DEVELOPMENT AND APPLICATION OF VECTOR MATHEMATICS FOR KINEMATIC ANALYSIS OF THREE-DIMENSIONAL MECHANISMS

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TABLE OF CONTENTS

TABLE OF CONTENTS CONT'D

LIST OF FIGURES

LIST OF FIGURES CONT'D

LIST OF FIGURES CONT'D

LIST OF TABLES

NOMENCLATURE

$\vec{C}_i$	Vector constant. Subscript may be dropped.
c_i	Scalar constant.
$D^n \vec{r}_{p_i p_j}$	The nth time derivative of $\vec{r}_{p_i p_j}$. $D\vec{r}_{p_i p_j}$ and $D^2\vec{r}_{p_i p_j}$ are the velocity and acceleration of point p_i relative to point p_j and the ground reference frame.
$D^n \vec{\omega}_{ij}$	The nth time derivative of $\vec{\omega}_{ij}$. $D\vec{\omega}_{ij}$ is the angular acceleration of body i relative to body j.
$(\vec{f}_{ij})_{p_k}$	Force exerted on body i, by body j, at point p_k. Drop the subscript p_k if only one force is exerted on i by j. Drop all subscripts if there is only one force in the linkage.
$\vec{g}_i$	Inertial Force exerted on link i ($\vec{g}_i \equiv - m_i D^2 \vec{r}_{c_i p_o}$)
$\vec{H}_i$	Moment of momentum of body i about its center of mass.
$\hat{i}, \hat{j}, \hat{k}$	Unit vectors of the ground reference frame.
m_i	Mass of link i.
$\vec{p}_i, \vec{q}_i, \vec{r}_i, \vec{s}_i, \vec{t}_i$	Dummy position vectors. Subscript may be dropped.
$\vec{r}_{p_i p_j}$	Position vector to point p_i from point p_j.

NOMENCLATURE CONT'D

θ_i	Azimuthal angle of a vector relative to $\hat{\lambda}_i$, $\hat{\mu}_i$, $\hat{\nu}_i$.
$\hat{\lambda}_i$, $\hat{\mu}_i$, $\hat{\nu}_i$	Unit vectors of the ith dummy reference frame. Orientation relative to $\hat{i}$, $\hat{j}$, $\hat{k}$ is always known.
μ_{ij}	Coefficient of Coulomb friction for translational motion of body i relative to body j. (Unitless)
ρ_{ij}	Coefficient of Coulomb friction for rotational motion of body i relative to body j. (Units of length)
$\vec{\sigma}_i$	Inertial Torque exerted on link i ($\vec{\sigma}_i \equiv -D\vec{H}_i$).
$(\vec{\tau}_{ij})_{p_k}$	Torque exerted on body i by body j at point p_k. Drop the p_k subscript if only one torque is exerted on i by j.
ϕ_i	Polar angle of a vector relative to $\hat{\lambda}_i$, $\hat{\mu}_i$, $\hat{\nu}_i$.
$\vec{\omega}_{ij}$	Angular velocity of body i relative to body j.

1.0 INTRODUCTION

Kinematic analysis is an old but important subject in many areas of engineering. It is a much simpler subject than dynamics because it assumes input motion instead of input force; problems can therefore be represented by algebraic equations instead of differential equations. From another viewpoint kinematics is an input to dynamics, because the kinetic and potential energy terms in Lagrange's Equations are kinematic expressions.

Vector analysis was created in the 1870's by J. Willard Gibbs [41] in response to the need for a succinct, natural mathematics for the problems of science and engineering. Perhaps this mathematics was partly inspired by problems in Gibbs' own thesis work--the first doctoral thesis in engineering in the United States: "On the Form of the Teeth of Wheels in Spur Gearing." [41] Vector analysis has become increasingly popular and is now part of the background of most scientists and engineers.

It seems inevitable that kinematic analysis should be pursued by vector methods. Almost every quantity involved in kinematics is a vector or the magnitude of a vector. (Angular quantities are exceptions, but all orders of their derivatives are vectors.) Most kinematic problems can be formulated as single or simultaneous vector equations, and these equations can usually be solved through use of vector operations.

However, conventional vector analysis has not been employed in kinematic analysis to nearly the extent possible. Instead, graphics, complex numbers, matrices, dual numbers, and quaternion algebra have been predominant tools.

Graphics and related methods have been important because they avoid detailed computation and provide a visual perspective. Much of this work has been done in Germany by Altman [2-4], Beyer [7-18], Federhofer [39], Hein [44], Keler [45], and others. With the advent of the digital computer the computational advantage of graphical methods is diminished. Of course, it has always been difficult to apply graphical methods to three-dimensional analysis.

Analysis and synthesis by conventional complex mathematics has been very successful for two-dimensional problems, partly because of the convenience of polar notation. Work by this method has been done by Freudenstein, McLarnan, Raven [61], Roth, Sandor, and others. Extension to three-dimensional analysis was suggested by Raven, but otherwise conventional complex mathematics has remained a two-dimensional tool.

Matrix methods have been developed and applied by Hartenberg, Denavit, and Uicker [26-31, 71-73]. A computer program, based on this mathematics, will obtain position, motion, and force solutions for the complete motion cycle of any single loop, three-dimensional mechanism connected by lower pairs. Two or three minutes (IBM 7090) are required for a cycle. The method has not been extended to complex spatial mechanisms and is more detailed than necessary for mechanisms of four links or less. Iteration is required for the position solutions, and interpretation of the matrix equations is difficult. However, it is a beautifully formulated approach and it affords the only immediately available numerical solution to an important category of mechanisms.

Dual numbers, quaternions, and other less familiar mathematics have been applied to three-dimensional mechanism analysis. Dimentberg [31-33], Denavit [26,27] and Dobrovolskii [35,36] have done work with dual numbers. More recently, Yang has developed an approach based on dual quaternions [79,80]. These approaches have advantages in the representation of spatial problems, but any advantages they may have for obtaining solutions have not been made clear.

The most difficult problem in kinematic analysis is determination of position; all other problems except frictional force analysis are linear. Physically, a mechanism may have two, four, eight, or many possible positions--depending on its complexity. Which position it actually takes depends on its initial assembly and subsequent behavior at locking positions. Because of these physical effects, probably the simplest position solution for any given mechanism is an algebraic polynomial of degree equal to the number of possible positions. No matter how the solution is formulated, there is the basic difficulty of obtaining the roots of this polynomial.

However, there are practical difficulties associated with advanced systems of mathematics. In obtaining polynomial solutions, it is of critical importance to fully exploit the symmetry of the problem. Otherwise a polynomial of artificially high degree must be generated. There is a danger that the mathematics itself will obscure the symmetry. Also, the very familiarity of the mathematics becomes important if no one method affords relative advantages in solution.

There have been many other contributions to three-dimensional kinematics besides those mentioned here. Several involve use of conventional vector mathematics--particularly those of Beggs [5], Kislitsin [47], Mangeron and Dragan [50-53], and Rim [64]. These do not overlap material in this thesis, but are included in the reference section as a convenience to those with general interest. In addition, there is probably a large amount of proprietary work that is unavailable. Several texts are included because of their usefulness as references.

The author's interest in three-dimensional kinematics was initiated by simultaneous exposure to the theory of vector analysis and Professor J. E. Shigley's practical observations on the "unit vector method" [67]. It has always been clear that mechanism problems can be represented by vector equations, but emphasis on the idea of factoring magnitude and unit vector leads to convenient means of solving the equations. Some of the author's preliminary work was published [21-23] and served as a basis for discussion and further generation of ideas.

Methods of analysis are described in later sections; a summary of the motives of analysis is appropriate here:

1) The wide range of current mechanism problems and the availability of the digital computer suggest a need for more uniform, familiar, definitive methods for kinematic analysis. The same approach should apply to two- and three-dimensional mechanisms, connected by lower or higher pairs into any number of loops. The mathematics should be simple and familiar, so that it can be interpreted, taught, and programmed easily. Solutions should be exact, or at most require iteration on only one variable.

2) Position solutions are inherently difficult because of their nonlinearity. However, direct solutions are important for purposes of interpretation, reliability, computation speed, and obtaining <u>all</u> the real roots. Emphasis is therefore placed on obtaining solutions as single polynomials. Optimum solutions to relatively simple, common conditions are derived and categorized. These include the Tetrahedron Solutions--a "complete" set of solutions to the single equation, sum of vectors is zero. The Tetrahedron Solutions apply to most practical three-dimensional mechanisms of four links or less. Most of the solutions are interpretable and can reasonably be evaluated by hand computation; all can be evaluated by the digital computer in hundredths of a second. More complicated conditions reduce to the problem of simultaneous solution of polynomials in two or more variables, even

when maximum use has been made of problem symmetry. An approach to these problems utilizing tensor formulation, the eliminant, and digital computation is described. At present capability any two low-degree polynomials can be reduced to a single resultant polynomial in a few minutes or seconds. Extension to the solution of more than two simultaneous polynomials of higher degree will require statistical methods, iteration, and/or a clearer insight to the nature of systems of polynomials.

3) Motion solutions are intrinsically linear. Moreover, they are dependent only on input motions of the same order, all lower order motions and position. This basic simplicity should be exploited, so that the means are clear for calculating velocity, acceleration, and higher order motions for any mechanism. In particular, direct differentiation of position solutions must be avoided.

4) Force solutions are intrinsically linear when the joints are frictionless. However, they are more detailed than motion solutions because two vector equations must be written for every link but one in the mechanism. A method for reducing these vector equations to a determinate set of simultaneous linear algebraic equations must be made clear. Also, specialized procedures for obtaining more interpretable solutions should be investigated.

2.0 METHOD OF ANALYSIS

2.1 Terminology and Basic Operations

The fundamentals of vector analysis are clearly explained in many texts. Notation and terminology vary; Table 2.1 and the nomenclature section (p. xi) explain that employed here. In particular, the symbols $\vec{a}$, $\hat{a}$, and a, respectively, denote vector, unit vector, and magnitude of $\vec{a}$. Table 2.2 defines kinematic terminology. Table 2.3 summarizes important vector relations.

2.2 Outline of Method

Sections 3.0 through 5.0 are devoted to an ordered development of vector methods for the analysis of linkages in general. Before this is begun the essentials of the approach will be illustrated via application to a simple planar example.

2.2.1 Problem Formulation

Consider the two-dimensional, offset, slider-crank mechanism shown in Figure 2.1:

Known Design Constants:

Vector: $\vec{r}_{p_1p_4}$

Unit vectors: $\hat{r}_{p_4p_3}$; all $\hat{\omega}_{ij}$, $D\hat{\omega}_{ij}$, $\hat{\tau}_{ij}$

TABLE 2.1

VECTOR TERMINOLOGY

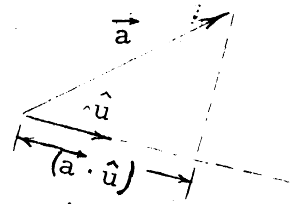
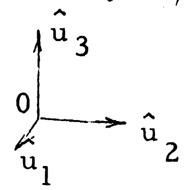
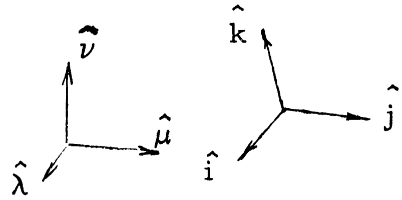
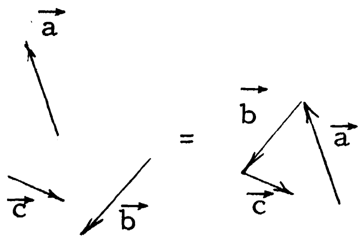
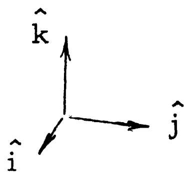
Item	Explanation	Representation
Addition and subtraction of vectors	Perform by adding or subtracting corresponding components	$\vec{a} \pm \vec{b} = (a_i \pm b_i)\hat{i} + (a_j \pm b_j)\hat{j} + (a_k \pm b_k)\hat{k}$
Component	Magnitude of the projection of a vector on the component direction.	
Coordinate frame	Three mutually perpendicular unit vectors (reference frame), plus a point considered as an origin.	
Dummy reference frame	Reference frame with orientation defined relative to ground reference frame. Fits natural geometry of problem.	
Equivalence	Two vectors are equal provided only that they have the same magnitude and direction. Vectors are unchanged by relocation with fixed orientation.	
Ground reference frame	Reference frame which is considered fixed. Express numerical results in terms of this frame.	
Magnitude	Scalar quantity (no direction). The magnitude of a vector is always positive. If a solution for magnitude is negative, the associated unit vector is reversed.	$a = (\vec{a} \cdot \vec{a})^{1/2}$

TABLE 2.1 CONT'D

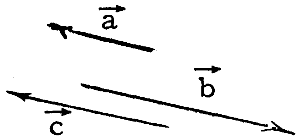
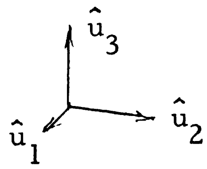
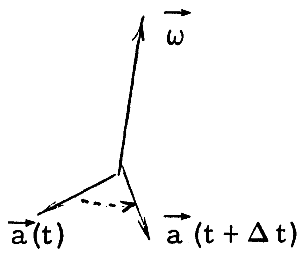
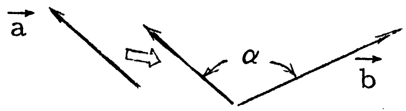
Item	Explanation	Representation
Parallel vectors	Vectors which are either co-directed or oppositely directed.	 $\vec{a} \parallel \vec{b} \parallel \vec{c}$
Reference frame	Three mutually perpendicular unit vectors. No origin. <u>Right-handed</u> if the rotation of the first unit vector into the second is co-directed with the third.	
Rotation	Angular displacement or motion. A vector quantity with unit vector directed perpendicularly to the instantaneous plane of motion. Magnitude positive if counter-clockwise; negative if clockwise.	
Scalar (dot) product	A scalar product of the magnitudes of two vectors and the cosine of the smallest angle between them.	$\vec{a} \cdot \vec{b} \equiv ab \cos \alpha$ $\vec{a} \cdot \vec{b} = (a_i b_i)+(a_j b_j)+(a_k b_k)$  $0 \leqq \alpha \leqq \pi$
Unit vector	The basic directional quantity. A vector of unit magnitude.	$\hat{a} \cdot \hat{a} = 1$ $\hat{a} = \sin \phi [\cos \theta \, \hat{\lambda} + \sin \theta \hat{\mu}] + \cos \phi \, \hat{\nu}$ $\hat{\lambda}, \hat{\mu}, \hat{\nu}$: right-hand dummy reference frame θ , ϕ: azimuthal and polar angles
Vector	Product of magnitude	$\vec{a} = a \, \hat{a}$

TABLE 2.1 CONT'D

Item	Explanation	Representation
Vector (cross)	A vector. Magnitude equals the product of the magnitudes of two vectors and the sine of the smallest angle between them. Unit vector directed according to positive rotation of first vector into second.	$\vec{a} \times \vec{b} \equiv (ab \sin \alpha)\, \hat{c}$ ($\hat{c}$ directed according to positive rotation of $\vec{a}$ into $\vec{b}$.) $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_i & a_j & a_k \\ b_i & b_j & b_k \end{vmatrix}$

TABLE 2.2

KINEMATIC TERMINOLOGY

Item	Definition
Dwell position	Mechanism position at which a zero velocity occurs with finite input velocity.
Force	Fundamental. Two uses: (1) Pure translational force; (2) Both translational and rotational force.
Link	Rigid body. A component of a mechanism.
Linkage	A system of interconnected links. Simple linkage if only one closed loop of links; complex linkage if more than one closed loop.
Locking position	Mechanism position at which output power is zero, regardless of the magnitude of input force.
Moment	Rotational force from both torque and $(\vec{f} \times \vec{r})$ terms.
Motion	All orders of the time derivative of position. Includes translational and angular velocity and acceleration.
Pair	Joint connecting and constraining relative motion between adjacent links. Higher pair: line or point contact. Lower pair: area contact.
Position	Instantaneous geometric configuration. Defined by position vectors between essential points.
Rotational or angular motion	Motion of one reference frame relative to another
Torque	Pure rotational force.
Translational motion	Motion of one point relative to another.

TABLE 2.3

VECTOR RELATIONS

Algebraic Operations

1a, b $\vec{a} \pm \vec{b} = (a_i \pm b_i)\hat{i} + (a_j \pm b_j)\hat{j} + (a_k \pm b_k)\hat{k}$

2 $\vec{a} \cdot \vec{b} = (a_i b_i) + (a_j b_j) + (a_k b_k)$

3 $$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_i & a_j & a_k \\ b_i & b_j & b_k \end{vmatrix} = (a_j b_k - a_k b_j)\hat{i} + (a_k b_i - a_i b_k)\hat{j} + (a_i b_j - a_j b_i)\hat{k}$$

4 $|\vec{a}| = +(\vec{a} \cdot \vec{a})^{1/2}$

Algebraic Identities

5 $\vec{a} = a\,\hat{a}$

6a, b $\vec{a} \cdot \vec{a} = a^2$; $\vec{a} \times \vec{a} = 0$

7a, b, c $\vec{a} + \vec{b} = \vec{b} + \vec{a}$; $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$; $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

8a $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

8b $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{c} \cdot \vec{b})\vec{a}$

9 $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$

10 $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$ $\qquad (\vec{a} \times \vec{b}) \cdot \vec{b} = 0$

11a $\vec{a} \cdot (\vec{b} \times \vec{c})$ changes sign if the cyclic order of the vectors is changed (e.g., $\vec{a}, \vec{b}, \vec{c}$ to $\vec{a}, \vec{c}, \vec{b}$). Otherwise, the value is unaffected by interchange of vectors and/or by exchange of cross and dot.

11b $[\vec{a} \cdot (\vec{b} \times \vec{c})]^2 = (abc)^2\{2(\hat{a} \cdot \hat{b})(\hat{b} \cdot \hat{c})(\hat{c} \cdot \hat{a}) - (\hat{a} \cdot \hat{b})^2 - (\hat{b} \cdot \hat{c})^2 - (\hat{c} \cdot \hat{a})^2 + 1\}$

TABLE 2.3 CONT'D

Differentiation Formulas

12 $D \equiv \frac{d}{dt}$; $D^n \equiv \frac{d^n}{dt^n}$ $n = 1, 2, 3 \ldots$

13 $D^n \vec{u} = (D^n u_i)\hat{i} + (D^n u_j)\hat{j} + (D^n u_k)\hat{k}$

14 $D\hat{u} = \vec{\omega} \times \hat{u}$

15 $D(\vec{u} + \vec{v}) = D\vec{u} + D\vec{v}$

16 $D(x\vec{u}) = (Dx)\vec{u} + x(D\vec{u})$

17 $D(\vec{u} \cdot \vec{v}) = (D\vec{u}) \cdot \vec{v} + \vec{u} \cdot (D\vec{v})$

18 $D(\vec{u} \times \vec{v}) = (D\vec{u}) \times \vec{v} + \vec{u} \times (D\vec{v})$

19 $D\vec{u} = D(u\,\hat{u}) = (Du)\hat{u} + (\vec{\omega} \times \vec{u})$

20 $D^2\vec{u} = D(D\vec{u}) = (D^2u)\hat{u} + \vec{\omega} \times (\vec{\omega} \times \vec{u}) + (D\vec{\omega} \times \vec{u}) + 2[\vec{\omega} \times (Du)\hat{u}]$

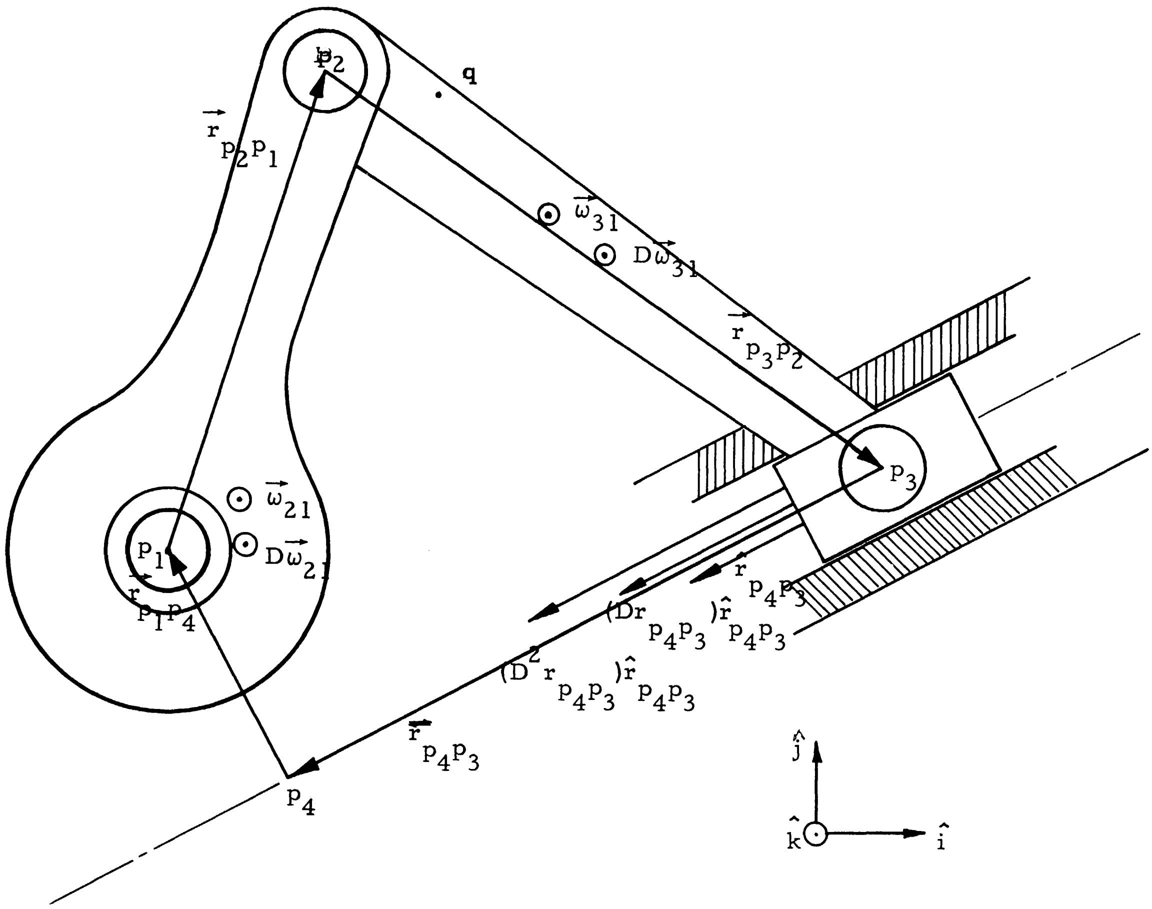


Figure 2.1 Planar Offset Slider-Crank Mechanism

Magnitudes: $r_{p_2p_1}$, $r_{p_3p_2}$

Pairs: (21), (32) hinge; (43) turn-slide

Mass distribution

Frictional characteristics (linear)

Negligible elastic effects

Known functions of time:

Vectors: $\vec{\omega}_{21}$, $D\vec{\omega}_{21}$, $\vec{\tau}_{21}$ (all in $\hat{k}$ direction)

Unit vector: $\hat{r}_{p_2p_1}$

Unknown:

Unit vector: $\hat{r}_{p_3p_2}$, all $\hat{f}_{ij}$

Magnitudes: $r_{p_4p_3}$; ω_{31}, $Dr_{p_4p_3}$; $D\omega_{31}$, $D^2r_{p_4p_3}$;

all f_{ij} and τ_{ij} (except τ_{21})

This is a kinematic problem, in which the forces are dependent on input positions and motions. Such problems are much simpler than dynamic problems (positions and motions dependent on input forces), and their solution can be obtained in an ordered, compartmented manner:

(1) Determine the unknown key positions ($\hat{r}_{p_4p_3}$, $r_{p_4p_3}$)

(2) Determine the unknown key velocities (ω_{31}, $Dr_{p_4p_3}$), regarding all positions as known and expressed as single symbols.

(3) Determine the unknown key accelerations $(D\omega_{31}, D^2 r_{p_4 p_3})$, regarding all positions and velocities as known and expressed as single symbols.

(4) Determine any higher order key motions desired, regarding all lower order motions as known.

(5) Determine the position and motion of any point in the mechanism (besides p_1, p_2, p_3, p_4), regarding the design of the individual links and the key positions and motions as known.

(6) Determine the force and torque exerted at the mechanism joints, regarding joint design and the key positions, velocities, and accelerations as known.

2.2.2 Position Solution

In general, mechanism position solutions are nonlinear and require solution of simultaneous vector and scalar equations. The present solution is of second degree and is determined from a single vector equation:

$$\vec{r}_{p_1 p_4} + \vec{r}_{p_2 p_1} + \vec{r}_{p_3 p_2} + \vec{r}_{p_4 p_3} = 0 \qquad (2.1)$$

At a given instant $\vec{r}_{p_1 p_4}$ and $\vec{r}_{p_2 p_1}$ are both known and can be summed into a single constant, $\vec{C}$. Vectors $\vec{r}_{p_3 p_2}$ and $\vec{r}_{p_4 p_3}$ are factored into magnitude and unit vector (a very frequent and convenient operation).

$$r_{P_3P_2}\hat{r}_{P_3P_2} + r_{P_4P_3}\hat{r}_{P_4P_3} + \vec{C} = 0 \tag{2.2}$$

$$\vec{C} \equiv \vec{r}_{P_1P_4} + \vec{r}_{P_2P_1} \tag{2.3}$$

Equation (2.2) contains two scalar unknowns, $r_{P_4P_3}$ and the angle defining $\hat{r}_{P_3P_2}$. The equation represents two scalar equations, being a two-dimensional vector equation, and is therefore a sufficient condition for the determination of $r_{P_4P_3}$ and $\hat{r}_{P_3P_2}$. The actual solution proceeds as follows:

$$r_{P_3P_2}\hat{r}_{P_3P_2} = -(r_{P_4P_3}\hat{r}_{P_4P_3} + \vec{C}) \tag{2.4}$$

Take the scalar product of each side of Equation (2.2) with itself, thereby eliminating $\hat{r}_{P_3P_2}$.

$$r^2_{P_3P_2} = r^2_{P_4P_3} + C^2 + 2(\vec{r}_{P_4P_3} \cdot \vec{C}) \tag{2.5}$$

Rearrange and factor,

$$r^2_{P_4P_3} + 2(\hat{r}_{P_4P_3} \cdot \vec{C})\, r_{P_4P_3} + (C^2 - r^2_{P_3P_2}) = 0 \tag{2.6}$$

From the quadratic formula,

$$r_{P_4P_3} = -(\hat{r}_{P_4P_3} \cdot \vec{C}) \pm [(\hat{r}_{P_4P_3} \cdot \vec{C})^2 - (C^2 - r^2_{P_3P_2})]^{1/2} \tag{2.7}$$

Vectors $\vec{r}_{P_4P_3}$ and $\vec{r}_{P_3P_2}$ can now be explicitly expressed in terms of known quantities.

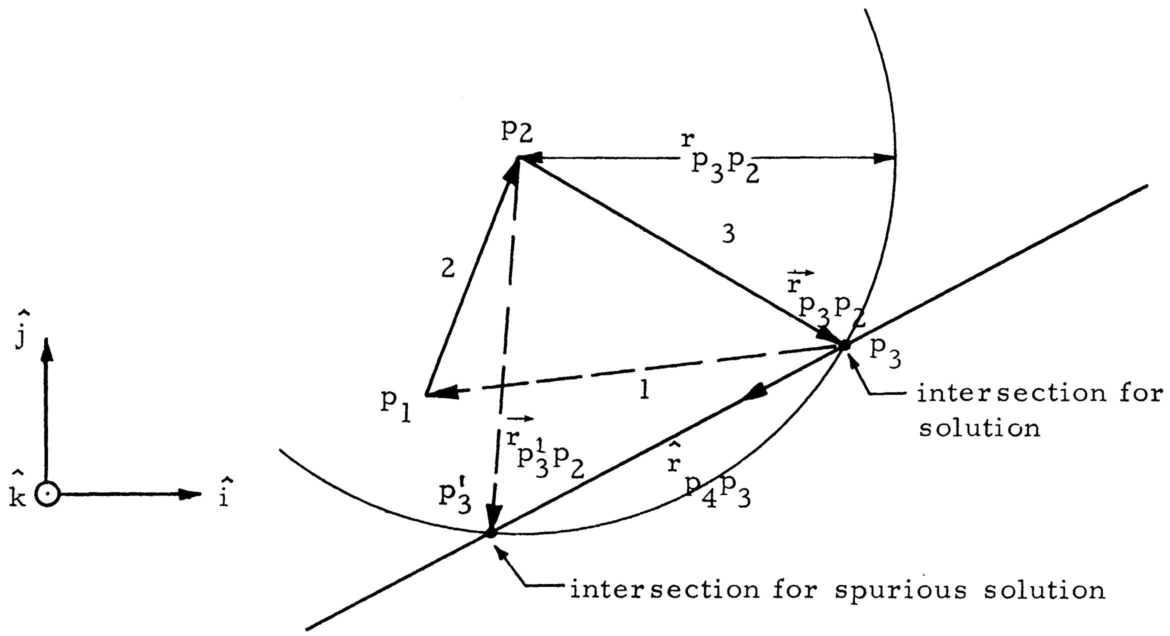


Figure 2.2 Graphical Solution to Two-Dimensional Offset Slider Crank-Mechanism

$$\vec{r}_{P_4P_3} = \left\{ -(\hat{r}_{P_4P_3} \cdot \vec{C}) \pm \left[(\hat{r}_{P_4P_3} \cdot \vec{C}) - (C^2 - r^2_{P_3P_2}) \right]^{1/2} \right\} \hat{r}_{P_4P_3} \tag{2.8}$$

$$\vec{r}_{P_3P_2} = \left\{ (\hat{r}_{P_4P_3} \cdot \vec{C}) \mp \left[(\hat{r}_{P_4P_3} \cdot \vec{C}) - (C^2 - r^2_{P_3P_2}) \right]^{1/2} \right\} \hat{r}_{P_4P_3} - \vec{C} \tag{2.9}$$

Observations:

(1) Graphically, the solutions to Equation (2.2) are obtained simply as the intersections of a straight line with a circle (Figure 2.2).

(2) Both the graphical and analytical solutions show that two physically real solutions exist--the first because of dual intersections, the second because of dual signs. More complicated mechanisms may have several physically real solutions. This poses the intrinsic difficulty that any solution in polynomial form must have a degree equal to or greater than the number of physically real solutions. The higher the degree of the polynomial, the more difficult it is to obtain the solutions. Of course, a given mechanism can only have a single instantaneous position, depending on how it was initially assembled and its subsequent behavior at locking positions.

(3) Complex roots (negative radicals in Equations (2.8) and (2.9)) indicate that the design parameters prohibit

TABLE 2.4

SOLUTIONS TO THE VECTOR TRIANGLE EQUATION

$$\vec{r} + \vec{s} + \vec{C} = 0$$

Case	Unknown	Known	Solution
1	$r,\ \theta_r$	$\vec{C},\ \vec{s}$	$\vec{r} = -(\vec{s} + \vec{C})$
2a	$r,\ s$	$\theta_r,\ \theta_s,\ \vec{C}$	$\vec{s} = \left[\dfrac{\vec{C}\cdot(\hat{r}\times\hat{k})}{\hat{r}\cdot(\hat{s}\times\hat{k})}\right]$
2b	$\theta_r,\ s$	$\vec{C},\ r,\ \theta_s$	$\vec{s} = \left[-(\vec{C}\cdot\hat{s}) \pm \{s^2 - [\vec{C}\cdot(\hat{s}\times\hat{k})]^2\}^{\frac{1}{2}}\right]\hat{s}$
2c	$\theta_r,\ \theta_s$	$\vec{C},\ r,\ s$	$\vec{s} = \pm\left[s^2 - \left(\dfrac{C^2+s^2-r^2}{2C}\right)^2\right]^{1/2}(\hat{C}\times\hat{k}) - \left(\dfrac{C^2+s^2-r^2}{2C}\right)\hat{C}$

assembly. Such a situation results here if $r_{p_3 p_2}$ is made small relative to C.

(4) In the analysis of simple planar mechanisms, an equation of the form of Equation (2.1) is very often the only equation that must be solved. There are only four unique arrangements of two unknown scalars in this equation, assuming that each unknown occurs in only one term. The solutions for each of these arrangements are summarized in Table 2.4. Note that the solution for the present mechanism corresponds to case 3.

(5) Conceivably, expressions for velocity and acceleration could be obtained by differentiating Equations (2.8) and (2.9) with respect to time. This would be difficult even for the explicit second degree solution obtained here. Explicit solutions to third and fourth degree polynomials are very detailed; for degrees higher than four they are theoretically impossible [75]. Thus, a more practical means for expressing motions is required.

2.3 Motion Solutions

Equation (2.1) can be differentiated as often as desired

$$D\vec{r}_{p_1 p_4} + D\vec{r}_{p_2 p_1} + D\vec{r}_{p_3 p_2} + D\vec{r}_{p_4 p_3} = 0 \tag{2.10}$$

$$D^2\vec{r}_{p_1 p_4} + D^2\vec{r}_{p_2 p_1} + D^2\vec{r}_{p_3 p_2} + D^2\vec{r}_{p_4 p_3} = 0 \tag{2.11}$$

In Equations 19 and 20, Table 2.3, the quantities $\vec{u}$ and $\vec{\omega}$ can be physically interpreted as the position vector $\vec{r}_{p_{i+1}p_i}$ and the angular velocity vector $\vec{\omega}_{i1}$. In three dimensions the unit vectors of angular velocity are time-dependent, and a certain development is required because of this. In two dimensions these vectors all have direction $\hat{k}$ (perpendicular to the plane of motion), and Equations 19 and 20, Table 2.3, become

$$D\vec{r}_{p_{i+1}p_i} = (Dr_{p_{i+1}p_i})\hat{r}_{p_{i+1}p_i} + \omega_{i1}(\hat{k} \times \vec{r}_{p_{i+1}p_i}) \tag{2.12}$$

$$\begin{aligned} D^2\vec{r}_{p_{i+1}p_i} &= [(D^2 r_{p_{i+1}p_i}) - \omega_{i1}^2 r_{p_{i+1}p_i}]\hat{r}_{p_{i+1}p_i} \\ &\quad + [(D\omega_{i1})(r_{p_{i+1}p_i}) + 2(\omega_{i1})(Dr_{p_{i+1}p_i})](\hat{k} \times \hat{r}_{p_{i+1}p_i}) \end{aligned} \tag{2.13}$$

To obtain a solution for the key velocities substitute Equation (2.12) into Equation (2.10) term by term. Note that physically $D\vec{r}_{p_1p_4}$, $Dr_{p_2p_1}$, $Dr_{p_3p_2}$, and ω_{41} are zero.

$$\omega_{21}(\hat{k} \times \vec{r}_{p_2p_1}) + \omega_{31}(\hat{k} \times \vec{r}_{p_3p_2}) + (Dr_{p_4p_3})\hat{r}_{p_4p_3} = 0 \tag{2.14}$$

There are only two unknowns in Equation (2.14): ω_{31} and $Dr_{p_4p_3}$. Equation (2.14) can be reduced to two scalar equations in these unknowns simply by taking the scalar product throughout--first with $\hat{i}$, then with $\hat{j}$. However, a more direct solution is obtained by taking scalar products throughout with $\hat{k} \times \hat{r}_{p_4p_3}$ and $\hat{r}_{p_3p_2}$.

$$\omega_{21}(\hat{k} \times \vec{r}_{P_2P_1}) \cdot (\hat{k} \times \hat{r}_{P_4P_3}) + \omega_{31}(\hat{k} \times \vec{r}_{P_3P_2}) \cdot (\hat{k} \times \hat{r}_{P_4P_3}) + 0 = 0 \tag{2.15}$$

$$\omega_{31} = - \frac{\omega_{21} r_{P_2P_1}}{r_{P_3P_2}} \frac{(\hat{r}_{P_2P_1} \cdot \hat{r}_{P_4P_3})}{(\hat{r}_{P_3P_2} \cdot \hat{r}_{P_4P_3})} \tag{2.16}$$

$$\omega_{21}[(\hat{k} \times \vec{r}_{P_2P_1}) \cdot \hat{r}_{P_3P_2}] + 0 + (Dr_{P_4P_3})(\hat{r}_{P_4P_3} \cdot \hat{r}_{P_3P_2}) = 0 \tag{2.17}$$

$$Dr_{P_4P_3} = - (\omega_{21} r_{P_2P_1}) \frac{[(\hat{k} \times \hat{r}_{P_2P_1}) \cdot \hat{r}_{P_3P_2}]}{(\hat{r}_{P_3P_2} \cdot \hat{r}_{P_4P_3})} \tag{2.18}$$

Equations (2.17) and (2.18) would have been much more detailed if $\hat{r}_{P_3P_2}$ had been expressed in full, via Equation (2.9). Instead, it is assumed to be completely known--having been determined in the position solution. Now, with ω_{31} and $Dr_{P_4P_3}$ determined, a very similar solution for the key accelerations can be obtained. Substitute Equation (2.13) into Equation (2.11) term by term.

$$-(\omega_{21}^2 r_{P_2P_1})\hat{r}_{P_2P_1} + (D\omega_{21})(r_{P_2P_1})(\hat{k} \times \hat{r}_{P_2P_1}) - (\omega_{31}^2 r_{P_3P_2})\hat{r}_{P_3P_2}$$

$$+ (D\omega_{31})(r_{P_3P_2})(\hat{k} \times \hat{r}_{P_3P_2}) + (D^2 r_{P_4P_3})\hat{r}_{P_4P_3} = 0 \tag{2.19}$$

For convenience, sum the three known terms into a single constant $\vec{C}_2$. The equation is then so similar to Equation (2.14) that the solution can be written by comparison. Formally, it is obtained from scalar products with the same two quantities $(\hat{k} \times \hat{r}_{P_4P_3})$ and $\hat{r}_{P_3P_2}$.

$$\vec{C}_2 + (D\omega_{31})(\hat{k} \times \vec{r}_{P_3P_2}) + (D^2 r_{P_4P_3})\hat{r}_{P_4P_3} = 0 \qquad (2.20)$$

$$C_2 \equiv -\omega_{21}^2 \vec{r}_{P_2P_1} - \omega_{31}^2 \vec{r}_{P_3P_2} + D\omega_{21}(\hat{k} \times \vec{r}_{P_2P_1}) \qquad (2.21)$$

$$D\omega_{31} = -\frac{\vec{C}_2 \cdot (\hat{k} \times \vec{r}_{P_3P_2})}{r_{P_3P_2}(\hat{r}_{P_3P_2} \cdot \hat{r}_{P_4P_3})} \qquad (2.22)$$

$$D^2 r_{P_4P_3} = \frac{-\vec{C}_2 \cdot \hat{r}_{P_3P_2}}{(\hat{r}_{P_3P_2} \cdot \hat{r}_{P_4P_3})} \qquad (2.23)$$

Suggested Generalizations:

(1) Kinematic motion solutions will always be linear. Equations (2.12) and (2.13) can never introduce unknown unit vectors into Equations such as (2.10) and (2.11), and the unknowns that are introduced occur in additive terms, not in products with each other.

