 = (\tau_{10}', \ldots, \tau_{k0}') \). We assume that \( b(\tau) \) is defined in a neighborhood \( V \)
of \( \tau_0 \) and that \( b(\tau) \) is of class \( C' \) in \( V \). The same reasoning as in (2.4)
yields the \( k+1 \) (finite) equations

\[
M(t_2) = 0, \quad \sum_{j=1}^{n} \lambda_j(t_2) b'_j(\tau_0) = 0, \quad s = 1, \ldots, k. \tag{3.4.6}
\]

Again the last \( k \) relations state that the \( n \)-vector \( \lambda_i(t_2) \), \( i = 1, \ldots, n \), is
orthogonal at 2 to the tangent plane \( l \) to the manifold \( S \) at the same point.

If \( S \) is given by \( n-k \) equations \( \beta_\sigma(x) = 0, \quad \sigma = 1, \ldots, k \), and then
\( dx_2 = (dx_2^1, \ldots, dx_2^n) \) is any vector satisfying

\[
\sum_j \beta_{\sigma x^j}(x_2) \ dx_2^j = 0, \quad \sigma = 1, \ldots, n-k, \tag{3.4.7}
\]

then, by the same reasoning as in (2.4), we see that (P4') yields \( M(t_2) = 0, \)
as well as
\[ \sum_{j=1}^{n} \lambda_j(t_2) dx_j^i = 0 \]  \hspace{1cm} (3.4.8)
for all \( dx_2 = (dx_2^1, \ldots, dx_2^n) \) satisfying (3.4.7). Again the latter restates that the \( n \)-vector \( \lambda_i(t_2) \), \( i = 1, \ldots, n \), is orthogonal to 2 to the plane \( i \) which is tangent to \( S \) at the same point. For autonomous problems, in all these cases (d1,2,3), \( M(t) \) is the constant zero in \([t_1, t_2]\).

(e) Let us consider the case concerning first end point \( l = (t_1, x_1) \) fixed, terminal time \( t_2 \) fixed, \( t_2 > t_1 \), and terminal point \( x_2 \) on a given set \( S \) in \( E_n \) (target) (case (c) of (1.6)). Let us consider first the case, say (e1), where \( S = E_n \) is the whole \( x \)-space. In other words, the target \( S \) is a fixed hyper-plane \( t = t_2 \) in the \( tx \)-space \( E_{n+1} \). In the illustration we have taken \( n = 2 \).

The same argument of (2.4) and the use of (P4') (instead of (P4)), yield the \( n \) (finite) equations
\[ \lambda_i(t_2) = 0, \quad i = 1, \ldots, n. \]

Let us consider the case, say (e2), where \( S \) is a given curve in \( E_n \), say \( S: \ x = b(\tau), \ \tau' \leq \tau \leq \tau'', \ \tau \ a \ parameter, \ b(\tau) = (b^{(\tau)}_1, \ldots, b^{(\tau)}_n), \) and assume that \( x \) hits the target \( S \) (at fixed time \( t_2 \)) at a point \( x_2 \) which is not an end point of \( S \), precisely, \( x_2 = x(t_2) = b(\tau_0) \) for some \( \tau_0, \ \tau' < \tau_0 < \tau''. \) The same
argument of (2.4) and \((P4')\) (instead of \((P4)\)) yield the only (finite) equation

\[
\sum_{j=1}^{n} \lambda_j(t_2) b_j^i(\tau_0) = 0, \tag{3.4.9}
\]

and this relation states that the n-vector \(\lambda_i(t_2)\), \(i = 1, \ldots, n\), is orthogonal at 2 to the line which is tangent to the curve S at the same point 2.

Let us consider now the case, say \((e3)\), where S is a given k-dimensional manifold in \(E_n\). If we assume that S is given parametrically as before, say \(S: x = b(\tau), \ b(\tau) = (b_1(\tau), \ldots, b_n(\tau)), \ \tau = (\tau_1, \ldots, \tau_k)\), then \(x_2 = x(t_2) = b(\tau_0)\) for some \(\tau_0 = (\tau_{10}, \ldots, \tau_{k0})\). We shall assume that \(b(\tau)\) is defined in a neighborhood \(V\) of \(\tau_0\) and that \(b(\tau)\) is of class \(C^1\) in \(V\). The same argument as in (2.4) and \((P4')\) (instead of \((P4)\)) yield the \(k\) (finite) equations

\[
\sum_{j=1}^{n} \lambda_j(t_2) b_j^s(\tau_0) = 0, \quad s = 1, \ldots, k. \tag{3.4.10}
\]

Again these equations state that the n-vector \(\lambda_i(t_2)\), \(i = 1, \ldots, n\), is orthogonal at 2 to the tangent plane \(l\) which is tangent to S at the same point.

