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EXISTENCE THEOREMS IN MULTIDIMENSIONAL PROBLEMS OF OTPTIMIZATION WITH DISTRIBUTED AND BOUNDARY CONTROLS

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1. INTRODUCTION

In the present paper, we consider multidimensional nonlinear problems of optimization of the Lagrange type involving a cost functional expressed by means of integrals on a fixed domain G in Euclidian space E^{ν}, $\nu \geqq 1$, and on its boundary, ∂G, and also involving state equations—usually partial differential equations—in G and on ∂G, and controls both in G (distributed controls), and on ∂G (boundary controls), while our state variable x is an element of a Banach space S. The state equations, both in G and on ∂G, are written in abstract functional analysis form, and hence may represent partial differential equations or more general functional relations. The state equations, both in G and on ∂G, may be written in the so-called "strong" form, or in the "weak" form, as usual in partial differential equations theory. This paper extends to the present situation the method and ideas of previous papers by Cesari [3abc], and particularly of [3e].

Let G be a fixed bounded open set in E^{ν}, $\nu \geq 1$, and let Γ be a given closed subset of ∂G on which we have a hyperarea measure μ. Let S be a Banach space of elements x, and let $\mathcal{L}$, $\mathcal{M}$, $\mathcal{J}$, $\mathcal{K}$ be operator on S, not necessarily linear, with values in the following spaces:

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$$\mathcal{L}: \; S \to (L_1(G))^r, \qquad \mathcal{J}: \; S \to (L_1(\Gamma))^{r'},$$

$$\mathcal{M}: \; S \to (L_1(G))^s, \qquad \mathcal{K}: \; S \to (L_1(\Gamma))^{s'},$$

where r,s,r' and s' are given positive integers.

For every $t = (t^1,\dots,t^\nu)$ in the closure of G, let A(t) be a nonempty closed subset of the y-space E^s. Let A be the set of all (t,y) such that $t \in cl(G)$ and $y = (y^1,\dots,y^s) \in A(t)$. For every $(t,y) \in A$, let U(t,y) be a nonempty subset of the u-space E^m, $u = (u^1,\dots,u^m)$. We define analogous sets on a closed subset Γ of ∂G as follows. For every $t \in \Gamma$, let B(t) be a nonempty closed subset of the $\mathring{y}$-space $E^{s'}$, $\mathring{y} = (\mathring{y}^1,\dots,\mathring{y}^{s'})$. Let B be the set of all $(t,\mathring{y})$ with $t \in \Gamma$ and $\mathring{y} \in B(t)$. For every $(t,\mathring{y}) \in B$, let $V(t,\mathring{y})$ be a nonempty subset of the v'-space $E^{m'}$, $v = (v^1,\dots,v^{m'})$.

We consider the problem of finding an element x of S, a measurable control vector $u(t) = (u^1,\dots,u^m)$, $t \in G$, and a μ-measurable control vector $v(t) = (v^1,\dots,v^{m'})$, $t \in \Gamma$, so as to minimize the cost functional:

$$I[x,u,v] \;=\; \int_G f_0(t,(\mathcal{M}x)(t),\, u(t))dt + \int_\Gamma g_0(t,(\mathcal{K}x)(t),\, v(t))d\mu \tag{1.1}$$

subject to the state equations:

$$(\mathcal{L}x)(t) \;=\; f(t,(\mathcal{M}x)(t),\, u(t)) \quad \text{a.e. in } G, \tag{1.2}$$

$$(\mathcal{J}x)(t) \;=\; g(t,(\mathcal{K}x)(t),\, v(t)) \quad \mu\text{-a.e. in } \Gamma, \tag{1.3}$$

and the constraints:

$$(\mathcal{M}x)(t) \in A(t), \qquad u(t) \in U(t,(\mathcal{M}x)(t)) \quad \text{a.e. in } G, \tag{1.4}$$

$$(\mathcal{K}x)(t) \in B(t), \qquad v(t) \in V(t,(\mathcal{K}x)(t)) \quad \mu\text{-a.e. in } \Gamma. \tag{1.5}$$

Here, $u'(t)$ is said to be a distributed control, and $v'(t)$ a boundary control. State equation (1.2) usually represents a system of partial differential equations, and (1.3) usually represents boundary data, or boundary controls, but may well represent a system of partial differential equations and related constraints and controls on the boundary ∂G of G. State equations (1.2) and (1.3) are said to be written in the <u>strong</u> form. We shall consider in § 6 also the problem of minimizing the cost functional (1.1) when (1.2) and (1.3) are written in the corresponding <u>weak</u> form as is usual in the theory of partial differential equations. The corresponding results are framed in the present general theory with no extra effort.

2. PRELIMINARIES

In order to state our lower closure and existence theorems, we will use C. B. Morrey's definition of a regular transformation of class K from his paper [8a].

Let X and Y be subsets of a Euclidian space E^n. A transformation $x = x(y)$ of X onto Y is said to be of <u>class K</u> provided it is one to one and continuous, and the functions $x = x(y)$ and $y = y(x)$ satisfy uniform Lipschitz conditions on each compact subset of X and Y, respectively. In addition, the transformation is said to be <u>regular</u> if the functions $x(y)$ and $y(x)$ satisfy uniform Lipschitz conditions on the whole of X and Y, respectively.

By a region, we mean an open connected set. If G is a region in E^ν, ∂G denotes its boundary and $\mathrm{cl}G$ its closure.

In stating our theorems, we will use properties of set valued functions.

We now give the definitions of two well known properties of set valued functions. We shall use the notations of § 1. Also, given a point $(t_o,y_o) \in A$ and a number $\delta > 0$ we denote by $N_\delta(t_o,y_o)$ the set of all points $(t,y) \in A$ at a distance $\leq \delta$ from (t_o,y_o).

For every $(t,y) \in A$ let $Q(t,y)$ be a subset of the z-space E^{r+1}, $z = (z^o,\ldots,z^r)$. We say that the sets $Q(t,y)$ have Kuratowski's upper semicontinuity property [6], or property (U), at the point $(t_o,y_o) \in A$ provided

$$Q(t_o,y_o) = \bigcap_{\epsilon>0} \text{cl } Q(t_o,y_o,\epsilon),$$

where

$$Q(t_o,y_o,\epsilon) = \bigcup_{(t,y) \in N_\epsilon(t_o,y_o)} Q(t,y).$$

We say that the sets $Q(t,y)$ have Cesari's upper semicontinuity property [3a], or property (Q), at $(t_o,y_o) \in A$, provided

$$Q(t_o,y_o) = \bigcap_{\epsilon>0} \text{cl co } Q(t_o,y_o,\epsilon).$$

We say that the sets $Q(t,y)$ have property (U), or (Q), in A if they have this property at every point $(t_o,y_o) \in A$. Sets having property (U) are closed, and sets having property (Q) are closed and convex. It was found useful to introduce intermediate properties $Q(\rho)$, $0 \leq \rho \leq r+1$, of variable sets (D. E. Cowles [4a]).

Let ρ be any integer, $0 \leq \rho \leq r+1$. We say that the sets $Q(t,y)$ have property $Q(\rho)$ at the point $(t_o,y_o) \in A$ provided for every $z_o = (z_o^o, z_o^1,\ldots, z_o^{r+1}) \in E^{r+1}$,

$$Q(t_o,y_o) \cap \{z = (z^o,\ldots,z^r) \in E^{r+1} | z^i = z^i_o,\ i = \rho,\ldots,r\}$$

$$= \bigcap_{\epsilon>0} \bigcap_{\beta>0} \text{cl co}(Q(t_o,y_o,\epsilon) \cap \{z \in E^{r+1} |\ |z^i - z^i_o| \leqq \beta,\ i = \rho,\ldots,r\}).$$

For $\rho = r+1$ we understand that the sets in braces in the first and second members of this relation coincide with E_r. Also note that if the sets $Q(t,y)$ have property $Q(\rho)$ at the point $(t_o,y_o) \in A$, then for every $z_o = (z^o_o, z^1_o, \ldots, z^{r+1}_o) \in E^{r+1}$ the set

$$Q(t_o,y_o) \cap \{z \in E^{r+1} | z^i = z^i_o,\ i = \rho,\ldots,r\}$$

is closed and convex as intersection of sets having the same property. Thus, for any two points

$$z_1 = (z^o_1,\ldots,z^{\rho-1}_1,\ z^\rho_o,\ldots,z^r_o) \in Q(t_o,y_o)\ ,$$

$$z_2 = (z^o_2,\ldots,z^{\rho-1}_2,\ z^\rho_o,\ldots,z^r_o) \in Q(t_o,y_o)\ ,$$

the points $\alpha z_1 \in (1-\alpha)z_2$ also belong to $Q(t_o,y_o)$, $0 \leq \alpha \leq 1$. Sets possessing this (partial) convexity property will be said to be ρ-convex.

(2.i) For any integer ρ, $0 \leq \rho \leq r$, property $Q(\rho+1)$ implies property $Q(\rho)$. Also, property $Q(r+1)$ holds if and only if property (Q) holds, and property $Q(0)$ holds if and only if property (U) holds.

For a proof of this statement see (D. E. Cowles [4a]).

For every point $(t,y) \in A$, let $Q(t,y)$ be a subset of E^{r+1}, $r \geqq 0$. We say that the sets $Q(t,y)$ have the "upper set property" on A provided $(t,y) \in$

A, $z_o = (z_o^o, z_o^1, \ldots, z_o^r) \in Q(t,y)$, $\bar{z}_o = (\bar{z}_o^o, z_o^1, \ldots, z_o^r) \in E^{r+1}$ with $\bar{z}_o^o \geq z_o^o$ implies $\bar{z}_o \in Q(t,y)$.

(2.ii) If the sets $Q(t,y)$ have the upper set property on A and also the property (U), then the same sets have property $Q(1)$ on A.

For the proof of this statement see (D. E. Cowles [4a]).

3. A LOWER CLOSURE THEOREM

Let G be a bounded measurable subset of E^ν, $\nu \geq 1$, whose boundary will be denoted by ∂G.

Let Γ_j, $j = 1,\ldots,N$, be subsets of ∂G, each of which is the image under a regular transformation t_j of class K of a bounded interval R_j' of $E^{\nu-1}$. Let Γ be a closed subset of $\cup_{j=1}^N \Gamma_j$, and let μ be a measure defined on $\cup_{j=1}^N \Gamma_j$. For each $j = 1,\ldots,N$, we assume that if e is a subset of Γ_j, measurable with respect to μ, then $E = t_j^{-1}(e)$ is measurable with respect to Lebesgue $(\nu-1)$-dimensional measure $|\ |$ on R_j'. Also, we assume that the converse is true, so that measurable sets on Γ_j and R_j' correspond under t_j, $j = 1,\ldots,N$. Finally, we assume that there is a constant $K > 1$ such that if $e = t_j(E)$ is μ-measurable, then

$$K^{-1}|E| \leq \mu(e) \leq K|E| \tag{3.1}$$

independently of $j = 1,\ldots,N$, and e. Since μ induces a measure on each set R_j' via the transformation t_j, $j = 1,\ldots,N$, we may define $J_j(\tau)$, $\tau \in R_j'$, as the function in $L_1(R_j')$ which satisfy the relations

$$\mu(t_j(E)) = \int_E J_j(\tau)d\tau \qquad (3.2)$$

for every measurable subset E of R_j', $j = 1,\dots,N$.

For every $t \in \mathrm{cl}G$, let $A(t)$ be a nonempty closed subset of the y-space E^s, $y = (y^1,\dots,y^s)$. Let A be the set of all points (t,y) with $t \in \mathrm{cl}G$ and $y \in A(t)$. For every $(t,y) \in A$ let $U(t,y)$ be a nonempty subset of the u-space E^m, $u = (u^1,\dots,u^m)$. Let M be the set of all $(t,y,u) \in E^\nu \times E^s \times E^m$ such that $(t,y) \in A$ and $u \in U(t,y)$.

For every $t \in \Gamma$, let $B(t)$ be a nonempty closed subset of the $\mathring{y}$-space $E^{s'}$, $\mathring{y} = (\mathring{y}^1,\dots,\mathring{y}^{s'})$. Let B be the set of all $(t,\mathring{y})$ with $t \in \Gamma$ and $\mathring{y} \in B(t)$. For every $(t,\mathring{y}) \in B$, let $V(t,\mathring{y})$ be a nonempty closed subset of the v-space $E^{m'}$, $v = (v^1,\dots,v^{m'})$. Let $\mathring{M}$ be the set of all $(t,\mathring{y},v) \in E^\nu \times E^{s'} \times E^{m'}$ with $(t,\mathring{y}) \in B$ and $v \in V(t,\mathring{y})$.