(2) The denominators of the motion solutions will be the same regardless of the order of motion. Thus in Equations (2.14), (2.19), and corresponding

higher order equations, the directions associated with terms of possible unknown magnitude will always correspond from order to order. Here ω_{21} and $D\omega_{21}$ have vector $(\hat{k} \times \vec{r}_{p_2p_1})$; ω_{31} and $D\omega_{31}$, $(\hat{k} \times \vec{r}_{p_3p_2})$; $Dr_{p_4p_3}$ and $D^2r_{p_4p_3}$, $\hat{r}_{p_4p_3}$. Mechanism locking positions can be identified by the zeros of the motion denominator. Here all output motions approach infinity, but input motion and output power approach zero.

2.4. Position and Motion of Any Point

With the essential positions and motions determined, the position and motion of any point fixed anywhere in the mechanism can be determined. For example, consider a point q fixed in link 3, Figure 2.3. The position of this point relative to link 3 is specified by the design of the link. That is, in the following equation $\vec{r}_{qp_2}$ is dependent upon $\hat{r}_{p_3p_2}$, by the design constants c_1 and c_2.

$$\vec{r}_{qp_2} = c_1\hat{r}_{p_3p_2} + c_2(\hat{k} \times \hat{r}_{p_3p_2}) \tag{2.24}$$

The vector $\vec{r}_{qp_1}$ can be determined by a vector sum of known vectors:

$$\vec{r}_{qp_1} = \vec{r}_{qp_2} + \vec{r}_{p_2p_1} \tag{2.25}$$

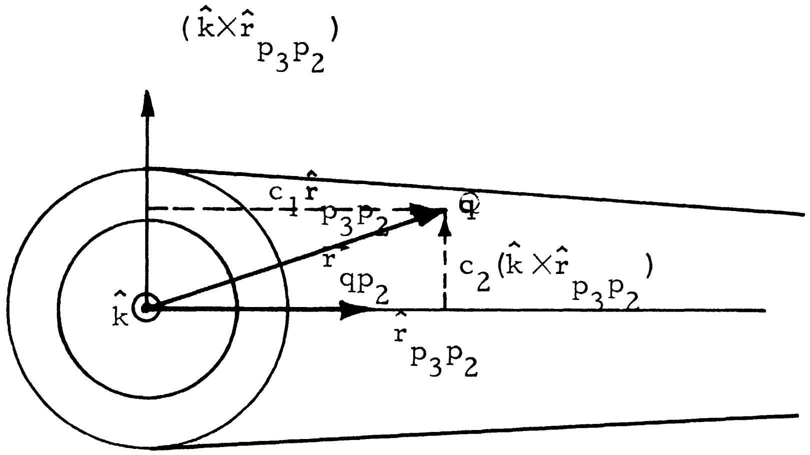


Figure 2.3 Dummy Reference Frame for Determination of Position and Motion of an Arbitrary Point, Given Essential Positions and Motions

Motion can be determined by differentiating Equation (2.25) and substituting Equations (2.12) and (2.13).

$$D\vec{r}_{qp_1} = D\vec{r}_{qp_2} + D\vec{r}_{p_2p_1} \tag{2.26}$$

$$D\vec{r}_{qp_1} = \omega_{31}(\hat{k} \times \vec{r}_{qp_2}) + \omega_{21}(\hat{k} \times \vec{r}_{p_2p_1}) \tag{2.27}$$

$$D^2\vec{r}_{qp_1} = D^2\vec{r}_{qp_2} + D^2\vec{r}_{p_2p_1} \tag{2.28}$$

$$\begin{aligned} D^2\vec{r}_{qp_1} = -\omega_{31}^2\vec{r}_{qp_2} + (D\omega_{31})(\hat{k} \times \vec{r}_{qp_2}) - \omega_{21}^2\vec{r}_{p_2p_1} \\ + (D\omega_{21})(\hat{k} \times \vec{r}_{p_2p_1}) \end{aligned} \tag{2.29}$$

2.5 Force Solutions

The force equilibrium solution to the mechanism of Figure 2.1 will be obtained assuming Coulomb friction in the pairs and significant link mass. This will illustrate that frictional effects can introduce nonlinearity and that inertial effects introduce only additional detail.

Figure 2.4 is an equilibrium diagram for the mechanism of Figure 2.1. The mechanism is driven against a known force, $-(f_{14o})\hat{s}$, by means of an input torque, $(\tau_{21i})\hat{k}$. The input torque varies in response to output, frictional and inertial forces.

Equilibrium conditions:

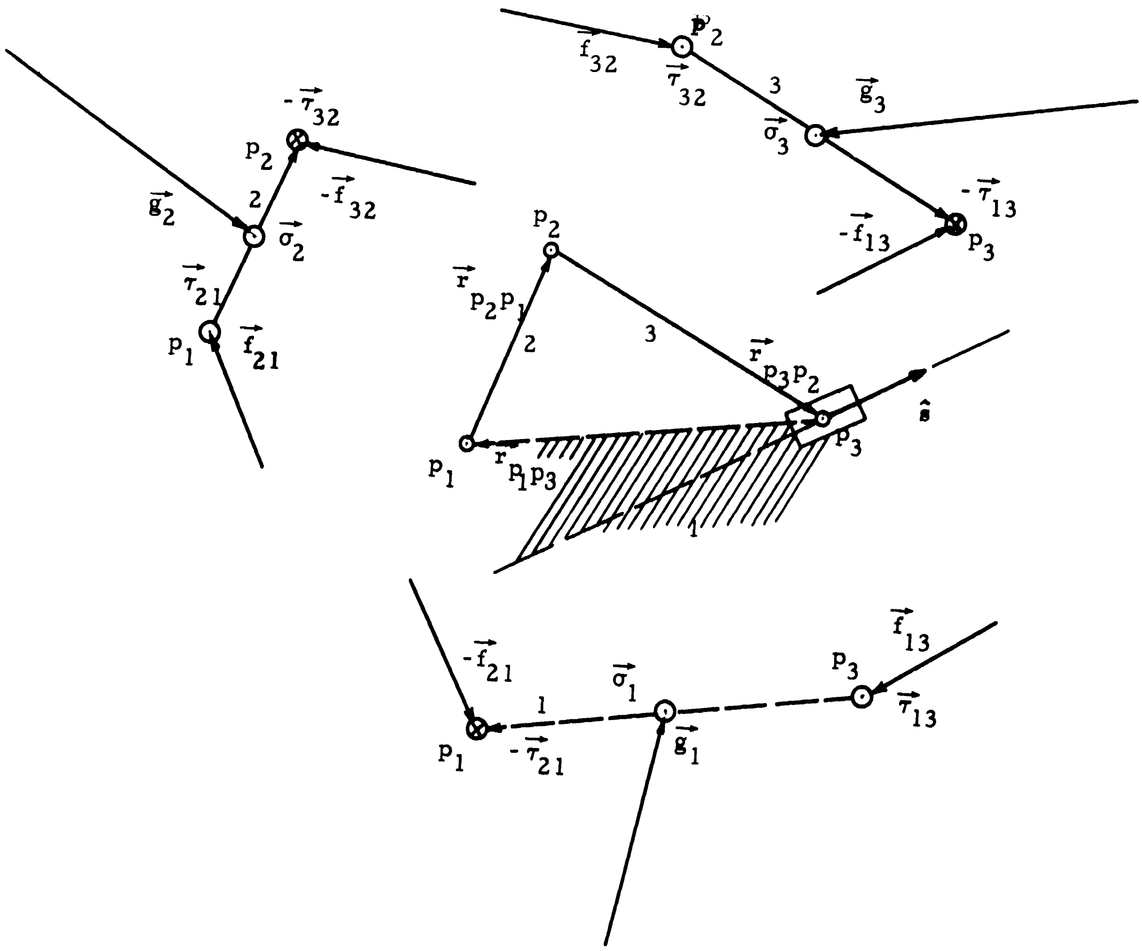


Figure 2.4 Equilibrium Diagram for an Offset Slider-Crank Mechanism

$$\vec{f}_{21} - \vec{f}_{32} + \vec{g}_2 = 0 \qquad (2.30)$$

$$\vec{f}_{32} - \vec{f}_{13} + \vec{g}_3 = 0 \qquad (2.31)$$

$$\vec{f}_{13} - \vec{f}_{21} + \vec{g}_1 = 0 \qquad (2.32)$$

$$\vec{\tau}_{21} - \vec{\tau}_{32} + (\vec{f}_{21} \times \vec{r}_{p_2p_1}) + (\vec{g}_2 \times \vec{r}_{p_2c_2}) + \vec{\sigma}_2 = 0 \qquad (2.33)$$

$$\vec{\tau}_{32} - \vec{\tau}_{13} + (\vec{f}_{32} \times \vec{r}_{p_3p_2}) + (\vec{g}_3 \times \vec{r}_{p_3c_3}) + \vec{\sigma}_3 = 0 \qquad (2.34)$$

$$\vec{\tau}_{13} - \vec{\tau}_{21} + (\vec{f}_{13} \times \vec{r}_{p_1p_3}) + (\vec{g}_1 \times \vec{r}_{p_1c_1}) + \vec{\sigma}_1 = 0 \qquad (2.35)$$

A force and moment equilibrium equation has been written for each link in the mechanism, including the ground link. Of the three equations in each set, only two are independent. A solution can be obtained from any four equations, two from each set. The terms $\vec{g}_i$ and $\vec{\sigma}_i$ are inertial forces and torques. These must be included even for the ground link.[1]

Pairs (21), (32), and (43) are affected by Coulomb friction. The direction of the frictional torque is opposite to the direction of rotation ($\hat{\tau}_{ij} = -\hat{\omega}_{ij}$) and the magnitude is proportional to the transmitted force.

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[1] Mathematically, any one equation in a set must be the sum of the other two. Physically, the acceleration of the ground link approaches zero, but its mass approaches infinity. The product, mass times acceleration, will in general be comparable in magnitude to the inertial terms of the other links.

$$\vec{\tau}_{21} = \vec{\tau}_{21i} - (\rho_{21} f_{21}) \hat{\omega}_{21} \tag{2.36}$$

$$\vec{\tau}_{32} = -(\rho_{32} f_{32}) \hat{\omega}_{32} \tag{2.37}$$

$$\vec{\tau}_{13} = -(\rho_{13} f_{13}) \hat{\omega}_{13} \tag{2.38}$$

The force $\vec{f}_{13}$ is the sum of three terms: output force, $(f_{13o})\hat{s}$; reaction force, $(f_{13r})(\hat{k} \times \hat{s})$; and frictional force, $-(\mu_{13} f_{13r})\hat{s}$.

$$\vec{f}_{13} = f_{13r}[(\hat{k} \times \hat{s}) - \mu_{13}\hat{s}] + (f_{13o})\hat{s} \tag{2.39}$$

If Equations (2.36) through (2.39) are substituted into Equations (2.30) through (2.35), the only unknowns in the resulting equations are $\vec{f}_{21}$, $\vec{f}_{32}$, τ_{21i} and f_{13r}. These amount to six scalar unknowns. Only four of the equations are independent. Of these, the two force vector equations are each two-dimensional $(\hat{i}, \hat{j})$; the two moment equations are each one-dimensional $(\hat{k})$. Thus, six scalar conditions are available for determining the six unknown scalars.

A procedure for reducing equilibrium conditions to simultaneous linear algebraic equations in a determinate number of unknowns is explained in Section 5.0. Such a reduction is impossible here because of the frictional terms. Instead, a specialized approach is taken. Use Equations (2.31) and (2.32) to express $\vec{f}_{32}$ and $\vec{f}_{21}$ in terms of $\vec{f}_{13}$. This effectively eliminates four unknown scalars at the expense of four scalar conditions.

$$\vec{f}_{32} = \vec{f}_{13} - \vec{g}_3 \tag{2.40}$$

$$\vec{f}_{21} = \vec{f}_{13} + \vec{g}_1 \tag{2.41}$$

Two unknown scalars remain: f_{13r} and τ_{21i}. Fortunately, τ_{21i} does not occur in Equation (2.34); to obtain f_{13r}, only one equation must be solved. Substitute Equations (2.37) and (2.38) into Equation (2.34).

$$-(\rho_{32}f_{32})\hat{\omega}_{32} + (\rho_{13}f_{13})\hat{\omega}_{13} + (\vec{f}_{13} \times \vec{r}_{p_3p_2}) + (\vec{g}_3 \times \vec{r}_{p_2c_3}) + \vec{\sigma}_3 = 0 \tag{2.42}$$

Nonlinearity is introduced when f_{32}, f_{13}, and $\vec{f}_{13}$ are expressed in terms of f_{13r}. Using Equations (2.31) and (2.39),

$$f_{32} = (\vec{f}_{32} \cdot \vec{f}_{32})^{1/2} = [f_{13}^2 - (2g_3)f_{13} + g_3^2]^{1/2} \tag{2.43}$$

$$f_{13} = (\vec{f}_{13} \cdot \vec{f}_{13})^{1/2} = [(1 + \mu_{13}^2)f_{13r}^2 - (2\mu_{13}f_{13o})f_{13r} + f_{13o}^2]^{1/2} \tag{2.44}$$

Substitute Equations (2.39), (2.43), and (2.44) into Equation (2.42), and take the scalar product with $\hat{k}$ throughout.

$$-\rho_{32}(\hat{\omega}_{32}\cdot\hat{k})\{[(1+\mu_{13}^2)f_{13r}^2-(2\mu_{13}f_{13o})f_{13r}+f_{13o}^2]$$

$$-2g_3[(1+\mu_{13}^2)f_{13r}^2-(2\mu_{13}f_{13o})f_{13r}+f_{13o}^2]^{1/2}+g_3^2\}^{1/2}$$

$$+\rho_{13}(\hat{\omega}_{13}\cdot\hat{k})[(1+\mu_{13}^2)f_{13r}^2-(2\mu_{13}f_{13o})f_{13r}+f_{13o}^2]^{1/2} \qquad (2.45)$$

$$+f_{13r}\{[(\hat{k}\times\hat{s})-\mu_{13}\hat{s}]\times\vec{r}_{p_3p_2}\}\cdot\hat{k}+f_{13o}[(\hat{s}\times\vec{r}_{p_3p_2})\cdot\hat{k}]$$

$$+[(\vec{g}_3\times\vec{r}_{p_2p_3})\cdot\hat{k}]+(\vec{\sigma}_3\cdot\hat{k})=0$$

If all frictional and inertial terms are retained, Equation (2.45) can at best be developed into a fourth degree polynomial in f_{13r}. If ρ_{32} is zero the problem is second degree; if both ρ_{32} and ρ_{13} are zero the problem is linear. The terms μ_{13}, g_3, and σ_3 have no influence on the degree of the solution.

<u>Suggested generalizations</u>:

(1) The equilibrium solutions for mechanisms with rigid links and no friction can always be obtained via a set of simultaneous linear algebraic equations. The number of simultaneous equations can be substantially reduced if the mechanism is simple, possibly to the point that the solution can easily be physically interpreted.

(2) Inertial terms introduce more detail, but the solution remains linear.

(3) Frictional terms may introduce nonlinearity if the magnitudes of the frictional forces are dependent on the magnitudes of the transmitted forces. This would be the case with journal bearings, but not with ideal roller bearings.

3.0 DIRECT SOLUTION OF THREE-DIMENSIONAL VECTOR EQUATIONS

Kinematic problems in position, motion, and force can usually be represented by single or simultaneous vector equations. The unknown quantities in these equations may not be distributed in a simple manner, especially in position problems. This section discusses two means by which direct (noniterative) solutions can be obtained: (1) use of symmetry, as in the Tetrahedron Solutions; (2) use of the eliminant, for more complicated problems. Examples are presented of the application of these techniques to position solutions of actual mechanisms.

3.1 Symmetry Solutions

The known unit vectors in a set of vector equations define a natural geometry or symmetry. Exploitation of this symmetry usually reduces the degree and number of algebraic polynomials in the eventual solution. Symmetry solutions are those which are obtained in general terms entirely by exploitation of symmetry. Typically such solutions are of first, second, fourth, or eighth degree in a single variable, and the expressions for the coefficients are interpretable.

Several important symmetry solutions are obtained and categorized in this section. Outlines of derivations and geometric interpretations are included to explain techniques of identifying and exploiting symmetry. When only a few quantities are known, symmetry is strong, because there is little conflict over which quantities should define the orientation of a dummy reference frame. When many quantities are known, more than one frame may be required and symmetry is weaker. Symmetry solutions to more than one or two vector equations are usually prohibitively difficult.

3.1.1 The Tetrahedron Solutions

The most common condition in kinematic analysis is the equation, sum of vectors equals zero:

$$\vec{r} + \vec{s} + \vec{t} + \vec{C} = 0 \tag{3.1}$$

Equation (3.1) is named the Vector Tetrahedron Equation because, geometrically, it outlines four of the six edges of a tetrahedron. This is analogous to the Vector Triangle Equation in two dimensions (Table 2.4). The utility is also analogous; in many situations a single Vector Tetrahedron Equation is either the only condition imposed or it can be solved independently of other conditions

Equation (3.1) is limited to four terms because, as a three-dimensional equation, it can determine only three scalar unknowns. These unknowns can be distributed throughout at most three vectors, provided no unknown occurs in more than one term. All other vectors must then be known and can be summed into the single vector constant $\vec{C}$.

TABLE 3.1

CATEGORIZATION OF SOLUTIONS TO THE VECTOR TETRAHEDRON EQUATION

$$\vec{r} + \vec{s} + \vec{t} + \vec{C} = 0$$

Case Number	Unknown	Known			Degree of Polynomial Solution
		Vectors	Unit Vectors	Scalars	
1	r, θ_r, ϕ_r	$\vec{C}$			1 (trivial)
2a	$r, \theta_r; s$	$\vec{C}$	$\hat{s}, \hat{\omega}_r$	ϕ_r	2
2b	$r, \theta_r; \theta_s$	$\vec{C}$	$\hat{\omega}_r, \hat{\omega}_s$	$\phi_r; s, \phi_s$	4
2c	$\theta_r, \phi_r; s$	$\vec{C}$	$\hat{s}$	r	2
2d	$\theta_r, \phi_r; \theta_s$	$\vec{C}$	$\hat{\omega}_s$	$r; s, \phi_s$	2
3a	$r; s; t$	$\vec{C}$	$\hat{r}, \hat{s}, \hat{t}$		1
3b	$r; s; \theta_t$	$\vec{C}$	$\hat{r}, \hat{s}, \hat{\omega}_t$	t, ϕ_t	2
3c	$r; \theta_s; \theta_t$	$\vec{C}$	$\hat{r}, \hat{\omega}_s, \hat{\omega}_t$	$s, \phi_s; t, \phi_t$	4
3d	$\theta_r; \theta_s; \theta_t$	$\vec{C}$	$\hat{\omega}_r, \hat{\omega}_s, \hat{\omega}_t$	$r, \phi_r; s, \phi_s; t, \phi_t$	8

Remarks:

(1) Unit vectors $\hat{\omega}_r$, $\hat{\omega}_s$, $\hat{\omega}_t$ are the known directions from which the known angles ϕ_r, ϕ_s, and ϕ_t are measured.

(2) Whenever any of the vectors $\vec{r}$, $\vec{s}$, or $\vec{t}$ are completely known they are added into the single constant $\vec{C}$.

Different solutions are obtained to Equation (3.1) for different distributions of the three unknowns. Vectors $\vec{r}$, $\vec{s}$, and $\vec{t}$ are expressed in spherical coordinates, so that the unknowns may be any three of the nine coordinates[2] $r, \theta_r, \phi_r;\ s, \theta_s, \phi_s;\ t, \theta_t, \phi_t$. (The angular coordinates can be measured from any known unit vector, not just from the ground reference frame.) It will be shown that there are only nine basic distributions of unknowns that lead to distinctly different solutions. These are called cases and are summarized in Table 3.1.

Two effects limit the number of cases in Table 3.1:

(1) The terms in Equation (3.1) are commutative. Combinations of unknowns such as $r; \theta_s; \theta_t$ and $\theta_r; s; \theta_t$ are therefore in the same case.

(2) When only one angle of a vector is unknown, it may be either the azimuthal or the polar angle. However, the same solution suffices for both situations. Combinations such as $\theta_r, \phi_r; \theta_s$ and $\theta_r, \phi_r; \phi_s$ are therefore in the same case. To see this, assume that a solution to Equation (3.1) has been obtained with θ_{r_1} unknown, ϕ_{r_1} known. The unit vector $\hat{r}_1$ may be written,

$$\hat{r}_1 = \left\{\sin \phi_{r_1} [\cos \theta_{r_1} \hat{\lambda}_1 + \sin \theta_{r_1} \hat{\mu}_1] + \cos \phi_{r_1} \hat{\nu}_1\right\} \quad (3.2)$$

[2] In listing coordinates, semicolons are used to separate coordinates from different vectors. Commas separate coordinates from the same vector.

Now assume a solution is desired for the situation ϕ_{r_2} unknown, θ_{r_2} known. The remaining two unknowns are the same as in the first solution. The unit vector $\hat{r}_2$ is written,

$$\hat{r}_2 = \{\sin \phi_{r_2} [\cos \theta_{r_2} \hat{\lambda}_1 + \sin \theta_{r_2} \hat{\mu}_1] + \cos \phi_{r_2} \hat{\nu}_1\} \tag{3.3}$$

Define a second dummy reference frame in terms of the first.

$$\hat{\mu}_2 \equiv [\cos \theta_{r_2} \hat{\lambda}_1 + \sin \theta_{r_2} \hat{\mu}_1] \tag{3.4}$$

$$\hat{\lambda}_2 \equiv \hat{\nu}_1 \tag{3.5}$$

$$\hat{\nu}_2 \equiv \hat{\lambda}_2 \times \hat{\mu}_2 \tag{3.6}$$

Unit vector $\hat{r}_2$ can now be expressed in exactly the same form as $\hat{r}_1$ in Equation (3.2).

$$\hat{r}_2 = \{\sin \frac{\pi}{2} [\cos \phi_{r_2} \hat{\lambda}_2 + \sin \phi_{r_2} \hat{\mu}_2] + (\cos \frac{\pi}{2}) \hat{\nu}_2 \} \tag{3.7}$$

Thus the same general solution for which θ_{r_1} was unknown will suffice for unknown ϕ_{r_2}, provided the following replacement of constants is made: $\phi_{r_1} \leftarrow \pi/2$, $\hat{\lambda}_1 \leftarrow \hat{\lambda}_2$, $\hat{\mu}_1 \leftarrow \hat{\mu}_2$, $\hat{\nu}_1 \leftarrow \hat{\nu}_2$. In fact, solutions for unknown ϕ are simple special cases of solutions for unknown θ, because of the simplifications introduced by the angle $\pi/2$.

Each of the Table 3.1 cases has its own symmetry. This symmetry is strong for cases in which few knowns enter (1, 2c, 2d) but becomes weaker as the number of known quantities increases (2b, 3c, 3d). As the symmetry weakens, the solutions become more difficult as indicated by the degree of the polynomial from which the solution is obtained.

An outline of the derivation and the solution for each of the nine cases will now be presented. The derivations are included to provide insight into the use of vector methods for identifying and exploiting symmetry. In most cases, if symmetry is ignored, a general solution is prohibitively difficult.

<u>Case 1</u>. r, θ_r, ϕ_r Unknown.

The unknowns all occur in the single vector $\vec{r}$. Vectors $\vec{s}$ and $\vec{t}$ are known and are added into $\vec{C}$. Equation (3.1) becomes

$$\vec{r} + \vec{C} = 0 \tag{3.8}$$

Solution:

$$\vec{r} = -\vec{C} \tag{3.9}$$

<u>Cases 2a - 2d</u>.

The unknowns are distributed throughout only two vectors $\vec{r}$ and $\vec{s}$. Vector $\vec{t}$ is known and is added into $\vec{C}$. The geometry of the individual cases is shown in Figures 3.1 through 3.4.

Equation (3.1) becomes

$$\vec{r} + \vec{s} + \vec{C} = 0 \tag{3.10}$$

2a. $r, \theta_r; s$ Unknown.

Expand Equation (3.10), expressing $\vec{r}$ in a dummy reference frame $\hat{\lambda}, \hat{\mu}, \hat{\nu}$.

$$r\{\sin\phi_r[\cos\theta_r\hat{\lambda} + \sin\theta_r\hat{\mu}] + \cos\phi_r\hat{\nu}\} + s\hat{s} + \vec{C} = 0 \qquad (3.11)$$

Define a unit vector, $\hat{p}$, perpendicular to vectors $\vec{C}$ and $\hat{s}$.

$$\hat{p} \equiv \frac{\vec{C} \times \hat{s}}{|\vec{C} \times \hat{s}|} \qquad (3.12)$$

Take the scalar product throughout Equation (3.11) with $\hat{p}$.

$$r\{\sin\phi_r[(\cos\theta_r)(\hat{\lambda}\cdot\hat{p}) + \sin\theta_r(\hat{\mu}\cdot\hat{p})] + \cos\phi_r(\hat{\nu}\cdot\hat{p})\} = 0 \quad (3.13)$$

Provided r is non-zero, Equation (3.13) is a condition involving only one unknown, θ_r. This condition can be made even simpler by suitably defining $\hat{\lambda}, \hat{\mu}, \hat{\nu}$. The angle ϕ_r between $\hat{r}$ and $\hat{\omega}_r$ is known; therefore, $\hat{\nu}$ is set equal to $\hat{\omega}_r$. However, $\hat{\lambda}$ can still be defined to cause the product $(\hat{\lambda}\cdot\hat{p})$ to be zero. The definition of $\hat{\mu}$ follows from that of $\hat{\nu}$ and $\hat{\lambda}$.

$$\hat{\nu} \equiv \hat{\omega}_r \qquad (3.14)$$

$$\hat{\lambda} \equiv \frac{\hat{p}\times\hat{\omega}_r}{|\hat{p}\times\hat{\omega}_r|} \qquad (3.15)$$

$$\hat{\mu} \equiv \hat{\nu}\times\hat{\lambda} = \frac{\hat{p} - (\hat{\omega}_r\cdot\hat{p})\hat{\omega}_r}{|\hat{p}\times\hat{\omega}_r|} \qquad (3.16)$$

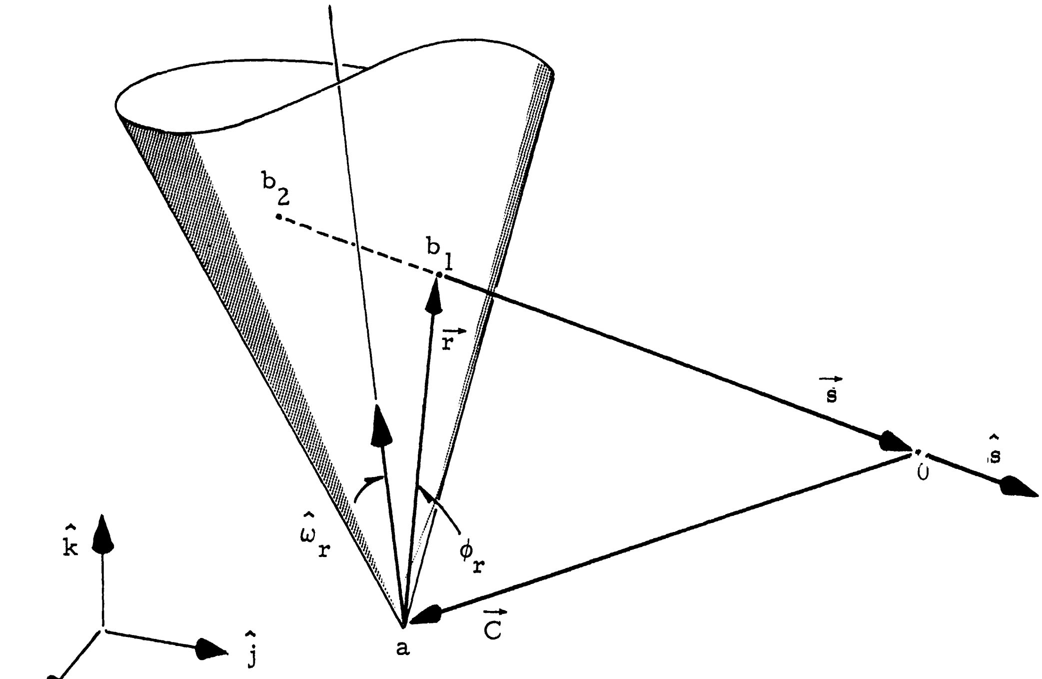


Figure 3.1 Case 2a. $r, \theta_r; s$ Unknown. Two Solutions Possible.

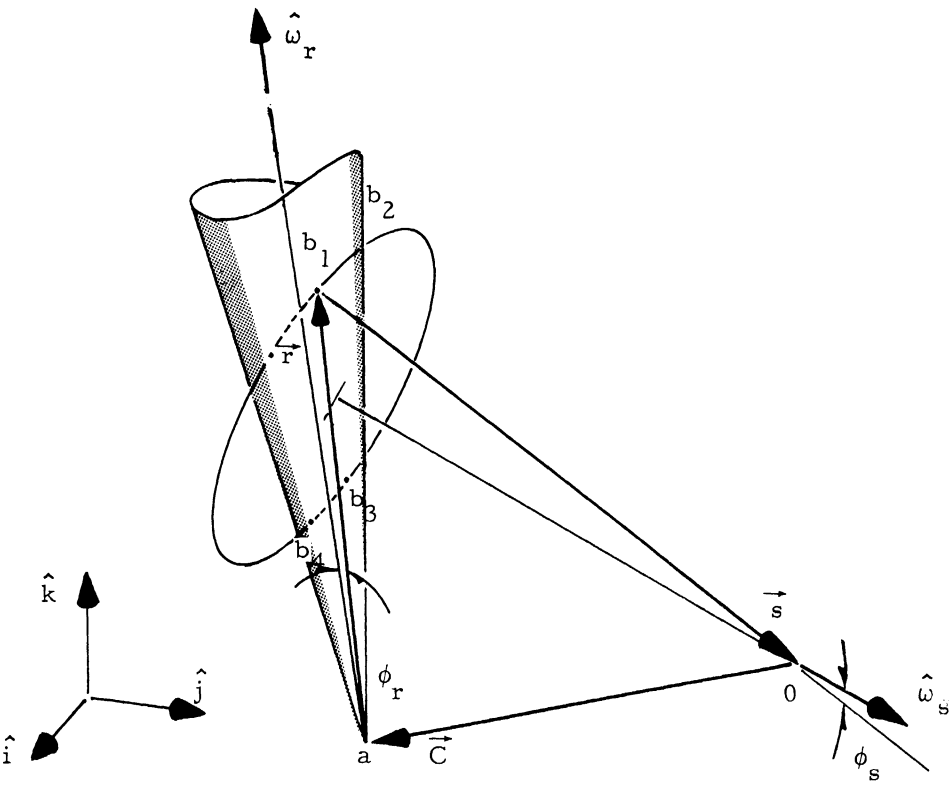


Figure 3.2 Case 2b. $r, \theta_r; \theta_s$ Unknown. Four Solutions Possible

Equation (3.13) can now be solved directly for $\sin \theta_r$ because the term involving $\cos \theta_r$ is zero.

$$\sin \theta_r = -\frac{(\hat{\nu} \cdot \hat{p})}{(\hat{\mu} \cdot \hat{p})} \cot \phi_r \tag{3.17}$$

An explicit expression for $\hat{r}$ can now be written in terms of the $\hat{\lambda}$, $\hat{\mu}$, $\hat{\nu}$ frame and the vector $\hat{p}$, using the form of Equation (3.2). Express $\cos \theta_r$ by means of Equation (3.17) and the identity $\cos \theta_r = \pm [1 - \sin^2 \theta_r]^{1/2}$.

Solution:

$$\hat{r} = \pm \left\{ \sin^2 \phi_r - \frac{(\hat{\nu} \cdot \hat{p})^2}{(\hat{\mu} \cdot \hat{p})^2} \cos^2 \phi_r \right\}^{1/2} \hat{\lambda} - \frac{(\hat{\nu} \cdot \hat{p})}{(\hat{\mu} \cdot \hat{p})} (\cos \phi_r) \hat{\mu} + (\cos \phi_r) \hat{\nu} \tag{3.18}$$

Unknowns r and s can now be obtained from cases 2a and 1, Table 2.4, where $\hat{k}$ is identified as $\hat{p}$.

$$\vec{r} = - \frac{[\vec{C} \cdot (\hat{p} \times \hat{s})]}{[\hat{r} \cdot (\hat{p} \times \hat{s})]} \hat{r} \tag{3.19}$$

$$\vec{s} = - (\vec{C} + \vec{r}) \tag{3.20}$$

2b. r, θ_r; θ_s <u>Unknown</u>.

In this case it is impossible to eliminate two of the unknowns with a single scalar product. Instead, two scalar products must be taken, each eliminating the same unknown, θ_r. A second unknown, r, is eliminated between the two equations resulting from the scalar products. Finally, the equation resulting from the elimination is transformed to a fourth degree polynomial in $\tan(\frac{\theta_s}{2})$.