Finally, if S is given by \(n-k\) equations \(\beta_0(x) = 0, \ \sigma = 1, \ldots, n-k\), and then \(dx_2 = (dx_2^1, \ldots, dx_2^n)\) is any n-vector satisfying equations (3.4.7), then the same argument as in (2.4) and \((P4')\) (instead of \((P4)\)) yield that the relation

\[
\sum_{j=1}^{n} \lambda_j(t_2) dx_2^j = 0 \tag{3.4.11}
\]

hold for every \(dx_2 = (dx_2^1, \ldots, dx_2^n)\) satisfying (3.4.7). Again this is a re-statement that the n-vector \(\lambda_j(x_2)\), \(j = 1, \ldots, n\), is orthogonal at 2 to the tangent plane \(l\) to S at the same point.
We finally remark that the assumption that the pair \( x, u \) be optimal, that is, \( I[x, u] \) has an absolute minimum at \( x, u \) in \( \Omega \), can be replaced by the assumption that the pair \( x, u \) is a strong relative minimum, that is, \( I[x, u] \leq I[\tilde{x}, \tilde{u}] \) for all pairs \( \tilde{x}, \tilde{u} \) whose trajectory \( \tilde{x} \) satisfies the same boundary conditions as \( x \) and lies in any given small neighborhood \( N_{\delta} \) of \( x \).

### 3.5 Hamilton-Jacobi partial differential equation

We shall consider here a domain \( V \) of the \( tx \)-space, \( V \subset A \), and \( V \) may be smaller than the original subset \( A \). We shall consider the Lagrange problem of the minimum of the functional

\[
I[x, u] = \int_{t_1}^{t_2} f_0(t, x(t), u(t)) dt
\]

with side conditions and constraints

\[
dx/dt = f(t, x(t), u(t)), \quad f = (f_1, \ldots, f_n),
\]

\[
(t, x(t)) \in V, \quad u(t) \in U,
\]

in the class \( \Omega \) of such pairs \( x(t), u(t), t_1 \leq t \leq t_2 \), transferring any given point \( l = (t_1, x_1) \) to a given target \( S \). If the minimum of \( I[x, u] \) in \( \Omega \) exists, and \( x_0(t), u_0(t), t_1 \leq t \leq t_2 \), is any optimal pair, then

\[
\omega(t_1, x_1) = I[x_0, u_0]
\]

\[
= \min_{(x, u) \in \Omega} I[x, u]
\]

does not depend upon the particular optimal pair \( x_0, u_0 \). Instead, we shall consider here what happens when we vary the initial point \( (t_1, x_1) \).
First we shall consider all points \((t_1, x_1)\) for which the problem above has a minimum.

Let \(3 = (\tilde{t}, \tilde{x})\) be any point of an optimal trajectory \(x_0(t), u_0(t), \)
\(t_1 \leq t \leq t_2\), say \(\tilde{x} = x_0(\tilde{t}), t_1 \leq \tilde{t} < t_2\). Clearly the point \(3 = (\tilde{t}, \tilde{x})\) can be
transferred to \(S\) in \(V\), since the pair \(x_0(t), u_0(t), \tilde{t} \leq t \leq t_2\), that is, the
restriction of \(x_0'u_0\) to the subinterval \([\tilde{t}, t_2]\), evidently performs the task.
Thus, the class \(\bar{\mathfrak{F}}\) of all (admissible) pairs \(x, u\) transferring \(3 = (\tilde{t}, \tilde{x})\) to \(S\) is
not empty.

(3.5.1) (Property of optimality). Every point \((\tilde{t}, \tilde{x})\) of the trajectory \(x_0', \)
\(\tilde{x} = x_0(\tilde{t}), t_1 \leq \tilde{t} < t_2\), is transferred to \(S\) in \(V\) optimally by the pair \(x_0(t), u_0(t), \tilde{t} \leq t \leq t_2\).

Proof. Let \(l = (t_1, x_1), 2 = (t_2, x(t_2))\). If the statement were not true, then
there would be another admissible pair \(x_1(t), u_1(t), \tilde{t} \leq t \leq t_1', \)
transferring \(3 = (\tilde{t}, \tilde{x})\) to some point, say \(4 = (t_1', x_1(t_1'))\) on \(S\), with, briefly, \(I_{34} < I_{32}\).
Let us consider now the admissible pair \(X(t), v(t), t_1 \leq t \leq t_1', \)
defined by \(X = x_0, v = u_0\) on \([t_1, \tilde{t}], X = x_1', v = u_1\) on \([\tilde{t}, t_2']\). Then \(X, v\) transfers
\(l = (t_1, x_1)\) to \(4 = (t_1', x_1(t_1'))\) on \(S\) with \(I_{14} = I_{13} + I_{34} < I_{13} + I_{32} = I_{12}\), a contradiction,
since \(x_0'u_0\) transfers \(l\) to \(2\) on \(S\) optimally. This proves (3.5.1).