Let $\tilde{f}(t,y,u) = (f_o,f) = (f_o,f_1,\dots,f_r)$ be a continuous $(r+1)$-vector function on M, and for every $(t,y) \in A$ let $\tilde{\tilde{Q}}(t,y)$ denote the set

$$\tilde{\tilde{Q}}(t,y) = \{\tilde{z} = (z^o,z) = (z^o,z^1,\dots,z^r) \in E^{r+1} \mid$$

$$z^o \geq f_o(t,y,u),\ z = f(t,y,u),\ u \in U(t,y)\}.$$

Let $\tilde{g}(t,\mathring{y},v) = (g_o,g) = (g_o,g_1,\dots,g_{r'})$ be a continuous $(r'+1)$-vector function on $\mathring{M}$, and for every $(t,\mathring{y}) \in B$, let $\tilde{\tilde{R}}(t,\mathring{y})$ denote the set

$$\tilde{\tilde{R}}(t,\mathring{y}) = \{\tilde{z} = (z^o,z) = (z^o,z^1,\dots,z^{r'}) \in E^{r'+1} \mid$$

$$z^o \geq g_o(t,\mathring{y},v),\ z = g(t,\mathring{y},v),\ v \in V(t,\mathring{y})\}.$$

We consider here the functional

$$I[y,\mathring{y},u,v] = \int_G f_o(t,y(t),u(t))dt + \int_\Gamma g_o(t,\mathring{y}(t),v(t))d\mu. \qquad (3.3)$$

In the lower closure theorem below we shall deal with sequences of functions all defined on G and Γ:

$$\begin{aligned}
z(t) &= (z^1,\dots,z^r), & z_k(t) &= (z_k^1,\dots,z_k^r),\\
y(t) &= (y^1,\dots,y^s), & y_k(t) &= (y_k^1,\dots,y_k^s),\\
u_k(t) &= (u_k^1,\dots,u_k^m), & & t \in G,\ k = 1,2,\dots,\\
\mathring{z}(t) &= (\mathring{z}^1,\dots,\mathring{z}^{r'}), & \mathring{z}_k(t) &= (\mathring{z}_k^1,\dots,\mathring{z}_k^{r'}),\\
\mathring{y}(t) &= (\mathring{y}^1,\dots,\mathring{y}^{s'}), & \mathring{y}_k(t) &= (\mathring{y}_k^1,\dots,\mathring{y}_k^{s'}),\\
v_k(t) &= (v_k^1,\dots,v_k^{m'}), & & t \in \Gamma,\ k = 1,2,\dots.
\end{aligned} \qquad (3.4)$$

(3.i) Theorem (a lower closure theorem). Let G be bounded and measurable, A, B, M, $\mathring{M}$ closed, $f_o(t,y,u)$, $f(t,y,u) = (f_1,\dots,f_r)$ continuous on M, $g_o(t,\mathring{y},v)$, $g(t,\mathring{y},v) = (g_1,\dots,g_{r'})$ continuous on $\mathring{M}$, and assume that for some integers ρ, ρ', $0 \le \rho \le r$, $0 \le \rho' \le r'$, the sets $\tilde{\tilde{Q}}(t,y)$ have property $Q(\rho+1)$ on A and the sets $\tilde{\tilde{R}}(t,\mathring{y})$ have property $Q(\rho'+1)$ on B. Let us assume that there are functions $\psi(t) \ge 0$, $t \in G$, $\psi \in L_1(G)$ and $\mathring{\psi}(t) \ge 0$, $t \in \Gamma$, $\mathring{\psi} \in L_1(\Gamma)$, such that $f_o(t,y,u) \ge -\psi(t)$ for all $(t,y,u) \in M$, and $g_o(t,\mathring{y},v) \ge -\mathring{\psi}(t)$ for all $(t,\mathring{y},v) \in \mathring{M}$. Let us assume that the functions $z^i(t)$, $z_k^i(t)$, $y^j(t)$, $y_k^j(t)$, $i = 1,\dots,r$, $j = 1,\dots,s$, are in $L_1(G)$, that the functions $u_k^\ell(t)$ are measurable on G, $\ell = 1,\dots,m$, that $f_o(t,y_k(t),u_k(t)) \in L_1(G)$, and that

$$y_k(t) \in A(t),\ u_k(t) \in U(t,y_k(t)),\ z_k^i(t)$$

$$= f_i(t,y_k(t),u_k(t)) \quad \text{a.e. on } G,\ k = 1,2,\ldots. \tag{3.5}$$

Let us assume that the functions $\mathring{z}^i(t)$, $\mathring{z}_k^i(t)$, $\mathring{y}^j(t)$, $\mathring{y}_k^j(t)$, $i = 1,\ldots,r'$, $j = 1,\ldots,s'$, are in $L_1(\Gamma)$, that the functions $v_k^\ell(t)$ are measurable in Γ, $\ell = 1,\ldots,m'$, that $g_o(t,\mathring{y}_k(t),v_k(t)) \in L_1(\Gamma)$, and that

$$\mathring{y}_k(t) \in B(t),\ v_k(t) \in V(t,\mathring{y}_k(t)),\ \mathring{z}_k^i(t)$$

$$= g_i(t,\mathring{y}_k(t),v_k(t)) \quad \mu\text{-a.e. on } \Gamma,\ k = 1,2,\ldots. \tag{3.6}$$

Finally, let us assume that as $k \to \infty$ we have

$$\begin{aligned}
&z_k^i(t) \to z^i(t) \text{ weakly in } L_1(G),\ i = 1,\ldots,\rho,\\
&z_k^i(t) \to z^i(t) \text{ strongly in } L_1(G),\ i = \rho+1,\ldots,r,\\
&y_k^j(t) \to y^j(t) \text{ strongly in } L_1(G),\ j = 1,\ldots,s,\\
&\mathring{z}_k^i(t) \to \mathring{z}^i(t) \text{ weakly in } L_1(\Gamma),\ i = 1,\ldots,\rho',\\
&\mathring{z}_k^i(t) \to \mathring{z}^i(t) \text{ strongly in } L_1(\Gamma),\ i = \rho'+1,\ldots,r',\\
&\mathring{y}_k^j(t) \to \mathring{y}^j(t) \text{ strongly in } L_1(\Gamma),\ j = 1,\ldots,s',\\
&\underline{\lim}_{k\to\infty} I[y_k,\mathring{y}_k,u_k,v_k] \le a_o < +\infty.
\end{aligned}$$

Then, $y(t) \in A(t)$ a.e. on G, $\mathring{y}(t) \in B(t)$ μ-a.e. on Γ, and there are measurable functions $u(t) = (u^1,\ldots,u^m)$, $t \in G$, and μ-measurable functions $v(t) = (v^1,\ldots,v^{m'})$, $t \in \Gamma$, such that $f_o(t,y(t),u(t)) \in L_1(G)$, $g_o(t,\mathring{y}(t),v(t)) \in L_1(\Gamma)$, and such that

$$u(t) \in U(t,y(t)), z^i(t) = f_i(t,y(t),u(t)),\ i = 1,\dots,r,\ \text{a.e. on } G,$$

$$v(t) \in V)t,\mathring{y}(t)), \mathring{z}^i(t) = g_i(t,\mathring{y}(t),v(t)),\ i = 1,\dots,r',\ \mu\text{-a.e. on } \Gamma,$$

$$I[y,\mathring{y},u,v] \leq a_o.$$

For the proof of this lower closure theorem see (L. Cesari [3e], D. E. Cowles [4b]).

Remark. In applications, it often occurs that the sets U and V are fixed and compact, or alternatively $U(t,y)$, $V(t,\mathring{y})$ are compact, equibounded, and have property (U) in A and B, respectively. Having assumed f_o, f, g_o, g continuous, then the sets $\tilde{\tilde{Q}}(t,y)$, $\tilde{\tilde{R}}(t,\mathring{y})$ certainly are compact and have property (U) in A and B, respectively. If convex, then they also have property (Q) (see [3a]). If ρ-convex, they have property $Q(\rho)$ (see [4a]).

4. AN EXISTENCE THEOREM FOR OPTIMIZATION PROBLEMS WITH STATE EQUATIONS IN THE STRONG FORM

In this section we should mainly use the same notations of § 3. For the sake of simplicity we shall denote by T the family of all measurable m-vector functions $u(t) = (u^1,\dots,u^m)$, $t \in G$, and by $\mathring{T}$ the family of all μ-measurable m'-vector functions $v(t) = (v^1,\dots,v^{m'})$, $t \in \Gamma$.

Let S be a Banach space of elements x with norm $\|x\|$, and let $\mathcal{L}$, $\mathcal{M}$, $\mathcal{J}$, $\mathcal{K}$ be operators, not necessarily linear, as described in § 1, or $\mathcal{L}$: $S \to (L_1(G))^r$, $\mathcal{M}$: $S \to (L_1(G))^s$, $\mathcal{J}$: $S \to (L_1(\Gamma))^{r'}$, $\mathcal{K}$: $S \to (L_1(\Gamma))^{s'}$. We shall discuss here the problem of optimization (1.1-5) of § 1.

A triple is said to be <u>admissible</u> (for the problem (1.1-5)) provided $x \in S$, $u \in T$, $v \in \mathring{T}$, $f_o(t,(\mathcal{M}x)(t),u(t)) \in L_1(G)$, $g_o(t,(\mathcal{K}x)(t),v(t)) \in L_1(\Gamma)$, and relations (1.2-5) hold.

A class Ω of admissible triples is said to be <u>closed</u> if the following occurs: if $(x_k,u_k,v_k) \in \Omega$, $k = 1,2,\dots$, $x_k \to x$ weakly in S as $k \to \infty$, $\underline{\lim}_{k\to\infty} I[x_k,u_k,v_k] \leq a < +\infty$, and there are admissible triples (x,u,v) such that $I[x,u,v] \leq a$, then there is also some triple $(x,\bar{u},\bar{v}) \in \Omega$, with $I[x,\bar{u},\bar{v}] \leq a$.

For a class Ω of admissible triples we denote by $\{x\}_\Omega$ the subset of S defined by $\{x\}_\Omega = \{x \in S \mid (x,u,v) \in \Omega \text{ for some } u \in T, v \in \mathring{T}\}$. Note that for $x \in S$, then $z(t) = (z^1,\dots,z^r) = (\mathcal{L}x)(t) \in (L_1(G))^r$ and we shall denote by $z^i(t) = (\mathcal{L}x)^i(t)$, $t \in G$, the i^{th} component of $\mathcal{L}x$. Analogously, $y(t) = (y^1,\dots,y^r) = (\mathcal{M}x)(t) \in (L_1(G))^s$ and we set $y^j(t) = (\mathcal{M}x)^j(t)$, $t \in G$, $j = 1,\dots,s$; $\mathring{z}(t) = (\mathring{z}^1,\dots,\mathring{z}^{r'}) = (\mathcal{J}x)(t) \in (L_1(\Gamma))^{r'}$ and we set $z^i(t) = (\mathcal{J}x)^i(t)$, $t \in \Gamma$, $i = 1,\dots,r'$; $\mathring{y}(t) = (\mathring{y}^1,\dots,\mathring{y}^{s'}) = (\mathcal{K}x)(t) \in (L_1(\Gamma))^{s'}$ and we set $\mathring{y}^j(t) = (\mathcal{K}x)^j(t)$, $t \in \Gamma$, $j = 1,\dots,s'$.

(4.i) <u>Existence Theorem</u>. Let G be bounded and measurable, A, B, M, $\mathring{M}$ closed, $f_o(t,y,u)$, $f(t,y,u) = (f_1,\dots,f_r)$ continuous on M, $g_o(t,\mathring{y},v)$, $g(t,\mathring{y},v) = (g_1,\dots,g_{r'})$ continuous on $\mathring{M}$, and assume that, for given integers ρ, ρ', $0 \leq \rho \leq r$, $0 \leq \rho' \leq r'$, the sets $\tilde{\tilde{Q}}(t,y)$ have property $Q(\rho+1)$ on A, and the sets $\tilde{\tilde{R}}(t,\mathring{y})$ have property $Q(\rho'+1)$ on B. Let us assume that there are functions $\psi(t) \geq 0$, $t \in G$, $\psi \in L_1(G)$, and $\mathring{\psi}(t) \geq 0$, $t \in \Gamma$, $\mathring{\psi} \in L_1(\Gamma)$, such that $f_o(t,y,u) \geq -\psi(t)$ for all $(t,y,u) \in M$, and $g_o(t,\mathring{y},v) \geq -\mathring{\psi}(t)$ for all $(t,\mathring{y},v) \in \mathring{M}$. Let us assume that, for every sequence x,x_k, $k = 1,2,\dots$, of elements of S with $x_k \to x$ weakly in S it occurs that there is some subsequence $[k_p]$ such that

$$\begin{aligned}
&(\mathcal{L}x_{k_p})^i \to (\mathcal{L}x)^i \text{ weakly in } L_1(G),\ i = 1,\dots,\rho,\\
&(\mathcal{L}x_{k_p})^i \to (\mathcal{L}x)^i \text{ strongly in } L_1(G),\ i = \rho+1,\dots,r,\\
&(\mathcal{M}x_{k_p})^j \to (\mathcal{M}x)^j \text{ strongly in } L_1(G),\ j = 1,\dots,s,\\
&(\mathcal{J}x_{k_p})^i \to (\mathcal{J}x)^i \text{ weakly in } L_1(\Gamma),\ i = 1,\dots,\rho',\\
&(\mathcal{J}x_{k_p})^i \to (\mathcal{J}x)^i \text{ strongly in } L_1(\Gamma),\ i = \rho'+1,\dots,r',\\
&(\mathcal{K}x_{k_p})^j \to (\mathcal{K}x)^j \text{ strongly in } L_1(\Gamma),\ j = 1,\dots,s',
\end{aligned} \tag{4.1}$$

as $p \to \infty$. Let Ω be a nonempty closed class of admissible triples (x,u,v) such that the set $\{x\}_\Omega$ is weakly sequentially relatively compact. Then, the functional (1.1), or $I[x,u,v]$, has an absolute minimum in Ω.

