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THE COUETTE FLOW BETWEEN TWO PARALLEL PLATES AS A FUNCTION OF THE KNUDSEN NUMBER

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I. INTRODUCTION

In a previous report[1] a formal solution was obtained for the heat transport through a gas between two parallel plates as a function of the Knudsen number d/λ (d is the distance between the plates, λ is the mean free path of the molecules). The starting point was the linearized Boltzmann equation, since it was assumed that the magnitude of the disturbance from equilibrium, measured by the ratio $\Delta T/T$ (ΔT = temperature difference between the plates, T = average temperature), was small.

In the following, the same method will be applied to the problem of the Couette flow of a gas between two parallel plates as a function of the Knudsen number d/λ. We will again assume that the Mach number which in this case is the ratio of the average flow velocity to the mean molecular velocity, is small, so that the disturbance of the equilibrium due to the moving plate is also small in this case.

Since the method and the general features of the solution are quite similar to those of the heat-transport problem, only an outline of the calculations will be presented in sections II, III, and IV. In section V some divergence difficulties will be discussed which also occur in the heat-transport problem, and which are due to the parallel-plate geometry.

[1]C. S. Wang Chang and G. E. Uhlenbeck, "The Heat Transport between Two Parallel Plates as Functions of the Knudsen Number", Univ. of Mich., Eng. Res. Inst., Project M999, Sept., 1953.

II. FORMULATION OF THE PROBLEM

We take the y-z plane halfway between the plates. The upper plate, situated at $x = d/2$, is stationary, while the lower plate at $x = d/2$ is moving in the z direction with a velocity w (measured in units of $\sqrt{m/2kT}$). The velocity, w, is assumed to be much smaller than unity so that terms of the order w^2 and higher will be neglected. The notation of the previous report will be used. The distribution function is written as

$$f = f_0 \left[1 + h(\vec{c}, x, w) \right] .$$

It is convenient to take for the zeroth approximation distribution function

$$f_0 = n \left(\frac{m}{2\pi kT} \right)^{3/2} e^{-C^2} ,$$

where $\vec{C} = \vec{c} - \vec{w}/2$, n is the equilibrium-number density, and T the equilibrium temperature, which are all constants. The Boltzmann equation is again

$$C_x \frac{\partial h}{\partial x} = nJ(h) \tag{1}$$

where J is the collision operator.

To formulate the boundary conditions we introduce the distribution functions for the molecules going up and down, i.e., f^+ and f^-, and the corresponding disturbances h^+ and h^-. In terms of the h's, the boundary conditions are

$$h^+\left(-\frac{d}{2}\right) = \alpha\left(B^+ + wC_z\right) + (1-\alpha)\, h^-\left(-\frac{d}{2}, -C_x\right) \qquad x > 0 \tag{2a}$$

and

$$h^-\left(+\frac{d}{2}\right) = \alpha\left(B^- - wC_z\right) + (1-\alpha)\, h^+\left(+\frac{d}{2}, -C_x\right) \qquad x < 0 \tag{2b}$$

where α is the accommodation coefficient and the constants B^+ and B^- are to be determined by the conditions

$$\int_{-\frac{d}{2}}^{+\frac{d}{2}} dx \int d\vec{c}\, e^{-c^2}\, h = 0 \,, \tag{3a}$$

$$\int d\vec{c}\, c_x\, e^{-c^2}\, h = 0 \,, \tag{3b}$$

expressing the conditions that the number of molecules per unit area between the plates is given (equal to nd) and that in the steady state there is no net flow in the x-direction. Up to terms linear in w we have the symmetry condition

$$h^{\pm}(c_x, x) = -h^{\mp}(-c_x, -x) \,, \tag{4}$$

and consequently

$$h(c_x, x) = -h(-c_x, -x) \,.$$

With the symmetry condition (4), Eq. (3a) is automatically satisfied, and the boundary conditions (2a) and (2b) are equivalent with

$$B^{+} = -B^{-} = B \,.$$

Thus our problem reduces to the solution of the integral differential equation (1), subjected to the boundary condition (2a) and the auxiliary conditions (3b) and (4). The physical quantities we are interested in are the drag on the upper plate, the velocity distribution as a function of x, and the velocity jumps at the plates.