We shall write here the Hamiltonian for the Lagrange problem above as
defined by (3.1.3) with the constant multiplier \(\lambda_0\) equal to one, or

\[
H(t, x, u, \lambda) = f_0(t, x, u) + \sum_{i=1}^{n} \lambda_i f_i(t, x, u). \tag{3.5.4}
\]

Let \(x(t), u(t), t_1 \leq t \leq t_2\), be any admissible pair transferring the
point \(l = (t_1, x_1)\) to a target \(S\) in \(V\). Thus \(x_1 = x(t_1), (t, x(t)) \in V, u(t) \in U, \)
\(t_1 \leq t \leq t_2\). Let us consider any point \(3 = (t, x(t))\) of the trajectory \(x\), and
let us take \(3\) to \(S\) along the same trajectory \(x\), that is, by the admissible
pair \(x(\tau), u(\tau), t \leq \tau \leq t_2\). Then the corresponding value of the functional
(3.5.5) below is a function of \(t\) in \([t_1, t_2]\).
(3.5.4) If there exists a function $\omega(t,x)$ of class $C^1$ in $V$ such that

$$\omega(t,x(t)) = \int_t^{t_2} f_0(\tau,x(\tau),u(\tau))d\tau, \quad t_1 \leq t \leq t_2,$$

then

$$\omega_t(t,x(t)) + H(t,x(t),u(t),\omega_x(t,x(t))) = 0, \quad t_1 \leq t \leq t_2.$$  

(3.5.6)

**Proof.** Since $\omega(t,x)$ is of class $C^1$ in $V$ and $x,u$ is admissible, we have

$$d\omega(t,x(t))/dt = \omega_t(t,x(t)) + \sum_{i=1}^{n} \omega_{x^i}(t,x(t))dx^i/dt$$

$$= \omega_t(t,x(t)) + \sum_{i=1}^{n} \omega_{x^i}(t,x(t))f_i(t,x(t),u(t)).$$  

(3.5.7)

On the other hand, by force of (3.5.5), we have

$$d\omega(t,x(t))/dt = -f_0(t,x(t),u(t))$$

and by comparison with (3.5.7) we have

$$\omega_t(t,x(t)) + f_0(t,x(t),u(t)) + \sum_{i=1}^{n} \omega_{x^i}(t,x(t))f_i(t,x(t),u(t)) = 0.$$  

(3.5.8)

In view of (3.5.4), this relation is exactly relation (3.5.6). Here $\omega_t$, $\omega_{x^i}$, $i = 1, \ldots, n$, are the first order partial derivatives of $\omega$, and $\omega_x$ denotes the $n$-vector $(\omega_{x^1}, \ldots, \omega_{x^n})$.

We shall assume below that $V$ is simply covered by a family of optimal trajectories $x(t)$, $t_1 \leq t \leq t_2$, that is, for each point $(\tilde{t}, \tilde{x}) \in V$, there is a well determined pair $x(t), u(t)$, $t_1 \leq t \leq t_2$, with $x(\tilde{t}) = \tilde{x}$, $t_1 \leq \tilde{t} \leq t_2$.  

transferring optimally \((t_1, x_1 = x(t_1))\) to the target \(S\) with \((t_2, x(t_2)) \in S\), and hence by (3.5.1), transferring optimally \((\bar{t}, \bar{x})\) to \(S\). In this situation the values \(u\) of the strategies \(u(t)\) become functions of \((t, x)\) in \(V\) by means of the equations

\[
u(\bar{t}, \bar{x}) = u(\bar{t}), \quad \bar{x} = x(\bar{t}),
\]

\((\bar{t}, \bar{x}) \in V\).

Also, a function \(\omega(t, x)\) is defined in \(V\) by means of the equations

\[
\omega(\bar{t}, \bar{x}) = \int_{\bar{t}}^{t_2} f_0(\tau, x(\tau), u(\tau))d\tau, \quad \bar{x} = x(\bar{t}).
\]

Thus

\[
\omega(t, x(t)) = \int_{t}^{t_2} f_0(\tau, x(\tau), u(\tau))d\tau, \quad t_1 \leq t \leq t_2,
\]

for each trajectory \(x(t), t_1 \leq t \leq t_2\), of the family covering \(V\). Here \(\omega(\bar{t}, \bar{x})\) is the optimal cost in transferring \((\bar{t}, \bar{x})\) to \(S\) in \(V\).

(3.5.iii) If \(\omega(t, x)\) is of class \(C^2\) in \(V\), then \(\omega\) satisfies the Hamilton-Jacobi partial differential equation

\[
\omega_t(t, x) = -H(t, x, u(t, x), \omega_x(t, x)), \quad (t, x) \in V. \tag{3.5.9}
\]

Also, on each trajectory in \(V\) we have

\[
\left(\frac{d}{dt}\right) \omega_x(t, x(t)) = -f_{01}(t, x(t), u(t))
\]

\[
- \sum_{\alpha=1}^{n} x_\alpha(t, x(t)) f_{\alpha 1}(t, x(t), u(t)). \tag{3.5.10}
\]
In other words, \( \lambda_i(t) = \omega_i(t,x(t)), \ i = 1, \ldots, n, \ t_1 \leq t \leq t_2 \) are Pontryagin’s multipliers of the trajectory \( x(t), \ t_1 \leq t \leq t_2 \), in \( V \).