In view of statements (2.i) and (2.ii), note that, for $\rho = r$ we actually require above that the sets $\tilde{\tilde{Q}}(t,y)$ have property (Q), and for $\rho = 0$ we actually require that the sets $\tilde{\tilde{Q}}(t,y)$ have property (U). Analogously, for $\rho' = r'$ we actually require that the sets $\tilde{\tilde{R}}(t,\mathring{y})$ have property (Q), for $\rho' = 0$ we require that the sets $\tilde{\tilde{R}}(t,\mathring{y})$ have property (U). In general, for $0 \leq \rho \leq r$, $0 \leq \rho' \leq r'$, properties $Q(\rho+1)$ and $Q(\rho'+1)$ represent intermediate requirements.

<u>Proof</u>. Let i be the infimum of $I[x,u,v]$ in the class Ω. Then i is finite, and we consider a minimizing sequence for I in Ω, that is, a sequence

(x_k,u_k,v_k), $k = 1,2,\ldots$, of admissible triples, all in Ω, with $I[x_k,u_k,v_k] \to i$ as $k \to \infty$. Since the set $\{x\}_\Omega$ is weakly sequentially relatively compact, there is some element $x \in S$ and some subsequence of $[x_k]$ which is weakly convergent to x. For the sake of simplicity we denote such a sequence by $\{k]$, and thus $x_k \to x$ weakly in S. As a consequence, there is a subsequence $[k_p]$ for which convergence relations (4.1) hold. We shall denote this subsequence again by $[k]$. By using the notations

$$z_k(t) = (\mathcal{L}x_k)(t), \quad y_k(t) = (\mathcal{M}x_k)(t), \quad z(t) = (\mathcal{L}x)(t),$$

$$y(t) = (\mathcal{M}x)(t), \quad t \in G,$$

$$\mathring{z}_k(t) = (\mathcal{J}x_k)(t), \quad \mathring{y}_k(t) = (\mathcal{K}x_k)(t), \quad \mathring{z}(t) = (\mathcal{J}x)(t),$$

$$\mathring{y}(t) = (\mathcal{K}x)(t), \quad t \in \Gamma,$$

we see that relations (1.2-5) imply

$$y_k(t) \in A(t),\ u_k(t) \in U(t,y_k(t)),\ z_k^i(t)$$

$$= f_i(t,y_k(t),u_k(t)) \quad \text{a.e. in } G,\ i = 1,\ldots,r,$$

$$\mathring{y}_k(t) \in B(t),\ v_k(t) \in V(t,\mathring{y}_k(t)),\ z_k^i(t)$$

$$= g_i(t,\mathring{y}_k(t),v_k(t)) \quad \mu\text{-a.e. in } \Gamma,\ i = 1,\ldots,r',\ k = 1,2,\ldots.$$

In addition, $z_k^i \to z^i$ weakly in $L_1(G)$, $i = 1,\ldots,\rho$, $z_k^i \to z^i$ strongly in $L_1(G)$, $i = \rho+1,\ldots,r$, $y_k^j \to y^j$ strongly in $L_1(G)$, $j = 1,\ldots,s$, $\mathring{z}_k^i \to \mathring{z}^i$ weakly in $L_1(\Gamma)$, $i = 1,\ldots,\rho'$, $\mathring{z}_k^i \to \mathring{z}^i$ strongly in $L_1(\Gamma)$, $i = \rho'+1,\ldots,r'$, $\mathring{y}_k^j \to \mathring{y}^j$

strongly in $L_1(\Gamma)$, $j = 1,\ldots,s'$. Finally, the sets $\tilde{\tilde{Q}}(t,y)$ satisfy property $Q(\rho+1)$ on A, and the sets $\tilde{\tilde{R}}(t,\mathring{y})$ satisfy property $Q(\rho'+1)$ on B. We can now apply lower closure theorem (3.i). Then, $y(t) = (\mathfrak{M}x)(t) \in A(t)$ a.e. on G, $\mathring{y}(t) = (\mathfrak{N}x)(t) \in B(t)$ μ-a.e. on Γ, and there are elements $u \in T$ and $v \in \mathring{T}$ such that

$$
\begin{aligned}
& u(t) \in U(t,y(t)),\ z^i(t) \\
& = f_i(t,y(t),u(t)),\ i = 1,\ldots,r, \quad \text{a.e. in } G, \\
& v(t) \in V(t,\mathring{y}(t)),\ \mathring{z}^i(t) \\
& = g_i(t,\mathring{y}(t),v(t)),\ i = 1,\ldots,r', \quad \mu\text{-a.e. on } \Gamma, \\
& f_o(t,y(t),u(t)) \in L_1(G), \quad g_o(t,\mathring{y}(t),v(t)) \in L_1(\Gamma), \\
& I[x,u,v] \le i,
\end{aligned}
\tag{4.2}
$$

that is, the triple (x,u,v) is admissible. Since Ω is a closed class of admissible triples, there is in Ω some admissible triples $(x,\bar{u},\bar{v})$ with $I[x,\bar{u},\bar{v}] \le i$, and relations (4.2) hold also for u,v replaced by $\bar{u},\bar{v}$. Since i is the infimum of I in Ω, we have $I[x,\bar{u},\bar{v}] \ge i$, and finally $I[x,\bar{u},\bar{v}] = i$.

Remark 1. Theorem (4.i) holds even if we replace the cost functional (1.1) by another analogous one with an added term $J[x]$, provided we know that $J[x]$ is lower semicontinuous functional on S with respect to weak convergence on S. That is, we need only require on J that $x_k \to x$ weakly in S implies $\underline{\lim}_{k\to\infty} J[x_k] \ge J[x]$.

Remark 2. In each of the theorems (3.i), (4.i), (4.ii) we could have assumed that G and Γ are each made up of a finite number of components on each of which there is a distinct system of state equations.

Remark 3. Of particular interest is the case where the element x of an admissible triple (x,u,v) uniquely determines the controls u and v. That is, (x,u,v), $(x,\bar{u},\bar{v})$ admissible implies $u = \bar{u}$ a.e. on G, and $v = \bar{v}$ μ-a.e. on Γ. In this situation, lower closure theorem (3.i) reduces to a lower semicontinuity theorem, and corresponding particular existence theorems could have been obtained by standard lower semicontinuity argument. This holds, for instance, for free problems of the calculus of variations and other problems.

Remark 4. As mentioned in the remark at the end of § 3, if U and V are fixed compact sets (or $U(t,y)$, $V(t,\mathring{y})$ are compact, equibounded, and have property (U)), then the sets $\tilde{\tilde{Q}}(t,y)$, $\tilde{\tilde{R}}(t,\mathring{y})$ certainly have property (U) and, if convex, have property (Q) (see [3a]). Also, an analogous statement holds for the intermediate properties $Q(\rho)$ in the sense that, if ρ-convex, than they have property $Q(\rho)$ (see [4a]).

Example 1. The following example, mentioned by Fichera [5], illustrates existence theorem (4.i). In this example the particular situation depicted in Remark 3 occurs, and therefore our lower closure theorem (3.i) reduces to a lower semicontinuity theorem (which includes Fichera's lower semicontinuity theorem). Let G be a bounded open subset of E^{ν} of class K, $\nu \geq 1$ (see § 2). Let μ be the hyperarea measure defined on the boundary $\Gamma = \partial G$ of G. Let

$W_p^\ell(G)$ be a Sobolev space on G for real p, $1 \leq p \leq +\infty$, and integer ℓ, $1 \leq \ell < +\infty$, and usual norm

$$\|x\| = \|x\|_{W_p^\ell(G)} = \left(\sum_{|\alpha|\leq\ell} \left(\int_G |D^\alpha x(t)|^p dt\right)^{2/p}\right)^{1/2},$$

where $D^\alpha x$ denotes the generalized partial derivative of x in G of order $\alpha = (\alpha_1,\ldots,\alpha_\nu)$, and $|\alpha| = \alpha_1+\ldots+\alpha_\nu$. Let X_o be the linear subset of $W_p^\ell(G)$ made up of all functions x which are continuous on $G \cup \partial G$, together with their partial derivatives of all orders. For $t \in \partial G$ let $(\mathfrak{K}x)(t)$ denote the vector

$$(\mathfrak{K}x)(t) = (\gamma x, \gamma\nabla x, \ldots, \gamma\nabla x^{m-1}),$$

where γf denotes the boundary values of f and ∇x^j, $0 \leq j \leq h-1$, denotes the vector of all partial derivatives $D^\alpha x$ of order $|\alpha| = j$. Let s' denote the total number of components of the vector $(\mathfrak{K}x)(t)$. For given real valued functions $a_\alpha(t)$, $t \in \partial G$, with $|\alpha| = \ell$, $a_\alpha \in L_\infty(\partial G)$, let $(\mathfrak{J}x)(t)$ denote the real valued function

$$(\mathfrak{J}x)(t) = \sum_{|\alpha|=m} a_\alpha(t)\, \gamma\, D^\alpha x(t),\ t \in \partial G.$$

Finally, let S be the completion of X_o with respect to the norm

$$|\|x\|| = \left(\|x\|^2_{W_p^\ell(G)} + \|\mathfrak{J}x\|^2_{L_p(\partial G)}\right)^{1/2}$$

From Sobolev space theory we know that $\mathfrak{K}x$ is defined on S. From the fact that S is the completion of X_o with respect to the norm above we conclude that $\mathcal{J}x$ also is defined on S.

We are concerned with the problem of the minimum in S of the functional

$$I[x] = \int_{\partial G} g_o(t,(\mathcal{K}x)(t),(\mathcal{J}x)(t))d\mu, \tag{4.3}$$

where g_o is a given continuous function on the closed set $\mathring{M} = \partial G \times E^s \times E'$.

This problem is immediately reducible to the form (1.1-5) by taking $\mathcal{L}x = 0$, $\mathcal{M}x = 0$, $f = 0$, by taking $B(t) = E^{s'}$ for every $t \in \Gamma = \partial G$, and $V(t,\mathring{y}) = E'$ for every $(t,\mathring{y}) \in B = \partial G \times E^{s'}$. Thus, there are no constraints on the control variable v, or $v \in V = E'$. We have now the problem of minimum of the type (1.1-5) with functional

$$I[x,v] = \int_\Gamma g_o(t,(\mathcal{K}x)(t),v(t))d\mu$$

and state equations (on the boundary)

$$(\mathcal{J}x)(t) = v(t), \quad \text{a.e. on } \Gamma = \partial G.$$

In the present situation, and using the notations of §§ 3 and 4, we have $r = s = m = 0$, s' as above, $r' = m' = 1$. The sets $\approx{R}$ are the sets

$$\begin{aligned}\tilde{\tilde{R}}(t,\mathring{y}) &= \{\tilde{z} = (\mathring{z},z) \in E^2 \mid \mathring{z} \geq g_o(t,\mathring{y},v),\ z = v \in E^1\} \\ &= \{(\mathring{z},v) \in E^2 \mid \mathring{z} \geq g_o(t,\mathring{y},v),\ v \in E^1\},\end{aligned}$$

and thus they are convex if and only if $g_o(t,\mathring{y},v)$ is convex in v for every $(t,\mathring{y}) \in B = \partial G \times E^{s'}$.

Finally, if we assume that there is some real-valued continuous function

$\Phi(\zeta)$, $0 \le \zeta < +\infty$, with

$$g_o(t,\mathring{y},v) \ge \Phi(|v|) \text{ for all } (t,\mathring{y},v) \in \mathring{M},$$

$$\Phi(\zeta)/\zeta \to +\infty \text{ as } \zeta \to +\infty, \tag{4.4}$$

then we know from (Cesari [3b]) that the sets $\approx{R}(t,\mathring{y})$ satisfy property (Q) on B. Note that, if $\mu = \text{Min}\, \Phi(\zeta)$, $0 \le \zeta < +\infty$, then relation $g_o(t,\mathring{y},v) \ge -\mathring{\psi}(t)$ holds for all $(t,\mathring{y},v) \in \mathring{M}$ with $\mathring{\psi} = -|\mu|$, a constant.