III. GENERAL SOLUTION

At first glance, this problem seems to be more complicated than the problem of the heat conduction because of the lack of axial symmetry around the x-axis in the present problem. It will be seen, however, that this is not a real difficulty. It is true, though, that in the development in eigenfunctions, we must now use the spherical harmonics instead of the Legendre polynomials used in Ref. (1). We write

$$\psi_{r\ell m} = N_{r\ell m}\, c^{\ell}\, S^{(r)}_{\ell+\frac{1}{2}}(c^2)\, P_{\ell}^{m}(\cos\theta)\, e^{im\phi} \,,$$

where Θ is the angle between the velocity $\vec{C}$ and the x-axis and ϕ is the azimuthal angle measured from the y-axis, and develop h according to

$$h = \sum_{r\ell m} a_{r\ell m}(x)\,\psi_{r\ell m} \tag{5}$$

Because of the reality of h one must have $a_{r\ell m} = a^*_{r\ell m}$. From the symmetry of the problem with regard to the y direction, it follows that h must be an even function of c_y, which has as a consequence that the $a_{r\ell m}$ for odd m must be pure imaginary The first few of the development coefficients $a_{r\ell m}$ are related to quantities with physical interest. For instance, the density and temperature are found from

$$n(x) = n\left[1 + \frac{a_{000}(x)}{\pi^{3/2} N_{000}}\right],$$

$$T(x) = T\left[1 - \frac{2}{3}\frac{a_{100}(x)}{\pi^{3/2} N_{100}}\right],$$

and the physical quantities (shearing stress, average velocity, and heat flux, all in the z-direction) in which we are especially interested are given by

$$p_{xz} = -\frac{nkT}{\pi^{3/2}}\,\frac{2}{3i}\,\frac{a_{021}}{N_{021}}, \tag{6a}$$

$$\overline{c_z} = \frac{w}{2} - \frac{a_{011}}{i\pi^{3/2} N_{011}}, \tag{6b}$$

$$q_z = nkT\sqrt{\frac{2kT}{m}}\,\frac{1}{i\pi^{3/2}}\left[\frac{a_{111}}{N_{111}} - \frac{5}{2}\frac{a_{011}}{N_{011}}\right]. \tag{6c}$$

From the conservation laws it follows that c_x, p_{xx}, p_{xy}, p_{xz}, and q_x are constants. We will require that $\overline{c}_x = 0$ (see eq. (3b)); p_{xy} is zero by symmetry. This implies that*

$$a_{010} = 0$$

$$a_{021} = \text{const.}$$

$$\frac{a_{020}}{N_{020}} - \frac{a_{100}}{2N_{100}} + \frac{3}{4}\frac{a_{000}}{N_{000}} = \text{const.}$$

$$a_{100} = \text{const.}$$

*There are some misprints in the corresponding formulas in the previous report, Ref. (1)

Introducing the development of h into the Boltzmann equation, we find that our problem reduces to the solution of an infinite set of homogeneous linear differential equations:

$$\frac{da_{r\ell m}}{dx} = n \sum_{r'\ell'm'} a_{r'\ell'm'} \left[\frac{1}{c_x} \psi^*_{r\ell m} , \psi_{r'\ell'm'} \right] \tag{7}$$

with the boundary conditions:

$$\begin{aligned} a^+_{r\ell m}\left(-\frac{d}{2}\right) &= \alpha \int d\vec{c}\, e^{-c^2} \psi^*_{r\ell m} \left(B + wC_z \right) \frac{1+\operatorname{sign} c_x}{2} \\ &\mp (1-\alpha)\, a^-_{r\ell m}\left(-\frac{d}{2}\right) \qquad \left(\ell - m = \begin{cases} \text{odd} \\ \text{even} \end{cases}\right) \end{aligned} \tag{8}$$

Finally, $a_{010} = 0$, and from the symmetry property eq. (4) it follows that

$$a_{r\ell m} = \begin{cases} \text{even} \\ \text{odd} \end{cases} \text{in } x \text{ according as } \ell - m \text{ is} \begin{cases} \text{odd} \\ \text{even.} \end{cases}$$

The square bracket in eq. (7) is different from zero only when $m = m'$, and $\ell - \ell'$ is odd. Thus coefficients with different values of m are not coupled.