Proof. From (3.5.11) we know that (3.5.9) is satisfied everywhere in \( V \). For every point \((t, x(t)) \in V\) and any fixed \( v \in U\), let us consider the constant control function \( \tilde{u}(\tau), \ t \leq \tau \leq t + \Delta t\), with constant value \( \tilde{u}(\tau) = v\), and let \( \tilde{x}(\tau), \ t \leq \tau \leq t + \Delta t\), the corresponding trajectory with initial value \( \tilde{x}(t) = x(t) \). The trajectory \( \tilde{x} \) certainly exists and is uniquely defined, and lies in \( V \) provided \( \Delta t > 0 \) is sufficiently small. Then \( x = x(t) \) is taken by \( \tilde{u} \) to a point \( x(t) + \Delta x = \tilde{x}(t + \Delta t) \), with \( \Delta x = (\Delta x^1, \ldots, \Delta x^n) \), and

\[ \Delta x_i = f_i(t, x(t), v) \Delta t + \epsilon_i \Delta t, \ i = 1, \ldots, n, \]  

(3.5.11)

where \( \epsilon_i \to 0 \) as \( \Delta t \to 0 \). Indeed \( \tilde{x} \) is certainly continuously differentiable in \([t, t + \Delta t]\), and \( \Delta x/\Delta t \to \tilde{x}'(t) = f(t, x(t), v) \) as \( \Delta t \to 0 \). Thus

\[ \Delta x_i/\Delta t \to f_i(t, x(t), v), \ i = 1, \ldots, n, \]  

as \( \Delta t \to 0 \), and (3.5.11) follows. The corresponding cost in transferring \( x = x(t) \) to \( x(t) + \Delta x \) is then given by the integral (3.5.12) below. By the continuity of \( f_0 \) and \( \tilde{x} \), we have

\[
\int_t^{t+\Delta t} f_0(\tau, \tilde{x}(\tau), v) d\tau = f_0(t, x(t), v) \Delta t + \epsilon_0 \Delta t, 
\]

(3.5.12)

where \( \epsilon_0 \to 0 \) as \( \Delta t \to 0 \). Now the point \( x(t) + \Delta x \) can be transferred to the target \( S \) optimally with a cost \( \omega(t + \Delta t, x(t) + \Delta x) \), and by Taylor formula

\[
\Delta \omega = \omega(t + \Delta t, x(t) + \Delta x) - \omega(t, x(t)) \]

\[
= \omega_t(t, x(t)) \Delta t + \sum_{\alpha=1}^{n} \omega_x(t, x(t)) \Delta x^\alpha + \epsilon' \Delta t,
\]
where $\epsilon' \to 0$ as $\Delta t \to 0$. By force of (3.5.11), we have now

$$\omega = \left[ \omega_t(t,x(t)) + \sum_{\alpha=1}^{n} \omega_{x,\alpha}(t,x(t))f_{\alpha}(t,x(t),v) \right] \Delta t + \epsilon'' \Delta t \quad (3.5.13)$$

where $\epsilon'' \to 0$ as $\Delta t \to 0$. Now, by force of (3.5.1) the cost of transferring $(t,x(t))$ of $S$ via $(t+\Delta t, x(t)+\Delta x)$ is certainly $\geq \omega(t,x(t))$, or

$$f_{\alpha}(t,x(t),v) \Delta t + \epsilon_0 \Delta t + \omega(t+\Delta t, x(t)+\Delta x) \geq \omega(t,x(t)). \quad (3.5.14)$$

By the use of (3.5.13), we have now

$$f_{\alpha}(t,x(t),v) + \omega_t(t,x(t)) + \sum_{\alpha=1}^{n} \omega_{x,\alpha}(t,x(t))f_{\alpha}(t,x(t),v) \Delta t \geq (-\epsilon_0 - \epsilon'') \Delta t, \quad (3.5.15)$$

where $\epsilon_0, \epsilon'' \to 0$ as $\Delta t \to 0^+$. By dividing (3.5.15) by $\Delta t > 0$ and taking the limit as $\Delta t \to 0$, we obtain

$$f_{\alpha}(t,x(t),v) + \omega_t(t,x(t)) + \sum_{\alpha=1}^{n} \omega_{x,\alpha}(t,x(t))f_{\alpha}(t,x(t),v) \geq 0 \quad (3.5.16)$$

for every $v \in U$ and every $(t,x(t)) \in V$. Let $g(t,x,u)$ denote the function