If $p > 1$, then for any sequence of elements x, x_k, $k = 1,2,\dots$, of S with $x_k \to x$ weakly in S as $k \to \infty$, certainly there is a subsequence $[k_p]$ such that $\mathfrak{J}x_{k_p} \to \mathfrak{J}x$ weakly in $L_p(\partial G)$ and $\mathfrak{K}x_{k_p} \to \mathfrak{K}x$ strongly in $(L_p(G))^{s'}$ as $k \to \infty$. We may take $y(t) = (\mathfrak{K}x)(t)$, $y_k(t) = (\mathfrak{K}x_k)(t)$, $t \in G$, and $z(t) = v(t) = (\mathfrak{J}x)(t)$, $z_k(t) = v_k(t) = (\mathfrak{J}x_k)(t)$, $t \in \partial G$, $k = 1,2,\dots$. By lower closure theorem (3.i) with $\rho = r = 0$, $\rho' = r' = 1$, we derive now the following lower semicontinuity theorem concerning the integral $I[x]$:

(a) If $g_o(t,\mathring{y},v)$ is continuous on $\mathring{M} = \partial G \times E^{s'} \times E'$ and convex in v for every $(t,\mathring{y}) \in B = \partial G \times E^s$, if there is a function $\Phi(\zeta)$, $0 \le \zeta < +\infty$, such that (4.4) holds, then the functional $I[x]$ is lower semicontinuous in S; precisely, x, $x_k \in S$, $x_k \to x$ weakly in S as $k \to \infty$, $\underline{\lim}_{k\to\infty} I[x_k] \le a_o < +\infty$ implies $I[x] \le a_o$.

Growth condition (4.4) can be disregarded if we know that the sets $\approx{R}(t,\mathring{y})$ satisfy property (Q) on B, and that $g_o(t,\mathring{y},v) \ge -\mathring{\psi}(t)$ for all

$(t,\mathring{y},v) \in \mathring{M}$ and some $\mathring{\psi} \geq 0$, $\mathring{\psi} \in L_1(\partial G)$. Also, note that under the assumed hypotheses $g_o(t,(\mathcal{K}x)(t),(\mathcal{J}(x))(t))$ is certainly μ-measurable on ∂G and $\geq -\mathring{\psi}(t)$ with $\mathring{\psi} \in L_1(G)$. Thus, the functional $I[x]$, or (4.3), is always defined in S, either finite, or $+\infty$.

Let Ω_N be the class of all $x \in S$ with $|||x||| \leq N$ and $I[x]$ finite. If i denotes the infimum of $I[x]$ in Ω_N, then, by force of $g_o \geq -\mathring{\psi}$, i is finite, and in the search of the minimum of $I[x]$ in Ω, we can restrict ourselves to the subclass Ω_N of all $x \in \Omega_N$ with $I[x] \leq i+1$. Any such class Ω_N' is obviously weakly compact in the topology of S, and weakly closed in the same topology. From (4.i) we may now derive the following existence statement:

(b) Under the conditions of (a) with $1 < p \leq +\infty$, the functional $I[x]$ has an absolute minimum in any nonempty class Ω_N.

Example 2. Let G be a connected bounded open subset of the $\zeta\eta$-plane E_2. If G is of class K, then the usual arc length measure s is defined on $\Gamma = \partial G$. We are concerned with the minimum of the functional

$$\begin{aligned} I[x,u,v] &= \iint_G f_o(\zeta,\eta,x,x_\zeta,x_\eta,u)\,d\zeta\,d\eta \\ &\quad + \int_{\partial G} g_o(\zeta,\eta,\gamma x,\gamma x_\zeta,\gamma x_\eta,v)\,ds, \end{aligned} \tag{4.5}$$

with state equations

$$x_{\zeta\zeta} + x_{\eta\eta} = f(\zeta,\eta,x,x_\zeta,x_\eta,u) \quad \text{a.e. in } G,$$

$$a(\zeta,\eta)\gamma x + b(\zeta,\eta)\gamma x_\zeta + c(\zeta,\eta)\gamma x_\eta$$

$$= g(\zeta,\eta,\gamma x,\gamma x_\zeta,\gamma x_\eta,v) \quad \text{s-a.e. on } \partial G \tag{4.6}$$

and constraints

$$u(\zeta,\eta) \in U(\zeta,\eta,x,x_\zeta,x_\eta) \quad \text{a.e. in } G,$$

$$v(\zeta,\eta) \in V(\zeta,\eta,\gamma x,\gamma x_\zeta,\gamma x_\eta) \quad \text{s-a.e. on } \partial G. \tag{4.7}$$

Here γf denotes the boundary values of f. This problem is immediately written in the form (1.1-5) by taking

$$\mathcal{L}x = x_{\zeta\zeta} + x_{\eta\eta}, \qquad \mathcal{M}x = (x,x_\zeta,x_\eta),$$

$$\mathcal{J}x = a\gamma x + b\gamma x_\zeta + c\gamma x_\eta, \qquad \mathcal{K}x = (\gamma x,\gamma x_\zeta,\gamma x_\eta),$$

$r = r' = 1$, $s = s' = 3$. We take for S the Sobolev space $S = W_p^2(G)$, $p > 1$, and we assume that the given functions a, b, c are of class $L_\infty(\partial G)$. Note that $(\mathcal{J}x)(\zeta,\eta)$ could be the normal derivative of x at $(\zeta,\eta) \in \partial G$ if only $a = 0$, and b, c the direction cosines of the normal to ∂G at (ζ,η)(s-a.e. on ∂G). Also, we take $A(\zeta,\eta) = E^3$, $B(\zeta,\eta) = B^3$, hence $A = (\mathrm{cl}G) \times E^3$, $B = (\partial G) \times E^3$.

For the sake of simplicity we assume $m = m' = 1$, so that, if $y = (y^1,y^2,y^3) \in E^3$, $\mathring{y} = (\mathring{y}^1,\mathring{y}^2,\mathring{y}^3) \in E^3$, then $U(\zeta,\eta,y)$ denotes a subset of E^1 for every $(\zeta,\eta,y) \in A$ and $V(\zeta,\eta,\mathring{y})$ a subset of E^1 for every $(\zeta,\eta,\mathring{y}) \in B$.

Finally, if M, $\mathring{M}$ are the corresponding sets, $M \subset (\mathrm{cl}G) \times E^3 \times E^1$, $\mathring{M} \subset (\partial G) \times E^3 \times E^1$, then f_o, f are real valued continuous functions on M, and g_o, g are real valued continuous functions on $\mathring{M}$. We consider here the sets

$$\tilde{\tilde{Q}}(\zeta,\eta,y) = \{(\mathring{z},z) \in E^2 \mid \mathring{z} \geq f_o(\zeta,\eta,y,u),$$

$$z = f(\zeta,\eta,y,u),\ u \in U(\zeta,\eta,y)\}$$

$$\tilde{\tilde{R}}(\zeta,\eta,\mathring{y}) = \{(\mathring{z},z) \in E^2 \mid \mathring{z} \geq g_o(\zeta,\eta,\mathring{y},v),$$

$$z = g(\zeta,\eta,\mathring{y},v),\ v \in V(\zeta,\eta,\mathring{y})\}$$

for every $(\zeta,\eta,y) \in A$ and for every $(\zeta,\eta,\mathring{y}) \in B$, respectively.

Note that, if $x_k \to x$ weakly in $S = W_p^2(G)$, then there is certainly a subsequence $[k_p]$ such that $\mathcal{L}x_{k_p} \to \mathcal{L}x$ weakly in $L_p(G)$, $\mathcal{M}x_{k_p} \to \mathcal{M}x$ strongly in $(L_p(G))^3$, $\mathcal{J}x_{k_p} \to \mathcal{J}x$ strongly in $L_1(\partial G)$, $\mathcal{K}x_{k_p} \to \mathcal{K}x$ strongly in $L_1(\partial G)$. An admissible triple is now a triple (x,u,v) with $x \in W_p^2(G)$, u measurable on G, v s-measurable on ∂G, satisfying (4.6), (4.7), and such that $f_o(\zeta,\eta,x,x_\zeta,x_\eta,u) \in L_1(G)$ and $g_o(\zeta,\eta,\gamma x,\gamma x_\zeta,\gamma x_\eta,v) \in L_1(\partial G)$.

We can now derive from (4.i) with $\rho = r$, $\rho' = 0$ the following existence statement:

(c) Let G be connected bounded open and of class K in E^2, let M, $\mathring{M}$ be closed, let f_o, f be real valued and continuous on M, and g_o, g real valued and continuous on $\mathring{M}$, and assume that the sets $\tilde{\tilde{Q}}(\zeta,\eta,y)$ satisfy property (Q) on $A = (\mathrm{cl}G) \times$

E^3 and the sets $\tilde{\tilde{R}}(\zeta,\eta,\mathring{y})$ satisfy property (U) on $B = (\partial G) \times E^3$. Let us assume that there are functions $\psi(\zeta,\eta) \geq 0$, $\psi \in L_1(G)$ and $\mathring{\psi}(\zeta,\eta) \geq 0$, $\mathring{\psi} \in L_1(\partial G)$ such that $f_o(\zeta,\eta,y,u) \geq -\psi(\zeta,\eta)$ and $g_o(\zeta,\eta,\mathring{y},v) \geq -\mathring{\psi}(\zeta,\eta)$ for all $(\zeta,\eta,y,u) \in M$ and $(\zeta,\eta,\mathring{y},v) \in \mathring{M}$, respectively. Let Ω be any nonempty class of admissible triples (x,u,v) for which the set $\{x\}_\Omega$ is norm bounded in $W_p^2(G)$, $p > 1$; that is, there is a constant N such that $(x,u,v) \in \Omega$ implies $\|x\|_p + \|x_\zeta\|_p + \|x_\eta\|_p \leq N$. Then, the cost functional (4.5) has an absolute minimum in Ω.

Remark 4. Many examples of optimization problems with distributed and boundary controls and state equations in the strong form are of the same general form of example 2 above. The equations $(\mathcal{L}x)(t) = f(t,(\mathcal{M}x)(t),u(t))$, $t \in G$, is a partial differential equation (or a system), and the equation $(\mathcal{J}x)(t) = g(t,(\mathcal{K}x)(t),v(t))$, $t \in \partial G$, represents a certain set of constraints on the boundary values of the state variables. The conditions of theorems (4.i) are usually satisfied with $\rho = r$ and $\rho' = 0$, that is, we require property (Q) on the sets $\tilde{\tilde{Q}}$ and property (U) on the sets $\tilde{\tilde{R}}$. One more example is in ([4a], no. 5, example 1).

In applications it often occurs that U and V are fixed compact sets (or alternatively $U(t,y)$, $V(t,\mathring{y})$ are compact equibounded variable sets satisfying property (U)). In either situation, the sets $\tilde{\tilde{Q}}(t,y)$, $\tilde{\tilde{R}}(t,\mathring{y})$ certainly have property (U) and, if convex, have property (Q). (Analogously, if ρ-convex, then they have property $Q(\rho)$ [4a].) A number of problems with U, V fixed compact sets have been considered in [3h], and existence theorem (4.i) may easily be applied.