Expand Equation (3.10), expressing $\vec{r}$ and $\vec{s}$ in terms of dummy reference frames $\hat{\lambda}_r$, $\hat{\mu}_r$, $\hat{\nu}_r$ and $\hat{\lambda}_s$, $\hat{\mu}_s$, $\hat{\nu}_s$. For convenience represent groups of known terms by single constants S and $\vec{C}_2$.

$$r\{\sin\phi_r[\cos\theta_r\hat{\lambda}_r + \sin\theta_r\hat{\mu}_r] + \cos\phi_r\hat{\nu}_r\} = - S[\cos\theta_s\hat{\lambda}_s + \sin\theta_s\hat{\mu}_s] - \vec{C}_2 \tag{3.21}$$

$$\hat{\nu}_r \equiv \hat{\omega}_r \tag{3.22}$$

$$S \equiv s\sin\phi_s$$

$$\vec{C}_2 \equiv \vec{C} + (s\cos\phi_s)\hat{\omega}_s \tag{3.23}$$

Eliminate θ_r for the first time by taking the scalar products of both sides of Equation (3.21) by themselves.

$$r^2 = S^2 + C_2^2 + 2S[(\vec{C}_2 \cdot \hat{\lambda}_s)\cos\theta_s + (\vec{C}_2 \cdot \hat{\mu}_s)\sin\theta_s] \tag{3.24}$$

Eliminate θ_r for the second time by taking the scalar product throughout Equation (3.21) with $\hat{\omega}_r$. The two terms involving θ_r will be zero, because $\hat{\lambda}_r \cdot \hat{\omega}_r$ and $\hat{\mu}_r \cdot \hat{\omega}_r$ are zero. (Unit vectors $\hat{\lambda}_r$, $\hat{\mu}_r$, $\hat{\nu}_r$ are mutually perpendicular and $\hat{\nu} \equiv \hat{\omega}_r$.) The term involving $\cos\theta_s$ can be made zero by defining $\hat{\lambda}_s$, $\hat{\mu}_s$, $\hat{\nu}_s$ as in Equations (3.26) through (3.28).

$$r \cos \phi_r = -[S(\hat{\mu}_s \cdot \hat{\omega}_r) \sin \theta_s + (C_2 \cdot \hat{\omega}_r)] \tag{3.25}$$

$$\hat{\nu}_s \equiv \hat{\omega}_s \tag{3.26}$$

$$\hat{\lambda}_s \equiv \frac{\hat{\omega}_r \times \hat{\omega}_s}{|\hat{\omega}_r \times \hat{\omega}_s|} \tag{3.27}$$

$$\hat{\mu}_s \equiv \hat{\nu}_s \times \hat{\lambda}_s = \frac{\hat{\omega}_r - (\hat{\omega}_s \cdot \hat{\omega}_r)\,\hat{\omega}_s}{|\hat{\omega}_r \times \hat{\omega}_s|} \tag{3.28}$$

Square both sides of Equation (3.25), then divide by $\cos \phi_r$. Subtract the resulting equation from Equation (3.24), to eliminate r. The difference is an equation involving only θ_s, in $\sin^2 \theta_s$, $\sin \theta_s$, and $\cos \theta_s$ terms. Transform these terms by the identities

$$\cos \theta_s = \frac{1 - u^2}{1 + u^2} \tag{3.29}$$

$$\sin \theta_s = \frac{2u}{1 + u^2} \tag{3.30}$$

$$u \equiv \tan \left(\frac{\theta_s}{2}\right) \tag{3.31}$$

A fourth degree polynomial in u is generated by multiplying throughout by $(1 + u^2)^2$:

Solution:

$$P_4 u^4 + P_3 u^3 + P_2 u^2 + P_1 u + P_0 = 0 \tag{3.32}$$

$$P_4 \equiv (\vec{C}_2 \cdot \omega_r^2) - \cos^2\phi_r[S^2 - 2S(\hat{\lambda}_s \cdot \vec{C}_2) + C_2^2] \tag{3.33}$$

$$P_3 \equiv 4S[(\hat{\mu}_s \cdot \hat{\omega}_r)(\vec{C}_2 \cdot \hat{\omega}_r) - \cos^2\phi_r(\hat{\mu}_s \cdot \vec{C}_2)] \tag{3.34}$$

$$P_2 \equiv 2[2S^2(\hat{\mu}_s \cdot \hat{\omega}_r)^2 + (\vec{C}_2 \cdot \hat{\omega}_r)^2 - \cos^2\phi_r(S^2 + C_2^2)] \tag{3.35}$$

$$P_1 = P_3 \tag{3.36}$$

$$P_0 \equiv (\vec{C}_2 \cdot \hat{\omega}_r)^2 - \cos^2\phi_r[S^2 + 2S(\hat{\lambda}_s \cdot \vec{C}_2) + C_2^2] \tag{3.37}$$

$$\vec{s} = s\left\{\frac{\sin\phi_s}{1 + u^2}[(1 - u^2)\hat{\lambda}_s + (2u)\hat{\mu}_s] + \cos\phi_s\hat{\omega}_s\right\} \tag{3.38}$$

$$\vec{r} = -(\vec{s} + \vec{C}) \tag{3.39}$$

2c. $\theta_r, \phi_r; s$ <u>Unknown</u>.

Rearrange Equation (3.10) explicitly in terms of $\vec{r}$ and eliminate $\hat{r}$ by taking the scalar product of both sides of the equation with themselves.

$$r^2 = s^2 + C^2 + 2sC(\hat{s} \cdot \hat{C}) \tag{3.40}$$

Equation (3.40) is a second degree polynomial in s and can be solved by means of the quadratic formula. The full vector $\vec{s}$ is expressed as the product of s and the known unit vector $\hat{s}$.

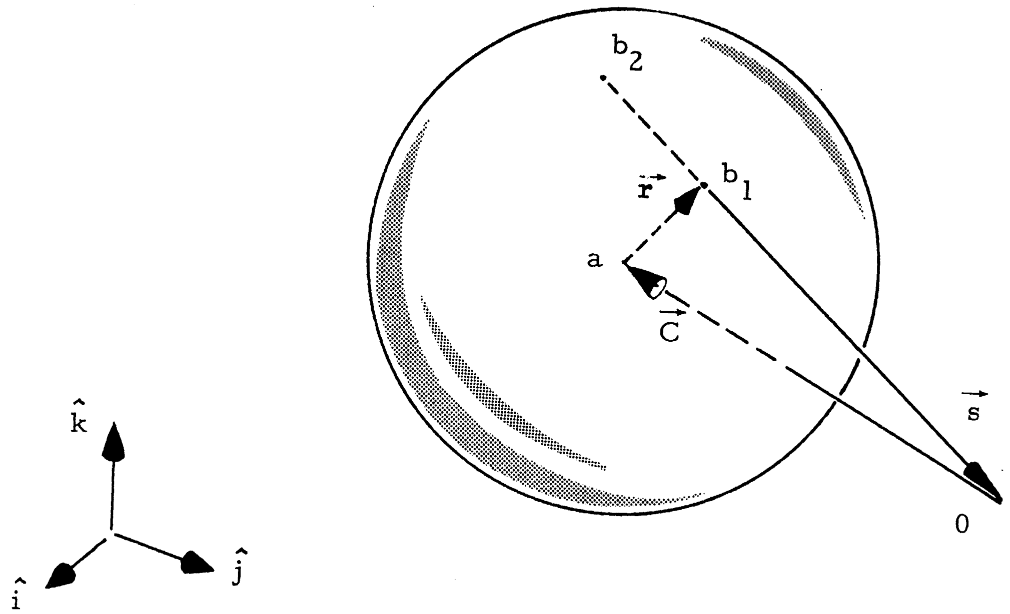


Figure 3.3 Case 2c. θ_r, ϕ_r; s Unknown. Two Solutions Possible.

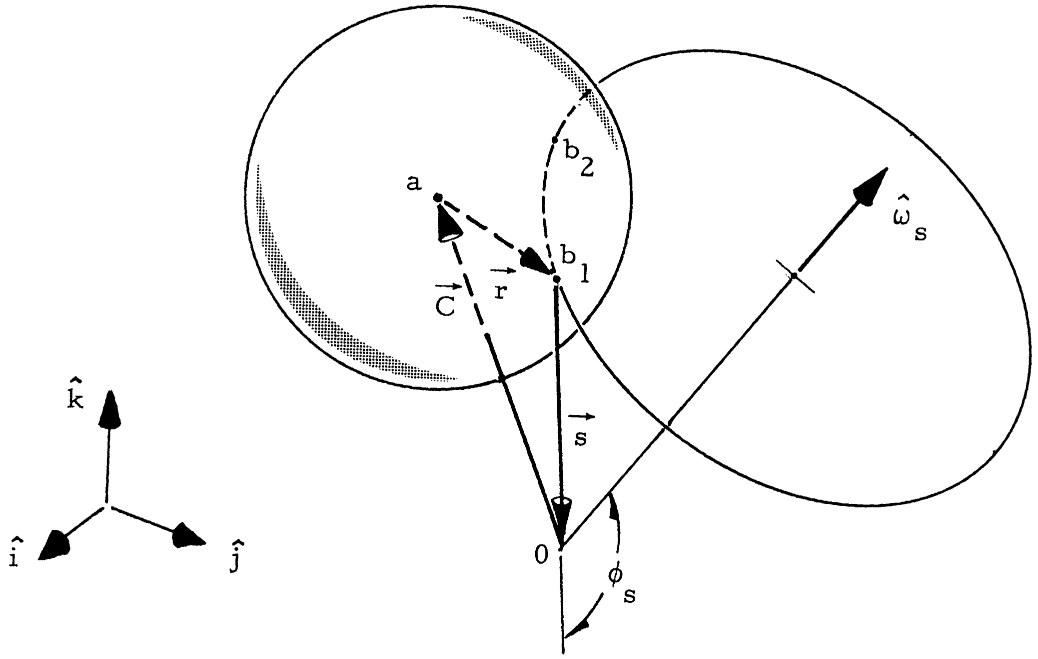


Figure 3.4 Case 2d. θ_r, ϕ_r; θ_s Unknown. Two Solutions Possible.

Solution:

$$\vec{s} = \left[- (\vec{C} \cdot \hat{s}) \pm \{r^2 - C^2[1 - (\hat{C} \cdot \hat{s})^2]\}^{1/2} \right] \hat{s} \tag{3.41}$$

$$\vec{r} = - (\vec{s} + \vec{C}) \tag{3.42}$$

2d. θ_r, ϕ_r; θ_s Unknown.

Obtain Equation (3.40) as in case 2c. Two conditions on $\hat{s}$ are now known:

$$(\hat{s} \cdot C) = [\frac{r^2 - s^2 - C^2}{2sC}] \tag{3.43}$$

$$(\hat{s} \cdot \hat{\omega}_s) = \cos \phi_s \tag{3.44}$$

The solution to a set of equations of this form is obtained in Section 3.1.2, Equation (3.131). Some rearrangements are made so that $\vec{s}$ can be evaluated with a minimum number of operations.

Solution:

$$\begin{aligned} \vec{s} = (s \cos \phi_s)\hat{\omega}_s + \frac{1}{[1 - (\hat{C} \cdot \hat{\omega}_s)^2]} \Bigg[& \pm \{s^2[1 - \cos^2 \phi_s][1 - (\hat{C} \cdot \hat{\omega}_s)^2] \\ & - [\frac{C^2 + s^2 - r^2 + 2s(\vec{C} \cdot \hat{\omega}_s)(\cos \phi_s)}{2C}]^2\}^{1/2} (\hat{C} \times \hat{\omega}_s) \\ & + [\frac{C^2 + s^2 - r^2 + 2s(\vec{C} \cdot \hat{\omega}_s)(\cos \phi_s)}{2C}] [(\hat{C} \times \hat{\omega}_s) \times \hat{\omega}_s] \Bigg] \end{aligned} \tag{3.45}$$

$$\vec{r} = - (\vec{s} + \vec{C}) \tag{3.46}$$

Cases 3a-3d.

Here the unknowns are distributed throughout all three vectors $\vec{r}$, $\vec{s}$, and $\vec{t}$. Therefore, Equation (3.1) is employed as stated. The geometry of the individual cases is shown in Figures 3.5 through 3.8 .

3a. r; s; t Unknown.

Restate Equation (3.1) with $\vec{r}$, $\vec{s}$, and $\vec{t}$ factored into magnitude and unit vector.

$$r\,\hat{r} + s\,\hat{s} + t\,\hat{t} + \vec{C} = 0 \tag{3.47}$$

Equation (3.47) can immediately be reduced to three simultaneous linear algebraic equations in three unknowns simply by taking scalar products throughout with any three known, non-parallel vectors, such as $\hat{i}$, $\hat{j}$, $\hat{k}$. However, for purposes of interpretation and ease of computation it is preferable to take these products with the vectors $(\hat{s} \times \hat{t})$, $(\hat{t} \times \hat{r})$ and $(\hat{r} \times \hat{s})$. The resulting equations will contain only r, s, and t, respectively.

Solution:

$$\vec{r} = \frac{-[\vec{C} \cdot (\hat{s} \times \hat{t})]\,\hat{r}}{[\hat{r} \cdot (\hat{s} \times \hat{t})]} \tag{3.48}$$

$$\vec{s} = \frac{-[\vec{C} \cdot (\hat{t} \times \hat{r})]\,\hat{s}}{[\hat{r} \cdot (\hat{s} \times \hat{t})]} \tag{3.49}$$

$$\vec{t} = \frac{-[\vec{C} \cdot (\hat{r} \times \hat{s})]\,\hat{t}}{[\hat{r} \cdot (\hat{s} \times \hat{t})]} \tag{3.50}$$

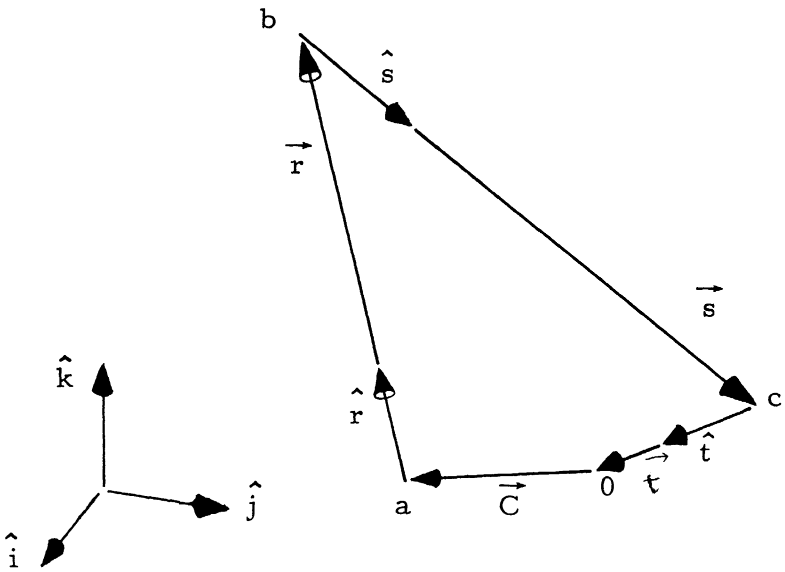


Figure 3.5 Case 3a. r, s, t Unknown.

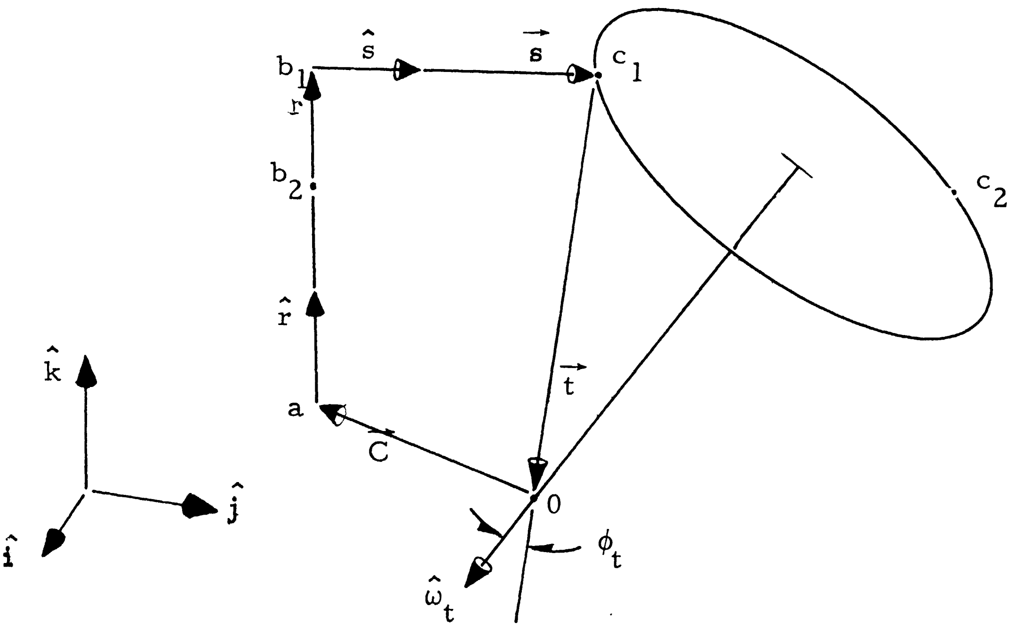


Figure 3.6 Case 3b. r, s, θ_t Unknown.
Two Solutions Possible.

with $\vec{r}$ and $\vec{s}$ known, $\vec{t}$ may be found most easily by

$$\vec{t} = -(\vec{r} + \vec{s} + \vec{C}) \tag{3.51}$$

3b. r; s; θ_t Unknown.

Expand Equation (3.1), expressing $\vec{t}$ in terms of a dummy reference frame $\hat{\lambda}$, $\hat{\mu}$, $\hat{\nu}$.

$$r\hat{r} + s\hat{s} + t\{\sin\phi_t[\cos\theta_t\hat{\lambda} + \sin\theta_t\hat{\mu}]\} + \vec{C}_2 = 0 \tag{3.52}$$

$$\vec{C}_2 \equiv (t\cos\phi_t)\hat{\omega}_t + \vec{C} \tag{3.53}$$

Define a unit vector, $\hat{p}$, perpendicular to $\hat{r}$ and $\hat{s}$.

$$\hat{p} \equiv \frac{\hat{r} \times \hat{s}}{|\hat{r} \times \hat{s}|} \tag{3.54}$$

Both r and s can be eliminated from Equation (3.52) by taking a scalar product throughout with $\hat{p}$. Moreover, the $\hat{\lambda}$, $\hat{\mu}$, $\hat{\nu}$, frame can be defined so that $\hat{\lambda} \cdot \hat{p}$ is zero and the term involving $\cos\theta_t$ drops out.

$$t(\sin\phi_t)(\sin\theta_t)(\hat{\mu} \cdot \hat{p}) + \vec{C}_2 \cdot \hat{p} = 0 \tag{3.55}$$

$$\hat{\nu} \equiv \hat{\omega}_t \tag{3.56}$$

$$\hat{\lambda} \equiv \frac{\hat{p} \times \hat{\omega}_t}{|\hat{p} \times \hat{\omega}_t|} \tag{3.57}$$

$$\hat{\mu} \equiv \hat{\nu} \times \hat{\lambda} = \frac{\hat{p} - (\hat{\omega}_t \cdot \hat{p})\hat{\omega}_t}{|\hat{p} \times \hat{\omega}_t|} \tag{3.57}$$

Equation (3.55) can be solved for $\sin \theta_t$, and $\cos \theta_t$ can be expressed by the identity $\cos \theta_t = \pm [1 - \sin^2 \theta_t]^{1/2}$. These expressions are substituted in the spherical coordinate expansion of $\hat{t}$ in the $\hat{\lambda}$, $\hat{\mu}$, $\hat{\nu}$ frame. A few rearrangements are made for computational convenience.

Solution:

$$\vec{t} = \frac{1}{|\hat{\omega}_t \times \hat{p}|^2} \Big[\pm \{[t\,|\hat{\omega}_t \times \hat{p}|\ \sin \phi_t]^2 - (\vec{C}_2 \cdot \hat{p})^2\}^{1/2} (\hat{\omega}_t \times \hat{p}) \tag{3.58}$$

$$[(\vec{C}_2 \cdot \hat{p})(\hat{\omega}_t \cdot \hat{p}) + t\,|\hat{\omega}_t \times \hat{p}|^2 (\cos \phi_t)(\sin \phi_t)]\hat{\omega}_t - (\vec{C}_2 \cdot \hat{p})\hat{p}$$

With $\vec{t}$ determined, Equation (3.1) reduces to the plane Vector Triangle Equation. Unknowns r and s can therefore be obtained from cases 2a and 1, Table 2.4, where $\hat{k}$ is identified as $\hat{p}$.

$$\vec{r} = - \frac{(\vec{t} + \vec{C}) \cdot (\hat{p} \times \hat{s})}{[\hat{r} \cdot (\hat{p} \times \hat{s})]} \tag{3.59}$$

$$\vec{s} = - (\vec{r} + \vec{t} + \vec{C}) \tag{3.60}$$

3c. r; θ_s; θ_t <u>Unknown</u>.

In this case, as in case 2b, it is impossible to eliminate two of the unknowns with a single scalar product. However, two scalar products can be taken to eliminate the unknown r. By careful definition of reference frames the first result will contain θ_s only in the form $\cos \theta_s$; the second only in the form $\sin \theta_s$. The unknown θ_s can be eliminated by squaring and adding. Finally, the equation

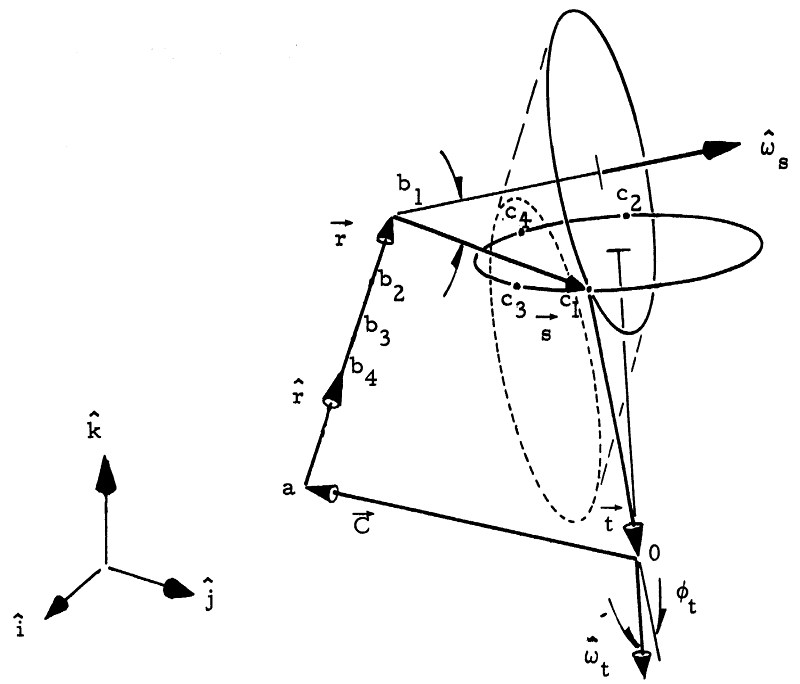


Figure 3.7 Case 3c. r, θ_s, θ_t Unknown. Four Solutions Possible.

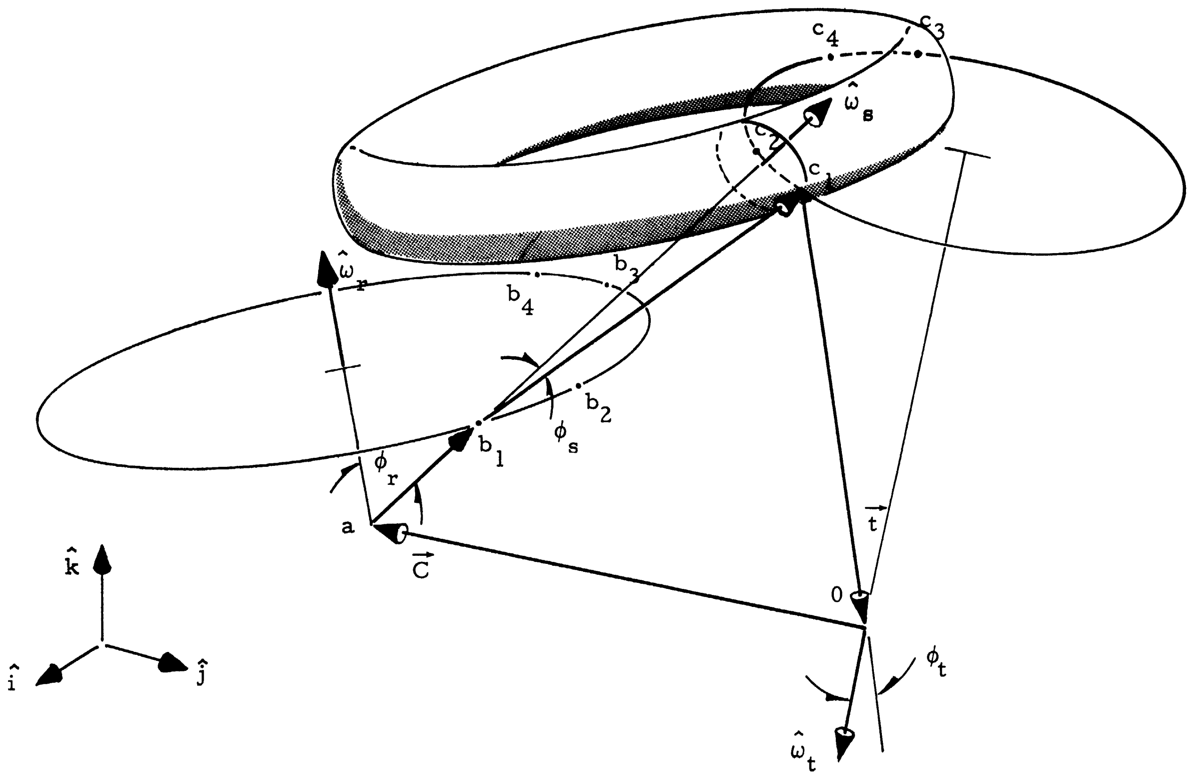


Figure 3.8 Case 3d. θ_r, θ_s, θ_t Unknown. Four Real Solutions Possible.

resulting from the addition is transformed to a fourth degree polynomial in $\tan\left(\frac{\theta_t}{2}\right)$.

Expand Equation (3.1), expressing $\vec{s}$ and $\vec{t}$ in terms of dummy reference frames $\hat{\lambda}_s$, $\hat{\mu}_s$, $\hat{\nu}_s$ and $\hat{\lambda}_t$, $\hat{\mu}_t$, $\hat{\nu}_t$. For convenience represent groups of known terms by single constants S, T, and $\vec{C}_2$.

$$r\hat{r} + S[\cos\theta_s\hat{\lambda}_s + \sin\theta_s\hat{\mu}_s] + T[\cos\theta_t\hat{\lambda}_t + \sin\theta_t\hat{\mu}_t] + \vec{C}_2 = 0 \tag{3.61}$$

$$S \equiv s\sin\phi_s \tag{3.62}$$

$$T \equiv t\sin\phi_t \tag{3.63}$$

$$\vec{C}_2 \equiv (s\cos\phi_s)\hat{\omega}_s + (t\cos\phi_t)\hat{\omega}_t + \vec{C} \tag{3.64}$$

Define the dummy reference frames and the vector $\hat{p}$ as follows:

$$\hat{\nu}_s \equiv \hat{\omega}_s \tag{3.65}$$

$$\hat{\lambda}_s \equiv \frac{\hat{r}\times\hat{\omega}_s}{|\hat{r}\times\hat{\omega}_s|} \tag{3.66}$$

$$\hat{\mu}_s \equiv \hat{\nu}_s\times\hat{\lambda}_s = \frac{\hat{r} - (\hat{\omega}_s\cdot\hat{r})\hat{\omega}_s}{|\hat{r}\times\hat{\omega}_s|} \tag{3.67}$$

$$\hat{\nu}_t \equiv \hat{\omega}_t \tag{3.68}$$

$$\hat{\lambda}_t \equiv \frac{\hat{\lambda}_s \times \hat{\omega}_t}{|\hat{\lambda}_s \times \hat{\omega}_t|} \tag{3.69}$$

$$\hat{\mu}_t \equiv \hat{\nu}_t \times \hat{\lambda}_t = \frac{\hat{\lambda}_s - (\hat{\omega}_t \cdot \hat{\lambda}_s)\hat{\omega}_t}{|\hat{\lambda}_s \times \hat{\omega}_t|} \tag{3.70}$$

$$\hat{p} \equiv \hat{r} \times \hat{\lambda}_s \tag{3.71}$$

Take scalar products throughout equation (3.62), first with $\hat{\lambda}_s$, then with $\hat{p}$.

$$S \cos \theta_s + T(\hat{\mu}_t \cdot \hat{\lambda}_s) \sin \theta_t + (\vec{C}_2 \cdot \hat{\lambda}_s) = 0 \tag{3.72}$$

$$S(\hat{\mu}_s \cdot \hat{p}) \sin \theta_s + T[(\hat{\lambda}_t \cdot \hat{p}) \cos \theta_t + (\hat{\mu}_t \cdot \hat{p}) \sin \theta_t] + (\vec{C}_2 \cdot \hat{p}) = 0 \tag{3.73}$$

Multiply through Equation (3.72) by $(\hat{\mu}_s \cdot \hat{p})$. Transfer the second and third terms in Equations (3.72) and (3.73) to the right side. Then square both sides of both equations and add to eliminate θ_s. The sum is an equation involving only θ_t, in $\sin^2 \theta_t$, $(\sin \theta_t \cos \theta_t)$, $\sin \theta_t$ and $\cos \theta_t$ terms. Transform these by the identities.

$$\cos \theta_t = \frac{1 - u^2}{1 + u^2} \tag{3.74}$$

$$\sin \theta_t = \frac{2u}{1 + u^2} \tag{3.75}$$

$$u = \tan \left(\frac{\theta_t}{2}\right) \tag{3.76}$$

A fourth degree polynomial in u is generated by multiplying throughout by $(1 + u^2)^2$.

Solution:

$$P_4 u^4 + P_3 u^3 + P_2 u^2 + P_1 u + P_0 = 0 \tag{3.77}$$

$$P_4 \equiv (\hat{\mu}_s \cdot \hat{p})^2[(\vec{C}_2 \cdot \hat{\lambda}_s)^2 - S^2] + [T(\hat{\lambda}_t \cdot \hat{p}) - (\vec{C}_2 \cdot \hat{p})]^2 \tag{3.78}$$

$$P_3 \equiv 4T\{(\hat{\mu}_s \cdot \hat{p})^2(\hat{\mu}_t \cdot \hat{\lambda}_s)(\vec{C}_2 \cdot \hat{\lambda}_s) + (\hat{\mu}_t \cdot \hat{p})[(\vec{C}_2 \cdot \hat{p}) - T(\hat{\lambda}_t \cdot \hat{p})]\} \tag{3.79}$$

$$P_2 \equiv 2\{(\hat{\mu}_s \cdot \hat{p})^2[(\vec{C}_2 \cdot \hat{\lambda}_s)^2 - S^2] + [(\vec{C}_2 \cdot \hat{p})^2 - T^2(\hat{\lambda}_t \cdot \hat{p})^2]$$
$$+ 2T^2[(\hat{\mu}_s \cdot \hat{p})^2(\hat{\mu}_t \cdot \hat{\lambda}_s)^2 + (\hat{\mu}_t \cdot \hat{p})^2]\} \tag{3.80}$$

$$P_1 \equiv 4T\{(\hat{\mu}_s \cdot \hat{p})^2(\hat{\mu}_t \cdot \hat{\lambda}_s)(\vec{C}_2 \cdot \hat{\lambda}_s) + (\hat{\mu}_t \cdot \hat{p})[(\vec{C}_2 \cdot \hat{p}) + T(\hat{\lambda}_t \cdot \hat{p})]\} \tag{3.81}$$

$$P_0 \equiv (\hat{\mu}_s \cdot \hat{p})^2[(\vec{C}_2 \cdot \hat{\lambda}_s)^2 - S^2] + [T(\hat{\lambda}_t \cdot \hat{p}) + (\vec{C}_2 \cdot \hat{p})]^2 \tag{3.82}$$

$$\vec{t} = t\{\frac{\sin \phi_t}{(1+u^2)}[1 - u^2)\hat{\lambda}_t + (2u)\hat{\mu}_t] + \cos \phi_t \hat{\omega}_t\} \tag{3.83}$$

Expand $\vec{s}$ in spherical coordinates in the $\hat{\lambda}_s$, $\hat{\mu}_s$, $\hat{\nu}_s$ frame, then substitute expressions for $\cos \theta_s$ and $\sin \theta_s$ from Equations (3.74) and (3.75).

$$\vec{s} = - \Big[[T(\hat{\mu}_t \cdot \hat{\lambda}_s) \sin \theta_t + (\vec{C}_2 \cdot \hat{\lambda}_s)]\hat{\lambda}_s \tag{3.85}$$
$$+ \frac{1}{(\hat{\mu}_s \cdot \hat{p})}\{T(\hat{\lambda}_t \cdot \hat{p}) \cos \theta_t + (\hat{\mu}_t \cdot \hat{p}) \sin \theta_t]$$
$$+ (\vec{C}_2 \cdot \hat{p})\}\hat{\mu}_s \Big] + (s \cos \phi_s)\hat{\omega}_s$$

$$\vec{r} = - (\vec{s} + \vec{t} + \vec{C}) \cdot \tag{3.86}$$

3d. θ_r; θ_s; θ_t Unknown.

This case has so little symmetry that three dummy reference frames are required, one each for $\vec{r}$, $\vec{s}$, and $\vec{t}$. A general solution can be obtained, but the difficulty involved suggests that a practical upper limit has been reached. Beyond this limit the exploitation of symmetry is helpful, but it is not sufficient for obtaining a complete solution. Additional tools are required, such as those of Denavit, Hartenburg, Razi, and Uicker [29, 63, 73], or the approach discussed in Section 3.2.

To obtain the present solution, three scalar products are taken. This yields three scalar trigonometric equations in $(\theta_r; \theta_s)$, $(\theta_r; \theta_t)$, and $(\theta_r; \theta_s; \theta_t)$, respectively. The second equation contains θ_t only in the form $\cos \theta_t$; the third only in $\sin \theta_t$. The unknown θ_t can therefore be eliminated by squaring and adding, although this causes a large build-up of terms. The equation formed by the sum and the first equation from the scalar product contain only θ_r and θ_s. The latter equation is simple in form and is solved for $\cos \theta_s$ in terms of $\cos \theta_r$; the result is then substituted for all θ_s terms in the former equation, using the identity, $\sin \theta_s = \pm [1 - \cos^2 \theta_2]^{1/2}$. The resulting equation contains only θ_s but must be squared again to eliminate square roots from the $\sin \theta_s$

substitution. The last equation is transformed to an eighth degree polynomial in $\tan\left(\frac{\theta_r}{2}\right)$. Even this tenuous solution would be prohibitively difficult without careful definition of the three dummy reference frames to minimize the number of terms and prepare for the elimination of θ_t and θ_s.