$$g(t,x,u) = f_{\alpha}(t,x,u) + \omega_t(t,x) + \sum_{\alpha=1}^{n} \omega_{x,\alpha}(t,x)f_{\alpha}(t,x,u),$$

where $(t,x) \in V, u \in U$. By comparison with (3.5.8) and (3.5.15), we see that $g(t,x,u) \geq 0$ for all $(t,x) \in V, v \in U$, and that $g(t,x(t),u(t)) = 0$. This can be interpreted by saying that, for $u = u(t)$, the function $g(t,x(u(t))$ has a minimum at $x = x(t)$. Since $(t,x)$ is an interior point of $V$ and $g$ possesses first order partial derivatives $\frac{\partial g}{\partial x_i}(t,x,u)$, $i = 1, \ldots, n$, we conclude that
\[ g_{\chi i}(t, x(t), u(t)) = 0, \quad i = 1, \ldots, n, \]
or
\[ f_{0\chi i}(t, x(t), u(t)) + \omega \chi i(t, x(t)) + \sum_{\alpha=1}^{n} \omega_{\chi x} \alpha i(t, x(t)) + \sum_{\alpha=1}^{n} \omega_{\chi x} \alpha i(t, x(t), u(t)) = 0, \]
\[ i = 1, \ldots, n. \quad (3.5.17) \]

Here \( f_\alpha = dx^\alpha/dt \), \( \alpha = 1, \ldots, n \), hence the sum of the second and third terms in (3.5.17) is \( (d/dt)\omega_{\chi i}(t, x(t)) \), and (3.5.17) becomes

\[ (d/dt)\omega_{\chi i}(t, x(t)) = -f_{0\chi i}(t, x(t), u(t)) - \sum_{\alpha=1}^{n} \omega_{\chi x} \alpha i(t, x(t)) + \sum_{\alpha=1}^{n} \omega_{\chi x} \alpha i(t, x(t), u(t)), \quad i = 1, \ldots, n. \]

Statement (3.5.iii) is thereby proved.

**Examples.** (a) Let us consider the same problem of minimum time of example (a) of (3.3), that is, the same exercise 1 of (2.4) with the functional \( t_2 \) written in Lagrange form. We have seen in example (a) of (3.3) that the Hamiltonian is \( H = \lambda_0 + \lambda_1 x + \lambda_2 u \), where we can take \( \lambda_0 = 1, \lambda_1 = \omega_x, \lambda_2 = \omega_y \). Thus, equation (3.3.9) reduces to \( 0 = 1 + y \omega_x + \omega_y \). Comparing with exercise 1 of (2.4), we know that \( u = -1 \), or \( u = 1 \) according as \((x, y)\) is above or below the switch line AOB. Thus, the Hamilton-Jacobi partial differential equation above the switch line is

\[ \omega_t + y \omega_x - \omega_y + 1 = 0. \]
and below the switch line is

\[ \omega_t + y \omega_x + \omega_y + 1 = 0. \]

We can verify easily that the optimum time \( \omega(x,y) \) given by \( \omega(x,y) = y + 2(2^{-1/2} y^2 + x)^{1/2} \) for \((x,y)\) above AOB, and \( \omega(x,y) = -y + 2(2^{-1/2} y^2 - x)^{1/2} \) for \((x,y)\) below AOB, certainly satisfy the Hamilton-Jacobi equation.

(b) Let us consider now problem (b) of (3.3), that is, the path of minimum length between two fixed points \( 1 = (t_1, x_1) \), \( 2 = (t_2, x_2) \), \( t_1 < t_2 \). We have seen there that the Hamiltonian is \( H = \lambda_0 (1 + u^2)^{1/2} + \lambda_1 u \). Here we can take \( \lambda_0 = 1 \), \( \omega_x = \lambda_1 \) so that equation (3.3.9) becomes \( \omega_t + (1 + u^2)^{1/2} + \omega_u = 0 \).

On the other hand, we have seen in example (b) of (3.3) that the relation between \( u \) and \( \lambda_1 \) (for \( \lambda_0 = 1 \)) is \( u(1 + u^2)^{-1/2} + \lambda_1 = 0 \). Thus, \( u(1 + u^2)^{-1/2} + \omega_x = 0 \); hence, \( -1 < \omega_x < 1 \), \( \omega_x \) and \( u \) have opposite signs, and solving with respect to \( u \) we have

\[ u = -\omega_x (1 - \omega_x)^{-1/2}, \quad 1 + u^2 = (1 - \omega_x)^{-1}. \]

By substitution, we find \( \omega_t + (1 - \omega_x)^{1/2} = 0 \), and finally the Hamilton-Jacobi equation is

\[ \omega_t^2 + \omega_x^2 = 1. \]

It is easy to verify that, for instance, \( \omega(t,x) = ((t-a)^2 + (x-b)^2)^{1/2} \), for \( a,b \) fixed, certainly satisfies this partial differential equation.
3.6 **Fields**

We shall consider here a domain $V$ as above, and we assume that

(a) There exists a family $(x(t), u(t), t_1 \leq t \leq t_2)$ of optimal pairs, and corresponding Pontryagin multipliers $\lambda(t) = (\lambda_1, \ldots, \lambda_n)$, each pair $x, u$ of the family transferring its initial point $l = (t_1, x_1)$ to a given target $S$, with trajectories $x$ lying in $V$ and filling $V$ once.