Example 3. In this example we wish to illustrate the use of the intermediate properties $Q(\rho)$. Let us consider the problem of the minimum of the cost functional

$$I[x,u_1,u_2,v] \;=\; \iint_G (x^2+x_\zeta^2+x_\eta^2+u_1^2+u_2^2(1-u_2)^2)\,d\zeta d\eta \;+\; \int_\Gamma (\gamma x-1)^2 ds,$$

with differential equations

$$x_{\zeta\zeta}+x_{\eta\eta} \;=\; u_1, \qquad x_\zeta + x_\eta \;=\; u_2, \qquad \text{a.e. in } G,$$

$$\gamma x_\zeta \;=\; \cos v, \qquad \gamma x_\eta \;=\; \sin v, \qquad \text{s-a.e. on } \Gamma = \partial G,$$

where $G = [(\zeta,\eta)|\zeta^2+\eta^2<1]$, Γ is the boundary of G, where γx, γx_ζ, γx_η denote the boundary values of x, x_ζ, x_η, and the control functions u_1, u_2, v have their values $(u_1,u_2) \in U = E_2$, $v \in V = E_1$. We want to minimize I in a class Ω of systems (x,u_1,u_2,v) with u_1, u_2 measurable in G, v measurable on Γ, x any element of the Sobolev class $W_2^2(G)$ satisfying all relations above, satisfying an inequality $\|x_{\zeta\zeta}\|_2 + \|x_{\zeta\eta}\|_2 + \|x_{\eta\eta}\|_2 \le M$, and for which I is finite. Here the constant M is assumed to be sufficiently large so that Ω is not empty. We may well consider only those elements of Ω for which $I \le N$ for some constant N. Here we have $f_o \ge 0$, $g_o \ge 0$, and we can take $\psi = 0$, $\mathring{\psi} = 0$. Also, we have $\mathcal{L}x = (x_{\zeta\zeta}+x_{\eta\eta},\ x_\zeta+x_\eta)$, $\mathfrak{M}x = (x,x_\zeta,x_\eta)$, $\mathcal{J}x = (\gamma x_\zeta,\gamma x_\eta)$, $\mathfrak{K}x = \gamma x$, $r = 2$, $s = 3$, $r' = 2$, $s' = 1$, $m = 2$, $m' = 1$. Then, for any sequence $[x_k]$ of elements $x \in \{x\}_\Omega$, certainly there is a subsequence, say still $[k]$ for the sake of simplicity, with $x_k \to x$ weakly in $S = W_2^2(G)$ for some $x \in S$, and $(\mathcal{L}x_k)^1 \to (\mathcal{L}x)^1$ weakly in $L_2(G)$, $(\mathcal{L}x_k)^2 \to (\mathcal{L}x)^2$ strongly in $L_2(G)$, $\mathcal{M}x_k \to \mathcal{M}x$ strongly in $(L_2(G))^3$, $\mathcal{J}x_k \to \mathcal{J}x$ strongly in $(L_2(\Gamma))^2$, $\mathcal{K}x_k \to \mathcal{K}x$ strongly in $L_2(\Gamma)$. We consider here the sets

$$\tilde{\tilde{Q}}(y_1,y_2,y_3) = [(z^o,z^1,z^2)|z^o \geq y_1^2+y_2^2+y_3^2+u_1^2+u_2^2(1-u_2)^2,$$

$$z^1 = u_1,\ z^2 = u_2,\ (u_1,u_2) \in E_2]$$

$$\tilde{\tilde{R}}(y) = [(z^o,z^1,z^2)|z^o \geq (y-1)^2,\ z^1 = \cos v,\ z^2 = \sin v,\ v \in E_1].$$

The sets $\tilde{\tilde{Q}} \subset E_3$ have property Q(2), the sets $\tilde{\tilde{R}} \subset E_3$ have property Q(1) and all have property (U), or Q(0). They are not convex, and do not have, therefore, property (Q), or Q(3). Nevertheless, existence theorem (4.i) applies with $\rho = 2$, $\rho' = 0$, and the problem under consideration has an absolute minimum in Ω.

5. ANOTHER EXISTENCE THEOREM FOR OPTIMIZATION PROBLEM WITH STATE EQUATIONS IN THE STRONG FORM

We now consider the case where the operators $\mathcal{L}$, $\mathcal{M}$, $\mathcal{J}$, $\mathcal{K}$ themselves depend on $x \in S$ and on suitable components of the controls, instead of depending on x alone as in § 4. Thus, the theorem we shall prove here is, for practical purposes, more general than theorem (4.i). Nevertheless, we shall prove it as a corollary of theorem (4.i).

We shall consider here additional spaces of distributed and boundary controls, T and $\mathring{T}$, with elements $u \in T$ and $v \in \mathring{T}$, respectively, both T and $\mathring{T}$ being given Banach spaces.

It may occur that ⓤ and ⓥ are vector functions $ⓤ(t) = (u^{m+1},\dots,u^{\bar m})$, $t \in G$, and $ⓥ(t) = (v^{m'+1},\dots,v^{\bar m'})$, $t \in \Gamma$, and in this case the control $\tilde u = (u, ⓤ)$ is an $\bar m$-vector function on G, and $\tilde v = (v, ⓥ)$ is an $\bar m'$-vector function on Γ. In any case, we write our controls as $\tilde u = (u, ⓤ)$ and $\tilde v = (v, ⓥ)$.

We are concerned here with the problem of the minimum of a functional

$$I[x,u,ⓤ,v,ⓥ] = \int_G f_o(t,(\mathcal{M}(x,ⓤ))(t),u(t))dt + \int_\Gamma g_o(t,(\mathcal{K}(x,ⓥ))(t),v(t))d\mu \tag{5.1}$$

subject to the state equations

$$(\mathcal{L}(x,ⓤ))(t) = f(t,(\mathcal{M}(x,ⓤ)(t),u(t)) \quad \text{a.e. in } G, \tag{5.2}$$

$$(\mathcal{J}(x,ⓥ))(t) = g(t,(\mathcal{K}(x,ⓥ)(t),v(t)) \quad \mu\text{-a.e. in } \Gamma, \tag{5.3}$$

and the constraints

$$(\mathcal{M}(x,ⓤ))(t) \in A(t),\ u(t) \in U(t,(\mathcal{M}(x,ⓤ))(t)) \quad \text{a.e. in } G, \tag{5.4}$$

$$(\mathcal{K}(x,ⓥ))(t) \in B(t),\ v(t) \in V(t,(\mathcal{K}(x,ⓥ))(t)) \quad \mu\text{-a.e. in } \Gamma. \tag{5.5}$$

As in § 1, G is a fixed bounded open set in E^ν, $\nu \geq 1$, and Γ a given closed subset of ∂G on which we have a hyperarea measure μ. Let S, Ⓣ, $\mathring{Ⓣ}$ be Banach spaces of elements x, ⓤ, ⓥ, and let $\mathcal{L}$, $\mathcal{M}$, $\mathcal{J}$, $\mathcal{K}$ be operators on $S \times Ⓣ$, $S \times \mathring{Ⓣ}$, not necessarily linear, with values in the following spaces:

$$\mathcal{L}: \; S \times (T) \to (L_1(G))^r, \qquad \mathcal{J}: \; S \times (\mathring{T}) \to (L_1(\Gamma))^{r'},$$

$$\mathcal{M}: \; S \times (T) \to (L_1(G))^s, \qquad \mathcal{K}: \; S \times (\mathring{T}) \to (L_1(\Gamma))^{s'},$$

where r, s, r', s' are given positive integers.

Let $A(t)$, A, $U(t,y)$, M and $B(t)$, B, $V(t,\mathring{y})$, $\mathring{M}$ be the sets defined in §§ 1 and 3, $M \subset (\mathrm{cl}G) \times E^s \times E^m$, $\mathring{M} \subset \Gamma \times E^{s'} \times E^{m'}$, $U(t,y) \subset E^m$, $V(t,\mathring{y}) \subset E^{m'}$; let $f_o(t,y,u)$, $f(t,y,u) = (f_1,\dots,f_r)$ be defined on M, and $g_o(t,\mathring{y},v)$, $g(t,\mathring{y},v) = (g_1,\dots,g_{r'})$ be defined on $\mathring{M}$. Let $\tilde{\tilde{Q}}(t,y) \subset E^{r+1}$ be the sets defined in § 3 for every $(t,y) \in A$, and let $\tilde{\tilde{R}}(t,\mathring{y}) \subset E^{r'+1}$ be the analogous sets also defined in § 3 for every $(t,\mathring{y}) \in B$.

As in § 4, we denote by T the set of all measurable m-vector functions $u(t) = (u^1,\dots,u^m)$, $t \in G$, and by $\mathring{T}$ the set of all measurable m'-vector functions $v(t) = (v^1,\dots,v^{m'})$, $t \in \Gamma$.

A triple $(x,\tilde{u},\tilde{v})$, or system $(x,u,(u),v,(v))$, is said to be <u>admissible</u> (for the problem (5.1-5)) provided $x \in S$, $u \in T$, $v \in \mathring{T}$, $(u) \in (T)$, $(v) \in (\mathring{T})$, $f_o(t,(\mathcal{M}(x,(u)))(t),u(t)) \in L_1(G)$, $g_o(t,(\mathcal{K}(x,(v)))(t),v(t)) \in L_1(\Gamma)$, and relations (5.2-5) hold.

A class Ω of admissible systems is said to be <u>closed</u> if the following occurs: if $(x_k,u_k,(u_k),v_k,(v_k)) \in \Omega$, $k = 1,2,\dots$, $x_k \to x$ weakly in S, $(u_k) \to (u)$ weakly in (T), $(v_k) \to (v)$ weakly in $(\mathring{T})$ as $k \to \infty$, $\underline{\lim}_{k\to\infty} I[x_k,u_k,(u_k),v_k,(v_k)] \le a < +\infty$, and there are admissible systems $(x,u,(u),v,(v))$ such that $I[x,u,(u),v,(v)] \le a$, then there are also systems $(x,\bar{u},(\bar{u}),\bar{v},(\bar{v})) \in \Omega$ with $I[x,\bar{u},(\bar{u}),\bar{v},(\bar{v})] \le a$.

For a class Ω of admissible systems (x, u, u, v, v) we denote by $\{x\}_\Omega$, $\{u\}_\Omega$, $\{v\}_\Omega$ the sets defined as $\{x\}_\Omega$ in § 4.

(5.i) Existence Theorem. Let G be bounded and measurable, A, B, M, $\mathring{M}$ closed, $f_o(t,y,u)$, $f(t,y,u) = (f_1,\dots,f_r)$ continuous on M, $g_o(t,\mathring{y},v)$, $g(t,\mathring{y},v) = (g_1,\dots,g_{r'})$ continuous on $\mathring{M}$, and assume that for given integers ρ, ρ', $0 \le \rho \le r$, $0 \le \rho' \le r'$, the sets $\tilde{\tilde{Q}}(t,y)$ have property $Q(\rho+1)$ on A, and the sets $\tilde{\tilde{R}}(t,\mathring{y})$ have property $Q(\rho'+1)$ on B. Let us assume that there are functions $\psi(t) \ge 0$, $t \in G$, $\psi \in L_1(G)$, and $\mathring{\psi}(t) \ge 0$, $t \in \Gamma$, $\mathring{\psi} \in L_1(\Gamma)$, such that $f_o(t,y,u) \ge -\psi(t)$ for all $(t,y,u) \in M$, and $g_o(t,\mathring{y},v) \ge -\mathring{\psi}(t)$ for all $(t,\mathring{y},v) \in \mathring{M}$. Let us assume that for every sequence x, u, v, x_k, u_k, v_k, $k = 1,2,\dots$, of elements, $x, x_k \in S$, $u, u_k \in T$, $v, v_k \in \mathring{T}$, with $x_k \to x$ weakly in S, $u_k \to u$ weakly in T, $v_k \to v$ weakly in $\mathring{T}$, it occurs that there is some subsequence $[k_p]$ such that

$$(\mathcal{L}(x_{k_p}, u_{k_p}))^i \to (\mathcal{L}(x, u))^i \text{ weakly in } L_1(G),\ i = 1,\dots,\rho,$$

$$(\mathcal{L}(x_{k_p}, u_{k_p}))^i \to (\mathcal{L}(x, u))^i \text{ strongly in } L_1(G),\ i = \rho+1,\dots,r,$$

$$(\mathfrak{M}(x_{k_p}, u_{k_p}))^j \to (\mathfrak{M}(x, u))^j \text{ strongly in } L_1(G),\ j = 1,\dots,s,$$

$$(\mathcal{J}(x_{k_p}, v_{k_p}))^i \to (\mathcal{J}(x, v))^i \text{ weakly in } L_1(\Gamma),\ i = 1,\dots,\rho',$$

$$(\mathcal{J}(x_{k_p}, v_{k_p}))^i \to (\mathcal{J}(x, v))^i \text{ strongly in } L_1(\Gamma),\ i = \rho'+1,\dots,r',$$

$$(\mathcal{K}(x_{k_p}, v_{k_p}))^j \to (\mathcal{K}(x, v))^j \text{ strongly in } L_1(\Gamma),\ j = 1,\dots,s',$$

as $p \to \infty$. Let Ω be a nonempty closed class of admissible systems $(x, u, u,$

v, ⓥ) such that $\{x\}_\Omega$, $\{ⓤ\}_\Omega$, $\{ⓥ\}_\Omega$ are weakly sequentially relatively compact. Then, the functional (5.1), or $I[x,u,ⓤ,v,ⓥ]$, has an absolute minimum in Ω.

<u>Proof</u>. Apply existence theorem (4.i) with S replaced by $S \times Ⓣ \times \mathring{Ⓣ}$ and x replaced by (x, u , v).

<u>Example</u> (a problem of evolution in strong form). Let G be a subset of the $t\tau$-space $E^{\nu+1}$, $\tau = (\tau^1,\dots,\tau^\nu)$, of the form $G = (o,T) \times G'$, where G' is an open bounded connected subset of E^ν of class K. Thus, $\nu+1$ replaces ν, and $\Gamma = \partial G$ is made up of three parts $\Gamma_1 = \{o\} \times G'$, $\Gamma_2 = (o,T) \times \partial G'$, $\Gamma_3 = \{T\} \times G'$. On Γ_1 and Γ_3 we have the Lebesgue ν-dimensional measure, or $|\ |_\nu$ (and we shall use the symbol $d\tau$ in integration). In Γ_2 we have the product measure $\sigma = |\ |_1 \times \mu$ of the one-dimensional measure on $[o,T]$ and of the hyperarea μ on the boundary $\partial G'$ of G' (and we shall use the symbol $dt\, d\mu$ in integration). Given a function x in G, we shall denote by γx the boundary values of x on $\Gamma = \partial G$, and specifically we shall denote by $\gamma_i x$ the boundary values of x on Γ_i, $i = 1,2,3$. We shall denote by T, $\mathring{T}_i$ the families of all measurable functions on G, Γ_i, $i = 1,2,3$, respectively.