Expand Equation (3.1), expressing $\vec{r}$, $\vec{s}$, and $\vec{t}$ in terms of dummy reference frames $\hat{\lambda}_r, \hat{\mu}_r, \hat{\nu}_r$; $\hat{\lambda}_s, \hat{\mu}_s, \hat{\nu}_s$; $\hat{\lambda}_t, \hat{\mu}_t, \hat{\nu}_t$. For convenience represent groups of known terms by single constants R, S, T, and $\vec{C}_2$.

$$R[\cos\theta_r\hat{\lambda}_r + \sin\theta_r\hat{\mu}_r] + S[\cos\theta_s\hat{\lambda}_s + \sin\theta_s\hat{\mu}_s] + T[\cos\theta_t\hat{\lambda}_t + \sin\theta_t\hat{\mu}_t] + \vec{C}_2 = 0 \tag{3.87}$$

$$R \equiv r\sin\phi_r \tag{3.88}$$

$$S \equiv s\sin\phi_s \tag{3.89}$$

$$T \equiv t\sin\phi_t \tag{3.90}$$

$$\vec{C}_2 \equiv r\cos\phi_r\hat{\omega}_r + s\cos\phi_s\hat{\omega}_s + t\cos\phi_t\hat{\omega}_t + \vec{C} \tag{3.91}$$

Define the dummy reference frames as follows:

$$\hat{\nu}_r \equiv \hat{\omega}_r \tag{3.92}$$

$$\hat{\mu}_r \equiv \frac{\hat{\omega}_r \times \hat{\omega}_t}{|\hat{\omega}_r \times \hat{\omega}_t|} \tag{3.93}$$

$$\hat{\lambda}_r \equiv \hat{\mu}_r \times \hat{\nu}_r = \frac{\hat{\omega}_t - (\hat{\omega}_r \cdot \hat{\omega}_t)\hat{\omega}_r}{|\hat{\omega}_r \times \hat{\omega}_t|} \tag{3.94}$$

$$\hat{\nu}_s \equiv \hat{\omega}_s \tag{3.95}$$

$$\hat{\mu}_s \equiv \frac{\hat{\omega}_s \times \hat{\omega}_t}{|\hat{\omega}_s \times \hat{\omega}_t|} \tag{3.96}$$

$$\hat{\lambda}_s \equiv \hat{\mu}_s \times \hat{\nu}_s = \frac{\hat{\omega}_t - (\hat{\omega}_s \cdot \hat{\omega}_t)\hat{\omega}_s}{|\hat{\omega}_s \times \hat{\omega}_t|} \tag{3.97}$$

$$\hat{\nu}_t \equiv \hat{\omega}_t \tag{3.98}$$

$$\hat{\mu}_t \equiv \frac{\hat{\omega}_s \times \hat{\omega}_t}{|\hat{\omega}_s \times \hat{\omega}_t|} \tag{3.99}$$

$$\hat{\lambda}_t \equiv \hat{\mu}_t \times \hat{\nu}_t = \frac{(\hat{\omega}_s \cdot \hat{\omega}_t)\hat{\omega}_t - \hat{\omega}_s}{|\hat{\omega}_s \times \hat{\omega}_t|} \tag{3.100}$$

Take scalar products throughout Equation (3.87) with $\hat{\omega}_t$, $\hat{\omega}_s$, and $\hat{\mu}_s$.

$$a_{11}\cos\theta_r + 0 + a_{13}\cos\theta_s + 0 + 0 + 0 + a_{17} = 0 \tag{3.101}$$

$$a_{21}\cos\theta_r + a_{22}\sin\theta_r + 0 + 0 + a_{25}\cos\theta t + 0 + a_{27} = 0 \tag{3.102}$$

$$a_{31}\cos\theta_r + a_{32}\sin\theta_r + 0 + a_{34}\sin\theta_s + 0 + a_{36}\sin\theta_t + a_{37} = 0 \tag{3.103}$$

The definitions of all constants, including the a_{ij}, are included in Table 3.2. The solution itself proceeds as described above. The equation resulting from the final squaring operation has the following form:

$$g_1 \cos^4 \theta_r + g_2 \cos^3 \theta_r \sin \theta_r + g_3 \cos^3 \theta_r + g_4 \cos^2 \theta_r \sin \theta_r$$
$$+ g_5 \cos^2 \theta_r + g_6 \cos \theta_r \sin \theta_r + g_7 \cos \theta_r + g_8 \sin \theta_r$$
$$+ g_9 = 0 \tag{3.104}$$

Transform this equation by the identities,

$$\cos \theta_r = \frac{1 - u^2}{1 + u^2} \tag{3.105}$$

$$\sin \theta_r = \frac{2u}{1 + u^2} \tag{3.106}$$

$$u \equiv \tan \left(\frac{\theta_r}{2}\right) \tag{3.107}$$

An eighth degree polynomial in u is generated by multiplying throughout the transformed equation by $(1 + u^2)^4$.

Solution:

$$P_8 u^8 + P_7 u^7 + P_6 u^6 + P_5 u^5 + P_4 u^4 + P_3 u^3 + P_2 u^2 + P_1 u + P_0 = 0 \tag{3.108}$$

$$P_8 \equiv g_1 - g_3 + g_5 - g_7 + g_9 \tag{3.109}$$

$$P_7 \equiv 2[-g_2 + g_4 - g_6 + g_8] \tag{3.110}$$

TABLE 3.2

DEFINITION OF CONSTANTS USED IN CASE 3d.

$R = r \sin \phi_r$ $\qquad S = s \sin \phi_s$ $\qquad T = t \sin \phi_t$

$$C_2 = r \cos \phi_r \hat{\omega}_r + s \cos \phi_s \hat{\omega}_s + t \cos \phi_t \hat{\omega}_t + \vec{C}$$

$$\hat{\nu}_r \equiv \hat{\omega}_r \qquad \hat{\mu}_r \equiv \frac{\hat{\omega}_r \times \hat{\omega}_t}{|\hat{\omega}_r \times \hat{\omega}_t|} \qquad \hat{\lambda}_r \equiv \frac{\hat{\omega}_t - (\hat{\omega}_r \cdot \hat{\omega}_t)\hat{\omega}_r}{|\hat{\omega}_r \times \hat{\omega}_t|}$$

$$\hat{\nu}_s \equiv \hat{\omega}_s \qquad \hat{\mu}_s \equiv \frac{\hat{\omega}_s \times \hat{\omega}_t}{|\hat{\omega}_s \times \hat{\omega}_t|} \qquad \hat{\lambda}_s \equiv \frac{\hat{\omega}_t - (\hat{\omega}_s \cdot \hat{\omega}_t)\hat{\omega}_s}{|\hat{\omega}_s \times \hat{\omega}_t|}$$

$$\hat{\nu}_t \equiv \hat{\omega}_t \qquad \hat{\mu}_t \equiv \frac{\hat{\omega}_s \times \hat{\omega}_t}{|\hat{\omega}_s \times \hat{\omega}_t|} \qquad \hat{\lambda}_t \equiv \frac{(\hat{\omega}_s \cdot \hat{\omega}_t)\hat{\omega}_t - \hat{\omega}_s}{|\hat{\omega}_s \times \hat{\omega}_t|}$$

$$a_{11} \equiv R|\hat{\omega}_r \times \hat{\omega}_t| \qquad a_{13} \equiv S|\hat{\omega}_s \times \hat{\omega}_t| \qquad a_{17} \equiv (\vec{C}_2 \cdot \hat{\omega}_t)$$

$$a_{21} \equiv \frac{R[(\hat{\omega}_s \cdot \hat{\omega}_t) - (\hat{\omega}_r \cdot \hat{\omega}_t)(\hat{\omega}_r \cdot \hat{\omega}_s]}{|\hat{\omega}_r \times \hat{\omega}_t|} \qquad a_{22} \equiv \frac{R[\hat{\omega}_r \cdot (\hat{\omega}_s \times \hat{\omega}_t)]}{|\hat{\omega}_r \times \hat{\omega}_t|}$$

$$a_{25} \equiv -T|\hat{\omega}_s \times \hat{\omega}_t| \qquad a_{27} \equiv (\vec{C}_2 \cdot \hat{\omega}_s)$$

$$a_{31} \equiv \frac{-R(\hat{\omega}_r \cdot \hat{\omega}_t)[\hat{\omega}_r \cdot (\hat{\omega}_s \times \hat{\omega}_t)]}{|\hat{\omega}_r \times \hat{\omega}_t|\,|\hat{\omega}_s \times \hat{\omega}_t|} \qquad a_{32} \equiv \frac{R[(\hat{\omega}_r \cdot \hat{\omega}_s) - (\hat{\omega}_r \cdot \hat{\omega}_t)(\hat{\omega}_s \cdot \hat{\omega}_t)}{|\hat{\omega}_r \times \hat{\omega}_t|\,|\hat{\omega}_s \times \hat{\omega}_t|}$$

$$a_{34} \equiv S \qquad a_{36} \equiv T \qquad a_{37} \equiv (\vec{C}_2 \cdot \hat{\mu}_s)$$

TABLE 3.2 CONT'D

$b_{31} \equiv -|\hat{\omega}_s \times \hat{\omega}_t| a_{31}$ $\quad$ $b_{32} \equiv -|\hat{\omega}_s \times \hat{\omega}_t| a_{32}$

$b_{34} \equiv -|\hat{\omega}_s \times \hat{\omega}_t| a_{34}$ $\quad$ $b_{37} \equiv -|\hat{\omega}_s \times \hat{\omega}_t| a_{37}$

$d_1 \equiv 4a_{11}^2$ $\quad$ $d_2 \equiv 8a_{11}a_{17}$ $\quad$ $d_3 \equiv 4(a_{17}^2 - b_{34}^2)$

$e_1 \equiv (b_{31}^2 - b_{32}^2)$ $\quad$ $e_2 \equiv 2b_{31}b_{32}$ $\quad$ $e_3 \equiv 2b_{31}b_{37}$

$e_4 \equiv 2b_{32}b_{37}$ $\quad$ $e_5 \equiv (b_{32}^2 + b_{37}^2)$

$f_1 \equiv (a_{21}^2 + b_{31}^2 - a_{11}^2 - a_{22}^2 - b_{32}^2)$ $\quad$ $f_2 \equiv 2(a_{21}a_{22} - b_{31}b_{32})$

$f_3 \equiv 2(a_{21}a_{27} + b_{31}b_{37} - a_{11}a_{17})$ $\quad$ $f_4 \equiv 2(a_{22}a_{27} + b_{32}b_{37})$

$f_5 \equiv (a_{27}^2 + b_{37}^2 + b_{34}^2 + a_{22}^2 + b_{32}^2 - a_{17}^2 - a_{25}^2)$

$g_1 \equiv d_1e_1 + f_1^2 - f_2^2$

$g_2 \equiv d_1e_2 + 2f_1f_2$

$g_3 \equiv d_1e_3 + d_2e_1 + 2f_1f_3 - 2f_2f_4$

$g_4 \equiv d_1e_4 + d_2e_2 + 2f_1f_4 + 2f_2f_3$

$g_5 \equiv d_1e_5 + d_2e_3 + d_3e_1 + 2f_1f_5 + f_3^2 - f_4^2 + f_2^2$

$g_6 \equiv d_2e_4 + d_3e_2 + 2f_2f_5 + 2f_3f_4$

$g_7 \equiv d_2e_5 + d_3e_3 + 2f_3f_5 + 2f_2f_4$

$g_8 \equiv d_3e_4 + 2f_4f_5$

$g_9 \equiv d_3e_5 + f_5^2 + f_4^2$

$$P_6 \equiv 2[-2g_1 + g_3 - g_7 + 2g_9] \tag{3.111}$$

$$P_5 \equiv 2[3g_2 - g_4 - g_6 + 3g_8] \tag{3.112}$$

$$P_4 \equiv 2[3g_1 - g_5 + 3g_9] \tag{3.113}$$

$$P_3 \equiv 2[-3g_2 - g_4 + g_6 + 3g_8] \tag{3.114}$$

$$P_2 \equiv 2[-2g_1 - g_3 + g_7 + 2g_9] \tag{3.115}$$

$$P_1 \equiv 2[g_2 + g_4 + g_6 + g_8] \tag{3.116}$$

$$P_0 = [g_1 + g_3 + g_5 + g_7 + g_9] \tag{3.117}$$

$$\vec{r} = r\left\{\frac{\sin \phi_r}{(1 + u^2)}\left[(1 - u^2)\hat{\lambda}_r + (2u)\hat{\mu}_r\right] + \cos \phi_r \hat{\omega}_r\right\} \tag{3.118}$$

$$\vec{s} = s\left\{\sin \phi_s[\cos \theta_s \hat{\lambda}_s + \sin \theta_s \hat{\mu}_s] + \cos \phi_s \hat{\omega}_s\right\} \tag{3.119}$$

In Equation (3.119), $\cos \theta_s$ and $\sin \theta_s$ are determined from Equation (3.101) and the equation resulting from the elimination of θ_t between Equations (3.102) and (3.103).

$$\cos \theta_s = \frac{- a_{17} - a_{11}(\cos \theta_r)}{a_{13}} \tag{3.120}$$

$$\sin \theta_s = \left\{\frac{a_{25}^2 - (a_{21}\cos \theta_r + a_{22}\sin \theta_r + a_{27})^2 - (b_{31}\cos \theta_r + b_{32}\sin \theta_r + b_{37})^2}{2b_{34}(b_{31}\cos \theta_r + b_{32}\sin \theta_r + b_{37})} - \frac{b_{34}^2(1 - \cos^2 \theta_s)}{2b_{34}(b_{31}\cos \theta_r + b_{32}\sin \theta_r + b_{37})}\right\} \tag{3.121}$$

$$\vec{t} = (\vec{r} + \vec{s} + \vec{C}) \tag{3.122}$$

3.1.2 Supplemental Solutions

In many position problems, conditions are imposed which cannot be stated in the form of Equation (3.1). These are difficult to categorize. However, two cases are of particular importance and their solution will be discussed here. The geometry of these solutions is shown in Figures 3.9 and 3.10.

(1) Two simultaneous scalar products containing an unknown unit vector.

$$(\hat{a} \cdot \hat{r}) = c_1 \tag{3.123}$$

$$(\hat{b} \cdot \hat{r}) = c_2 \tag{3.124}$$

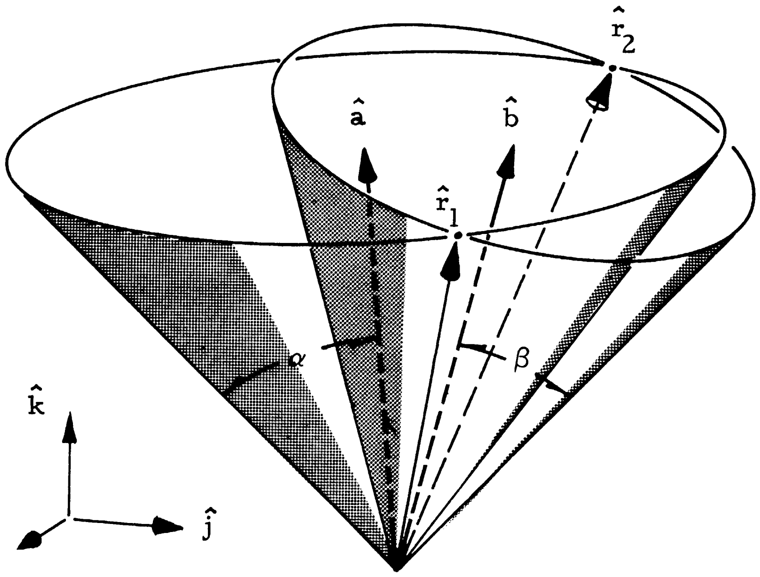
Here $\hat{a}$ and $\hat{b}$ are known unit vectors; c_1 and c_2 are known scalar constants. Unit vector $\hat{r}$ is completely unknown.

Expand $\hat{r}$ in spherical coordinates in a dummy $\hat{\lambda}$, $\hat{\mu}$, $\hat{\nu}$ reference frame:

$$\hat{r} = \sin \phi_r[\cos \theta_r \hat{\lambda} + \sin \theta_r \hat{\mu}] + \cos \phi_r \hat{\nu} \tag{3.125}$$

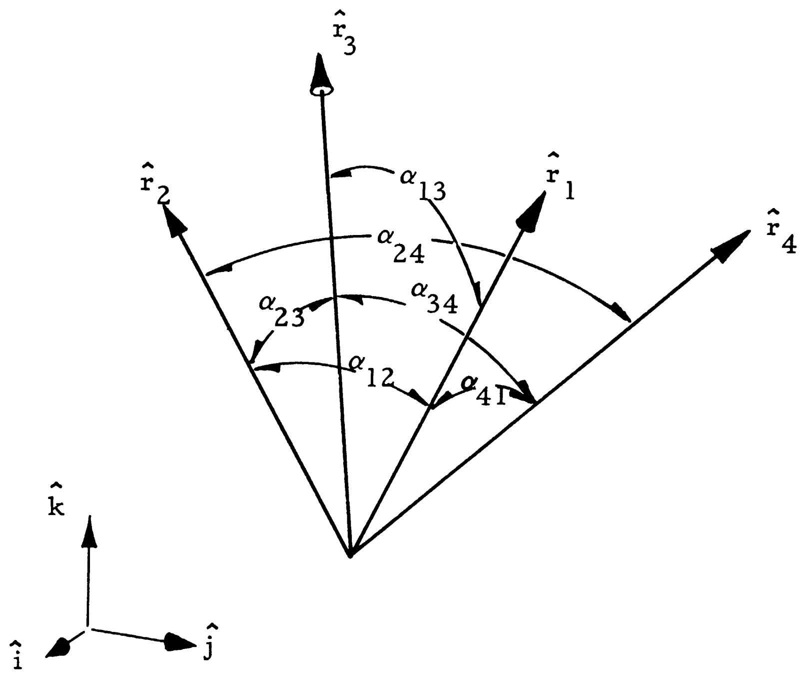
Define $\hat{\nu}$ so that only $\cos \phi_r$ will remain when the scalar product indicated in Equation (3.124) is performed. Define μ so that the $\hat{\mu}$ term will be zero in the scalar product of Equation (3.123).

$$\hat{\nu} \equiv \hat{b} \tag{3.126}$$



$$\alpha \equiv \cos^{-1}(c_1)$$
$$\beta \equiv \cos^{-1}(c_2)$$

Figure 3.9 Geometry of Solution for Simultaneous Scalar Products



$$\alpha_{12} = \cos^{-1}(c_{12})$$
$$\alpha_{23} = \cos^{-1}(c_{23})$$
$$\alpha_{34} = \cos^{-1}(c_{34})$$
$$\alpha_{41} = \cos^{-1}(c_{41})$$
$$\alpha_{24} = \text{unknown}$$

Figure 3.10 Geometry of Solution for Five Scalar Products between Four Unit Vectors, Sixth Product Unknown

$$\hat{\mu} \equiv \frac{(\hat{a} \times \hat{b})}{|\hat{a} \times \hat{b}|} \tag{3.127}$$

$$\hat{\lambda} \equiv \hat{\mu} \times \hat{\nu} = \frac{(\hat{a} \cdot \hat{b})\hat{b} - \hat{a}}{|\hat{a} \times \hat{b}|} \tag{3.128}$$

Perform the scalar products,

$$- \sin \phi_r \cos \theta_r \, |\hat{a} \times \hat{b}| + \cos \phi_r (\hat{a} \cdot \hat{b}) = c_1 \tag{3.129}$$

$$\cos \phi_r = c_2 \tag{3.130}$$

Equations (3.129) and (3.130) are the source of expressions for $\sin \theta_r$, $\cos \theta_r$; $\sin \phi_r$, $\cos \phi_r$. These are substituted back into Equation (3.125).

Solution:

$$\hat{r} = \frac{1}{[1 - (\hat{a} \cdot \hat{b})^2]^{1/2}} \{[c_2(\hat{a} \cdot \hat{b}) - c_1]\hat{\lambda} \pm [2(c_1)(c_2)(\hat{a} \cdot \hat{b}) - c_1^2 - c_2^2 - (\hat{a} \cdot \hat{b})^2 + 1]^{1/2} \hat{\mu}\} + c_2 \hat{b} \tag{3.131}$$

(2) Five scalar products between four unit vectors, sixth product unknown.

Given:

$$(\hat{r}_1 \cdot \hat{r}_2) = c_{12} \tag{3.132}$$

$$(\hat{r}_2 \cdot \hat{r}_3) = c_{23} \tag{3.133}$$

$$(\hat{r}_3 \cdot \hat{r}_4) = c_{34} \tag{3.134}$$

$$(\hat{r}_4 \cdot \hat{r}_1) = c_{41} \tag{3.135}$$

$$(\hat{r}_1 \cdot \hat{r}_3) = c_{13} \tag{3.136}$$

Determine $(\hat{r}_2 \cdot \hat{r}_4)$ in terms of the constants c_{ij}.

Using Equation (3.131), first express $\hat{r}_2$ in terms of $\hat{r}_1$, $\hat{r}_3$, c_{12}, c_{23}, and c_{13}; then express $\hat{r}_4$ in terms of $\hat{r}_1$, $\hat{r}_3$, c_{34}, c_{41}, and c_{13}.

$$\hat{r}_2 = \frac{1}{[1 - c_{13}^2]^{1/2}} \{ [c_{23} c_{13} - c_{12}] \hat{\lambda}_1 \pm [2c_{12}c_{23}c_{13} - c_{12}^2 - c_{23}^2 - c_{13}^2 + 1]^{1/2} \hat{\mu}_1 + c_{23}\hat{r}_3 \tag{3.137}$$

$$\hat{\mu}_1 \equiv \frac{\hat{r}_1 \times \hat{r}_3}{|\hat{r}_1 \times \hat{r}_3|} \tag{3.138}$$

$$\hat{\lambda}_1 \equiv \frac{(\hat{r}_1 \cdot \hat{r}_3)\hat{r}_3 - \hat{r}_1}{|\hat{r}_1 \times \hat{r}_3|} \tag{3.139}$$

$$\hat{r}_4 = \frac{1}{[1 - c_{13}^2]^{1/2}} \{ [c_{41}c_{13} - c_{34}]\hat{\lambda}_2 \pm [2c_{34}c_{41}c_{13} - c_{34}^2 - c_{41}^2 - c_{13}^2 + 1]^{1/2} \hat{\mu}_2 + c_{34}\hat{r}_3 \tag{3.140}$$

$$\hat{\mu}_2 = \hat{\mu}_1 \tag{3.141}$$

$$\hat{\lambda}_2 = \hat{\lambda}_1 \tag{3.142}$$

Obtain $(\hat{r}_2 \cdot \hat{r}_4)$ as the scalar product of Equations (3.137) and (3.140).

Solution:

$$\hat{r}_2 \cdot \hat{r}_4 = \frac{1}{[1 - c_{13}^2]}\{[c_{23}c_{13} - c_{12}][c_{41}c_{13} - c_{34}] \pm [2c_{12}c_{23}c_{13} - c_{12}^2 - c_{23}^2 - c_{13}^2 + 1]^{1/2}[2c_{34}c_{41}c_{13} - c_{34}^2 - c_{41}^2 - c_{13}^2 + 1]^{1/2}\} + c_{23}c_{34} \tag{3.143}$$

3.2 The Eliminant

The solution of difficult vector equations is likely to require simultaneous solution of algebraic polynomials. This is suggested by the solutions to cases 2b, 3c, and 3d of the Tetrahedron Equation. In each of these cases the solution can be obtained as a polynomial in a single unknown. However, in more complicated problems it might only be feasible to reduce the problem to two polynomials, each in the same two unknowns. The direct mathematical approach to obtaining roots in such a situation involves use of the eliminant (also known as the resultant determinant or Sylvester's determinant). Several texts on algebra derive and discuss the eliminant [75]; only its use will be described here.

Development of the eliminant approach as a major tool for the solution of three or more simultaneous polynomials is yet to be achieved. This will probably require use of statistical or iterative methods, because of the astronomical number of operations involved in an exact procedure. The present development is a reasonable tool for simultaneous solution of two polynomials of low degree, but is primarily an exploratory device.

Consider the problem of obtaining the same y root in each of the following two equations.

$$f_m y^m + f_{m-1} y^{m-1} + \ldots + f_1 y + f_0 = 0 \tag{3.144}$$

$$g_n y^n + g_{n-1} y^{n-1} + \ldots + g_1 y + g_0 = 0 \tag{3.145}$$

A necessary and sufficient condition for this is that the following determinant be zero:

$$\begin{array}{c} \\ \\ n \\ \\ \\ \\ m \\ \\ \\ \end{array} \overbrace{\begin{vmatrix} f_m & f_{m-1} & f_{m-2} & \cdots & f_0 & & \\ 0 & f_m & f_{m-1} & f_{m-2} & & f_0 & \\ \vdots & & & & & & \\ & & f_m & f_{m-1} & f_{m-2} & \cdots & f_0 \\ g_n & g_{n-1} & g_{n-2} & \cdots & g_0 & & \\ 0 & g_n & g_{n-1} & g_{n-2} & \cdots & g_0 & \\ \vdots & & & & & & \\ & & g_n & g_{n-1} & g_{n-2} & \cdots & g_0 \end{vmatrix}}^{t = m + n} = 0 \tag{3.146}$$

In Equation (3.146) all blanks should be filled with zeros. For example, when $m = 3$ and $n = 2$, Equation (3.146) becomes

$$\begin{vmatrix} f_3 & f_2 & f_1 & f_0 & 0 \\ 0 & f_3 & f_2 & f_1 & f_0 \\ g_2 & g_1 & g_0 & 0 & 0 \\ 0 & g_2 & g_1 & g_0 & 0 \\ 0 & 0 & g_2 & g_1 & g_0 \end{vmatrix} = 0 \tag{3.147}$$

In Equation (3.146), the quantities f_i and g_i may be functions of any number of variables $(u, v, w, x, \ldots)$. However, for present purposes let the f_i all be kth degree polynomials in x; the g_i all ℓth degree polynomials in x. The result of expanding Equation (3.146) is therefore a single polynomial in x, of degree $(nk + ml)$. However, the number of multiplications involved in the expansion prohibits doing it by hand, unless all but a few of the coefficients of the f_i and g_i polynomials are zero. This presents two alternatives, both of which require the digital computer.

(1) Determine the roots of Equation (3.146) by iteration on the determinant itself. Equation (3.146) is identical in form to the determinant that must be solved in an eigenvalue problem, such as that of a multidegree-of-freedom vibrating system. Well-developed iterative methods exist for determining the roots [60].

(2) Derive a tensor expression for the coefficients of the resultant polynomial and evaluate the coefficients exactly or statistically. Determine the roots by iteration on the resultant polynomial.

The latter approach is taken here because it seems easier to extend to the solution of more than two simultaneous polynomials.

In tensor notation [69], a determinant $|h^i_{\cdot j}|$ is written

$$|h^i_{\cdot j}| = \epsilon^{i_1 i_2 \cdots i_{(t)}} h^1_{\cdot i_1} h^2_{\cdot i_2} \cdots h^{(t)}_{\cdot i_{(t)}} \tag{3.148}$$

$$t \equiv m + n \tag{3.149}$$

$\epsilon^{i_1 i_2 \cdots i_{(t)}}$ is a permutation symbol and takes the following values:

(1) +1 when $i_1 i_2 \cdots i_{(t)}$ is an even permutation of the numbers 1, 2, ..., t.

(2) -1 when $i_1 i_2 \cdots i_{(t)}$ is an odd permutation of the numbers 1, 2, ..., t.

(3) 0 in all other cases. (Other cases occur when there are two or more duplicate integers among the $i_1 i_2 \cdots i_{(t)}$).

In Equation (3.148), each of the $h^j_{\cdot i_{(j)}}$ terms is a polynomial in x. (The zero terms are regarded as polynomials with all zero coefficients.)

$$h^j_{\cdot i_{(j)}} = c^j_{\cdot i_{(j)} k_{(j)}} x^{k_{(j)}} \tag{3.150}$$

The $c^j_{i_{(j)}k_{(j)}}$ correspond to coefficients in the f or g polynomials (Equations (3.144) and (3.145)), depending on whether $i_{(j)}$ exceeds or is less than m, in Equation (3.146). Substitute Equation (3.150) into Equation (3.148).

$$|h^i_j| = \epsilon^{i_1 i_2 \cdots i_{(t)}} c^1_{i_1 k_1} c^2_{i_2 k_2} \cdots c^{(t)}_{i_{(t)} k_{(t)}} x^{k_1} x^{k_2} \cdots x^{k_{(t)}} \tag{3.151}$$

The $2t$ summations indicated in Equation (3.151) may be carried out by whatever procedure is most convenient. The following approach is designed to compute the individual coefficients of the resultant polynomial in the course of the summation.

(1) Increment an integer p, in steps of one, over the range $0 \leqq p \leqq (nk + m\ell)$.

(2) At each value of p, evaluate Equation (3.151) for every combination of $k_1, k_2, \ldots, k_{(t)}$ totaling p. Each such evaluation can be performed by a standard determinant routine. This is because the $x^{k_{(j)}}$ terms can be factored out as x^p, and because the remaining terms have the form of Equation (3.148) when the $k_{(j)}$ are fixed.

(3) For each value of p, sum all of the determinant values obtained in (2). This sum is the pth coefficient of the resultant polynomial in x.

(4) When p has completed its range, all coefficients of the resultant polynomial in x will be available. Solve this polynomial for all $(nk + m\ell)$ x roots (a standard routine may be employed). Substitute each of these roots, one at a time, back into both Equations (3.144) and (3.145), to evaluate the f and g coefficients. Then for each x root, Equations (3.144) and (3.145) are solved for their several roots. Ordinarily one and only one pair of the y roots from Equations (3.144) and (3.145) will match, for any one c root. The x root and the matching y root will constitute one of the $(nk + m\ell)$ solutions to Equations (3.144) and (3.145).

A computer subprogram written to perform this work is described in Appendix A.3.3. Results from two example problems are presented in Table 3.3.

In theory there is no limit to the degree or the number of simultaneous polynomials that can be solved by an approach of this kind. The present program has no theoretical restriction on the degrees of the polynomials. (This is suggested by Example 2, Table 3.3.) More than two simultaneous polynomials can be solved by a Gauss-Jordon kind of reduction process, pairing one polynomial against all the others to eliminate an unknown throughout, then repeating until all but one unknown is eliminated. Of course, the eliminant (Equation 3.151))

TABLE 3.3

EXAMPLES OF SOLUTION OF SIMULTANEOUS POLYNOMIALS

Example 1. Execution Time = 5.12 sec.

$$(3x^2 + 2x + 1)y^2 + (6x^2 - 1x + 3)y + (x^2 + 5x - 1) = 0$$

$$(x^2 + 5x - 3)y^2 + (3x^2 + 1x + 2)y + (x^2 - 2x + 1) = 0$$

Deg. of Coeff.	Coefficient of Resultant x Polynomial	Root No.	x roots: real part / imag. part	y roots: real part / imag. part
8	-5.000000×10^{0}	1	9.866105×10^{0}	-3.017123×10^{-1}
7	9.100000×10^{1}		0	0
6	-5.820000×10^{2}	2	2.876584×10^{-1}	-1.809936×10^{-1}
5	1.794000×10^{3}		0	0
4	-1.089000×10^{3}	3,4	3.811290×10^{0}	-1.463227×10^{0}
3	2.220000×10^{1}		$\pm3.748579\times10^{0}$	$\mp3.414708\times10^{-1}$
2	4.400001×10^{1}	5,6	-3.283790×10^{-1}	8.034719×10^{-1}
1	1.610000×10^{2}		$\pm3.626602\times10^{1}$	$\mp3.931192\times10^{-2}$
0	-5.100000×10^{1}	7,8	5.402081×10^{-1}	-7.141178×10^{-1}
			$\pm4.833163\times10^{-1}$	$\pm8.829509\times10^{-1}$

TABLE 3.3 CONT'D

Example 2. **Execution Time** = 49.6 sec.

$$(7x - 9)y^4 + (0x + 5)y^3 + (-7x + 2)y^2 - (-3x - 9)y + (-5x + 2) = 0$$

$$(2x^3 + 5x^2 + 0x - 5)y^2 + (5x^3 + 1x^2 + 5x - 1)y + (7x^3 + 6x^2 - 3x + 5) = 0$$

Deg. of Coeff.	Coefficient of Resultant x Polynomial	Root No.	x roots real part imag. part	y roots real part imag. part
14	1.323100×10^5	1	-3.301638×10^0	-1.053711×10^0
13	2.024010×10^5		0	0
12	-9.715408×10^5	2	-7.172048×10^{-1}	8.797837×10^{-1}
11	-6.582907×10^4		0	0
10	1.989712×10^6	3,4	-1.232323×10^0	1.432858×10^{-1}
9	-3.348428×10^5		$\pm4.584016\times10^{-1}$	$\pm7.117867\times10^{-1}$
8	-1.238512×10^6	5,6	1.483417×10^0	-7.254179×10^{-1}
7	1.371025×10^6		$\pm6.324904\times10^{-1}$	$\pm1.445311\times10^0$
6	-4.126216×10^5	7,8	6.743634×10^{-1}	5.733167×10^{-1}
5	8.232079×10^4		$\pm3.781570\times10^0$	$\pm1.454501\times10^0$
4	-1.085100×10^5	9,10	3.279200×10^{-1}	-1.994073×10^{-2}
3	3.719302×10^4		$\pm7.156668\times10^{-1}$	$\mp2.938320\times10^{-1}$
2	2.529590×10^5	11,12	-3.084983×10^{-1}	$+9.671565\times10^{-1}$
1	-1.522590×10^5		$\pm6.224477\times10^{-1}$	$\mp3.212334\times10^{-1}$
0	2.468700×10^4	13,14	2.996680×10^{-1}	-9.472472×10^{-1}
			$\pm9.074256\times10^{-2}$	$\pm9.358603\times10^{-3}$

is required for each pairing rather than the multiply-through-and-subtract-operation used with simultaneous linear equations. The coefficients of the eliminant are polynomials of more than one variable, until the very last elimination. There are probably other more efficient procedures, but this is at least one possibility.

In practice, there are several effects which severely limit application:

(1) Time required for exact computation. The coefficients of the resultant polynomial are computed by successive evaluation of determinants. Even when only two simultaneous polynomials must be solved, a total of $(k + 1)^n(\ell + 1)^m$ determinants must be evaluated, each of size $(m + n) \times (m + n)$. The approximate IBM 7090 time required per determinant is,

$$\Delta t = 56(m + n)^3 + 72(m + n)^2 + 320(m + n) + 157 \tag{3.152}$$

(Δt in microseconds)

Figure 3.11 is a plot of the time required for determinant evaluation alone, versus m, for several combinations of n, k, ℓ. Significant additional time is required for the iterative solution of the single-variable polynomials obtained in the procedure. Clearly, the computational time can become excessive. It may be that the computation time limitation can be substantially eased by a Monte Carlo procedure. Equation (3.151) is then evaluated for many sets of indices, chosen

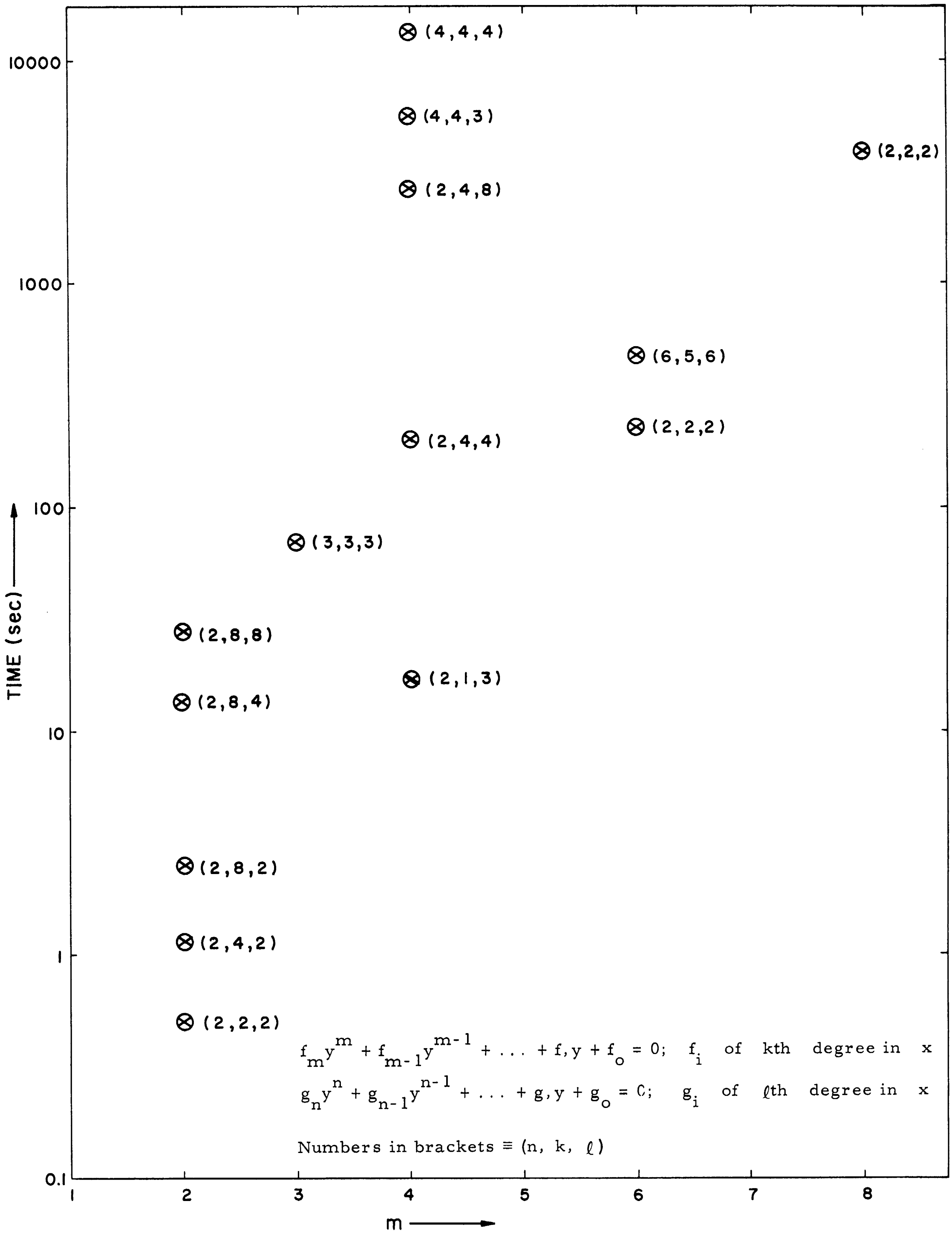


Figure 3.11 Computer Time Required for Exact Computation of Determinants in Simultaneous Solution of Two Polynomials

at random. The coefficients of the resultant polynomial are then statistical estimates (except for the very high and very low degree coefficients, which can be obtained exactly with the evaluation of only a few determinants). Approximate roots can be obtained, then used as initial approximations for an iterative procedure.