By this we mean first of all that for each point $(\bar{t}, \bar{x}) \in V - S$ there exists one and only one pair $x(t), u(t), t_1 \leq t \leq t_2$, of the family with $t_1 \leq \bar{t} \leq t_2$, $\bar{x} = x(\bar{t})$. Also, we assume that each pair $x(t), u(t), t_1 \leq t \leq t_2$, being optimal, satisfies Pontryagin's necessary condition, with multipliers $\lambda(t)$, so that necessary conditions (P'1-4) of (3.1) are actually satisfied.

Under the assumption (a), then, for each point $(\bar{t}, \bar{x}) \in V - S$ there is a well determined system $x(t), u(t), \lambda(t), t_1 \leq t \leq t_2$, with $\bar{x} = x(\bar{t})$, $t_1 \leq \bar{t} \leq t_2$, and hence a well determined $m$-vector $u(\bar{t})$ which we shall denote by $u(\bar{t}, \bar{x})$, and a well determined $n$-vector $\lambda(\bar{t})$ which we shall denote by $\lambda(\bar{t}, \bar{x})$.

Thus, we have two functions $u(t, x), \lambda(t, x)$, $(t, x) \in V$. In other words, to each point $(t, x) \in V$ there is assigned a well determined control $u = u(t, x)$, and the Pontryagin multipliers $\lambda(t, x)$. Obviously, the two examples (a), (b) of (2.3) lead to such a situation. It is said that a control synthesis has been performed in $V$.

Very often, in particular in the two examples mentioned above, such a synthesis can be performed in the whole tx-space $E_{n+1}$ (or at least in the whole subset $A$ of this space in which the problem has been defined in the
first place), each pair \( x(t), u(t) \) being the unique optimal pair transferring each point \((t, x(t))\) to \( S \) optimally in \( E_{n+1} \) (or \( A \)). However, it may well occur that a synthesis can be performed (if any) only in a rather small part \( V \) of \( E_{n+1} \) (or \( A \)), and there is no guarantee that an analogous synthesis can be performed in some set \( V' \) larger than \( A \), or that the optimal trajectories \( x \) filling \( V \) are still optimal when we enlarge \( V \).

Note that, a synthesis having been performed in \( V \), the cost functional \( \omega(\bar{t}, \bar{x}) \) to transfer optimally \((\bar{t}, \bar{x})\) to \( S \) in \( V \) is given by

\[
\omega(\bar{t}, \bar{x}) = \int_{\bar{t}}^{t_2} f_0(t, x(t), u(t)) dt,
\]

where \( x(t), u(t), t_1 \leq t \leq t_2' \) is the only pair of the given family with \( t_1 \leq \bar{t} \leq t_2, \bar{x} = x(\bar{t}) \). Thus, we have now three functions \( u(t, x), \lambda(t, x), \omega(t, x) \) defined in \( V \).

Having performed a synthesis in the region \( V \) as above, we shall say that \( V \) is a **field** provided

(b) \( u(t, x) \) is of class \( C^1 \) in \( V \)-\( S \), and \( \lambda(t, x), \omega(t, x) \) are of class \( C^2 \) in \( V \), and \( \lambda(t, x) = \omega_x(t, x) \), or \( \lambda_i = \omega_{x_i} \), \( i = 1, \ldots, n \).

We shall also assume, as usual, that \( f_0(t, x, u), \ldots, f_n(t, x, u) \) are continuous functions in \((t, x, u)\), with continuous first order partial derivatives

\[
f_{jx_i}, \quad j = 0, 1, \ldots, n, \quad i = 1, \ldots, n.
\]

A first property of a field (under hypothesis (a)) follows from Pontryagin necessary conditions (P'1-4), and principle of optimality (3.5.1). Indeed,
each point \((\bar{z}, \bar{x}) \in V\) is transferred optimally to \(S\) in \(V\) by solving the differential system and initial condition

\[
dx/dt = f(t,x,u(t,x)), \quad x(\bar{t}) = \bar{x}.
\]

Indeed now, this system has a unique solution satisfying the initial condition \(x(\bar{t}) = \bar{x}\), and this solution must, therefore, coincide with the given trajectory \(x(t)\) since

\[
dx(t)/dt = f(t,x(t),u(t)), \quad u(t) = u(t,x(t)).
\]