We shall denote by $S_p^\ell(G)$, $1 < p \leq +\infty$, $\ell \geq 1$, the space of all real valued functions $x(t,\tau)$, $(t,\tau) \in G$, such that $\partial x/\partial t$ and all $D_\tau^\alpha x$, $\alpha = (\alpha_1,\dots,\alpha_\nu)$, $0 \leq |\alpha| \leq \ell$, exist as generalized derivatives, and are all in $L_p(G)$. We shall make $S_p^\ell(G)$ a Banach space by means of the norm

$$\|x\| = \|x\|_{S_p^\ell(G)}$$

$$= \left[\left(\int_G |\partial x/\partial t|^p dt d\tau\right)^{2/p} + \sum_{0\le|\alpha|\le\ell} \left(\int_G |D_\tau^\alpha x|^p dt d\tau\right)^{2/p}\right]^{1/2}.$$

These spaces $S_p^\ell(G)$ have been studied by J. P. Aubin [1], who has proved for these spaces weak compactness properties similar to those for the Sobolev spaces. Each element $x \in S_p^\ell(G)$ possesses boundary values $\gamma_1 x$, $\gamma_3 x$ on Γ_1, Γ_3, respectively, and all $\gamma_2 D_\tau^\alpha x$, $0 \le |\alpha| \le \ell-1$, on Γ_2.

We are concerned with the minimum of a functional

$$I[x,u,u,v_2,v_2,v_3] = \int_G f_o(t,\tau,(\mathfrak{M}x)(t,\tau),u(t,\tau))dtd\tau$$

$$+ \int_{\Gamma_2} g_o(t,\tau,(\mathfrak{K}x)(t,\tau),v_2(t,\tau))dtd\mu$$

$$+ \int_{\Gamma_3} g_o(\tau,(\mathfrak{K}x)(T,\tau),v_3(\tau))d\tau$$

with state equations (in the strong form)

$$\mathfrak{L}(x,u)(t,\tau) = f(t,\tau,(\mathfrak{M}x)(t,\tau),u(t,\tau)) \quad \text{a.e. on } G,$$

$$\mathfrak{J}(x,v_2)(t,\tau) = g(t,\tau,(\mathfrak{K}x)(t,\tau),v_2(t,\tau)) \quad \sigma\text{-a.e. on } \Gamma_2,$$

and constraints

$$(\mathfrak{M}x)(t,\tau) \in A(t,\tau),\ u(t,\tau) \in U(t,\tau,(\mathfrak{M}x)(t,\tau)) \quad \text{a.e. on } G,$$

$$(\mathfrak{K}x)(t,\tau) \in B_2(t,\tau),\ v_2(t,\tau) \in V_2(t,\tau,(\mathfrak{K}x)(t,\tau)) \quad \sigma\text{-a.e. on } \Gamma_2,$$

$$(\mathfrak{K}x)(T,\tau) \in B_3(\tau),\ v_3(\tau) \in V_3(\tau,(\mathfrak{K}x)(T,\tau)) \quad \text{a.e. on } \Gamma_3,$$

and no boundary conditions on Γ_1 (in other words, the initial values $v_1(\tau) = x(o,\tau)$ are free and $v_1(\tau)$, $\tau \in \Gamma_1$, is itself a control, therefore.

We take now $x \in S = S_p^{\ell}(G)$, $u \in T$, $v_2 \in \mathring{T}_2$, $v_3 \in \mathring{T}_3$, and $u \in T$, $v_2 \in \mathring{T}_2$, where T and $\mathring{T}_2$ are weakly closed subsets of $L_p(G)$ and $L_p(\Gamma_2)$, respectively, and both are norm bounded (in the norms of $L_p(G)$ and $L_p(\Gamma_2)$). Above $\mathcal{M}$, $\mathcal{K}$, $\mathcal{L}$, $\mathcal{J}$ are operators, not necessarily linear, say $\mathcal{M}$: $S \to (L_p(G))^s$, $\mathcal{K}$: $S \to (L_p(\Gamma))^{s'}$, $\mathcal{L}$: $S \times T \to (L_p(G))^r$, $\mathcal{J}$: $S \times \mathring{T}_2 \to (L_p(\Gamma_2))^{r'}$. We assume that $x_k \to x$ weakly in S, $u_k \to u$ weakly in $L_p(G)$, $v_{2k} \to v_2$ weakly in $L_p(\Gamma_2)$ implies that $\mathcal{M}x_k \to \mathcal{M}x$ strongly in $(L_p(G))^s$, $\mathcal{K}x_k \to \mathcal{K}x$ strongly in $(L_p(\Gamma))^s$, $\mathcal{L}(x_k, u_k) \to \mathcal{L}(x, u)$ weakly in $(L_p(\Gamma_2))^{r'}$, $\mathcal{J}(x_k, v_{2k}) \to \mathcal{J}(x, v_2)$ weakly in $L_p(\Gamma_2)^{r'}$. Theorem (5.1) now applies easily.

For instance, we can take $r = r' = 1$, $m = m' = 1$, $T = 0$, $p = \infty$, $\ell = 2$, and

$$\mathcal{L}x = \partial x/\partial t - \Sigma_{i=1}^{\nu} \partial^2 x/(\partial\tau^i)^2,$$

$$\mathcal{J}(x, v_2) = \Sigma_{i=1}^{\nu} a_i(t,\tau)\, \gamma_2(\partial x/\partial\tau^i) + v_2(t,\tau)\, \gamma_2 x(t,\tau).$$

Here $T_2 \subset L_\infty(\Gamma_2)$, and the coefficients a_i are given elements of $L_\infty(\Gamma_2)$.

6. AN EXISTENCE THEOREM FOR OPTIMIZATION PROBLEMS WITH STATE EQUATIONS IN THE WEAK FORM

We shall now consider the case, mentioned in § 1, where the state equations (1.2), (1.3) are written in the weak form as is customary in partial differential equations theory.

We shall use the general notations of the previous sections. Let W denote a normed space of test functions $w = (w_1, w_2)$, where $w_1 \in (L_q(G))^r$, $w_2 \in (L_q(\Gamma))^{r'}$, and $p^{-1} + q^{-1} = 1$, with $1 \leq p \leq +\infty$, $1 \leq q \leq +\infty$, and the usual conventions hold. We shall assume that the norms $\|w_1\|_q$ of w_1 in $(L_q(G))^r$ and $\|w_2\|_q$ of w_2 in $(L_q(\Gamma))^{r'}$ are related to the norm $\|w\|_W$ of $w = (w_1, w_2)$ in W by a relation of the form

$$\|w_1\|_q + \|w_2\|_q \leq K\|w\|_W, \tag{6.1}$$

where K is a constant. We shall denote by W* the dual space of W. We shall deal here with only three operators, $\mathfrak{M}$, $\mathfrak{K}$ as in § 5, and $\mathfrak{F}$ replacing both $\mathcal{L}$ and $\mathcal{J}$:

$$\mathfrak{M}: \quad S \times T \to (L_1(G))^s, \qquad \mathfrak{K}: \quad S \times \mathring{T} \to (L_1(\Gamma))^{s'},$$

$$\mathfrak{F}: \quad S \times T \times \mathring{T} \to W^*.$$

For every $x \in S$, $u \in T$, $u \in T$, $v \in \mathring{T}$, $v \in \mathring{T}$, we consider now the operator h, or $h(x, u, u, v, v)$, $h: \quad W \to E'$, defined by

$$hw = \int_G f(t, (\mathfrak{M}(x, u))(t), u(t)) w_1(t) dt$$

$$+ \int_\Gamma g(t, (\mathfrak{K}(x, v))(t), v(t)) w_2(t) d\mu,$$

where fw_1 and gw_2 denote inner products in E^r and $E^{r'}$, respectively.

Instead of state equations (5.2), (5.3) we shall now consider the unique state equation in the weak form

$$\mathfrak{F} = h, \text{ or} \tag{6.2}$$

$$\mathfrak{J}w = hw \quad \text{for all } w \in W, \text{ or} \tag{6.3}$$

$$\mathfrak{J}(x, (u), (v))(w_1, w_2) = \int_G f(t, (\mathfrak{M}(x, (u)))(t), u(t)) w_1(t) dt$$

$$+ \int_\Gamma g(t, (\mathfrak{K}(x, (v)))(t), v(t)) w_2(t) d\mu$$

$$\text{for all } (w_1, w_2) \in W. \tag{6.4}$$

Note that relation (6.1) implies

$$(L_p(G))^r \times (L_p(\Gamma))^{r'} \subset W^*,$$

and, as mentioned, $\mathfrak{J}$: $S \times (T) \times (\mathring{T}) \to W^*$. In most applications, however, we shall have

$$\mathfrak{J}: \; S \times T \times \mathring{T} \to (L_p(G))^r \times (L_p(\Gamma))^{r'} \subset W^*,$$

and the actual determination of W^* will be irrelevant.

We understand here that the present unique state equation (6.3) is the weak form of state equations (5.2), (5.3). In other words, in any particular situation $\mathfrak{J}$ and W must be so chosen so that any solution of the equations (5.2), (5.3) (strong form) is necessarily a solution of (6.3).

Thus, we are interested here in the problem of the minimum of the functional

$$I[x, u, (u), v, (v)] = \int_G f_o(t, (\mathfrak{M}(x, (u)))(t), u(t)) dt$$

$$+ \int_\Gamma g_o(t, (\mathfrak{K}(x, (v)))(t), v(t)) d\mu, \tag{6.5}$$

with state equation (in the weak form)

$$\mathfrak{J}w = hw \quad \text{for all } w \in W, \tag{6.6}$$

and constraints

$$(\mathfrak{M}(x, u))(t) \in A(t), \; u(t) \in U(t, (\mathfrak{M}(x, u))(t)) \quad \text{a.e. in } G, \tag{6.7}$$

$$(\mathfrak{K}(x, v))(t) \in B(t), \; v(t) \in V(t, (\mathfrak{K}(x, v))(t)) \quad \mu\text{-a.e. in } \Gamma. \tag{6.8}$$

In the present situation we shall require suitable growth conditions, or condition (H):

(H) For $p = 1$ we assume that, given $\varepsilon > 0$, there are functions $\phi_\varepsilon \geq 0$, $\phi_\varepsilon \in L_1(G)$, and $\mathring{\phi}_\varepsilon \geq 0$, $\mathring{\phi}_\varepsilon \in L_1(\Gamma)$, such that

$$|f(t,y,u)| \leq \phi_\varepsilon(t) + \varepsilon f_o(t,y,u),$$

for all $(t,y,u) \in M$,

$$|g(t,\mathring{y},v)| \leq \mathring{\phi}_\varepsilon(t) + \varepsilon g_o(t,\mathring{y},v)$$

for all $(t,\mathring{y},v) \in \mathring{M}$.

If $p > 1$ we assume that there are functions $\phi_o \geq 0$, $\phi_o \in L_1(G)$, and $\mathring{\phi}_o \geq 0$, $\mathring{\phi}_o \in L_1(\Gamma)$, and constants $a > 0$, $b > 0$, such that

$$|f(t,y,u)|^p \leq \phi_o(t) + af_o(t,y,u)$$

for all $(t,y,u) \in M$,

$$|g(t,\mathring{y},v)|^p \leq \mathring{\phi}_o(t) + bg_o(t,\mathring{y},v)$$

for all $(t,\mathring{y},v) \in \mathring{M}$.

This condition, for $p = 1$, has been systematically used by Cesari [3be] as a suitable extension of previous more restrictive growth hypotheses used by Tonelli and McShane.

A triple $(x,\tilde{u},\tilde{v})$, or system $(x,u,ⓤ,v,ⓥ)$, is now said to be admissible provided $x \in S$, $u \in T$, $ⓤ \in Ⓣ$, $v \in \mathring{T}$, $ⓥ \in (\mathring{T})$, relations (6.6), (6.7), (6.8) hold, $f_o(t,(\mathfrak{M}(x,ⓤ))(t),u(t)) \in L_1(G)$, $g_o(t,(\mathfrak{K}(x,ⓥ))(t),v(t)) \in L_1(\Gamma)$. Also, we require that $f(t,(\mathfrak{M}(x,ⓤ))(t),u(t)) \in (L_p(G))^r$ and that $g(t,(\mathfrak{K}(x,ⓤ))(t),v(t)) \in (L_p(\Gamma))^{r'}$. In the existence theorem below, however, this last requirement will be a consequence of property (H). We shall now consider nonempty closed classes Ω of admissible systems $(x,u,ⓤ,v,ⓥ)$, where the definition of closedness is analogous to the ones in §§ 4 and 5.