(2) Round-off error. This has not been a problem for exact computation (involving degrees less than 3 or 4) performed so far. However, such difficulty is expected in view of the amount of multiplication and subtraction that takes place.

(3) Excessive degree of the resultant polynomial. For simultaneous solution of only two polynomials the resultant polynomial has degree $p = nk + m\ell$. In general, p will rapidly increase with increase in the number of polynomials to be solved. For $p > 1000$ the iteration time to obtain the roots may become a serious limitation. Perhaps larger polynomials must be represented by power series approximations in a smaller number of terms, before being used in the eliminant.

3.3 Application

3.3.1 Direct use of the Tetrahedron Solutions

Figure 3.12 is a schematic of the front independent suspension and steering system of a conventional automobile. To achieve good handling characteristics the automotive designer must have a means

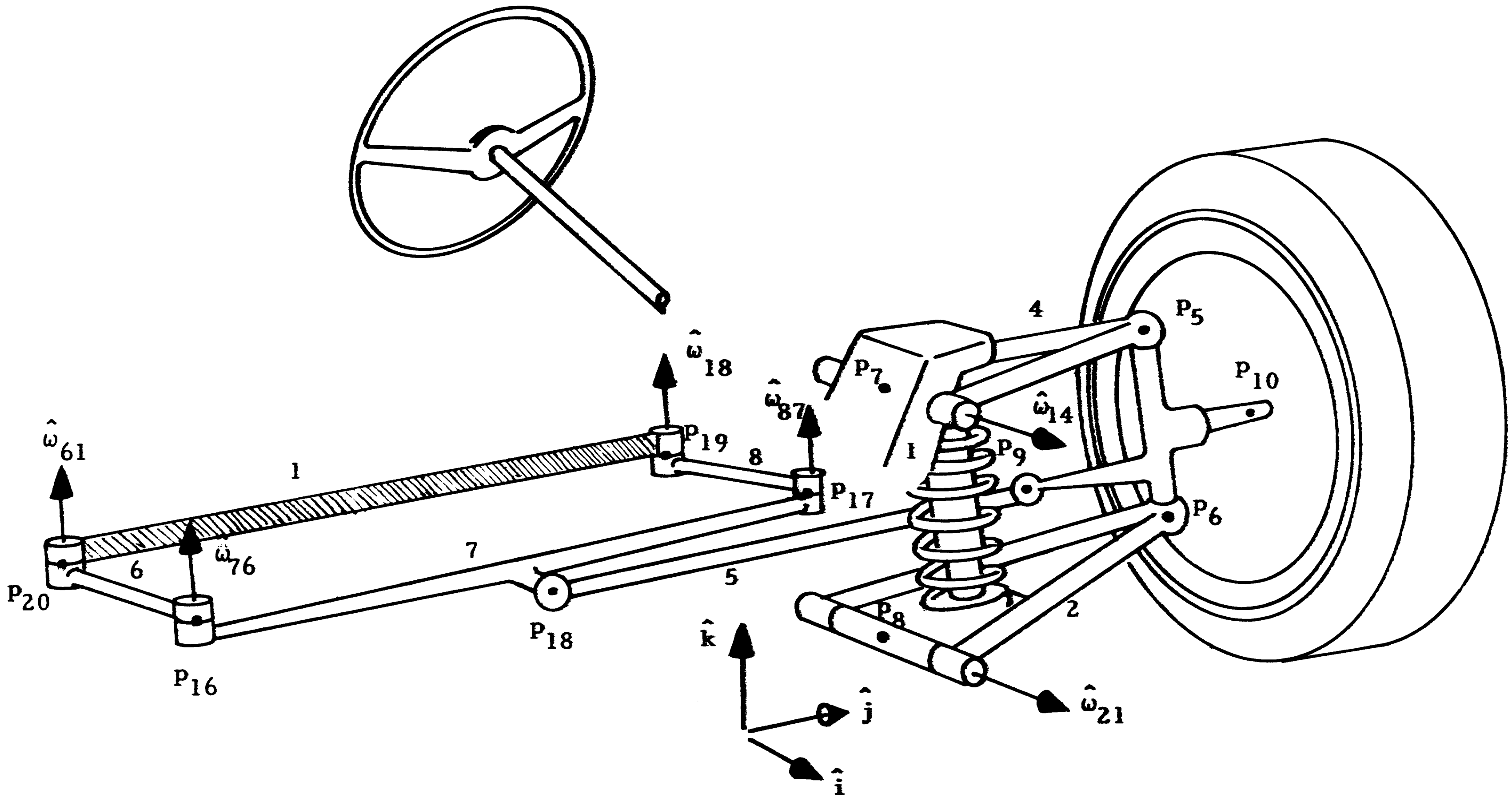


Figure 3.12 Schematic of a Front Independent Suspension and Steering System

of calculating all the position vectors in this sytem, given a steering angle and some input suspension parameter.

Assume that the instantaneous position of the automobile frame relative to the road is known. If this is so, vectors $\vec{r}_{p_8p_7}$, $\vec{r}_{p_{20}p_{19}}$, $\vec{r}_{p_{19}p_8}$, $\hat{\omega}_{21}$, $\hat{\omega}_{41}$, $\hat{\omega}_{61}$, $\hat{\omega}_{18}$ are all known. For evaluating the performance of the linkage system itself, the two input quantities can be the azimuthal angles of the rotations of $\vec{r}_{p_{16}p_{20}}$ and $\vec{r}_{p_6p_8}$ about their respective axes, $\hat{\omega}_{61}$ and $\hat{\omega}_{18}$. These angles determine $\vec{r}_{p_{16}p_{20}}$ and $\vec{r}_{p_6p_8}$ because the corresponding magnitudes and polar angles are known from design.

The solution can begin with loop p_{20}, p_{16}, p_{17}, p_{19}. As a three-dimensional linkage this loop is over-determined. However, it is very nearly two-dimensional in its motion, and the hinge pairs at p_{16} and p_{17} are slightly elastic. The solution is obtained from case 2d of the Tetrahedron Solutions, because only $\hat{r}_{p_{17}p_{16}}$ and the azimuthal angle of $\vec{r}_{p_{19}p_{17}}$ are unknown. The vector $\vec{r}_{p_{18}p_{16}}$ is then determined from $\hat{r}_{p_{17}p_{16}}$, $\hat{\omega}_{76}$, and design constants c_1, c_2, and c_3.

$$\vec{r}_{p_{18}p_{16}} = c_1 \hat{r}_{p_{17}p_{16}} + c_2 \hat{\omega}_{76} + c_3 \frac{(\hat{r}_{p_{17}p_{16}} \times \hat{\omega}_{76})}{|\hat{r}_{p_{17}p_{16}} \times \hat{\omega}_{76}|} \qquad (3.153)$$

Case 2d also solves the loop p_6, p_5, p_7, p_8 because the only unknowns there are $\hat{r}_{p_5p_6}$ and the azimuthal angle of $\vec{r}_{p_7p_5}$. This determines $\vec{r}_{p_5p_6}$ and $\vec{r}_{p_7p_5}$, but cannot determine the rotation of link 3 in the two spherical pairs at p_6 and p_5. In fact, this loop would thereby be underdetermined, if not for the constraint from link 5.

A third loop p_{18}, p_{11}, p_6 can be solved, using information from the results of the first two solutions. Here the unknowns are $\hat{r}_{p_{11}p_8}$ and the θ of $\vec{r}_{p_6p_{11}}$, and case 2d again applies. Two king-pin unit vectors, $\hat{r}_{p_5p_6}$ and $\hat{r}_{p_{11}p_6}$, are now known. The spindle vector $\vec{r}_{p_{10}p_9}$ can therefore be determined from these unit vectors and the design constants c_4, c_5, c_6.

$$\vec{r}_{p_{10}p_9} = c_4\hat{r}_{p_5p_6} + c_5\hat{r}_{p_{11}p_6} + c_6 \frac{(\hat{r}_{p_5p_6} \times \hat{r}_{p_{11}p_6})}{|\hat{r}_{p_5p_6} \times \hat{r}_{p_{11}p_6}|} \tag{3.154}$$

All unknown vectors have now been determined. However, other quantities such as camber, castor, and toe angle are of importance to the automotive designer. These can easily be calculated from the vector output as follows:

$$\text{Camber angle} = \cos^{-1}\left[\frac{(\vec{r}_{p_{10}p_9} \times \hat{i}) \cdot \hat{k}}{|\vec{r}_{p_{10}p_9} \times \hat{i}|}\right] \tag{3.155}$$

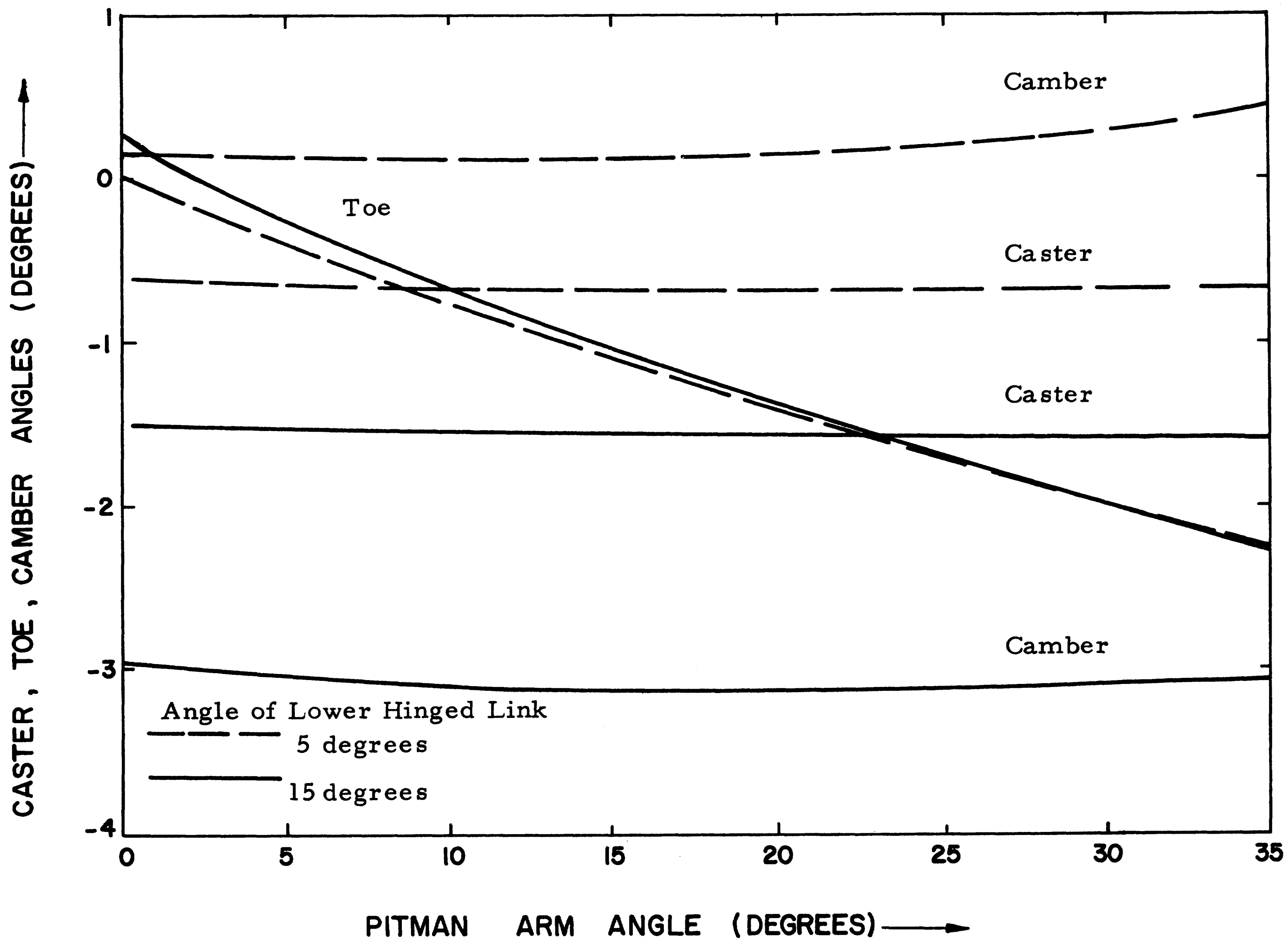


Figure 3.13 Variation of Camber, Castor and Toe Angle vs. Angles of Pitman Arm and Lower Hinged Link

$$\text{Castor angle} = \cos^{-1}\left\{\frac{[\vec{r}_{P_5P_6} - (\vec{r}_{P_5P_6} \cdot \hat{j})\hat{j}] \cdot \hat{k}}{|\vec{r}_{P_5P_6} - (\vec{r}_{P_5P_6} \cdot \hat{j})\hat{j}|}\right\} \tag{3.156}$$

$$\text{Toe angle} = \cos^{-1}[\hat{r}_{P_{10}P_9} \cdot (-\hat{j})] \tag{3.157}$$

A computer program was written on the basis of these solutions to evaluate all unknown positions for several input sets of the azimuthal angles of $\vec{r}_{P_{16}P_{20}}$ and $\vec{r}_{P_6P_8}$. Design constants were evaluated from the "static position" of a 1962 Ford Galaxy. Figure 3.13 shows families of curves based on results of the computer program.[3/] Camber, castor, and toe angles are plotted versus Pitman arm angle, for constant values of the azimuthal angle of the lower hinged link. Of course, much more information than shown is required in actual design work. Figure 3.13 only suggests that concrete results are in fact obtained and that high accuracy is required to correctly predict the small variations in angles.

3.3.2 A Four-Bar Linkage with Turn-Slide Pairs

Figure 3.14 shows a three-dimensional four-bar linkage with one hinge pair and three turning and sliding pairs. The axes of the pairs are skew and all links are "bent." There are several reasons why this linkage was chosen as an example.

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[3] Most of the programming of this computer work was performed by Mr. De Witt Cooper of the IBM Automotive and Machine Design Project, and his assistance here is especially appreciated.

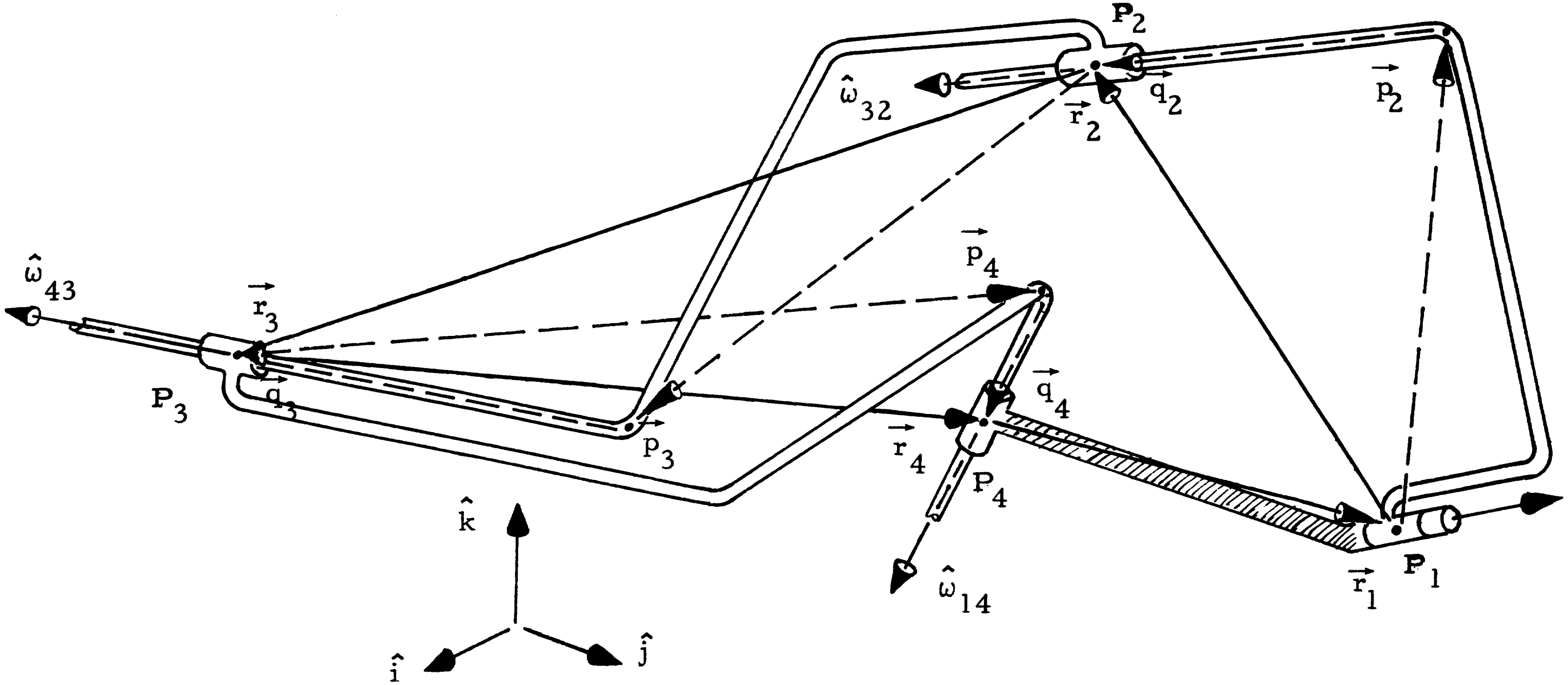


Figure 3.14 Three-Dimensional, Four-Bar Linkage with One Hinge Pair and Three Turn-Slide Pairs.

(1) The linkage has so far found no use in practical design, to the author's knowledge. It may be that availability of its position, motion, and force solutions--in terms of conventional mathematics--will encourage such application. Only the position solution is derived in this section. Motion and force solutions are derived in Sections 4.0 and 5.0.

(2) Solution of combined vector and scalar conditions is required. This suggests a source of difficulty in linkages with higher numbers of links, but does not prevent a relatively simple solution here.

(3) Position, velocity, and force solutions have been explored by A. T. Yang, by use of quaternion mathematics [79, 80]. This may provide a common basis of comparison between the use of quaternion and ordinary vector mathematics in the analysis of three-dimensional mechanisms.

(4) All terms are present in the motion analysis, because the link motion involves both relative rotation and relative slide. Likewise power is transmitted through the three turn-slide joints by both force and torque, so that the force equilibrium analysis has a generalized nature. The example is therefore valuable as a check against the

vector approach itself, especially against the acceleration and force equilibrium methods.

Three conditions govern the position of the linkage in Figure 3.14:

$$\vec{r}_1 + (\vec{p}_2 + \vec{q}_2) + (\vec{p}_3 + \vec{q}_3) + (\vec{p}_4 + \vec{q}_4) = 0 \tag{3.158}$$

$$(\hat{\omega}_{32} \cdot \hat{\omega}_{43}) = c_1 \tag{3.159}$$

$$(\hat{\omega}_{43} \cdot \hat{\omega}_{41}) = c_2 \tag{3.160}$$

In Equations (3.159) and (3.160), c_1 and c_2 are design constants. For example, c_1 can be determined from bench measurements of link 3 before the linkage is assembled.

Fortunately Equations (3.159) and (3.160) can be solved independently of Equation (3.158); otherwise a symmetry solution would not be feasible. The input angle is θ_{21} (the azimuthal angle of rotation of $\vec{p}_2$ about $\hat{\omega}_{21}$). This determines $\vec{p}_2$, because p_2 and the polar angle ϕ_{21} are already known.

$$\vec{p}_2 = p_2\{\sin \phi_{21}[\cos \theta_{21}\hat{\lambda}_{21} + \sin \theta_{21}\hat{\mu}_{21}] + \cos \phi_{21}\hat{\nu}_{21}\} \tag{3.161}$$

$$\hat{\nu}_{21} \equiv \hat{\omega}_{21} \tag{3.162}$$

$$\hat{\lambda}_{21} \equiv \frac{\vec{r}_1 \times \hat{\omega}_{21}}{|\vec{r}_1 \times \hat{\omega}_{21}|} \tag{3.163}$$

$$\hat{\mu}_{21} \equiv (\hat{\nu}_{21} \times \hat{\lambda}_{21}) \tag{3.164}$$

The orientation of $\hat{\omega}_{32}$ relative to $\vec{p}_2$ and $\hat{\omega}_{21}$ is known from the design of link 2. With $\vec{p}_2$ and $\hat{\omega}_{21}$ both known, $\hat{\omega}_{32}$ is also known.

$$\hat{\omega}_{32} = \sin \phi_{32}[\cos \theta_{32}\hat{\lambda}_{32} + \sin \theta_{32}\hat{\mu}_{32}] + \cos \phi_{32}\hat{\nu}_{32} \tag{3.165}$$

$$\hat{\nu}_{32} \equiv \hat{\omega}_{21} \tag{3.166}$$

$$\hat{\lambda}_{32} \equiv \frac{\hat{p}_2 \times \hat{\omega}_{21}}{|\hat{p}_2 \times \hat{\omega}_{21}|} \tag{3.167}$$

$$\hat{\mu}_{32} \equiv \hat{\nu}_{32} \times \hat{\lambda}_{32} \tag{3.168}$$

(θ_{32} and ϕ_{32} are design constants.)

The unit vector $\hat{\omega}_{14}$ is known, because it is fixed in orientation relative to ground. Equations (3.159) and (3.160) therefore have the form required for solution by Equation (3.131), because only $\hat{\omega}_{43}$ is unknown, of the three unit vectors $\hat{\omega}_{32}$, $\hat{\omega}_{43}$, $\hat{\omega}_{14}$.

Determine $\hat{\omega}_{43}$ from Equation (3.131). The plus or minus sign is chosen plus if $(\hat{\omega}_{43} \cdot \hat{\mu})$ is plus, minus if $(\hat{\omega}_{43} \cdot \hat{\mu})$ is minus--for the mechanism as initially assembled.

In Equation (3.158), vectors $\vec{r}_2$, $\vec{r}_3$, and $\vec{r}_4$ have been expressed as sums of component vectors: $(\vec{p}_2 + \vec{q}_2)$, $(\vec{p}_3 + \vec{q}_3)$, $(\vec{p}_4 + \vec{q}_4)$. This is done because the pair axes are skew. The $\vec{q}_i$ vectors are defined parallel to the slide axes, $\hat{\omega}_{i+1,i}$, but are unknown in length. The $\vec{p}_i$ vectors are directed to any given point on the axes of slide, from the preceding pair. With $\hat{\omega}_{32}$, $\hat{\omega}_{43}$ and $\hat{\omega}_{14}$ determined, the vectors $\vec{p}_2$, $\vec{p}_3$ and $\vec{p}_4$ are all known. This is suggested by the following equations:

$$\vec{p}_2 = c_{21}\hat{\omega}_{21} + c_{22}\frac{(\hat{\omega}_{21} \times \hat{\omega}_{32})}{|\hat{\omega}_{21} \times \hat{\omega}_{32}|} \tag{3.169}$$

$$\vec{p}_3 = c_{31}\hat{\omega}_{32} + c_{32}\frac{(\hat{\omega}_{32} \times \hat{\omega}_{43})}{|\hat{\omega}_{32} \times \hat{\omega}_{43}|} \tag{3.170}$$

$$\vec{p}_4 = c_{41}\hat{\omega}_{43} + c_{42}\frac{(\hat{\omega}_{43} \times \hat{\omega}_{14})}{|\hat{\omega}_{43} \times \hat{\omega}_{14}|} \tag{3.171}$$

(c_{21}, c_{22}, c_{31}, c_{32}, c_{41}, c_{42} are design constants.)

Equation (3.158) can be restated,

$$q_2\hat{\omega}_{32} + q_3\hat{\omega}_{43} + q_4\hat{\omega}_{14} + \vec{C} = 0 \tag{3.172}$$

$$\vec{C} \equiv \vec{r}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4 \tag{3.173}$$

Equation (3.172) has the form of case 3a of the Tetrahedron Solutions, because only q_2, q_3, and q_4 are unknown. Vectors $\vec{q}_2$, $\vec{q}_3$, $\vec{q}_4$ are determined from Equation (3.173), then $\vec{r}_2$, $\vec{r}_3$, $\vec{r}_4$ are found from the sums, $(\vec{p}_i + \vec{q}_i)$.

TABLE 3.4

INPUT PARAMETERS FOR EXAMPLE FOUR-BAR LINKAGE WITH ONE HINGE PAIR AND THREE TURN-SLIDE PAIRS

Design Constants:

$\vec{r}_1 = 0.0\hat{i} + 1.0\hat{j} + 0.0\hat{k}$ in

$\hat{\omega}_{21} = \frac{1}{\sqrt{26.0}} (1.0\hat{i} + 3.0\hat{j} + 4.0\hat{k})$

$\hat{\omega}_{14} = \frac{1}{\sqrt{38.0}} (-1.0\hat{i} - 1.0\hat{j} - 6.0\hat{k})$

$(\vec{p}_2 \cdot \hat{\omega}_{21}) = 3.0$ in, $\vec{p}_2 \cdot \frac{(\hat{\omega}_{21} \times \hat{\omega}_{32})}{|\hat{\omega}_{21} \times \hat{\omega}_{32}|} = -2.0$ in, $\cos^{-1}(\hat{\omega}_{21} \cdot \hat{\omega}_{32}) = .5$ radians

$(\vec{p}_3 \cdot \hat{\omega}_{32}) = 2.0$ in $\vec{p}_3 \cdot \frac{(\hat{\omega}_{32} \times \hat{\omega}_{43})}{|\hat{\omega}_{32} \times \hat{\omega}_{43}|} = -4.0$ in $\cos^{-1}(\hat{\omega}_{32} \cdot \hat{\omega}_{43}) = 1.2$ radians

$(\vec{p}_4 \cdot \hat{\omega}_{43}) = 1.0$ in $\vec{p}_4 \cdot \frac{(\hat{\omega}_{43} \times \hat{\omega}_{14})}{|\hat{\omega}_{43} \times \hat{\omega}_{14}|} = 3.0$ in $\cos^{-1}(\hat{\omega}_{43} \cdot \hat{\omega}_{14}) = 1.9$ radians

Initial position, motion, and torque on link 2:

$\theta_{21i} = 0.0$ radians, measured from $\hat{\lambda}_2$ in the $+\hat{\nu}_2$ direction:

$$\hat{\lambda}_2 \equiv \frac{\vec{r}_1 \times \hat{\omega}_{21}}{|\vec{r}_1 \times \hat{\omega}_{21}|}, \quad \hat{\mu}_2 \equiv \hat{\omega}_{21} \times \hat{\lambda}_2 , \quad \hat{\nu}_2 \equiv \hat{\omega}_{21}$$

$\omega_{21i} = .1$ radians/sec.

$D\omega_{21i} = .01$ radians/(sec)^2

$\tau_{21i} = 2.0$ lb_f-in

Relation between output force and torque:

$\tau_{14o} = c_1 f_{14o} + c_2$

$c_1 = 4.0, \quad c_2 = 2.5$

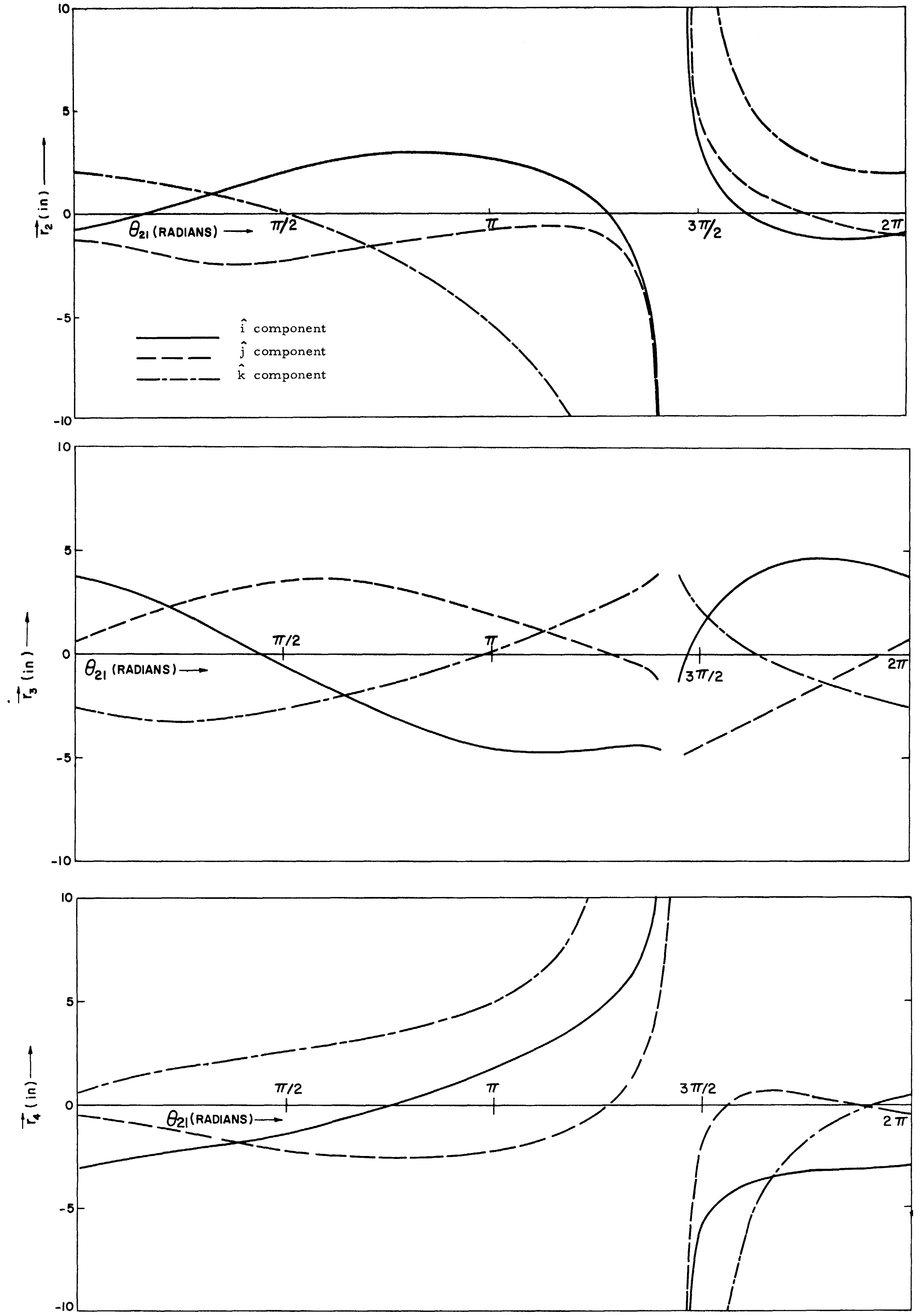


Figure 3.15 Variation of Position Vectors $\vec{r}_1$, $\vec{r}_3$, $\vec{r}_4$ for a Complete Cycle of the Mechanism of Figure 3.14

Figure 3.15 shows the variation of the $\hat{i}$, $\hat{j}$, $\hat{k}$ components of position vectors $\vec{r}_2$, $\vec{r}_3$, $\vec{r}_4$ as the mechanism of Figure 3.14 is moved through a complete cycle. Input parameters for this example are summarized in Table 3.4. Computations were performed by digital computer, using a program written to evaluate and check the position, motion, and force in any mechanism of this type.

The position singularities evident in Figure 3.15 occur when $\hat{\omega}_{32}$, $\hat{\omega}_{43}$, and $\hat{\omega}_{14}$ become co-planar ($\theta_{21} = 1.40\pi$, 1.43π). In this situation the quantity $[\hat{\omega}_{32} \cdot (\hat{\omega}_{43} \times \hat{\omega}_{14})]$ is zero, and this quantity happens to be the denominator of the expressions for q_2, q_3, q_4. Geometrically the singularities occur when the two cones in Figure 3.9 become tangent (unit vectors $\hat{a}$, $\hat{r}$, $\hat{b}$ correspond to $\hat{\omega}_{32}$, $\hat{\omega}_{43}$, $\hat{\omega}_{14}$, respectively). There is a region between the two tangent positions in which the mechanism cannot exist; here this is only about five degrees wide.

The motions and forces of this mechanism are discussed in Sections 4.0 and 5.0. Motion singularities occur at the same place as position singularities.

4.0 MOTION

4.1 Development

4.1.1 Vector Loop Equations

In kinematic problems each order of motion is dependent only on motion quantities having the same or lower order. This fact allows compartmenting the entire solution of a mechanism's motion. First, the position problem (zeroth order motion) is solved, as discussed in Section 3.0. These solutions are usually non-linear and often difficult to obtain, but they always proceed from single or simultaneous conditions of the following form:

$$\left[\sum_{i=1}^{n-1} r_{p_{i+1}p_i}\right] + \vec{r}_{p_1p_n} = 0 \qquad (4.1)$$

$$(\hat{\omega}_{i,i-1}) \cdot (\hat{\omega}_{j,j-1}) = c_{ij} \qquad (4.2)$$

A direct approach to obtaining motion solutions is differentiation of the actual solutions to Equations (4.1) and (4.2). This is rejected for reasons stated earlier (p. 21). Instead, Equation (4.1) itself is differentiated and a companion condition on angular velocity is stated and differentiated. For mth order motion, the forms of these conditions are

$$\left[\sum_{i=2}^{n} D^{m-1}\vec{\omega}_{i,i-1}\right] + D^{m-1}\vec{\omega}_{p_1p_n} = 0 \qquad (4.3)$$

$$\left[\sum_{i=1}^{n-1} D^m \vec{r}_{p_{i+1}p_i}\right] + D^m \vec{r}_{p_1 p_n} = 0 \tag{4.4}$$

Equation (4.3) with $m = 1$ is a statement that the sum of the relative angular velocities of a linkage loop is zero. An appropriate form of it is written for every closed loop in the mechanism considered. For illustration, consider the single loop in Figure 4.1. Body 2 rotates relative to body 1 with angular velocity $\vec{\omega}_{21}$. Similarly, body 3 rotates relative to body 2 with $\vec{\omega}_{32}$. ($\vec{\omega}_{32}$ is the angular velocity of body 3 measured relative to a dummy reference frame fixed in body 2.) Proceeding around the loop, finally body 1 rotates with angular velocity $\vec{\omega}_{1n}$ relative to n. A vectorial angular position statement of this form cannot be made [43]. Thus Equation (4.3) must be considered the fundamental angular condition. Equation (4.1) is the fundamental linear condition. In three-dimensional motion these conditions are independent.

4.1.2 Expressions for derivatives

Solutions for successive orders of motion can proceed from equations of the form (4.3) and (4.4) if general expressions for the vector derivatives are substituted term by term.