Also the Pontryagin multipliers \(\lambda(t) = (\lambda_1, \ldots, \lambda_n)\) already associated with the optimal pair \(x,u\) satisfies the differential system

\[
d\lambda_i/dt = -\sum_j f_{jx_i}(t,x,u(t,v)), \quad \lambda(\bar{t}) = \lambda(\bar{t}, \bar{x}).
\]

Indeed, now this system has a unique solution satisfying the initial condition \(\lambda(\bar{t}) = \lambda(\bar{t}, \bar{x})\), and this solution must, therefore, coincide with the given system \(\lambda(t)\) of Pontryagin multipliers since

\[
d\lambda(t)/dt = -\sum_j f_{jx_i}(t,x(t),u(t)), \quad u(t) = u(t,x(t)).
\]

By Pontryagin's necessary condition (P3'), we have now

\[
H(t,x,u(t,x),\lambda(t,x)) = \min_{u \in U} H(t,x,u,\lambda(t,x)), \quad (t,x) \in V,
\]

or, by force of (b), also

\[
H(t,x,u(t,x),\omega_x(t,x)) = \min_{u \in U} H(t,x,u,\omega_x(t,x)), \quad (t,x) \in V - S.
\]
Also, by force of (3.5), we know that \( \omega(t,x) \) satisfies the Hamilton-Jacobi equation

\[ \omega_t(t,x) + H(t,x,u(t),x) = 0, \quad (t,x) \in V-S. \]

**Problems**

1. We have seen that the Hamilton-Jacobi partial differential equation of the integral length \( \int (1 + x^2)^{1/2} \, dt \) is

\[ v_t^2 + v_x^2 = 1. \]

(a) Verify this equation for a field of parallel straight lines.

(b) Verify this equation for a field of straight lines converging to a point.

2. Write exercise 2 of (2.3) as a Lagrange problem \( f_0 = 1 \), and write the corresponding Hamilton-Jacobi partial differential equation.

3. Same as number 2 for exercise 3 of (2.3).

3.7 **A sufficient condition for Lagrange problems**

We shall consider here a domain \( V \) of the \( tx \)-space \( E_n \), and the Lagrange problem of the minimum of the functional

\[ I[x,u] = \int_{t_1}^{t_2} f_0(t,x(t),u(t)) \, dt, \]

with side conditions and constraints

\[ \frac{dx}{dt} = f(t,x(t),u(t)), \quad f = (f_1, \ldots, f_n), \]

\[ (t,x(t)) \in V, \quad u(t) \in U, \]
in the class \( \Omega \) of pairs \( x(t), u(t) \), \( t_1 \leq t \leq t_2 \), transferring a given point \( l = (t_1, x_1) \in V \) to a target \( S \subset V \).

Let \( H(t,x,u,\lambda) \) be the Hamiltonian of the Lagrange problem above which we shall write with the multiplier \( \lambda_0 = 1 \):

\[
H(t,x,u,\lambda) = f_0(t,x,u) + \sum_{i=1}^{n} \lambda_i f_i(t,x,u).
\]

We shall assume that

(a) For each \( (t,x) \in V \) and \( \lambda \in E_n \), the function \( H(t,x,u,\lambda) \), as a function of \( u \) in \( U \), has a unique absolute minimum at \( u = \hat{u}(t,x,\lambda) \).

\[
H(t,x,\hat{u}(t,x,\lambda),\lambda) < H(t,x,u,\lambda)
\]

for all \( u \in U \), \( u \neq \hat{u}(t,x,\lambda) \), and for all \( (t,x) \in V \), \( \lambda \in E_n \).

(b) There exists a function \( \omega(t,x) \) of class \( C^1 \) in \( V \) satisfying the partial differential equation

\[
\omega_t(t,x) + H(t,x,\hat{u}(t,x,\omega_x(t,x)),\omega_x(t,x)) = 0,
\]

where \( \omega_t = \partial \omega / \partial t \), \( \omega_x = (\omega_{x_1}, \ldots, \omega_{x_n}) \) is the \( n \)-vector of the first order partial derivatives of \( \omega \) with respect to \( x_1, \ldots, x_n \); and

\[
\omega(t,x) = 0 \quad \text{for} \ (t,x) \in S.
\]

(c) A pair \( x_0(t), u_0(t), t_1 \leq t \leq t_2 \), in the class \( \Omega \), has the property
\[ u_0(t) = \tilde{u}(t, x_0(t), \omega_x(t, x_0(t))). \] (3.7.3)

\((3.7.1)\) Statement. Under hypotheses (a), (b), and (c) we have \( I[x_0, u_0] \leq I[x, u] \) for every pair \( x, u \) in \( \Omega \).