(6.i) <u>Existence Theorem</u>. Let G be bounded and measurable, A, B, M, $\mathring{M}$ closed, $f_o(t,y,u)$, $f(t,y,u) = (f_1,\ldots,f_r)$ continuous on M, $g_o(t,\mathring{y},v)$, $g(t,\mathring{y},v) = (g_1,\ldots,g_{r'})$ continuous on $\mathring{M}$, and assume that the sets $\tilde{\tilde{Q}}(t,y)$ have property (Q) on A, and the sets $\tilde{\tilde{R}}(t,\mathring{y})$ have property (Q) on B. Let us assume that there are functions $\psi(t) \geq 0$, $t \in G$, $\psi \in L_1(G)$, and $\mathring{\psi}(t) \geq 0$, $t \in \Gamma$, $\mathring{\psi} \in L_1(\Gamma)$, such that $f_o(t,y,u) \geq -\psi(t)$ for all $(t,y,u) \in M$, and $g_o(t,\mathring{y},v) \geq -\mathring{\psi}(t)$ for all $(t,\mathring{y},v) \in M$. Let us assume that relation (6.1) holds, and that growth condition (H) is satisfied. Let us assume that, for every sequence x, ⓤ, ⓥ, x_k, (u_k), (v_k), $k = 1,2,\ldots$, of elements x, $x_k \in S$,

u, $u_k \in T$, v, $v_k \in \mathring{T}$ with $x_k \to x$ weakly in S, $u_k \to u$ weakly in T, $v_k \to v$ weakly in $\mathring{T}$, it occurs that there is some subsequence $[k_p]$ such that

$$\mathfrak{M}(x_{k_p}, u_{k_p}) \to \mathfrak{M}(x, u) \text{ strongly in } (L_1(G))^s,$$

$$\mathfrak{K}(x_{k_p}, v_{k_p}) \to \mathfrak{K}(x, v) \text{ strongly in } (L_1(\Gamma))^{s'},$$

$$\mathfrak{J}(x_{k_p}, u_{k_p}, v_{k_p})w \to \mathfrak{J}(x, u, v)w \text{ for every } w \in W, \tag{6.9}$$

as $p \to \infty$. Let Ω be a nonempty closed class of admissible systems (x,u,u, v,v) such that $\{x\}_\Omega$, $\{u\}_\Omega$, $\{v\}_\Omega$ are weakly sequentially relatively compact. Then the functional (6.1), or $I[x,u,u,v,v]$, has an absolute minimum in Ω.

The hypothesis concerning $\mathfrak{J}$ above can be reworded by saying that $\mathfrak{J}(x_{k_p}, u_{k_p}, v_{k_p}) \to \mathfrak{J}(x, u, v)$ in the weak star topology on W*.

Proof. As usual let i be the infimum of $I[x,u,u,v,v]$ in the class Ω. Since $f_o \geq -\psi$, $g_o \geq -\mathring{\psi}$ and Ω is nonempty, $\underline{\underline{i}}$ is finite. Let $(x_k, u_k, u_k, v_k, v_k)$, $k = 1,2,\ldots$, be a sequence with $I[x_k, u_k, u_k, v_k, v_k] \to i$ as $k \to \infty$, and we may well assume that

$$\underline{\underline{i}} \leq I[x_k, u_k, u_k, v_k, v_k] \leq \underline{\underline{i}} + k^{-1} \leq \underline{\underline{i}}+1, \quad k = 1,2,\ldots.$$

Since we have assumed that the sets $\{x\}_\Omega$, $\{u\}_\Omega$, $\{v\}_\Omega$ are weakly sequentially relatively compact, there is a subsequence, say still [k] for the sake of simplicity such that $x_k \to x$ weakly in S, $u_k \to u$ weakly in

(T), $(v_k) \to (v)$ weakly in $(\overset{\circ}{T})$ as $k \to \infty$. We may even assume that the subsequence has been so chosen that limit relations (6.9) hold. Let

$$z_k(t) = f(t, \mathcal{M}(x_k, (u_k))(t), u_k(t)), \ t \in G,$$

$$\overset{\circ}{z}_k(t) = g(t, \mathcal{K}(x_k, (v_k))(t), v_k(t)), \ t \in \Gamma, \ k = 1,2,\ldots.$$

By growth condition (H) and $I[x_k, u_k, (u_k), v_k, (v_k)] \leq i+1$ for all k, we see that, if $p > 1$, the functions $z_k(t)$ and $\overset{\circ}{z}_k(t)$, $k = 1,2,\ldots$, are equibounded in the norms of $(L_p(G))^r$ and $(L_p(\Gamma))^{r'}$, respectively. If $p = 1$ it follows from an argument of Cesari [3be] that the same functions $z_k(t)$ and $\overset{\circ}{z}_k(t)$ are equiabsolutely integrable in G and Γ, respectively. In any case, there exists a subsequence, say still $[k]$ for the sake of simplicity, and elements $z(t)$, $t \in G$, $z \in (L_p(G))^r$, and $\overset{\circ}{z}(t)$, $t \in \Gamma$, $\overset{\circ}{z} \in (L_p(\Gamma))^{r'}$, such that $z_k \to z$ weakly in $(L_p(G))^r$ and $\overset{\circ}{z}_k \to \overset{\circ}{z}$ weakly in $(L_p(\Gamma))^{r'}$. In other words,

$$f(t, \mathcal{M}(x_k, (u_k))(t), u_k(t)) \to z(t) \text{ weakly in } (L_p(G))^r,$$

$$g(t, \mathcal{K}(x_k, (v_k))(t), v_k(t)) \to \overset{\circ}{z}(t) \text{ weakly in } (L_p(\Gamma))^{r'}, \tag{6.10}$$

as $k \to \infty$, while

$$\lim_{k\to\infty} I[x_k, u_k, (u_k), v_k, (v_k)] = i, \tag{6.11}$$

$$(\mathcal{M}(x_k, (u_k)))(t) \in A(t), \ u_k(t) \in U(t, (\mathcal{M}(x_k, (u_k)))(t)) \quad \text{a.e. in } G,$$

$$(\mathcal{K}(x_k, (v_k)))(t) \in B(t), \ v_k(t) \in V(t, (\mathcal{K}(x_k, (v_k)))(t)) \quad \mu\text{-a.e. in } \Gamma. \tag{6.12}$$

If $w = (w_1, w_2)$ is any element of W then, by relation (6.1) we know that $w_1 \in (L_q(G))^r$ and $w_2 \in (L_q(\Gamma))^{r'}$, hence

$$\int_G z_k(t)w_1(t)dt \to \int_G z(t)w_1(t)dt, \quad \int_\Gamma \mathring{z}_k(t)w_2(t)d\mu \to \int_\Gamma \mathring{z}(t)w_2(t)d\mu$$

as $k \to \infty$. Finally, by the definition of the operator h, we have

$$h(x_k, u_k, (u_k), v_k, (v_k))w \to \int_G z(t)w_1(t)dt + \int_\Gamma \mathring{z}(t)w_2(t)d\mu \qquad (6.13)$$

as $k \to \infty$, for every $w = (w_1, w_2) \in W$. By hypothesis we have also

$$\mathfrak{F}(x_k, (u_k), (v_k))w \to \mathfrak{F}(x, (u), (v))w \qquad (6.14)$$

as $k \to \infty$, again for every $w \in W$. Here each system $(x_k, u_k, (u_k), v_k, (v_k))$, $k = 1,2,\ldots$, is admissible, hence the first members of (6.13) and (6.14) are equal for every $w \in W$. From (6.13) and (6.14), by comparison we obtain

$$\mathfrak{F}(x, (u), (v))w = \int_G z(t)w_1(t)dt + \int_\Gamma \mathring{z}(t)w_2(t)d\mu \qquad (6.15)$$

for every $w \in W$.

Now relations (6.10), (6.11), (6.12) show that we can apply lower closure theorem (3.i) with S replaced by $S \times (T) \times (\mathring{T})$ and with $\rho = r$, $\rho = r'$. Here the sets $\tilde{\tilde{Q}}(t,y)$ have property (Q) on A, hence property $Q(r+1)$ by force of (1.i). Analogously, the sets $\tilde{\tilde{R}}(t,\mathring{y})$ have property (Q) on B, hence property $Q(r'+1)$. We conclude that $(\mathfrak{M}(x, (u)))(t) \in A(t)$ a.e. in G, that $(\mathfrak{N}(x, (v)))(t) \in B(t)$ μ-a.e. in Γ, and that there are elements $u \in T$, $v \in \mathring{T}$ such that

$u(t) \in U(t,(\mathfrak{M}(x, ⓤ))(t)),\ z(t)$

$= f(t,(\mathfrak{M}(x, ⓤ))(t),u(t))$ a.e. in G,

$v(t) \in V(t,(\mathfrak{K}(x, ⓥ))(t)),\ \mathring{z}(t)$

$= g(t,(\mathfrak{K}(x, ⓥ))(t),v(t))$ μ-a.e. in Γ,

$f_o(t,(\ (x, ⓤ))(t),u(t)) \in L_1(G)$,

$g_o(t,(\ (x, ⓥ))(t),v(t)) \in L_1(\Gamma)$,

$$I[x,u, ⓤ ,v, ⓥ] \leq i. \tag{6.16}$$

Relations (6.15) and (6.16) show, by comparison, that

$$\mathfrak{J}(x, ⓤ , ⓥ)w = h(x,u, ⓤ ,v, ⓥ)w \quad \text{for all } w \in W.$$

Thus, the system $(x,u, ⓤ ,v, ⓥ)$ is admissible, and since Ω is closed, the same system belongs to Ω, and $I[x,u, ⓤ ,v, ⓥ] \geq i$. Thus, $I = i$, and existence theorem (6.i) is thereby proved.

Remark. Remarks 1, 2, and 3 of § 4 apply to the present theorem (6.i) as well.

Example. Let G be an open bounded connected subset of E^ν, $\nu \geq 1$, of class K. We are concerned with the minimum of a functional

$$I[x,u] = \int_G f_o(t,(\mathfrak{M}x)(t),u(t))dt, \tag{6.17}$$

with no boundary conditions, with state equations which we want to be a weak form of

$$\sum_{i=1}^{\nu} \partial^2 x/(\partial t^i)^2 = f(t,(\mathcal{M}x)(t),u(t)), \tag{6.18}$$

and with constraints

$$(\mathcal{M}x)(t) \in A(t), \; u(t) \in U(t,(\mathcal{M}x)(t)). \tag{6.19}$$

Here x and u are functions on G. Thus, $g_o = 0$, we have no boundary condition on x, we can take $g = 0$, $\mathcal{J} = 0$, $\mathcal{K} = 0$, and need make no references to Γ, B, V, $\mathring{M}$. In this problem x and u are functions on G.

We shall think of W as simply being $C_o^\infty(G)$ and write elements $w = (w_1,o) \in W$ as simply $w_1(t)$, $t \in G$, with $w_1 \in C_o^\infty(G)$. As a weak form of (6.18) we take now:

$$-\int_G \sum_{i=1}^{\nu} (\partial x/\partial t^i)(\partial w_1/\partial t^i)dt = \int_G f(t,(\mathcal{M}x)(t),u(t))w_1(t)dt \tag{6.20}$$

for all $w_1 \in C_o^\infty(G)$. It is easy to verify that any (strong) solution x, u of (6.18), say $x \in W_1^2(G)$, $u \in T$, is certainly a solution of (6.20).

We take $x \in W_1^1(G)$, $p = 1$, $u \in T$, (that is, u measurable in G). We can take in $W = C_o^\infty(G)$ any topology which is stronger than the topology on $C_o^\infty(G)$ induced by the $W_\infty^1(G)$ norm. For instance:

$$\|w_1\|_W = \mathrm{Max}|w_1(t)| + \sum_{i=1}^{\nu} \mathrm{Max}|\partial w_1/\partial t^i|,$$

where each Max is taken in G. Then $\|w_1\|_\infty = \|w_1\|_{L_\infty(G)} \le \|w\|_W$, and thus (6.1) holds with $K = 1$, and $W \subset L_\infty(G)$.