By definition, a vector is differentiated according to Equation 13, Table 2.3, even if the vector is a unit vector. However, an operator $\vec{\omega}$ can be defined such that for a given unit vector, $\hat{u} = \hat{u}(t)$:

$$D\hat{u} = \vec{\omega} \times \hat{u} \tag{4.5}$$

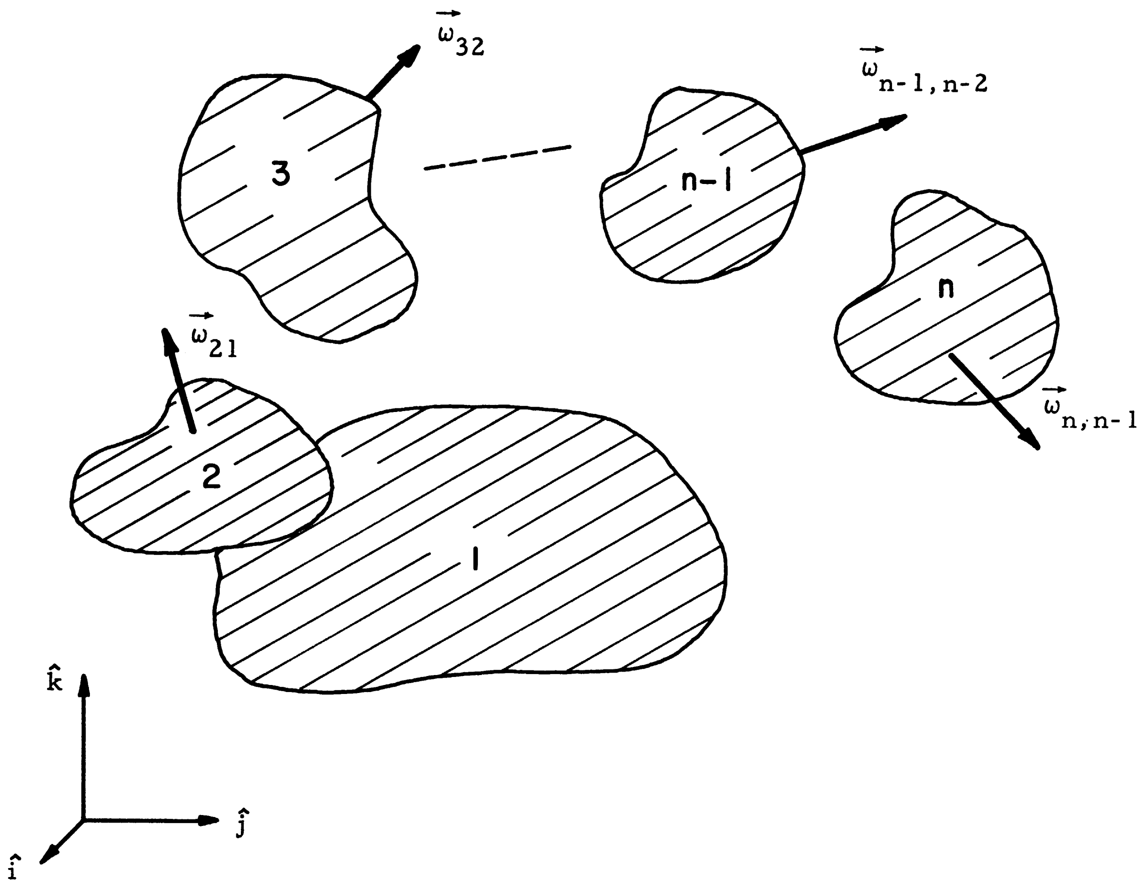


Figure 4.1 Relative rotation of
n rigid bodies

In mechanism analysis $\hat{u}$ usually corresponds to the direction of a position vector $(\hat{r}_{p_i p_j})$ or an axis of rotation $(\hat{\omega}_{ij})$; $\vec{\omega}$ corresponds to the angular velocity of $\hat{u}$ relative to ground $(\vec{\omega}_{i1})$.

For present purposes Equation (4.5) is specifically stated,

$$D\hat{u} = \vec{\omega}_{i1} \times \hat{u} \tag{4.6}$$

Here $\hat{u}$ is any unit vector, fixed in a rigid body i. Body i rotates relative to ground with angular velocity $\vec{\omega}_{i1}$.

Equations 19 and 20, Table 2.3, are derived as follows, using Equation (4.6) and Equations 16 and 18, Table 2.3:

$$u = u\hat{u} \tag{4.7}$$

$$D\vec{u} = (Du)\hat{u} + u(D\hat{u}) \tag{4.8}$$

$$D\vec{u} = (Du)\hat{u} + (\vec{\omega} \times \vec{u}) \tag{4.9}$$

$$D^2\vec{u} = (D^2u)\hat{u} + (Du)(D\hat{u}) + (D\vec{\omega} \times \vec{u}) + (\vec{\omega} \times D\vec{u}) \tag{4.10}$$

$$D^2\vec{u} = (D^2u)\hat{u} + \vec{\omega} \times (\vec{\omega} \times \vec{u}) + (D\vec{\omega} \times \vec{u}) + 2[\vec{\omega} \times (Du)\hat{u}] \tag{4.11}$$

Higher order derivatives can be obtained similarly, although the number of terms increases rapidly with the order. In most kinematic analysis only the first and second derivatives are of interest.

Replace $\vec{u}$ with $\vec{r}_{p_{i+1}p_i}$ and $\vec{\omega}$ with $\vec{\omega}_{i1}$ in Equations (4.9) and (4.11).

$$D\vec{r}_{p_{i+1}p_i} = (Dr_{p_{i+1}p_i})\hat{r}_{p_{i+1}p_i} + (\vec{\omega}_{i1} \times \vec{r}_{p_{i+1}p_i}) \tag{4.12}$$

$$D^2\vec{r}_{p_{i+1}p_i} = (D^2 r_{p_{i+1}p_i})\hat{r}_{p_{i+1}p_i} + \vec{\omega}_{i1} \times (\vec{\omega}_{i1} \times \vec{r}_{p_{i+1}p_i}) \tag{4.13}$$

$$+ (D\vec{\omega}_{i1} \times \vec{r}_{p_{i+1}p_i}) + 2[\vec{\omega}_{i1} \times (Dr_{p_{i+1}p_i})\hat{r}_{p_{i+1}p_i}]$$

In Equations (4.12) and (4.13) the terms $\vec{\omega}_{i1}$ and $D\vec{\omega}_{i1}$ are more usefully expressed as summations of relative rotations because the unit vectors of the relative rotations (axes of rotation) are always available from the position solution.

From Equation (4.3), with i replaced by j and n replaced by i,

$$\vec{\omega}_{i1} = \sum_{j=2}^{i} (\omega_{j,j-1})(\hat{\omega}_{j,j-1}) \tag{4.14}$$

Now differentiate Equation (4.14) term by term, using Equation (4.6).

$$D\vec{\omega}_{i1} = \sum_{j=2}^{i}[(D\omega_{j,j-1})\hat{\omega}_{j,j-1} + (\vec{\omega}_{j-1,1} \times \vec{\omega}_{j,j-1})] \tag{4.15}$$

Substitute Equation (4.14) into Equation (4.12) and Equation (4.15) into Equation (4.13).

$$D\vec{r}_{p_{i+1}p_i} = (Dr_{p_{i+1}p_i})\hat{r}_{p_{i+1}p_i} + [\sum_{j=2}^{i} (\omega_{j,j-1})(\hat{\omega}_{j,j-1})] \times \vec{r}_{p_{i+1}p_i} \tag{4.16}$$

$$D^2\vec{r}_{p_{i+1}p_i} = (D^2 r_{p_{i+1}p_i})\hat{r}_{p_{i+1}p_i} + [\sum_{j=2}^{i} (D\omega_{j,j-1})\hat{\omega}_{j,j-1}] \times \vec{r}_{p_{i+1}p_i}$$

$$+ \vec{\omega}_{i1} \times (\vec{\omega}_{i1} \times \vec{r}_{p_{i+1}p_i}) + 2[\vec{\omega}_{i1} \times (Dr_{p_{i+1}p_i})\hat{r}_{p_{i+1}p_i}]$$

$$+ [\sum_{j=3}^{i} (\vec{\omega}_{j-1,1} \times \vec{\omega}_{j,j-1})] \times \vec{r}_{p_{i+1}p_i} \tag{4.17}$$

4.1.3 Solution Procedure

In general a motion solution for a given mechanism will proceed in the following steps:

(1) At the instant of time considered, assume that all $\vec{r}_{p_{i+1}p_i}$ and $\hat{\omega}_{i,i-1}$ vectors are known. These are the pair-to-pair and axis of rotation vectors that are determined in the position solution. In addition a number of input motions must be given equal to the degree of freedom of the linkage.

(2) Write Equations (4.3) and (4.4) once for every independent linkage loop in the mechanism. In this step a velocity solution is sought and m equals one $(D^o\vec{\omega}_{i,i-1} \equiv \vec{\omega}_{i,i-1})$.

(3) Substitute Equations (4.14) and (4.16) respectively for every term in the equations obtained in step (2).

(4) Identify all unknown quantities. These are necessarily either $\omega_{i,i-1}$ terms, $Dr_{p_{i+1}p_i}$ terms, or both. In a determinate problem, the total number of these unknowns must equal the number of independent loops multiplied by six.

(5) A set of simultaneous linear algebraic equations in the unknown terms can be obtained by taking scalar products throughout each of the vector equations with any three known, non-parallel unit vectors.

(6) Now assume that all $\vec{r}_{p_{i+1}p_i}$, $\vec{\omega}_{i,i-1}$, and $D\vec{r}_{p_{i+1}p_i}$ vectors are known, at the instant of time considered.

(7) Write Equations (4.3) and (4.4) as in step (2), except that now an acceleration solution is sought, and m equals two.

(8) Substitute Equations (4.15) and (4.17), respectively, for every term in the equations obtained in step (7). The result can be reduced to a set of simultaneous linear algebraic equations, just as in step (5). The unknowns will be either $D\omega_{i,i-1}$ terms, $D^2 r_{p_{i+1}p_i}$ terms, or both, equal in number to the unknowns in step (4).

This procedure provides all the essential unknown linear and angular velocities and accelerations. If any other motions are sought, they can easily be determined in terms of known position vectors and the essential motions already obtained. Higher order motions can be obtained by an identical procedure, except that the derivatives analogous to Equations (4.15) and (4.17) become much more detailed. However, the motion solution for any order of motion will always be linear and of the same form as that of all the other orders.

4.2 Application

Consider again the mechanism shown in Figure 3.14. The position solution has been obtained (Section 3.32). The velocity conditions from Equations (4.3) and (4.4) are

$$\vec{\omega}_{21} + \vec{\omega}_{32} + \vec{\omega}_{43} + \vec{\omega}_{14} = 0 \tag{4.18}$$

$$D\vec{r}_1 + D(\vec{p}_2 + \vec{q}_2) + D(\vec{p}_3 + \vec{q}_3) + D(\vec{p}_4 + \vec{q}_4) = 0 \tag{4.19}$$

In Equation (4.18) $\vec{\omega}_{21}$ is an input quantity and the unit vectors $\hat{\omega}_{32}$, $\hat{\omega}_{43}$, $\hat{\omega}_{14}$ are all known from the position solution. Equation (4.16) is substituted term-by-term into Equation (4.19), and zero quantities are dropped out. ($D\vec{r}_1$ and the factors Dp_2, Dp_3, Dp_4 are zero.) Equations (4.18) and (4.19) become

$$\omega_{32}\hat{\omega}_{32} + \omega_{43}\hat{\omega}_{43} + \omega_{14}\hat{\omega}_{14} + \vec{\omega}_{21} = 0 \tag{4.20}$$

$$\begin{aligned}(Dq_2)\hat{\omega}_{32} &+ (Dq_3)\hat{\omega}_{43} + (Dq_4)\hat{\omega}_{14} + (\vec{\omega}_{21} \times \vec{r}_2) \\ &+ (\vec{\omega}_{21} + \omega_{32}\hat{\omega}_{32}) \times \vec{r}_3 + (\vec{\omega}_{21} + \omega_{32}\hat{\omega}_{32} + \omega_{43}\hat{\omega}_{43}) \times \vec{r}_4 = 0\end{aligned} \tag{4.21}$$

The unknown quantities in Equations (4.20) and (4.21) are ω_{32}, ω_{43}, ω_{14}, Dq_2, Dq_3, Dq_4. A set of six simultaneous linear algebraic equations in these six unknowns can easily be obtained by taking scalar products throughout Equations (4.20) and (4.21) with the unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$. Such a procedure can always be applied, regardless of the number of mechanism loops or kinds of pairs. Here, the solution is even simpler because ω_{32}, ω_{43}, ω_{14} can be determined immediately from Equation (4.20) by means of the case 3a Tetrahedron Solution (Equations (3.48)-(3.50). When these quantities are substituted into Equation (4.21) the only remaining unknowns are Dq_2, Dq_3, Dq_4. Equation (4.21) can then also be solved by case 3a.

The numerators and denominators of these solutions have interesting physical interpretations. Zero numerators identify dwells; a zero denominator identifies a locking position. All solutions have the same denominator, $[\hat{\omega}_{32} \cdot (\hat{\omega}_{43} \times \hat{\omega}_{14})]$. The acceleration solution is completely analogous. Equations (4.3) and (4.4) are written

$$D\vec{\omega}_{21} + D\vec{\omega}_{32} + D\vec{\omega}_{43} + D\vec{\omega}_{14} = 0 \tag{4.22}$$

$$D^2\vec{r}_1 + D^2(\vec{p}_2 + \vec{q}_2) + D^2(\vec{p}_3 + \vec{q}_3) + D^2(\vec{p}_4 + \vec{q}_4) = 0 \tag{4.23}$$

In Equation (4.22), $(D\omega_{21})$ is an input quantity, and in both equations all positions and velocities are now regarded as known. Substitute Equations (4.15) and (4.17) term-by-term.

$$\begin{aligned}(D\omega_{32})\hat{\omega}_{32} + (D\hat{\omega}_{43})\hat{\omega}_{43} + (D\omega_{14})\hat{\omega}_{14} + (D\omega_{21})\hat{\omega}_{21} \\ + (\vec{\omega}_{21} \times \vec{\omega}_{32}) + (\vec{\omega}_{31} \times \vec{\omega}_{43}) = 0\end{aligned} \tag{4.24}$$

$$(D^2q_2)\hat{\omega}_{32} + (D^2q_3)\hat{\omega}_{43} + (D^2q_4)\hat{\omega}_{14} + (D\omega_{21})(\hat{\omega}_{21} \times \vec{r}_2)$$

$$+ [(D\,\omega_{21})\hat{\omega}_{21} + (D\,\omega_{32})\hat{\omega}_{32}] \times \vec{r}_3$$

$$+ [(D\omega_{21})\hat{\omega}_{21}) + (D\omega_{32})\,\hat{\omega}_{32} + (D\omega_{43})\hat{\omega}_{43}] \times \vec{r}_4 + \vec{\omega}_{21} \times (\vec{\omega}_{21} \times \vec{r}_2)$$

$$+ \vec{\omega}_{31} \times (\vec{\omega}_{31} \times \vec{r}_3) + \vec{\omega}_{41} \times (\vec{\omega}_{41} \times \vec{r}_4) + 2[\vec{\omega}_{21} \times (Dq_2)\hat{\omega}_{32}]$$

$$+ 2[\vec{\omega}_{31} \times (Dq_3)\hat{\omega}_{43}] + (\vec{\omega}_{21} \times \vec{\omega}_{32})\vec{r}_3$$

$$+ [(\vec{\omega}_{31} \times \vec{\omega}_{43}) + (\vec{\omega}_{21} \times \vec{\omega}_{32})] \times \vec{r}_4 = 0 \qquad (4.25)$$

The unknown quantities in Equations (4.24) and (4.25) are $D\omega_{32}$, $D\omega_{43}$, $D\omega_{14}$, D^2q_2, D^2q_3, D^2q_4. The equations are lengthy, but only the first three terms in Equation (4.24) and the first six terms in Equation (4.25) contain unknowns. The remaining terms in each equation can all be summed into a single vector constant. The solution is most simply obtained by using Case 3a to determine $D\omega_{32}$, $D\omega_{43}$, and $D\omega_{14}$ from Equation (4.24), then using case 3a again to determine D^2q_2, D^2q_3, D^2q_4 from Equation (4.25). Again, the denominator of all solutions is $[\hat{\omega}_{32} \cdot (\hat{\omega}_{43} \times \hat{\omega}_{14})]$.

Figure 4.2 shows the variation of the $\hat{i}$, $\hat{j}$, $\hat{k}$ components of velocities $D\vec{r}_2$, $D\vec{r}_3$, $D\vec{r}_4$, as the mechanism of Figure 3.14 is moved through a complete cycle. Similarly, Figure 4.3 shows the variation of accelerations $D^2\vec{r}_2$, $D^2\vec{r}_3$, $D^2\vec{r}_4$. Input parameters

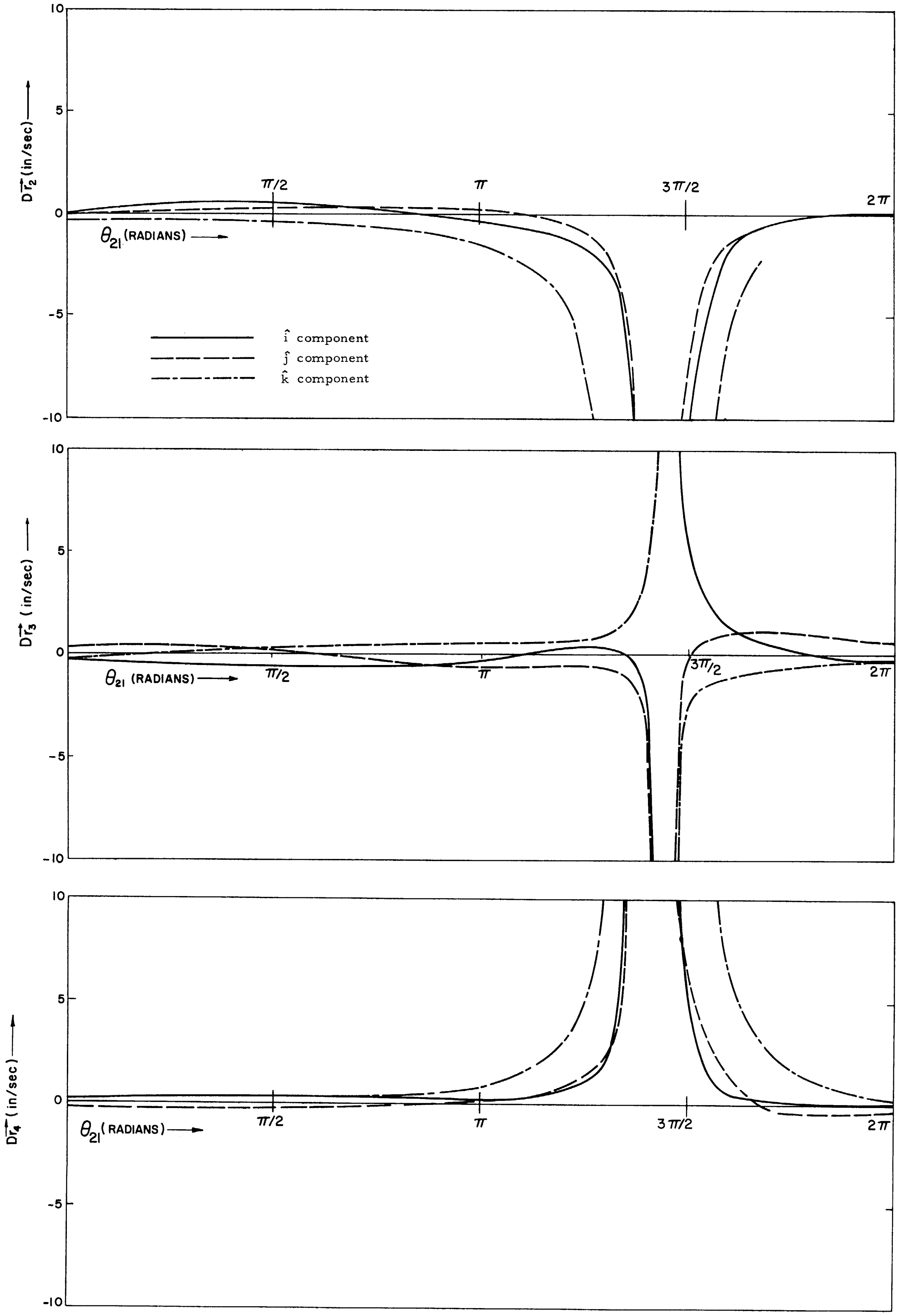


Figure 4.2 Variation of Velocities $D\vec{r}_2$, $D\vec{r}_3$, $D\vec{r}_4$ for a Complete Cycle of the Mechanism of Figure 3.14

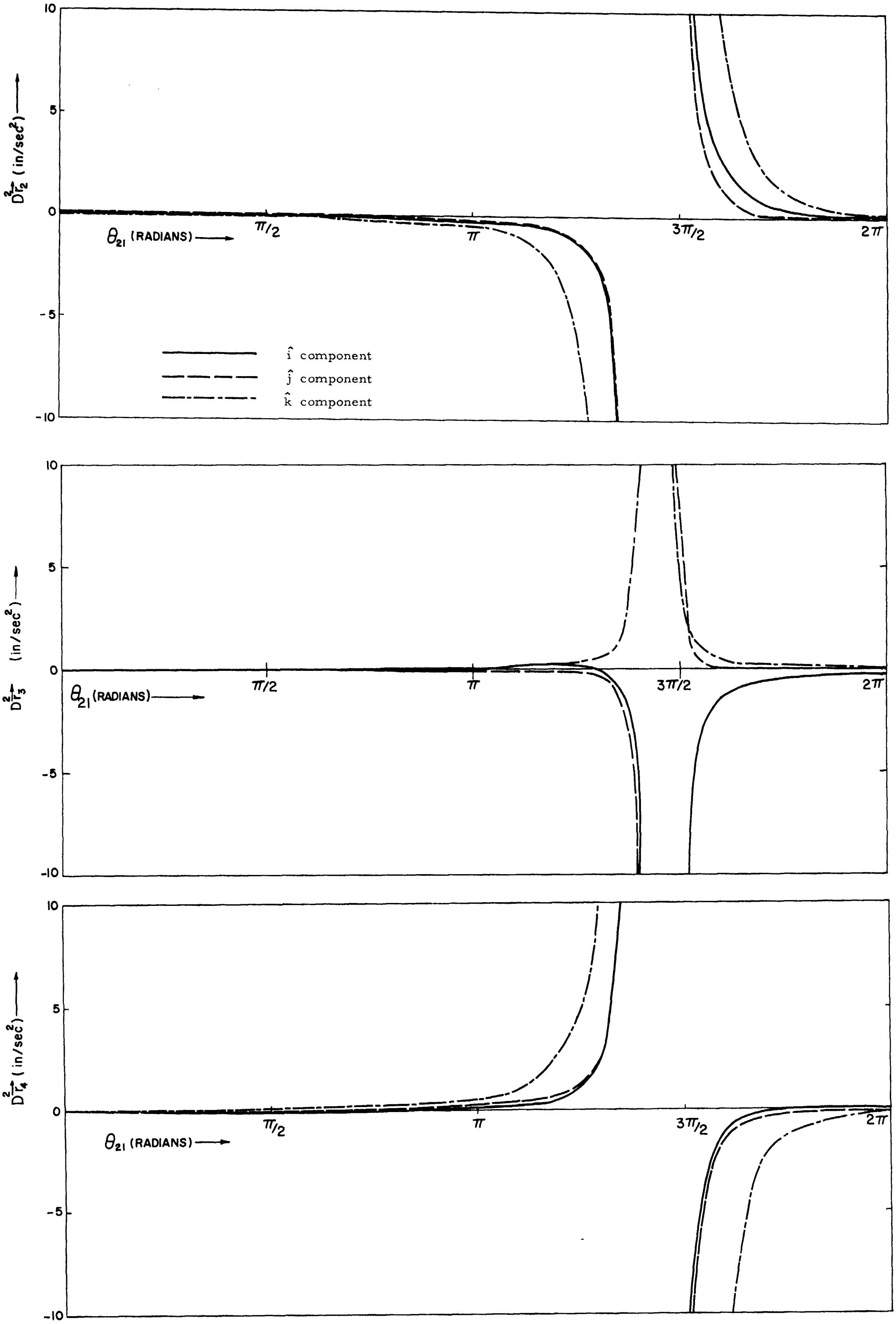


Figure 4.3 Variation of Accelerations $D^2\vec{r}_2$, $D^2\vec{r}_3$, $D^2\vec{r}_4$ for a Complete Cycle of the Mechanism of Figure 3.14

are summarized in Table 3.4. The singularities evident in all the motions occur when $\hat{\omega}_{32}$, $\hat{\omega}_{43}$, and $\hat{\omega}_{14}$ become co-planar. In this situation $[\hat{\omega}_{32} \cdot (\hat{\omega}_{43} \times \hat{\omega}_{14})]$ is zero. In other mechanisms of this type the $[\hat{\omega}_{32} \cdot (\hat{\omega}_{43} \times \hat{\omega}_{14})]$ discontinuity may not be present.

A dwell occurs in output angular velocity when the quantity $[\hat{\omega}_{21} \cdot (\hat{\omega}_{32} \times \hat{\omega}_{43})]$ becomes zero. ($\theta_{21} = 1.37\pi$). However, this cannot be seen in Figure 4.2 because the output translational velocity remains finite.

5.0 FORCE

5.1 Development

Mechanisms are inherently statically determinant because they are designed so that the number of degrees of freedom equals the number of input motions. Mechanism force analysis is therefore simpler than that of most structures. The elastic constants of the individual members need not be considered; instead, the analysis only requires knowledge of design, position, motion, mass distribution, frictional characteristics, and input force.

Force solutions are obtained from the vector equations of force and moment equilibrium applied to each link of the given mechanism. It will be shown that without friction the entire set of these equations will always reduce to a set of simultaneous, linear, algebraic equations. The condition of equal power transmission at each joint can also be applied, but is usually insufficient for obtaining all unknown forces and torques. Instead, it is used as a check.

Consider the ith link of a mechanism having n links, joined in any number of loops. One or more forces and torques can be exerted on this link by any of the others. The conditions of force and moment equilibrium of the ith link are

$$\left[\sum_{j=1}^{n} \sum_{k=1}^{m} (\vec{f}_{ij})_{p_k}\right] + \vec{g}_i = 0 \tag{5.1}$$

$$\left\{\sum_{j=1}^{n} \left[\sum_{\ell=1}^{q} (\vec{\tau}_{ij})_{p_\ell} + \sum_{k=1}^{m} ((\vec{f}_{ij})_{p_k} \times \vec{r}_{p_i p_k})\right]\right\} + \vec{\sigma}_i + (\vec{g}_i \times \vec{r}_{p_i c_i}) = 0 \tag{5.2}$$

Interpretation:

(1) The inner summation in Equation (5.1) allows the jth link alone to exert up to m forces on link i, at points p_k, $k = 1,2,3,\ldots,m$. For example, a single floating link may be subjected to several different forces from the ground link.

(2) The forces discussed in (1) are present in moment terms in Equation (5.2). The "moment arm" vectors $\vec{r}_{p_i p_k}$ are always directed to the same point p_i but originate at the different points of application of forces, p_k.

(3) The inner summation on $(\vec{\tau}_{ij})_{p_k}$ in Equation (5.2) allows the jth link alone to exert up to q torques on link i, at points p_ℓ, $\ell = 1, 2, 3, \ldots, q$.

(4) The outer summations in Equations (5.1) and (5.2) sum the forces and moments exerted on link i by all the other links in the system. Many of these terms will be zero; ordinarily only the links adjacent to link i can exert forces or torques on link i.

(5) The terms $\vec{g}_i$ and $\vec{\sigma}_i$ represent inertial forces and torques. These are dependent only on position, motion, and mass distribution and are therefore known. The "moment arm" vector $\vec{r}_{p_i c_i}$ is directed to point p_i from the center of mass c_i.

(6) Exactly twice as many force and torque terms as necessary will result when Equations (5.1) and (5.2) are written for every link in a mechanism. For example, the equations for a link 6 may have a term, $(\vec{f}_{65})_{p_3}$, representing the force exerted on link 6 by link 5 at point p_3. But then the equations for link 5 will have a term equal in magnitude and opposite in direction: $(\vec{f}_{56})_{p_3}$. To avoid this dual notation the subscripts are reversed on every term for which i is less than j, and a negative sign is placed before the term. The term $(\vec{f}_{56})_{p_3}$ is therefore written $-(\vec{f}_{65})_{p_3}$. (In this convention the ground link must have two numbers, 1 and $n+1$, so that if $n = 8$, the subscripts in a term such as $(\vec{f}_{18})_{p_6}$ would not be reversed.)

A total of $2n$ equations are obtained by writing Equations (5.1) and (5.2) once for every link in a given n-link mechanism. Only $n-1$ of the force equations, and $n-1$ of the moment equations are independent; the nth equation in each set is the sum of all the others. For convenience, the solution may proceed with the $n-1$ simplest equations from each set, even if these happen to include the equation of equilibrium for the ground link. Regardless, a total of $2n-2$

simultaneous vector equations must be solved. In three dimensions these represent $6n - 6$ scalar conditions. The effect of frictionless higher or lower pairs is to prohibit transmission of force and torque in one, two, or three directions, depending on the design of the pair. Table 5.1 summarizes these restrictions for several common pairs, and states consistent expressions for the transmitted force and torque. When transmission in only one direction is prohibited, the corresponding unknown three-dimensional vector is reduced to an unknown two-dimensional vector. The two-dimensional vector is expressed in rectangular coordinates in a plane perpendicular to the prohibited direction. When transmission in each of two directions is prohibited, there is a reduction to a one-dimensional vector, directed perpendicularly to the two prohibited directions. Finally, when transmission in three different directions is prohibited the vectors of transmitted force and torque must be zero.

There will remain exactly $6n - 6$ scalar unknowns to match the $6n - 6$ scalar conditions, once all pair effects and output conditions have been included. These unknowns can occur only in force or torque quantities because all positions and motions are predetermined. In the absence of friction, angular coordinates are never unknown; they are either known from the position solutions and pair restrictions or are included in vectors which are entirely unknown. Thus, all

Table 5.1 Restraints Introduced by Pair Design

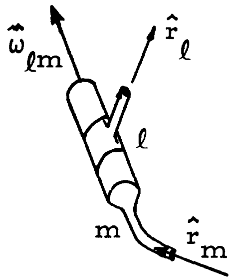
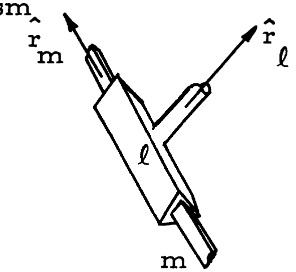
Pairs	Restrictions	Expressions
Hinge 	$\vec{f}_{\ell m}$: none $\vec{\tau}_{\ell m}$: $(\hat{\tau}_{\ell m} \cdot \hat{\omega}_{\ell m}) = 0$ $\vec{\tau}_{\ell m}$: $(\hat{\tau}_{\ell m} \cdot \hat{\omega}_{\ell m}) = 0$	$\vec{f} = (f_{\ell m})_i \hat{i} + (f_{\ell m})_j \hat{j} + (f_{\ell m})_k \hat{k}$ $\vec{\tau}_{\ell m} = (\tau_{\ell m})_\lambda \hat{\lambda} + (\tau_{\ell m})_\mu \hat{\mu}$ $\hat{\lambda}, \hat{\mu}$ satisfy $(\hat{\lambda} \cdot \hat{\omega}_{\ell m}) = 0, \quad (\hat{\mu} \cdot \hat{\omega}_{\ell m}) = 0, \quad (\hat{\lambda} \cdot \hat{\mu})^2 \neq 1$ Example: $\hat{\lambda} \equiv \dfrac{\hat{r}_j \times \hat{\omega}_{\ell m}}{\lvert \hat{r}_j \times \hat{\omega}_{\ell m} \rvert}; \quad \hat{\mu} \equiv (\hat{\omega}_{\ell m} \times \hat{\lambda})$
Prism 	$\vec{f}_{\ell m}$: $(\hat{f}_{\ell m} \cdot \hat{r}_m) = 0$ $\vec{\tau}_{\ell m}$: none	$\vec{f}_{\ell m} = (f_{\ell m})_\lambda \hat{\lambda} + (f_{\ell m})_\mu \hat{\mu}$ $\hat{\lambda}, \hat{\mu}$ satisfy $(\hat{\lambda} \cdot \hat{r}_m) = 0, \quad (\hat{\mu} \cdot \hat{r}_m) = 0, \quad (\hat{\lambda} \cdot \hat{\mu})^2 \neq 1$ Example: $\hat{\lambda} \equiv \dfrac{\hat{r}_\ell \times \hat{r}_m}{\lvert \hat{r}_\ell \times \hat{r}_m \rvert}; \quad \hat{\mu} \equiv \hat{r}_m \times \hat{\lambda}$ $\vec{\tau}_{ij} = (\tau_{\ell m})_i \hat{i} + (\tau_{\ell m})_j \hat{j} + (\tau_{\ell m})_k \hat{k}$

Table 5.1 (Cont'd)

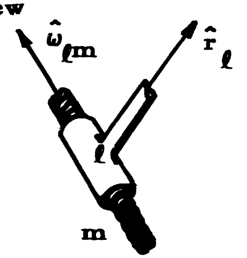
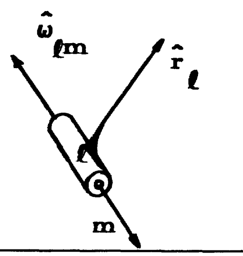
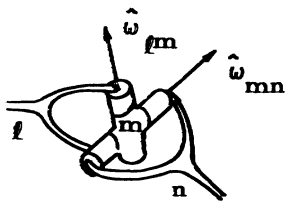
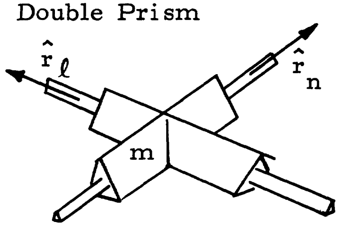
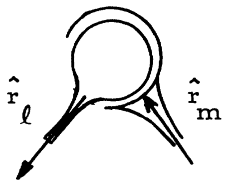
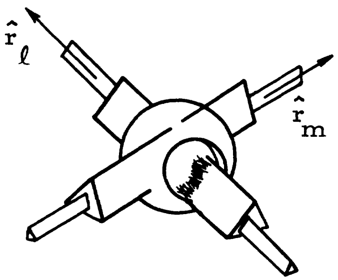
Screw 	$(\vec{f}_{\ell m} \cdot \hat{\omega}_{\ell m}) + C(\vec{\tau}_{\ell m} \cdot \hat{\omega}_{\ell m}) = 0$ $C \equiv$ **design constant**	$\vec{f}_{\ell m} = (f_{\ell m})_\lambda \hat{\lambda} + (f_{\ell m})_\mu \hat{\mu} + C(\tau_{\ell m})_\nu \hat{\nu}$ $\vec{\tau}_{\ell m} = (\tau_{\ell m})_\lambda \hat{\lambda} + (\tau_{\ell m})_\mu \hat{\mu} + (\tau_{\ell m})_\nu \hat{\nu}$ $\hat{\lambda}, \hat{\mu}, \hat{\nu}$ satisfy $\hat{\nu} \equiv \hat{\omega}_{\ell m}$, $(\hat{\lambda} \cdot \hat{\omega}_{\ell m}) = 0$, $(\hat{\mu} \cdot \hat{\omega}_{\ell m}) = 0$, $(\hat{\lambda} \cdot \hat{\mu})^2 = 1$
Turn-Slide 	$\vec{f}_{\ell m}$: $(\hat{f}_{\ell m} \cdot \hat{\omega}_{\ell m}) = 0$ $\vec{\tau}_{\ell m}$: $(\hat{\tau}_{\ell m} \cdot \hat{\omega}_{\ell m}) = 0$	$\vec{f}_{\ell m} = (f_{\ell m})_\lambda \hat{\lambda} + (f_{\ell m})_\mu \hat{\mu}$ $\vec{\tau}_{\ell m} = (\tau_{\ell m})_\lambda \hat{\lambda} + (\tau_{\ell m})_\mu \hat{\mu}$ $\hat{\lambda}, \hat{\mu}$ satisfy $(\hat{\lambda} \cdot \hat{\omega}_{\ell m}) = 0$, $(\hat{\mu} \cdot \hat{\omega}_{\ell m}) = 0$, $(\lambda \cdot \hat{\mu})^2 \neq 1$
Universal 	$\vec{f}_{\ell n}$: none $\vec{\tau}_{\ell n}$: $(\hat{\tau}_{\ell n} \cdot \hat{\omega}_{\ell m}) = 0$ $(\hat{\tau}_{\ell n} \cdot \hat{\omega}_{mn}) = 0$	$\vec{f}_{\ell n} = (f_{\ell n})_i \hat{i} + (f_{\ell n})_j \hat{j} + (f_{\ell n})_k \hat{k}$ $\vec{\tau}_{\ell n} = \tau_{\ell n} \left[\dfrac{\hat{\omega}_{\ell m} \times \hat{\omega}_{mn}}{\lvert \hat{\omega}_{\ell m} \times \hat{\omega}_{mn} \rvert} \right]$

Table 5.1 (Cont'd)

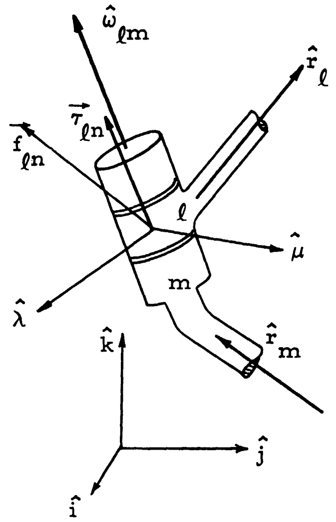
Double Prism 	$\vec{f}_{\ell n}$: $(\hat{f}_{\ell n} \cdot \hat{r}_\ell) = 0$ $(\hat{f}_{\ell n} \cdot \hat{r}_n) = 0$ $\vec{\tau}_{\ell n}$: none	$\vec{f}_{\ell n} = f_{\ell n}\left[\dfrac{\hat{r}_\ell \times \hat{r}_n}{\lvert\hat{r}_\ell \times \hat{r}_n\rvert}\right]$ $\vec{\tau}_{\ell n} = (\tau_{\ell n})_i \hat{i} + (\tau_{\ell n})_j \hat{j} + (\tau_{\ell n})\hat{k}$
Spherical 	$\vec{f}_{\ell m}$: none $\vec{\tau}_{\ell m}$: $\tau_{\ell m} = 0$	$\vec{f}_{\ell m} = (f_{\ell m})_i \hat{i} + (f_{\ell m})_j \hat{j} + (f_{\ell m})_k \hat{k}$ $\vec{\tau}_{\ell m} = 0$
Combined Spherical Double Prism 	$\vec{f}_{\ell m}$: $(\hat{f}_{\ell m} \cdot \hat{r}_\ell) = 0$ $(\hat{f}_{\ell m} \cdot \hat{r}_m) = 0$ $\vec{\tau}_{\ell m}$: $\tau_{\ell m} = 0$	$\vec{f}_{\ell m} = f_{\ell \hat{m}}\left[\dfrac{\hat{r}_\ell \times \hat{r}_m}{\lvert\hat{r}_\ell \times \hat{r}_m\rvert}\right]$ $\vec{\tau}_{\ell m} = 0$

scalar unknowns can be included as the rectangular coordinates of one-, two-, or three-dimensional vectors. In general, the one- and two-dimensional vectors must be expressed in dummy reference frames, defined in terms of the $\hat{i}$, $\hat{j}$, $\hat{k}$ ground frame. The three-dimensional vectors may be expressed directly in the ground frame because, regardless of the reference frame, three coordinates will be unknown. Several examples of expressions are included in Table 5.1. Once all $6n-6$ scalar unknowns have been identified as rectangular coordinates, the $2n-2$ vector equations containing them can be reduced to $6n-6$ linear algebraic equations by taking scalar products throughout with any three non-parallel vectors.