Proof. Let \( x(t), u(t), t_1 \leq t \leq t_3 \), be any pair in \( \Omega \). Thus \( x, u \), as well as \( x_0, u_0 \), transfer \( l = (t_1^1, x_1) \) to \( S \) in \( V \), hence

\[ x_1 = x_0(t_1) = x(t_1), \quad (t_2, x_0(t_2)) \in S, \quad (t_3, x(t_3)) \in S. \]

By hypothesis (b) we have

\[
\omega_t(t, x) + f_0(t, x, \tilde{u}(t, x, \omega_x(t, x))) + \sum_{i=1}^{n} \omega_i(t, x) f_i(t, x, \tilde{u}(t, x, \omega_x(t, x))) = 0 \tag{3.7.4}
\]

for all \((t, x) \in V\). By taking \( x = x_0(t) \) in (3.7.4) and integrating from \( t_1 \) to \( t_2 \), we have

\[
\int_{t_1}^{t_2} \left( f_0(t, x_0(t), \tilde{u}(t, x_0(t), \omega_x(t, x_0(t)))) + \omega_t(t, x_0(t)) \right) + \sum_{i=1}^{n} \omega_i(t, x_0(t)) f_i(t, x_0(t), \tilde{u}(t, x_0(t), \omega_x(t, x_0(t)))) dt = 0,
\]

where

\[ u_0(t) = \tilde{u}(t, x_0(t), \omega_x(t, x_0(t))), \]

\[ dx_0^i/dt = f_i(t, x_0(t), u_0(t)), \quad i = 1, \ldots, n, \]

and hence
\[ I[x_0^0, u_0^0] + \int_{t_1}^{t_2} \left[ \omega_t(t, x_0(t))dt + \sum_{i=1}^{n} \omega_i(t, x_i(t))dx_i \right] = 0. \]

The expression in brackets is now the differential of \( \omega \) taken along \( x_0(t) \).

Hence

\[ I[x_0^0, u_0^0] + \omega(t_2, x(t_2)) - \omega(t_1, x(t_1)) = 0, \]

and

\[ I[x_0^0, u_0^0] = \omega(t_1, x_1), \quad (3.7.5) \]

since \( \omega(t_2, x(t_2)) = 0 \) and \( x(t_1) = x_1 \).

By hypothesis (a) we have now for \((t, x) \in V \) and \( \lambda = \omega_x(t, x) \),

\[ \omega_t(t, x) + f_0(t, x, u) + \sum_{i=1}^{n} \omega_i(t, x)f_i(t, x, u) \geq \omega_t(t, x) \]

\[ + f_0(t, x, \tilde{u}(t, x, \omega_x(t, x))) + \sum_{i=1}^{n} \omega_i(t, x)f_i(t, x, \tilde{u}(t, x, \omega_x(t, x))) = 0, \quad (3.7.6) \]

where the second member is zero because of (b).

By taking \( x = x(t), u = u(t) \) in \((3.7.6)\) and integrating from \( t_1 \) to \( t_3 \), we have

\[ \int_{t_1}^{t_3} \left[ f_0(t, x(t), u(t)) + \omega_t(t, x(t)) \right. \]

\[ + \sum_{i=1}^{n} \omega_i(t, x(t))f_i(t, x(t), u(t)))dt \geq 0, \quad (3.7.7) \]

where \( dx_i = f_i(t, x(t), u(t)), \; i = 1, \ldots, n \), and hence

\[ I[x, u] + \int_{t_1}^{t_3} [\omega_t(t, x(t))dt + \sum_{i=1}^{n} \omega_i(t, x(t))dx_i] \geq 0. \]
The expression in brackets is the differential of \( \omega \) taken along \( x(t) \).

Hence

\[
I[x,u] + \omega(t_3,x(t_3)) - \omega(t_1,x(t_1)) \geq 0, \tag{3.7.8}
\]

\[
I[x,u] \geq \omega(t_1,x_1), \tag{3.7.9}
\]

since \( \omega(t_3,x(t_3)) = 0, x_1 = x(t_1) \). By comparison of (3.7.5) and (3.7.9), we obtain

\[
\omega(t_1,x_1) = I[x_0,u_0] \leq I[x,u].
\]

**Remark.** If \( V \) is the original domain of the given Lagrange problem, then statement (i) gives a global sufficiency condition.

**Remark.** In order to obtain global sufficiency conditions, or at least, in order to enlarge as much as possible the domain \( V \) in \( E_n \) on which statement (i) is valid, the following remark is important. Assume that \( V \) can be divided into finitely many nonoverlapping subdomains \( V_1, V_2, \ldots, V_N \) by means of smooth \( n \)-dimensional manifolds so that (1) \( \omega(t,x) \) is continuous in \( V \); (2) \( \omega(t,x) \) satisfies (3.7.9) in each of the subdomains \( V_s, s = 1, 2, \ldots, N \). Then statement (i) still holds. We can even assume that \( V \) is the union of countably many subdomains \( V_j, j = 1, 2, \ldots \), as above, under the additional hypothesis that any compact subset \( K \) in \( V \) intersects at most finitely many \( V_s \) (in other words, the subdomains \( V_s \) cluster at most on the boundary of \( V \) (or at infinity)).