Let $\mathfrak{M}$: $S \to (L_1(G))^s$ be any operator such that x, $x_k \in S$, $k = 1,2,\ldots$, and $x_k \to x$ weakly in S implies $\mathfrak{M}x_k \to \mathfrak{M}x$ strongly in $(L_1(G))^s$. Let $A(t)$, A, $U(t,y)$, M be defined as usual and closed, with $r = 1$, $m = 1$, let $f_o(t,y,u)$, $f(t,y,u)$ be real valued continuous functions on M, and let $\tilde{\tilde{Q}}(t,y)$ be the sets

$$\tilde{\tilde{Q}}(t,y) = \{(\overset{\circ}{z},z) \in E^2 | \overset{\circ}{z} \geq f_o(t,y,u),\ z$$

$$= f(t,y,u),\ u \in U(t,y)\}.$$

We assume that the sets $\tilde{\tilde{Q}}(t,y)$ have property (Q) in A. We shall assume that there is some function $\psi \geq 0$, $\psi \in L_1(G)$ such that $f_o(t,y,u) \geq -\psi(t)$ for all $t \in G$. We shall also assume that growth condition (H) holds for f_Q and f with $p = 1$. Here $\mathfrak{F}$: $S \to W^*$ is defined by

$$\mathfrak{F}(x)w_1 = \int_G \Sigma_{i=1}^{\nu} (\partial x/\partial t^i)(\partial w_1/\partial t^i)dt .$$

Now, if x, $x_k \in S = W_1^1(G)$, $k = 1,2,\ldots$, and $x_k \to x$ weakly in $S = W_1^1(G)$, then $\partial x_k/\partial t^i \to \partial x/\partial t^i$ as $k \to \infty$ weakly in $L_1(G)$, $i = 1,\ldots,\nu$, hence $(\mathfrak{F}x_k)w_1 \to (\mathfrak{F}x)w_1$ for every $w_1 \in L_\infty(G)$ and then certainly for every $w_1 \in W = C_o^\infty(G)$. Thus, the hypothesis required on $\mathfrak{F}$ in (6.i) is satisfied.

A pair (x,u) is here admissible provided $x \in W_1^1(G)$, u is measurable in G, $(\mathfrak{M}x)(t) \in A(t)$ and $u(t) \in U(t,(\mathfrak{M}x)(t))$ a.e. in G, $f_o(t,(\mathfrak{M}x)(t),u(t)) \in L_1(G)$, and relation (6.20) holds for every $w_1 \in W = C_o^\infty(G)$.

Note that hypothesis (H) (for p = 1) certainly assures that also $f(t,(\mathcal{M}x)(t),u(t)) \in L_1(G)$.

We shall take for Ω a nonempty closed class of admissible pairs (x,u). Theorem (6.i) now guarantees that the functional (6.17) with state equation (6.20) (in weak form) and constraints (6.19) has an absolute minimum in Ω.

7. APPLICATIONS TO PROBLEMS OF OPTIMIZATION WITH AN EVOLUTION EQUATION IN WEAK FORM

In this section we apply theorem (6.i) to problems of optimization with an evolution equation in the weak form.

Example 1. We are using in this and following examples the notations of the example in § 5. We are concerned with the problem of the minimum of the functional

$$I[x,u,v_1,v_2,v_3] = \int_G f_o(t,\tau,(\mathcal{M}x)(t,\tau),u(t,\tau))dtd\tau + \int_{\Gamma_3} g_o(\tau,(\mathcal{K}x)(\tau),v_3(\tau))d\tau \qquad (6.21)$$

with no boundary conditions on Γ_1 and Γ_3, with a state equation (concerning Γ_2 and G) which will be a suitable weak form of the system of equations

$$\partial x/\partial t - \sum_{i=1}^{\nu} \partial^2 x/(\partial\tau^i)^2 = f(t,\tau,(\mathcal{M}x)(t,\tau),u(t,\tau)) \text{ in } G, \qquad (6.22)$$

$$\partial x/\partial n + v_2(t,\tau)\gamma_2 x(t,\tau) = 0 \quad \text{in } \Gamma_2. \tag{6.23}$$

and constraints

$$(\mathfrak{M}x)(t,\tau) \in A(t,\tau),\ u(t,\tau) \in U(t,\tau,(\mathfrak{M}x)(t,\tau)) \quad \text{a.e. in } G, \tag{6.24}$$

$$(\mathfrak{K}x)(\tau) \in B(\tau),\ v_3(\tau) \in V(\tau,(\mathfrak{K}x)(\tau)) \quad \text{a.e. in } \Gamma_3. \tag{6.25}$$

Here x, u: $G \to E^1$, v_2: $\Gamma_2 \to E^1$, v_3: $\Gamma_3 \to E^1$, denote real valued functions, x state variable, u, v_2, v_3 controls. In other words, we are interested in the determination of a function $x(t,\tau)$ in G (in particular, of its initial values, say $v_1(\tau) = x(o,\tau)$ on Γ_1), and of controls $u(t,\tau)$ in G, $v_2(t,\tau)$ in Γ_2, $v_3(\tau)$ in Γ_3, such that the functional (6.21) has its minimum value, under constraints (6.24), (6.25), and a suitable weak form of state equations (6.22), (6.23).

We take for W the space of all pairs $w = (\omega,\gamma\omega)$, where $\omega \in C^\infty(\mathrm{cl}G)$. As a weak form of (6.22-23) we take the equation

$$\begin{aligned} &\int_G \Sigma_{i=1}^{\nu} (\partial x/\partial\tau^i)(\partial\omega/\partial\tau^i)\,dt\,d\tau + \int_G (\partial x/\partial\tau)\omega(t,\tau)\,dt\,d\tau \\ &+ \int_{\Gamma_2} v_2(t,\tau)\gamma_2 x(t,\tau)\gamma_2\omega(t,\tau)\,dt\,d\mu \\ &= \int_G f(t,\tau,(\mathfrak{M}x)(t,\tau),u(t,\tau)\omega(t,\tau)\,dt\,d\tau \quad \text{for all } \omega \in C^\infty(G) \end{aligned} \tag{6.26}$$

Here $\mathfrak{J}w$, or $\mathfrak{J}(x,v_2)w$, that is, the operator $\mathfrak{J}$, is defined by the first member of (6.26).

It is easy to verify that any strong solution x, u, v_2 of (6.22-23), say with $x \in S_1^2$, $u \in T$, $v_2 \in L_\infty(\Gamma_2)$, is certainly a solution of (6.26) for all $\omega \in C^{\infty}(G)$.

We shall take here $x \in S = W_1^1(G)$, $p = 1$, $u \in T$, $v_2 \in \mathring{T}_2$ with $\mathring{T}_2$ a weakly closed subset of $L_\infty(\Gamma_2)$, which is bounded in the norm of $L_\infty(\Gamma_2)$ (in other words, the elements $v_2 \in \mathring{T}_2 \subset L_\infty(\Gamma_2)$ are equibounded in Γ_2). We take in W any topology which is stronger than the topology on $C^\infty(G) \times C^\infty(\Gamma)$ induced by the topologies of $W_\infty^1(G)$ and $L_\infty(\Gamma)$, say

$$\|w\|_W = \|(\omega,\gamma\omega)\|_W = \operatorname{Max}|\omega(t,\tau)| + \Sigma_{i=1}^{\nu} \operatorname{Max}|\partial\omega/\partial\tau^i|$$

where all Max are taken in clG. Then, (6.1) holds with $q = \infty$, $K = 2$, and $W \subset W_\infty^1(G) \times L_\infty(\Gamma)$.

Let $\mathfrak{M}$: $S \to (L_1(G))^s$ and $\mathfrak{K}$: $S \to (L_1(\Gamma_3))^{s'}$ be operators such that x, $x_k \in S = W_1^1(G)$, $k = 1,2,\dots$, and $x_k \to x$ weakly in S implies $\mathfrak{M}x_k \to \mathfrak{M}x$ strongly in $(L_1(G))^s$, $\mathfrak{K}x_k \to \mathfrak{K}x$ strongly in $(L_1(\Gamma_3))^{s'}$.

Note that $x_k \to x$ weakly in $S = W_1^1(G)$ implies that $x_k \to x$ strongly in $L_1(G)$, $\partial x_k/\partial t \to \partial x/\partial t$, $\partial x_k/\partial\tau^i \to \partial x/\partial\tau^i$ weakly in $L_1(G)$, $i = 1,\dots,\nu$, and $\gamma x_k \to \gamma x$ strongly in $L_1(\Gamma)$. If v_2, v_{2k}, $k = 1,2,\dots$, are elements of $\mathring{T}_2 = L_\infty(\Gamma_2)$, with $v_{2k} \to v_2$ weakly in $L_\infty(\Gamma_2)$, then $v_{2k}(\gamma_2 x_k) \to v_2(\gamma_2 x)$ weakly in $L_\infty(\Gamma_2)$. Finally, from the definition of $\mathfrak{J}(x, v_2)$, we conclude that $\mathfrak{J}(x_k, v_{2k})w \to \mathfrak{J}(x, v_2)w$ as $k \to \infty$ for every $w = (\omega,\gamma\omega) \in W$, as requested.

For any $(t,\tau) \in G$ let $A(t,\tau) \subset E^s$ be a given set, let A, $U(t,\tau,y)$, M be defined as usual, A and M closed, let $f_o(t,\tau,u)$, $f(t,\tau,y,u)$ be real valued continuous functions on M. Let $\tilde{\tilde{Q}}(t,\tau,y)$ be the sets

$$\tilde{\tilde{Q}}(t,\tau,y) = \{(\mathring{z},z) \in E^2 | \mathring{z} \geq f_o(t,\tau,y,u),\ z$$

$$= f(t,\tau,y,u),\ u \in U(t,\tau,y)\},$$

and let us assume that these sets have property (Q) on A. Also, in harmony with (6.i), we assume that there is a function $\psi(t,\tau) \geq 0$, $\psi \in L_1(G)$ with $f_o(t,\tau,y,u) \geq -\psi(t,\tau)$ for all $(t,\tau,y,u) \in M$, and that for every $\varepsilon > 0$ there is some function $\phi_\varepsilon(t,\tau) \geq 0$, $\psi_\varepsilon \in L_1(G)$, such that $|f(t,\tau,y,u)| \leq \phi_\varepsilon(t,\tau) + \varepsilon f_o(t,\tau,y,u)$ for all $(t,\tau,y,u) \in M$.

For any $(T,\tau) \in \Gamma_3$, that is, $\tau \in \mathrm{cl}G'$, let $B(\tau) \subset E^{s'}$ be a given set, let B, $V(\tau,\mathring{y})$, $\mathring{M}$ be defined as usual, B and $\mathring{M}$ closed, let $g_o(\tau,\mathring{y},v_3)$ be real valued and continuous on $\mathring{M}$. We assume that there is a real valued function $\psi_3(\tau) \geq 0$, $\psi_3 \in L_1(G')$ with $g_o(\tau,\mathring{y},v_3) \geq -\psi_3(\tau)$ for all $(\tau,\mathring{y},v_3) \in \mathring{M}$. Here $g = 0$, hence the corresponding sets $\tilde{\tilde{R}}$ are half straight lines, and certainly are convex and satisfy property (Q) and the corresponding part of condition (H) is also satisfied. On Γ_1 and Γ_2 we have both $g = 0$, $g_o = 0$, and no further discussion is needed.

A system (x,u,v_2,v_3) is here admissible provided $x \subset W_1^1(G)$, u is measurable in G, v_3 measurable in Γ_3, $v_2 \in \mathring{T}_2 \subset L_\infty(\Gamma_2)$, $f_o(t,\tau,(\mathfrak{M}x)(t,\tau),u(t,\tau)) \in L_1(G)$, $g_o(\tau,(\mathfrak{K}x)(\tau),v_3(\tau)) \in L_1(\Gamma_3)$, and relations (6.24), (6.25), (6.26) hold. Because of hypothesis (H), $f(t,\tau,(\mathfrak{M}x)(t,\tau),u(t,\tau))$ also is in $L_1(G)$. We shall take for Ω a nonempty closed class of admissible systems. Theorem (6.i) now guarantees that the functional (6.21) with state equation (6.26) (in weak form) and constraints (6.24), (6.25) has an absolute minimum in Ω.

Example 2. The same as example 1 where now we are concerned with the minimum of the functional (6.21), with constraints (6.24), (6.26), state equation which will be a suitable weak form of (6.22) in G, and no boundary conditions on Γ_1, Γ_2, Γ_3. Here x, u: $G \to E^1$, v_3: $\Gamma_3 \to E^1$, denote real valued functions, x state variable, u, v_3 controls. In other words, we are concerned with the determination of a function $x(t,\tau)$ in G (in particular, of its initial values, say $v_1(\tau) = x(o,\tau)$ on σ_1, and lateral values $v_2(t,\tau) = x(t,\tau)$ on Γ_2), and of controls $u(t,\tau)$ in G and $v_3(\tau)$ on Γ_3, such that the functional (6.21) has its minimum value under constraints (6.24), (6.25) and suitable weak form of (6.22).

Here we take for W the space of all pairs $w = (\omega, o)$ where $\omega \in C_o^\infty(G)$. As a weak form of (6.22) we take

$$\int_G \sum_{i=1}^{\nu} (\partial x/\partial \tau^i)(\partial \omega/\partial \tau^i)\,dt\,d\tau + \int_G (\partial x/\partial t)\,\omega(t,\tau)\,dt\,d\tau$$

$$= \int_G f(t,\tau,(\mathfrak{M}x)(t,\tau),u(t,\tau))\,\omega(t,\tau)\,dt\,d\tau$$

for all $\omega \in C_o^\infty(G)$. The discussion of this example is similar to example 1.

Examples similar to the ones above have been considered by J. L. Lions [7ab].

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