The solution of the $2n-2$ vector equations can usually be simplified if an approach specialized to the particular mechanism is employed. Frequently unknown vectors can be eliminated by substitution or subtraction of one equation from another. This may lead to solutions that can easily be physically interpreted.

Pair friction is likely to introduce unknown angular coordinates. Two such circumstances are shown in Figure 5.1. This may cause non-linearity in the force solutions in the same way that non-linearity was caused in the Tetrahedron Solutions (Section 3.1.1). For simple mechanisms a specialized solution may be obtained in polynomial form by techniques discussed in Section 3.0. For any

Hinge Pair



Without friction: $(f_{\ell m})_i, (f_{\ell m})_j, (f_{\ell m})_k, (\tau_{\ell m})_\lambda, (\tau_{\ell m})_\mu$ unknown

$$\vec{f}_{\ell m} = (f_{\ell m})_i \hat{i} + (f_{\ell m})_j \hat{j} + (f_{\ell m})_k \hat{k}$$

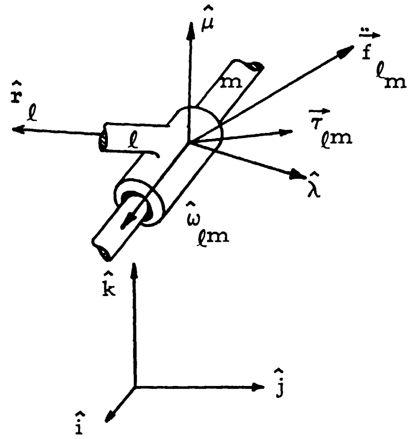
$$\vec{\tau}_{\ell m} = (\tau_{\ell m})\hat{\lambda} + (\tau_{\ell m})\hat{\mu}$$

With friction: $f_{\ell m}, \theta_f, \phi_f, (\tau_{\ell m})_\lambda, (\tau_{\ell m})_\mu$ unknown

$$\vec{f}_{\ell m}' = f_{\ell m}[\sin \phi_f(\cos \theta_f \hat{\lambda} + \sin \theta_f \hat{\mu}) + \cos \phi_f \hat{\omega}_{\ell m}]$$

$$\vec{\tau}_{\ell m} = (\tau_{\ell m})_\lambda \hat{\lambda} + (\tau_{\ell m})_\mu \hat{\mu} - (\rho_{\ell m} f_{\ell m}) \hat{\omega}_{\ell m}$$

Turn-Slide Pair



Without friction: $(f_{\ell m})_\lambda, (f_{\ell m})_\mu, (\tau_{\ell m})_\lambda, (\tau_{\ell m})_\mu$ unknown

$$\vec{f}_{\ell m} = (f_{\ell m})_\lambda \hat{\lambda} + (f_{\ell m})_\mu \hat{\mu}$$

$$\vec{\tau}_{\ell m} = (\tau_{\ell m})_\lambda \hat{\lambda} + (\tau_{\ell m})_\mu \hat{\mu}$$

With friction: $f_{\ell m}, \theta_f, (\tau_{\ell m})_\lambda, (\tau_{\ell m})_\mu$ unknown

$$\vec{f}_{\ell m} = f_{\ell m}\{+(1-\mu_{\ell m}^2)^{1/2}[\cos \theta_f \hat{\lambda} + \sin \theta_f \hat{\mu}] - \mu_{\ell m} \hat{\omega}_{\ell m}\}$$

$$\vec{\tau}_{\ell m} = (\tau_{\ell m})_\lambda \hat{\lambda} + (\tau_{\ell m})_\mu \hat{\mu} - (\rho_{\ell m} f_{\ell m}) \hat{\omega}_{\ell m}$$

Figure 5.1 Effect of Pair Friction on Introducing Unknown Angular Coordinates mechanism an iterative solution can be attempted, using the frictionless solution as an initial approximation.

5.2 Application

Consider the mechanism of Figure 3.14. Position and motion solutions have already been obtained (Sections 3.3.2 and 4.2) and all information required for a force and torque analysis is available. To minimize detail the links are assumed without mass; to retain linearity the pairs are assumed frictionless.

Conditions of force and moment equilibrium:

$$\vec{f}_{21} - \vec{f}_{32} = 0 \tag{5.3}$$

$$\vec{f}_{32} - \vec{f}_{43} = 0 \tag{5.4}$$

$$\vec{f}_{43} - \vec{f}_{14} = 0 \tag{5.5}$$

$$\vec{f}_{14} - \vec{f}_{21} = 0 \tag{5.6}$$

$$\vec{\tau}_{21} - \vec{\tau}_{32} + (\vec{f}_{21} \times \vec{r}_2) = 0 \tag{5.7}$$

$$\vec{\tau}_{32} - \vec{\tau}_{43} + (\vec{f}_{32} \times \vec{r}_3) = 0 \tag{5.8}$$

$$\vec{\tau}_{43} - \vec{\tau}_{14} + (\vec{f}_{43} \times \vec{r}_4) = 0 \tag{5.9}$$

$$\vec{\tau}_{14} - \vec{\tau}_{21} + (\vec{f}_{14} \times \vec{r}_1) = 0 \tag{5.10}$$

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$$\vec{\tau}_{32} - \vec{\tau}_{43} + (\vec{f}_{32} \times \vec{r}_3) = 0 \tag{5.8}$$

$$\vec{\tau}_{43} - \vec{\tau}_{14} + (\vec{f}_{43} \times \vec{r}_4) = 0 \tag{5.9}$$

$$\vec{\tau}_{14} - \vec{\tau}_{21} + (\vec{f}_{14} \times \vec{r}_1) = 0 \tag{5.10}$$

The forces and torques of the pairs at p_2 and p_3 are prohibited from transmission parallel to the respective axes of rotation, $\hat{\omega}_{32}$ and $\hat{\omega}_{43}$. From Table 5.1, these restrictions may be expressed as follows:

$$(\hat{f}_{32} \cdot \hat{\omega}_{32}) = 0 \tag{5.18}$$

$$(\hat{\tau}_{32} \cdot \hat{\omega}_{32}) = 0 \tag{5.19}$$

$$(\hat{f}_{43} \cdot \hat{\omega}_{43}) = 0 \tag{5.20}$$

$$(\hat{\tau}_{43} \cdot \hat{\omega}_{43}) = 0 \tag{5.21}$$

Now all unknown force and torque vectors can be expressed in one-, two-, or three-dimensional rectangular coordinates, consistent with the number of prohibited directions.

One-dimensional vectors:

$$\vec{f}_{14o} = (f_{14o})\hat{\omega}_{14} \tag{5.22}$$

$$\vec{\tau}_{14o} = (\tau_{14o})\hat{\omega}_{14} \tag{5.23}$$

Two-dimensional vectors:

$$\vec{\tau}_{21r} = (\tau_{21r\lambda})\hat{\lambda}_{21} + (\tau_{21r\mu})\hat{\mu}_{21} \tag{5.24}$$

$$\vec{f}_{32} = (f_{32\lambda})\hat{\lambda}_{32} + (f_{32\mu})\hat{\mu}_{32} \tag{5.25}$$

$$\vec{\tau}_{32} = (\tau_{32\lambda})\hat{\lambda}_{32} + (\tau_{32\mu})\hat{\mu}_{32} \tag{5.26}$$

$$\vec{f}_{43} = (f_{43\lambda})\hat{\lambda}_{43} + (f_{43\mu})\hat{\mu}_{43} \tag{5.27}$$

$$\vec{\tau}_{43} = (\tau_{43\lambda})\hat{\lambda}_{43} + (\tau_{43\mu})\hat{\mu}_{43} \tag{5.28}$$

$$\vec{f}_{14r} = (f_{14r\lambda})\hat{\lambda}_{14} + (f_{14r\mu})\hat{\mu}_{14} \tag{5.29}$$

$$\vec{\tau}_{14r} = (\tau_{14r\lambda})\hat{\lambda}_{14} + (\;_{14r\mu})\hat{\mu}_{14} \tag{5.30}$$

Three-dimensional vectors:

$$\vec{f}_{21} = (f_{21i})\hat{i} + (f_{21j})\hat{j} + (f_{21k})\hat{k} \tag{5.31}$$

The $\hat{\lambda}_{ij}$, $\hat{\mu}_{ij}$ unit vectors in Equations (5.24) through (5.30) are known. They are only required to be perpendicular to the corresponding $\hat{\omega}_{ij}$ vector and non-parallel to each other. For example, $\hat{\lambda}_{21}$ and $\hat{\mu}_{21}$ can be expressed,

$$\hat{\lambda}_{21} \equiv \frac{\vec{a} \times \hat{\omega}_{21}}{|\vec{a} \times \hat{\omega}_{21}|} \tag{5.32}$$

$$\hat{\mu}_{21} \equiv \hat{\omega}_{21} \times \frac{(\vec{a} \times \hat{\omega}_{21})}{|\vec{a} \times \hat{\omega}_{21}|} \tag{5.33}$$

Here $\vec{a}$ is any known vector with direction other than $\pm\hat{\omega}_{21}$.

There is a total of 19 scalar unknowns in Equations (5.22) through (5.31). But Equations (5.3) through (5.10) only provide

$(6 \times 4) - 6 = 18$ independent scalar conditions. Thus one additional condition is required. For example, there might be a linear relation between τ_{140} and f_{140}.

$$\tau_{140} = c_1 f_{140} + c_2 \tag{5.34}$$

Substitute Equations (5.22) through (5.31) into Equations (5.3) through (5.10). Choose the simplest three vector equations from each of the two sets and take scalar products throughout with $\hat{i}$, $\hat{j}$, $\hat{k}$ or any three non-parallel unit vectors. Including Equation (5.34), the result will be nineteen linear algebraic equations in nineteen scalar unknowns.

The foregoing procedure has the advantage of generality and can easily be programmed for computer solution. However, a much more interpretable, specialized solution can be obtained for this particular problem. From Equations (5.3) through (5.6),

$$\vec{f} \equiv \vec{f}_{21} = \vec{f}_{32} = \vec{f}_{43} = \vec{f}_{14} \tag{5.35}$$

From conditions (5.18) and (5.20)

$$(\hat{f} \cdot \hat{\omega}_{32}) = 0 \tag{5.36}$$

$$(\hat{f} \cdot \hat{\omega}_{43}) = 0 \tag{5.37}$$

This defines the direction of $\hat{f}$:

$$\hat{f} = \frac{(\hat{\omega}_{32} \times \hat{\omega}_{43})}{|\hat{\omega}_{32} \times \hat{\omega}_{43}|} \tag{5.38}$$

Rewrite Equations (5.7) through (5.10), substituting according to Equations (5.12), (5.15), and (5.35).

$$\vec{\tau}_{21i} + \vec{\tau}_{21r} - \vec{\tau}_{32} + (\vec{f} \times \vec{r}_2) = 0 \tag{5.39}$$

$$\vec{\tau}_{32} - \vec{\tau}_{43} + (\vec{f} \times \vec{r}_3) = 0 \tag{5.40}$$

$$\vec{\tau}_{43} - \vec{\tau}_{14o} - \vec{\tau}_{14r} + (\vec{f} \times \vec{r}_4) = 0 \tag{5.41}$$

$$\vec{\tau}_{140} + \vec{\tau}_{14r} - \vec{\tau}_{21i} + (\vec{f} \times \vec{r}_1) = 0 \tag{5.42}$$

Only three of Equations (5.39) through (5.42) are independent. Here, Equation (5.39) is dropped. Equations (5.40) through (5.42) can be added so that $\vec{\tau}_{43}$, $\vec{\tau}_{14r}$, and $\vec{\tau}_{21r}$ occur only once each in an entire set of three equations.

$$\vec{\tau}_{32} - \vec{\tau}_{43} + [\vec{f} \times \vec{r}_3] = 0 \tag{5.43}$$

$$\vec{\tau}_{32} - \vec{\tau}_{14o} - \vec{\tau}_{14r} + [\vec{f} \times (\vec{r}_3 + \vec{r}_4)] = 0 \tag{5.44}$$

$$\vec{\tau}_{32} - \vec{\tau}_{21i} - \vec{\tau}_{21r} + [\vec{f} \times (\vec{r}_3 + \vec{r}_4 + \vec{r}_1)] = 0 \tag{5.45}$$

Equation (5.43) is a restatement of (5.40); Equation (5.44) is the sum of (5.40) and (5.41); Equation (5.45) is the sum of (5.40), (5.41) and (5.42). Vectors $\vec{\tau}_{43}$, $\vec{\tau}_{14r}$ and $\vec{\tau}_{21r}$ are all two-dimensional and can be eliminated by taking the scalar product throughout Equations (5.43), (5.44), and (5.45), with $\hat{\omega}_{43}$, $\hat{\omega}_{14}$, and $\hat{\omega}_{21}$, respectively (ref. Equations (5.21), (5.17), and (5.13)). This eliminates six scalar unknowns and yields three scalar equations. No further scalar products need be taken.

$$(\vec{\tau}_{32} \cdot \hat{\omega}_{43}) + f\{[\hat{f} \times \vec{r}_3] \cdot \hat{\omega}_{43}\} = 0 \tag{5.46}$$

$$-\tau_{14o} + (\vec{\tau}_{32} \cdot \hat{\omega}_{14}) + f\{[\hat{f} \times (\vec{r}_3 + \vec{r}_4)]\,\hat{\omega}_{14}\} = 0 \tag{5.47}$$

$$-\tau_{21i} + (\vec{\tau}_{32} \cdot \hat{\omega}_{21}) + f\{[\hat{f} \times (\vec{r}_3 + \vec{r}_4 + \vec{r}_1)] \cdot \hat{\omega}_{21}\} = 0 \tag{5.48}$$

These equations contain only four scalar unknowns: $\tau_{32\lambda}$, $\tau_{32\mu}$, τ_{14o}, and f. An additional condition is required, such as Equation (5.34). Whatever the additional condition is, it is most likely to relate τ_{14o} and f_{14o}, not τ_{14o} and f. Because of Equations (5.14) and (5.16), f may be replaced in Equations (5.46) through (5.48) by

$$f = \frac{f_{14o}}{(\hat{f} \cdot \hat{\omega}_{14})} \tag{5.49}$$

The problem has nowbeen reduced to four simultaneous linear algebraic equations in four unknowns, provided the additional condition has the form of Equation (5.34). However, the additional condition may be non-linear, or a more interpretable solution may be sought. This requires reduction of Equations (5.46) through (5.48).

Define:

$$\hat{\tau}_{32} = (\cos\theta)\hat{\lambda} + (\sin\theta)\hat{\mu} \tag{5.50}$$

$$\hat{\mu} \equiv \frac{\hat{\omega}_{32} \times \hat{\omega}_{43}}{|\hat{\omega}_{32} \times \hat{\omega}_{43}|} \tag{5.51}$$

$$\hat{\lambda} \equiv \hat{\mu} \times \hat{\omega}_{32} = \frac{-(\hat{\omega}_{32} \cdot \hat{\omega}_{43})\hat{\omega}_{32} + \hat{\omega}_{43}}{|\hat{\omega}_{32} \times \hat{\omega}_{43}|} \tag{5.52}$$

$$T_3 \equiv \frac{[\hat{f} \times \vec{r}_3] \cdot \hat{\omega}_{43}}{(\hat{f} \cdot \hat{\omega}_{14})} \tag{5.53}$$

$$T_4 \equiv \frac{[\hat{f} \times (\vec{r}_3 + \vec{r}_4)] \cdot \hat{\omega}_{14}}{(\hat{f} \cdot \hat{\omega}_{14})} \tag{5.54}$$

$$T_1 \equiv \frac{[\hat{f} \times (\vec{r}_3 + \vec{r}_4 + \vec{r}_1)] \cdot \hat{\omega}_{21}}{(\hat{f} \cdot \hat{\omega}_{14})} \tag{5.55}$$

Equation (5.46) is solved for τ_{32}:

$$\tau_{32} = \frac{-T_3(f_{14o})}{|\hat{\omega}_{32} \times \hat{\omega}_{43}| \cos\theta} \tag{5.56}$$

Substitute Equation (5.56) for τ_{32} in Equations (5.47) and (5.48), then eliminate f_{14o} by subtraction.

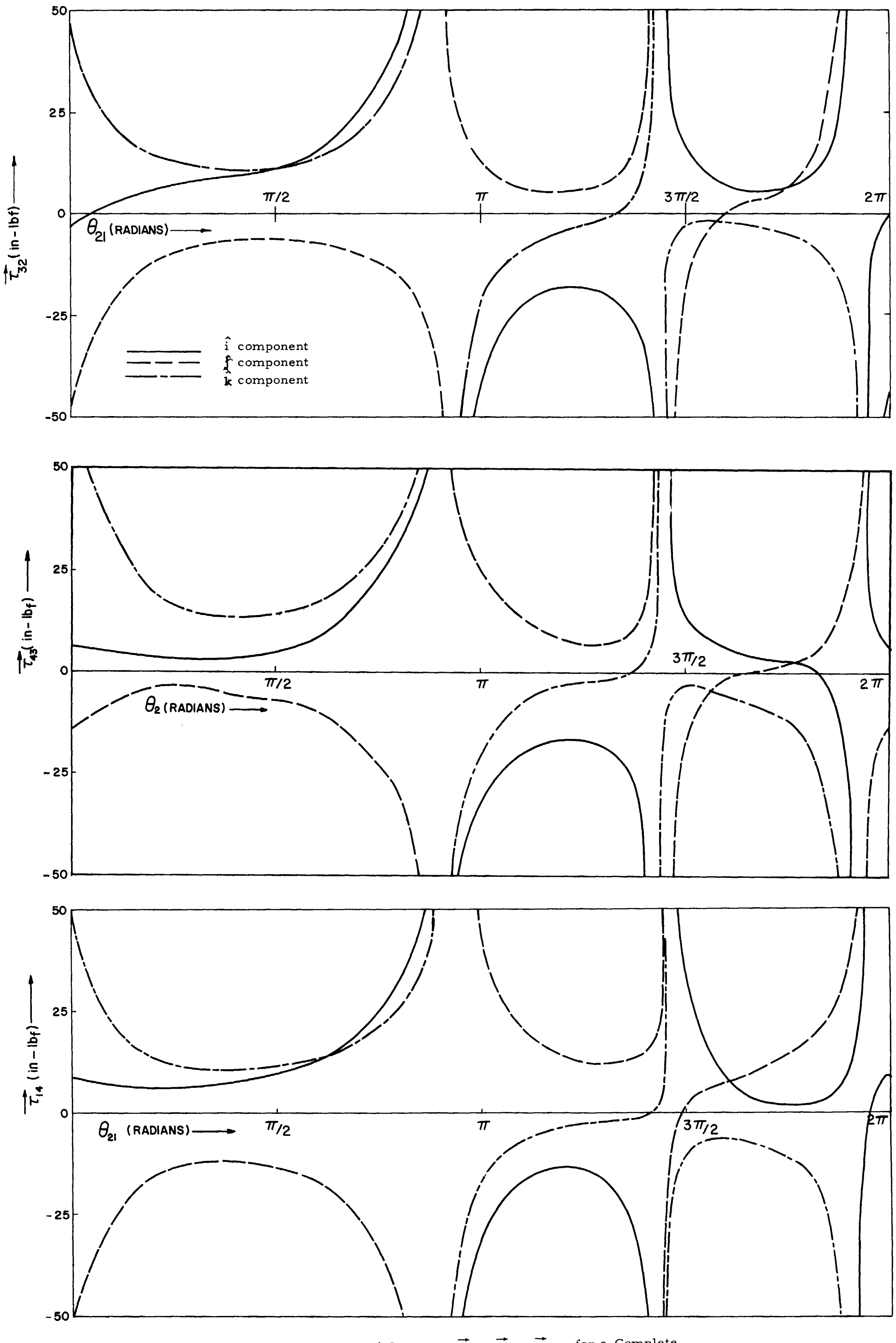

Figure 5.2 Variation of Torques $\vec{\tau}_{32}$, $\vec{\tau}_{43}$, $\vec{\tau}_{14}$ for a Complete Cycle of the Mechanism of Figure 3.14

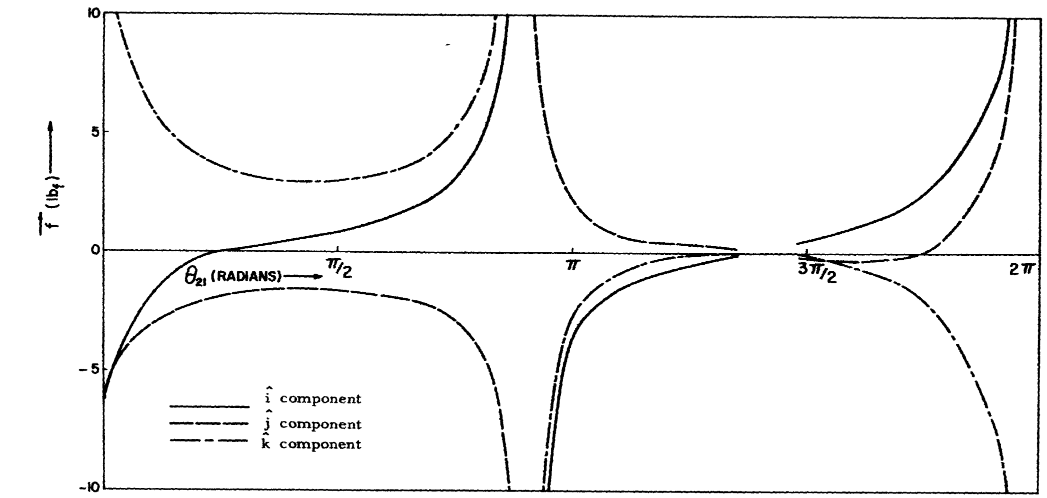

Figure 5.3 Variation of Transmitted Force for a Complete Cycle of the Mechanism of Figure 3.14

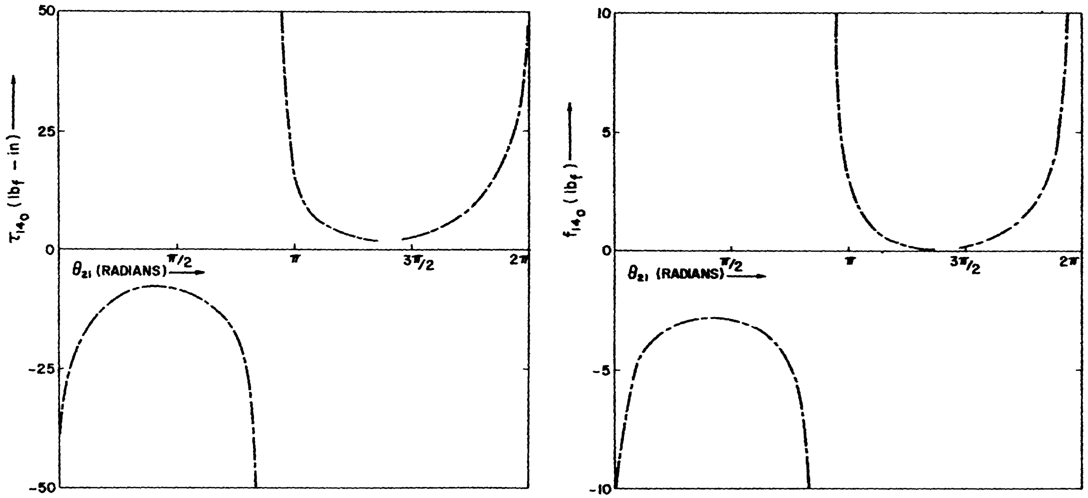

Figure 5.4. Variation of Output Torque and Force for a Complete Cycle of the Mechanism of Figure 3.14.

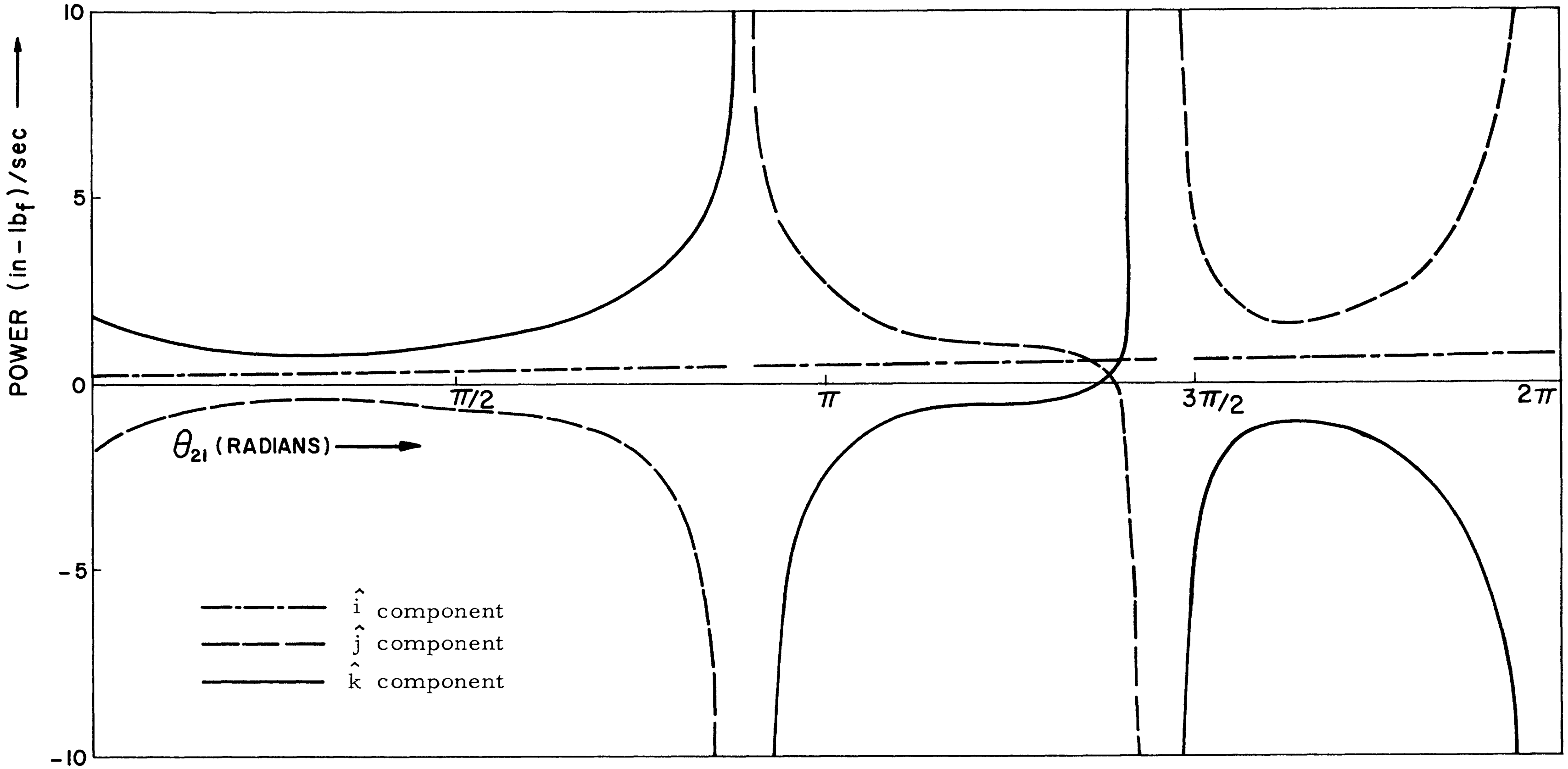

Figure 5.5 Variation of Output Power Transmitted in Rotational and Translational Motion for a Complete Cycle of the Mechanism of Figure 3.14

$$-\tau_{14o} + a_1 \tau_{21i} + a_2 f_{14o} = 0 \tag{5.57}$$

$$a_1 \equiv \left[\frac{\hat{\mu} \cdot \hat{\omega}_{14}}{\hat{\mu} \cdot \hat{\omega}_{21}}\right] \tag{5.58}$$

$$a_2 \equiv \left[\frac{-T_3\left[(\hat{\lambda} \cdot \hat{\omega}_{14}) - \dfrac{(\hat{\mu} \cdot \hat{\omega}_{14})(\hat{\lambda} \cdot \hat{\omega}_{21})}{(\hat{\mu} \cdot \hat{\omega}_{21})}\right]}{|\hat{\omega}_{32} \times \hat{\omega}_{43}|} + T_4 - \frac{(\hat{\mu} \cdot \hat{\omega}_{14})}{(\hat{\mu} \cdot \hat{\omega}_{21})} T_1\right] \tag{5.59}$$

Equation (5.57) is independent of the form of the additional condition between τ_{14o} and f_{14o}. Now even if τ_{14o} has nonlinear dependence on f_{14o} a solution can be obtained by substituting into Equation (5.57) and solving the result by iteration. If the additional condition has the form of Equation (5.34) the solution for f_{14o} is

$$f_{14o} = \frac{a_1 \tau_{21i} - c_2}{c_1 - a_2} \tag{5.60}$$

All other unknown forces and torques can be obtained by evaluating the equations leading to Equation (5.57).

Figures 5.2 and 5.3 show the variation of the $\hat{i}$, $\hat{j}$, $\hat{k}$ components of torques $\vec{\tau}_{32}$, $\vec{\tau}_{43}$, $\vec{\tau}_{14}$ and force $\vec{f}$. Figure 5.4 shows the variation of output torque and force, τ_{14o} and f_{14o}; Figure 5.5 shows the variation of output power transmitted in rotational and translational motion. Input parameters are summarized in Table 3.4.

Singularities occur in force, torque, and power when the denominator of Equation (5.60) becomes zero ($\theta_{21} = .90\pi,\ 1.95\ \pi$ in Figures 5.2 through 5.5). These singularities can be eliminated by positively increasing the constant, c_1. Additional singularities occur in $\vec{\tau}_{32}$, $\vec{\tau}_{43}$, $\vec{\tau}_{14}$ when the quantity $[\hat{\omega}_{32} \cdot (\hat{\omega}_{43} \times \hat{\omega}_{14})]$ becomes zero ($\theta_{21} = 1.40\,\pi,\ 1.43\,\pi$). In this situation it can be shown that $\cos\theta$ becomes zero, in Equation (5.56). This causes $\vec{\tau}_{32}$, $\vec{\tau}_{43}$, and $\vec{\tau}_{14}$ to approach infinity but does not affect $\vec{f}$, τ_{14_o}, or f_{14_o}.

At all times input and output power must be equal. However, output power is the algebraic sum of power transmitted by force plus power transmitted by torque; individually these powers may take large absolute values, even though their algebraic sum always equals the input power. When $[\hat{\omega}_{32} \cdot (\hat{\omega}_{43} \times \hat{\omega}_{14})]$ becomes zero, both output powers become infinite because of the corresponding singularities in velocity (Figure 4.2).

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APPENDIX A

COMPUTER PROGRAMMING

A system of simple computer subprograms was written to facilitate evaluation of vector expressions. From these, larger subprograms were constructed for evaluation of specific solutions obtained in this thesis. All work described here was performed in the MAD (Michigan Algorithmic Decoder) language [57] and was processed at the University of Michigan Computing Center on an IBM 7090 computer. However, many of the solutions in this thesis are being programmed in Fortran.[4/]

A. Preliminary

Languages such as Fortran or MAD allow scalar expressions to be written directly, in terms of scalar operations such as equality, addition, subtraction, multiplication, and addition. For example, the following statement is a permissible part of a program in either Fortran or MAD:

$$X = (2.*(A + (B*C)))/D \tag{A.1}$$

It is possible to extend this facility to allow vector expressions to be written directly, even when they involve operations such as vector

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[4] This work is being performed by the IBM Automotive and Machine Design Project, Dearborn, Michigan.

equality, vector addition, vector subtraction, the vector scalar product, and the vector cross product--besides the usual scalar operations. This facility requires modification of the language itself and is not available in Fortran, although it was recently (December, 1963) made available in MAD [57]. A permissible statement of this kind appears in MAD as

VX = (2. *(VA + (VB. X. VC)))/D (A.2)

Here, VA, VB, VC, and VX are declared as vectors (both in a dimension statement and a special mode statement) and .X. is defined as the vector cross product. Operators +, *, and / perform according to the type of quantities surrounding them. Thus, the operator * performs multiplication of a vector by a scalar, although in other circumstances it performs the vector scalar product or the product of two scalars.

Vector expressions may also be programmed by use of subprograms to represent the individual vector operations. The former scheme was employed in this thesis, primarily because it was immediately available in both MAD (via the external function) and in Fortran (via the function subprogram). Here, an operation such as the vector cross product requires a separate statement:

EVCP. (VB, VC, V1) (A.3)

In statement (A.3) the external function EVCP. performs the vector cross product VB X VC and stores the result in vector V1. EVCP. itself can assume only scalar values. EVCP. is therefore assigned no value, since the only output quantity involved is a vector. The program shown in Figure A.1 is needed to define EVCP. and the binary deck corresponding to the compiled program must accompany any other program in which EVCP. is employed.

To program an entire solution, a sequence of statements is required. For example, the following sequence is required for Equation (A.2)

```
EVCP. (VB, VC, V1)
EVA2. (VA, V1, V2)
EVFP. (V2, 2., V3)
EVFQ. (V3, D, VX)
```

Here EVA2., EVFP., and EVFQ., respectively, perform addition of two vectors, multiplication of a vector by a scalar, and division of a vector by a scalar. Other larger expressions can be programmed in essentially the same way, as in Figures A.2, A.3, and A.4.

A.2 Conventions

It was found helpful to employ conventions regarding categorization of external functions, names of variables and external functions, ordering of arguments, etc. These conventions are outlined here and assumed for all subsequent discussion of programming. They are consistent with the rules of MAD programming but are in no other way a required part of the MAD language.

A.2.1 Categorization of external functions according to task

(1) Basic. Perform elementary vector operations. Do not rely on any other external functions.

(2) Intermediate. Perform relatively simple, frequently occurring vector operations. Rely on basic external functions.

(3) Special. Obtain the solutions to frequently occurring vector and scalar equations.

(4) Auxiliary. Perform conventional tasks in scalar mathematics. Support special external functions.

A.2.2 Names of variables and external functions

By the rules of the MAD language, the names of variables and external functions may be from one to six letters or digits, the first of which must be a letter. External function names must be followed by a period. Within these restrictions the following conventions were followed.

A.2.2 1 Variables The first letter designates the type of quantity represented by the variable according to Table A.1.

The full name for the magnitude of a vector is the vector name with the leading V omitted. Derivatives begin with a D followed by an integer specifying the order of differentiation. (If the integer is omitted the order is one.) Dummy variable vectors and scalars are often written $V\alpha$ and $X\alpha$, where α is some integer. Otherwise, the letters in a variable name are chosen to correspond to the symbols and subscripts used in the solutions being programmed.

A.2.2.2 External functions

The first letter specifies the language in which the function is programmed. For MAD, this letter is E (external function). The second letter specifies the type of output quantity, according to Table A.1. If there is more than one type of output quantity, the letter which is highest in Table A.1 is used. Remaining letters and digits are used to suggest the task performed by the function. The total number of letters and digits in a function name is as follows:

basic and intermediate functions: four

special functions: five

auxilliary functions: four to six

A.2.3 Order of external function arguments

(1) Input quantities are listed first; output quantities second. The last letter in the name of an output argument is an X.

(2) Within (1), types of quantities are ordered according to Table A.1.

(3) Within (1) and (2), identical types of quantities are ordered alphabetically, then numerically.

Example:

EMG3C. (VC, UPS, UPT, UR, RPS, RPT, S, T, MRX, MSX, MTX)

TABLE A.1

MEANING OF LETTERS IN NAMES

Letter	Type of Quantity
M	An array other than an ordinary vector or unit vector. May have any number of subscripts.
V	Ordinary three-component vector. A linear array.
U	Ordinary three-component unit vector. A linear array.
D	Angle in degrees.
R	Angle in radians.
S	Ordinary scalar.
E	Small scalar used in error or comparison test
I, J, K	Integers.
Unlisted letters	Ordinary scalar.

Here the first eight quantities are input the last three output. Of the input quantities, VC is a vector; UPS, UPT, and UR are unit vectors; RPS and RPT are angles in radians; S and T are ordinary scalars. The output quantities MRX, MSX and MTX are all four-by-four matrices.

A.2.4 Value of an external function

(1) Whenever an external function has scalar output and satisfactory operation is certain, the function is assigned the value of one of the scalar outputs.

(2) Whenever an external function has only array or vector output the function is assigned a value as follows:

Satisfactory operation certain: No value assigned

Normal operation: 0.

Solution at least partly indeterminate: 1. to 9.

Solution complex: 10. to 19.

Solution approaches $\pm\infty$: 20. to 29.

A 2.5 Storage of arrays and vectors

In MAD, a storage location is automatically reserved for the zeroth element of a linear array. However, as programmed here, all basic external functions operate only on the first, second, and third elements of vectors. Therefore, the zeroth element of an ordinary vector is always unused. However, whenever n vectors are assembled (for convenience) into a single large array, the first element of the array is assigned the decimal value of n. The 4ith elements, i = 1, 2, 3, . . . , are then unused.

A multi-dimensional array can be manipulated by use of a single "linear" subscript, regardless of the number of subscripts it is considered to have. Of course, it can also be manipulated by its multiple subscripts.

A.3 Description of External Functions

Brief descriptions of all external functions programmed for this thesis are included below. They are arranged alphabetically within the categories: basic, intermediate, special, and auxiliary. Normal operation has been checked numerically for all functions. Enough functions are included to meet the needs of routine vector programming and most of the more specialized situations that have arisen in this thesis.

A complete listing of all programs was not included because of limitations of space and clarity. However, sample listings of each category of functions are shown in Figures A.1, A.2, A.3, and A.4.

Where the meaning of function arguments and function returns is clear from convention, no definition is given.

A.3.1 Basic functions

EMEQ

Purpose. Equate one matrix to another

Call. EMEQ. (MA, MX)

Arguments

MA Known array having linear subscript I, $0 \leqq I \leqq 4M - 1$, where M is the number of vectors stored in MA. MA(0) = decimal M.

MX Array having the same linear subscript range as MA

EMV1., EMV2., EMV4.

Purpose

EMV1: Insert a vector in the Nth row of a two-subscript array having any number of rows and four columns.

EMV2, EMV4: Compose a 2 x 4 or 4 x 4 array of 2 or 4 vectors.

Call

EMV1. (VA,N,MX)

EMV2. (VA1,VA2,MX)

EMV4. (VA1,VA2,VA3,VA4, MX)

Arguments

VA,VA1,VA2,VA3,VA4 Vectors to be stored in MX

N Row of MX into which VA is inserted

MX An m x 4, 2 x 4 or 4 x 4 array having integer linear subscripts 0 to 4m - 1, 7, or 15, respectively

ESDP.

Purpose. Compute scalar product of two vectors

Call. ESDP. (VA, VB, X)

Arguments

VA, VB

X Scalar product of VA and VB

Function return. X

EVA2., EVA3., EVN2., EVS2.

Purpose

EVA2. Add two vectors

EVA3.. Add three vectors

EVN2. Negate the sum of two vectors

EVS2. Subtract one vector from another

Call

EVA2. (VA, VB, VX)

EVA3. (VA, VB, VC, VX)

EVN2. (VA, VB, VX)

EVS2.. (VA, VB, VX)

Arguments

VA, VB, VC (In EVS2., VB is subtracted from VA.)

VX Vector result

EVCP

Purpose. Compute vector cross product

Call. EVCP. (VA, VB, VX)

Arguments

VA, VB

VX The cross product, VA x VB.

EVEQ.

Purpose. Equate one vector to another

Call. EVEQ. (VA, VX)

Arguments

VA

VX

EVFP.

Purpose

EVFP. Multiply a vector by a scalar.

Call. EVFP. (VA, B, VX)

Arguments

VA

B

VX Product of VA and B.

A.3.2 Intermediate functions

EIUU.

Purpose. Compare two unit vectors

Call. EIUU.(UA, UB, EU, IX)

Arguments

UA, UB Unit vectors to be compared

EU Decimal constant specifying acceptable error. $UA = \pm UB$ whenever $1. - |UA \cdot UB| < EU$

IX = 0: $UA \neq +UB$

IX = 1: $UA = -UB$

IX = 2: $UA = UB$

Function return IX

EIVV.

Purpose. Compare two vectors

Call. EIVV.(VA,VB,EU,EM,IX)

Arguments

VA,VB Vectors to be compared

EU,EM Decimal constants, specifying acceptable errors.

$UA = \pm UB$ whenever $1. - |UA \cdot UB| < EU.$

$A = B$ whenever $\left| \frac{2.(A - B)}{(A + B)} \right| < EM$

$IX = 0$:	$UA \neq \pm UB,\ A \neq B$
$IX = 1$:	$UA \neq \pm UB,\ A = B$
$IX = 2$:	$UA = - UB,\ A \neq B$
$IX = 3$:	$UA = UB \quad A \neq B$
$IX = 4$:	$UA = -UB,\ A = B$
$IX = 5$:	$UA = UB,\ A = B$

Function return. IX

ERUF

Purpose. Compute the azimuthal and polar angles of a unit vector

Call. ERUF.(UA,UL,UM,UN,RAX,RPX)

Arguments

UA Unit vector with angles RAX, RPX

UL, UM, UN Right-hand reference frame

RAX Azimuthal angle of UA measured from UL, positively increasing in the UN direction of rotation.

$0 \leqq \text{RAX} < 2\pi$

RPX Polar angle of UA measured from UN.

$0 \leqq \text{RPX} \leqq \pi$

Function return. RPX

ERUU.

Purpose. Compute the angle between two unit vectors

Call. ERUU.(UA,UB,RX)

Arguments

UA, UB

RX Angle between UA and UB

$0 \leqq \text{RX} \leqq \pi$

Function return. RX

ESAV.

Purpose. Compute the absolute value and square of the absolute value of a vector.

Call. ESAV. (VA, X, SQX)

Arguments

VA

X Absolute value of VA

SQX Square of X

Function return. X

ESRP.

Purpose. Compute the ratio of two scalar triple products

Call. ESRP. (VAN, VBN, VCN, VAD, VBD, VCD, X)

Arguments

VAN, VBN, VCN Vectors in numerator

V VAD, VBD, VCD Vectors in denominator

X $\dfrac{\text{VAN} \cdot (\text{VBN} \times \text{VCN})}{\text{VAD} \cdot (\text{VBD} \times \text{VCD})}$

Function return

0. Normal operation

1. VAN · (VBN x VCN) = 0., VAD · (VBD x VCD) = 0.

20. VAN · (VBN x VCN) $\neq$ 0., VAD · (VBD x BCD) = 0.

ESTP.

Purpose. Compute the scalar triple product

Call. ESTP. (VA, VB, VC, X)

Arguments

VA, VB, VC

X VA· (VB x VC)

Function return. X

EUMV

Purpose. Compute the unit vector and absolute value (magnitude) of a vector

Call. EUMX. (VA, UX, X)

Arguments

VA

UX Unit vector of VA, consistent with a positive magnitude of VA

X Absolute value or magnitude of VA

Function return. X

Remark. Whenever X = 0., UX is assigned the value 1., 0., 0.

EURF.

Purpose. Compute a right-hand reference frame from two vectors

Call. EURF.(VA, VB, I, ULX, UMX, UNX)

Arguments

ULX, UMX, and UNX are computed from VA and VB, according to the value of I:

I	ULX	UMX	UNX
1, 5	$\frac{VA \times VB}{\lvert VA \times VB \rvert}$	(UNX) x (ULX)	UA
2, 6	$\frac{VB \times VA}{\lvert VA \times VB \rvert}$	(UNX) x (ULX)	UA
3, 7	(UMX) x (UNX)	$\frac{VA \times VB}{\lvert VA \times VB \rvert}$	UA
4, 8	(UMX) x (UNX)	$\frac{VB \times VA}{\lvert VA \times VB \rvert}$	UA

$1 \leqq I \leqq 4$ VA and VB are checked for the possibilities

$A = 0.$, $B = 0.$, and $UA = \pm UB$

$5 \leqq I \leqq 8$ VA and VB are not checked

Function return

0. Normal operation and/or $5 \leqq I \leqq 8$

1. $A = 0.$ and $B = 0.$

2. $A = 0.$ and $B \neq 0.$

3. $A \neq 0.$ and $B = 0.$

4. $A \neq 0$, $B \neq 0$, but $UA = \pm UB$

Remarks.

1. EIVV. is employed to compare VA and VB, with $EM = EU = 10^{-6}$.

2. When the function return is other than 0., ULX, UMX, and UNX are assigned values according to the following table:

Function Return	ULX	UMX	UNX
1.	1.,0.,0.	0.,1.,0.	0.,0.,1.
2.	(UMX)x(UNX)	$\frac{UB \times VI}{\lvert UB \times VI \rvert}$	UB
3.	(UMX)x(UNX)	$\frac{UA \times VI}{\lvert UA \times VI \rvert}$	UA
4.	(UMX)x(UNX)	$\frac{UA \times VI}{\lvert UA \times VI \rvert}$	UA = UB

$VI(1) \equiv UNX(1)$

$VI(2) \equiv UNX(2) + 1.$

$VI(3) \equiv UNX(3)$

EU2R.

Purpose. Compute a unit vector from its azimuthal and polar angles

Call. EU2R.(UL, UM, UN, RA, RP, UX)

Arguments

UL, UM, UN	Unit vectors of a right-hand reference frame.
RA	Azimuthal angle of UX measured from UL, positively increasing in the UN direction of rotation. $0 \leqq RA < 2\pi$
RP	Polar angle of UX measured from UN. $0 \leqq RP \leqq \pi$
UX	sin RP [(cos RA)UL + (sin RA) UM] + (cos RP)UN

EVFQ.

Purpose. Divide a vector by a scalar

Call. EVFQ. (VA, B, VX)

Arguments

VA

B

VX Quotient of VA and B, VA/B

Function return

0. Normal operation

1. $A = 0.$, $B = 0.$

20. $A \neq 0.$, $B = 0.$

Remark. When the function return is 1., VA is assigned the value VA = 1., 1., 1.

EVTP.

Purpose. Compute the vector triple product

Call. EVTP. (VA, VB, VC, I, VX)

Arguments

VA, VB, VC

I

VX $\quad I = 1 : VX = VA \times (VB \times VC)$

$\quad I = -1: VX = (VA \times VB) \times VC$

A.3.3. Special Functions

The Tetrahedron Functions

Purpose. Solution of the three-dimensional equation

$$VR + VS + VT + VA = 0$$

Each vector is expressed in spherical coordinates (magnitude, azimuthal angle, and polar angle) with angles measured from known unit vectors. The nine Tetrahedron Functions solve the equation for all possible combinations of three unknown coordinates out of the nine coordinates of VR, VS, and VT. VA is always known. Cases in which one coordinate is functionally dependent on another are excluded.

Calls

EVG1A. (VC, VRX)

EMG2A. (VC, UPR, US, RPR, MRX, MSX)

EMG2B. (VC, UPR, UPS, RPR, RPS, S, MRX, MSX)

EMG2C. (VC, US, R, MRX, MSX)

EMG2D. (VC, UPS, RPS, R, S, MRX, MSX)

EVG3A. (VC, UR, US, UT, VRX, VSX, VTX)

EMG3B. (VC, UPT, UR, US, RPT, T, MRX, MSX, MTX)

EMG3G. (VC, UPS, UPT, UR, RPS, RPT, S, T, MRX, MSX, MTX)

EMG3D. (VC, UPR, UPS, UPT, RPR, RPS, RPT, R, S, T, MRX, MSX, MTX)

Arguments

VC = VS + VT + VA	(EVG1A)
= VT + VA	(EMG2A, B, C, D)
= VA	(EVG3A;EMG3B, C, D)
UPR, UPS, UPT	Unit vectors from which polar angles RPR, RPS, RPT are measured
UR, US, UT	Unit vectors of VR, VS, VT
RPR, RPS, RPT	Polar angles of VR, VS, VT measured from UPR, UPS, UPT. $0. \leqq$ RPR, RPS, RPT $\leqq \pi$.
R, S, T	Magnitudes of VR, VS, VT
VRX, VSX, VTX	Unique solutions of VR, VS, VT
MRX, MSX, MTX	Multiple solutions of VR, VS, VT stored as arrays.
	VR = MRX(4I-3), MRX(4I-2), MRX(4I-1)
	VS = MSX(4I-3), MSX(4I-2), MSX(4I-1)
	VT = MTX(4I-3), MTX(4I-2), MTX(4I-1)

Each complete solution VR, VS, VT corresponds to a particular value of the integer I. Elements MRX(0) = MSX(0) = MTX(0) = N where N is the decimal number of complete solutions obtained.

EMG2A, 2C, 2D, 3B:	N = 2.
EMG2B, 3C:	N = 2. or 4.
EMG3D	N = 2. or 4. or 8.

In special cases two or more solutions may be identical

Use

Each tetrahedron function obtains VR, VS, VT for the corresponding combination of unknown spherical coordinates in the following table:

Function	Unknown VR			Spherical VS		Coordinates VT	
EVG1A.	R	RAR	RPR				
EMG2A.	R	RAR		S			
EMG2B.	R	RAR			RAS		
EMG2C.		RAR	RPR	S			
EMG2D.		RAR	RPR		RAS		
EVG3A.	R			S		T	
EMG3B	R			S			RAT
EMG3C.	R				RAS		RAT
EMG3D.		RAR			RAS		RAT

(1) VR, VS, VT are dummy variables and may appear in any order in the equation VR + VS + VT + VA = 0.

(2) A polar angle of a vector is known if the vector maintains a known angle from <u>any</u> known unit vector. Both the known angle (RPR, RPS, or RPT) and the known unit vector (UPR, UPS, or UPT) are entered as arguments in the appropriate function.

(3) A case in which a polar angle is the only unknown angle in a given vector can always be transformed to a case in which an azimuthal angle is the only unknown and the polar angle is $\frac{\pi}{2}$. Thus, let RPR1 be unknown and let RAR1 be known, as measured from ULR1 with positive rotation in the UNR1 direction.

Transform, regarding an azimuthal angle RAR2 as unknown.

$$RPR2 = \frac{\pi}{2} \quad \text{measured from} \quad UPR2$$

$$UPR2 = (\sin RAR1)(ULR1) - (\cos RAR1)(UNR1 \times ULR1)$$

(4) Solutions are always obtained in terms of the full vectors which contain the unknown coordinates. The unknown coordinates can then be obtained individually from the full vectors.

(5) When multiple solutions are obtained a test may be necessary to isolate physically realistic solutions.

Function returns

EVG1A.	None	
EMG2A.	0.	Normal operation
	1.	C = 0. and UR = -US. MRX and MSX indeterminate
	2.	C = 0. and UR = +US. MRX and MSX indeterminate
	10.	C = 0. MRX and MSX complex
	11.	MRX and MSX complex

EMG2B	0.	Normal operation
	2.,3 ,4.	Failure to solve polynomial. Function returns from EMRP.
	5.	UPR, UPS, VC2 parallel. MRX and MSX indeterminate
	10.	MRX and MSX complex
EMG2C.	0.	Normal operation
	10.	MRX, MSX complex
EMG2D.	0.	Normal operation
	1.	VC, UPS parallel. MRX, MSX indeterminate.
	2.	C = 0. MRX, MSX indeterminate.
	10.	VC, UPS parallel. MRX, MSX complex.
	11.	C = 0. MRX, MSX complex.
	12.	MRX, MSX complex.
EVG3A.	0.	Normal operation
	1.	UR, US parallel. VRX, VSX indeterminate; VTX determinate.
	2.	UR, UT parallel. VRX, VTX indeterminate; VSX determinate.
	3.	US, UT parallel. VSX, VTX indeterminate; VRX determinate.
	4.	UR, US, UT parallel. VRX, VSX, VTX indeterminate.
	5.	UR, US, UT co-planar. VRX, VSX, VTX indeterminate.

EMG3B.	0.	Normal operation
	1.	UR, US parallel. RX, SX indeterminate. MRX, MSX assigned value (RX + SX)UR. MTX determinate.
	2.	(UR x US), UPT parallel. MRX, MSX, MTX indeterminate
	10.	UR, US parallel. MRX, MSX, MTX complex
	11.	(UR x US), UPT parallel. MRX, MSX, MTX complex
	12.	MRX, MSX, MTX complex
EMG3C.	0.	Normal operation
	2.,3.,4.	Failure to solve polynomial. Function returns from EMRP.
	5.	VC, UPS, UPT, UR parallel. MSX, MTX indeterminate; MRX determinate.
	10.	MRX, MSX, MTX complex.
	11.	VC, UPS, UPT, UR parallel. MRX, MSX, MTX complex.
EMG3D.	0.	Normal operation
	2.,3.,4.	Failure to solve polynomial. Function return from EMRP.
	5.	UPR, UPS, UPT parallel. VR, VS, VT concentric. MRX, MSX, MTX indeterminate.
	6	UPR, UPS, UPT parallel. VR, VS, VT not co-planar. MRX, MSX, MTX indeterminate.
	7	UPR, UPS, UPT parallel. VC2[5/], R sin RPR, S sin RPS, T sin RPT zero. MRX, MSX, MTX indeterminate.

---

[5] VC2 ≡ VC + R sin RPR + S sin RPS + T sin RPT. Modify this definition by dropping any term containing an unknown Only in EMG3D are all four terms present.

8. UPR, UPS, UPT parallel. VR, VS, VT co-planar. MRX, MSX, MTX ideterminate.

10. No real roots from EMRP. MRX, MSX, MTX complex.

11. UPR, UPS, UPT parallel. VR, VS, VT co-planar. R sin RPR, S sin RPS, T sin RPT zero. MRX, MSX, MTX complex.

12. UPR, UPS, UPT parallel. VR, VS, VT co-planar. MRX, MSX, MTX complex.

13. UPR, UPS, UPT parallel. VR, VS, VT co-planar. Either R sin RPR, S sin RPS or T sin RPT finite. MRX, MSX, MTX complex.

20. Denominator term zero in solution for MSX. MRX determinate. MTX not determined.

Approximate Execution Time

Function	Time (sec.)
EVG1A	less than .02
EMG2A	less than .02
EMG2B	.08
EMG2C	less than .02
EMG2D	.02
EVG3A	less than .02
EMG3B	.02
EMG3C	.05
EMG3D[6/]	2.0

[6] This time will probably be substantially reduced by an improved polynomial routine being written to replace ZER2. (ref. EMRP.)

EMRES

Purpose. Simultaneous solution of two real-coefficient polynomials each in two unknowns. The solution is obtained by determining the coefficients of the resultant polynomial and solving this polynomial by iteration on a single variable.

Call. EMRES. (A, B, MM, KM, NM, LM, X, Y)

Arguments

A, B Two-dimensional arrays of the real coefficients of two polynomials of the following form:

$$
\begin{aligned}
&[A(MM, KM)X^{KM} + A(MM, KM-1)X^{KM-1} + \ldots + A(MM, 0)]Y^{MM} \\
+ &[A(MM-1, KM)X^{KM} + A(MM-1, KM-1)X^{KM-1} + \ldots + A(MM-1, 0)]Y^{MM-1} \\
&\vdots \\
+ &[A(0, KM)X^{KM} + A(0, KM-1)X^{KM-1} + \ldots + A(0, 0)] = 0
\end{aligned}
$$

$$
\begin{aligned}
&[B(NM, LM)X^{LM} + B(NM, LM-1)X^{LM-1} + \ldots + B(NM, 0)]Y^{NM} \\
+ &[B(NM-1, LM)X^{LM} + B(NM-1, LM-1)X^{LM-1} + \ldots + B(NM-1, 0)]Y^{NM-1} \\
&\vdots \\
+ &[B(0, LM)X^{LM} + B(0, LM-1)X^{LM-1} + \ldots + B(0, 0)] = 0
\end{aligned}
$$

KM, LM Integers specifying maximum degree of X in the A and B polynomials.

MM, NM Integers specifying maximum degree of Y in the A and B polynomials.

X, Y Linear arrays of the complex X and Y roots of the resultant polynomial.

Real part of Ith root = X(2I-1), Y(2I-1)

Imag. part of Ith root = X(2I), Y(2I)

Limitations

To obtain the coefficients of the resultant polynomial this function evaluates $(KM+1)^{NM}(LM+1)^{MM}$ determinants, each of size (MM + NM) x (MM + NM). Computation time can become excessive (Section 3.2).

EUUUU.

Purpose. Solve the set of equations

$$(UA \cdot UX) = C$$

$$(UB \cdot UX) = D$$

Call. EUUUU. (UA, UB, C, D, UXI, UX2)

Arguments

UA, UB	Known unit vectors
C, D	Known scalars $C \leqq 1$, $D \leqq 1$.
UX1, UX2	Unknown unit vectors. Each is a solution to the original two equations.

Function return

0. Normal operation
1. UA = UB, C = D. UX1, UX2 indeterminate
2. UA = UB, C = -D. UX1, UX2 indeterminate

10. UA = UB, $C \neq \pm D$. UX1, UX2 complex

11. UX1, UX2 complex

Approximate execution time: < .02 sec.

A.3.4 Auxiliary Functions

CMPWR1.

Purpose. Raise a complex number to a power. (Used by EMRES.)

Call. .CMPWR1(REA,IMA,P,REX,IMX)

Arguments

REA,IMA	Real and imaginary parts of complex number A
P	Decimal power to which A is raised.
REX, IMX	Real and imaginary parts of A^P

EDETZ.

Purpose Accept the rows of a square matrix from a three-dimensional array and compute the determinant. Identify zero rows and columns and in such cases equate the determinant to zero without numerical evaluation (Used by EMRES.)

Call. EDETZ. (MQ, KI, TM)

Arguments

MQ Three-dimensional array, MQ(I, J, KI(I))

$$1 \leq I \leq TM$$

$$1 \leq J \leq TM$$

$$0 \leq KI(I) \leq KKI(I)$$

KI KI(I) and KKI(I) are linear arrays of integers:

$$1 \leq I \leq TM, \quad 0 \leq KI(I) \leq KKI(I)$$

TM Span of the row and column subscripts of MQ

Function Return. Value of the determinant.

EMCP., EMRP.

Purpose. Compute roots of a polynomial with real coefficients, using ZER2.[7] Accepts coefficients subscripted according to powers of the associated variable. Defines the degree of the polynomial as the subscript of the highest non-zero coefficient. EMCP. retains all roots obtained by ZER2.; EMRP. retains only the real roots. (EMCP. used by EMRES; EMRP by EMG2B, EMG3C, EMG3D, EMRES.)

Call

EMCP. (MP, D, MZ)

EMRP. (MP, D, MX)

Arguments

MP Linear array of the real coefficients of the polynomial $MP(D)X^{D} + MP(D-1)X^{D-1} + \ldots + MP(0) = 0$

D Degree of the polynomial (integer)

MX Linear array of the real roots of the polynomial. MX(0) is the decimal number of real roots in MX.

Ith root = MX(I)

MZ Linear array of the complex roots of the polynomial

Real part of ith root = MX(2I - 2)

Imag. part of Ith root = MZ(2I - 1)

$1 \leqq I \leqq D$

---

[7] ZER2. is a standard SHARE subroutine for obtaining the real and complex roots of a polynomial having complex coefficients.

Function return

0.	Normal operation
2.	Arguments out of range in ZER2.
3.	Impossible for ZER2. to locate all of the roots within 250 iterations. (EMCP. only)
-3.	Impossible for ZER2. to locate all of the roots within 250 iterations. Assume that all real roots have been obtained and continue (EMRP. only).
4.	Division by zero in ZER2.
10.	No real roots obtained (EMRP. only).

ESDR

Purpose. Convert degrees to radians, radians to degrees.

Call. ESDR. (DA, RA, I)

Arguments

DA	Angle in degrees	
RA	Angle in radians	
I	$I > 0$:	Convert degrees to radians
		RA = .0174532928 * DA
	$I \leq 0$:	Convert radians to degrees
		DA = 57.295778 * RA

```
$COMPILE MAD,EXECUTE,DUMP,PRINT OBJECT,PUNCH OBJECT              EVCP 001
         RCROSS PRODUCT OF TWO VECTORS     VX=VAXVB
          EXTERNAL FUNCTION(VA,VB,VX)                                    EVCP0001
          ENTRY TO EVCP.                                                 EVCP0002
          VX(1) = VA(2)*VB(3) - VA(3)*VB(2)                              EVCP0003
          VX(2) = VA(3)*VB(1) - VA(1)*VB(3)                              EVCP0004
          VX(3) = VA(1)*VB(2) - VA(2)*VB(1)                              EVCP0005
          FUNCTION RETURN                                                EVCP0006
          END OF FUNCTION                                                EVCP0007
```

Figure A.1 Basic External Function (Subprogram) for Performing the Vector Cross Product

```
$COMPILE MAD,EXECUTE,DUMP,PRINT OBJECT,PUNCH OBJECT              EIUU 001
         RCOMPARE TWO UNIT VECTORS FOR EQUALITY                          EIUU0001
          EXTERNAL FUNCTION(UA,UB,EU,IX)                                 EIUU0002
          ENTRY TO EIUU.                                                 EIUU0003
          INTEGER IX                                                     EIUU0004
          ESDP.(UA,UB,X)                                                 EIUU0005
          WHENEVER (1. -.ABS.(X)).GE.EU                                  EIUU0006
          IX=0                                                           EIUU0007
          OTHERWISE                                                      EIUU0008
          WHENEVER X.G.0.                                                EIUU0009
          IX=2                                                           EIUU0010
          OTHERWISE                                                      EIUU0011
          IX=1                                                           EIUU0012
          END OF CONDITIONAL                                             EIUU0013
          END OF CONDITIONAL                                             EIUU0014
          FUNCTION RETURN IX                                             EIUU0015
          END OF FUNCTION                                                EIUU0016
```

Figure A.2 Intermediate External Function for Comparing Two Unit Vectors

```
$COMPILE MAD,DUMP,PRINT OBJECT,PUNCH OBJECT                     EMG2D001
        RCASE 2D OF SUM OF VECTORS EQUAL ZERO (RAR,RPR,RAS UNKNOWN)    EMG2D
         EXTERNAL FUNCTION (VC,UPS,RPS,R,S,MRX,MSX)                     EMG2D
         DIMENSION V1(3),V2(3),V3(3),V4(3),V5(3),V6(3),V7(3),V8(3),VR1EMG2D
        1(3),VR2(3),VS1(3),VS2(3),UC(3)                                EMG2D
         ENTRY TO EMG2D.                                                EMG2D
         EUMV.(VC,UC,C)                                                 EMG2D
         ESDP.(UC,UPS,Y1)                                               EMG2D
         Y2=1.-Y1.P.2                                                   EMG2D
         WHENEVER Y2.L.10E-6, TRANSFER TO A0001                         EMG2D
         WHENEVER .ABS.(2.*C/(R+S)).L.10E-6, TRANSFER TO A0002          EMG2D
         SSQ=S.P.2                                                      EMG2D
         Y3=S*COS.(RPS)                                                 EMG2D
         Y4=(SSQ-Y3.P.2)*Y2                                             EMG2D
         Y5=(C.P.2+SSQ-R.P.2+2.*C*Y1*Y3)/(2.*C)                         EMG2D
         Y6=Y4-Y5.P.2                                                   EMG2D
         WHENEVER Y6 .L. 0., TRANSFER TO A0003                          EMG2D
         Y7=+SQRT.(Y6)/Y2                                               EMG2D
         Y8=Y5/Y2                                                       EMG2D
         EVCP.(UC,UPS,V1)                                               EMG2D
         EVCP.(V1,UPS,V2)                                               EMG2D
         EVFP.(UPS,Y3,V5)                                               EMG2D
         EVFP.(V2,Y8,V6)                                                EMG2D
         EVA2.(V5,V6,V3)                                                EMG2D
         EVFP.(V1,Y7,V4)                                                EMG2D
         EVA2.(V3,V4,VS1)                                               EMG2D
         EVS2.(V3,V4,VS2)                                               EMG2D
         EVN2.(VC,VS1,VR1)                                              EMG2D
         EVN2.(VC,VS2,VR2)                                              EMG2D
         EMV2.(VR1,VR2,MRX)                                             EMG2D
         EMV2.(VS1,VS2,MSX)                                             EMG2D
         FUNCTION RETURN 0.                                             EMG2D
A0001    WHENEVER Y1.G.0.                                               EMG2D
         Y9=+2.*C*S*Y1                                                  EMG2D
         OTHERWISE                                                      EMG2D
         Y9=-2.*C*S*Y1                                                  EMG2D
         END OF CONDITIONAL                                             EMG2D
         WHENEVER (.ABS.(R.P.2-(S.P.2+C.P.2+Y9)))/(R.P.2+S.P.2+C.P.2) EMG2D
        1.L.10E-4                                                       EMG2D
         PRINT COMMENT $0VC PARALLEL TO UPS IN EVG2D.  MRX AND MSX INDEMG2D
        1ET.  FCT. RET. 1.$                                             EMG2D
         FUNCTION RETURN 1.                                             EMG2D
         OTHERWISE                                                      EMG2D
         PRINT COMMENT $0VC PARALLEL TO UPS IN EVG2D.  MRX AND MSX IMAEMG2D
        1G.  FCT. RET. 10.$                                             EMG2D
         FUNCTION RETURN 10.                                            EMG2D
         END OF CONDITIONAL                                             EMG2D
A0002    WHENEVER  .ABS.(2.*(R-S)/(R+S)).L.10E-4                        EMG2D
         PRINT COMMENT $0C=0 IN EVG2D.  MRX AND MSX INDET.  FCT. RET. EMG2D
        12.$                                                            EMG2D
         FUNCTION RETURN 2.                                             EMG2D
         OTHERWISE                                                      EMG2D
         PRINT COMMENT $0C=0 IN EVG2D. MRX AND MSX IMAG.  FCT. RET. 11EMG2D
        1.$                                                             EMG2D
         FUNCTION RETURN 11.                                            EMG2D
         END OF CONDITIONAL                                             EMG2D
A0003    PRINT COMMENT $0Y5 IMAG. IN EMG2D. MRX AND MSX IMAG.  FCT. REEMG2D
        1T. 12.$                                                        EMG2D
         FUNCTION RETURN 12.                                            EMG2D
         END OF FUNCTION                                                EMG2D
```

Figure A.3 Special External Function for Evaluating the Case 2d. Solution of the Vector Tetrahedron Equation

```
$COMPILE MAD,DUMP,PRINT OBJECT,PUNCH OBJECT,EXECUTE           EMRP 001
          RCOMPUTE REAL ROOTS OF A POLYNOMIAL. (USES ZER2.)           EMRP
           EXTERNAL FUNCTION (MP,D,MX)                                EMRP
           ENTRY TO EMRP.                                             EMRP
           DIMENSION A(200),R(200)                                    EMRP
           INTEGER I,II,IR,J,D                                        EMRP
           MX(0)=0.                                                   EMRP
           THROUGH A0001, FOR I=D,-1,MP(I) .NE. 0. .OR. I .L. 1       EMRP
A0001      D=D-1                                                      EMRP
           THROUGH A0002,FOR I=0,1,I.G.D                              EMRP
           A(2*I+1)=MP(D-I)                                           EMRP
A0002      A(2*I+2)=0.                                                EMRP
           EXECUTE ZER3.(500)                                         EMRP
           W=ZER2.(D,A(1),R(1),A0010)                                 EMRP
A0010      WHENEVER W .E. 2. .OR. W .E. 4., TRANSFER TO A0004         EMRP
           J=1                                                        EMRP
           THROUGH A0005,FOR II=2,2,II.G.D*2                          EMRP
           IR=II-1                                                    EMRP
           WHENEVER R(II).E.0.,TRANSFER TO A0003                      EMRP
           WHENEVER .ABS.(R(II)/(.ABS.(R(IR))+.ABS.(R(II)))).L.10E-6, EMRP
          1TRANSFER TO A0003                                          EMRP
A0005      CONTINUE                                                   EMRP
           TRANSFER TO A0025                                          EMRP
A0003      MX(J)=R(IR)                                                EMRP
           MX(0)=J                                                    EMRP
           J=J+1                                                      EMRP
           TRANSFER TO A0005                                          EMRP
A0025      WHENEVER J .G. 1 .AND. W .E. 1., FUNCTION RETURN 0.        EMRP
           WHENEVER J .G. 1 .AND. W .E. 3., FUNCTION RETURN -W        EMRP
           FUNCTION RETURN 10.                                        EMRP
A0004      FUNCTION RETURN W                                          EMRP
           END OF FUNCTION                                            EMRP
```

Figure A.4 **Auxiliary External Function for Evaluating a Polynomial**