Since for $|m|>1$ the integral in the boundary condition (8) is zero, it is clear that the equations for $a_{r\ell m}$ with $|m|>1$ are completely homogeneous, so that the only solution will be zero. This is also the case for $m = 0$, because in eq. (8) the quantity B is also an unknown and the inhomogeneous part is only the term containing wC_z. This term is also zero for $m = 0$. Hence one may conclude*

$$B = 0 ,$$

$$a_{r\ell m} = 0 , \qquad \text{for } m \neq \pm 1 .$$

We use the same procedure as was employed in Ref. (1). One must distinguish between even and odd values of ℓ. Eliminating $a_{r(2\ell+1)1}$ one obtains an infinite set of second-order homogeneous differential equations for $a_{r(2\ell)1}$, which, with the use of the symmetry conditions, has as solutions:

$$a_{r2\ell 1} = b^{(0)}_{r2\ell 1} + \sum_i b^{(i)}_{r2\ell 1} \cosh np_i x , \tag{9}$$

*This is in contrast to the case of heat conduction. It is also physically evident that a temperature gradient will produce a change of the density n near the plate, while this is not the case for a velocity gradient.

where the p_i's are the positive and nonzero solutions of the secular determinant

$$\Delta = \left| p^2 \delta_{rr''} \delta_{\ell\ell''} - \sum_{r'\ell'} \left[\frac{1}{c_x} \psi^*_{r2\ell 1}, \psi_{r'2\ell'+1\,1} \right] \left[\frac{1}{c_x} \psi^*_{r'2\ell'+1\,1}, \psi_{r''2\ell''1} \right] \right| = 0 ,$$

and where in the sets of the $b_{r(2\ell)1}$ one set can be taken as arbitrary, for which we choose $b^{(0)}_{021}$ and $b^{(i)}_{121}$. Furthermore, from the constancy of p_{xz} it follows that $b^{(i)}_{021} = 0$ for all i.

The coefficients $a_{r(2\ell+1)1}$ are now completely determined and one finds

$$a_{r2\ell+1\,1} = nx \sum_{r'\ell'} \left[\frac{1}{c_x} \psi^*_{r2\ell+1\,1}, \psi_{r'2\ell'1} \right] b^{(0)}_{r'2\ell'1}$$
$$+ \sum_i \frac{1}{p_i} \sinh np_i x \sum_{r'\ell'} \left[\frac{1}{c_x} \psi^*_{r2\ell+1\,1}, \psi_{r'2\ell'1} \right] b^{(i)}_{r'2\ell'1} . \qquad (10)$$

The remaining unknown constants $b^{(0)}_{021}$ and $b^{(i)}_{121}$ have to be determined from the boundary conditions. For this, one needs first the $a^{\pm}_{r(2\ell)1}$, for which one obtains:

$$a^{\pm}_{r2\ell 1} = \sum_{r'\ell'} \left(\frac{1}{2} \delta_{rr'} \delta_{\ell\ell'} \pm n T^{+}_{r2\ell 1; r'2\ell'1} x \right) b^{(0)}_{r'2\ell'1} + \qquad (11)$$
$$+ \sum_{r'\ell'} \sum_i \left(\frac{1}{2} \delta_{rr'} \delta_{\ell\ell'} \cosh np_i x \pm \frac{1}{p_i} \sinh np_i x \, T^{+}_{r2\ell 1; r'2\ell'1} \right) b^{(i)}_{r'2\ell'1}$$

where

$$T^{+}_{r2\ell 1; r'2\ell'1} \equiv \left[\frac{1}{c_x} \psi_{r2\ell 1} \frac{1 + \operatorname{sign} c_x}{2} , \psi_{r'2\ell'1} \right] .$$

Putting eq. (11) in the boundary conditions, one is led to the set of inhomogeneous linear equations

$$w \int d\vec{c}\, e^{-c^2} \psi^*_{r2\ell 1} c_z \frac{1 + \operatorname{sign} c_x}{2} = \left[\frac{2-\alpha}{2\alpha} - \frac{nd}{2} \sum_{r'\ell'} T^{+}_{r2\ell 1; r'2\ell'1} B_{r'2\ell'1; 021} \right] b^{(0)}_{021}$$
$$+ \sum_i \left[\frac{2-\alpha}{2\alpha} \cosh \frac{np_i d}{2} - \frac{1}{p_i} \sinh \frac{np_i d}{2} \sum_{r'\ell'} T^{+}_{r2\ell 1, r'2\ell'1} B^{(i)}_{r'2\ell'1; 121} \right] b^{(i)}_{121} \qquad (12)$$

where the known constant matrices $\mathbb{B}$ are defined by

$$b^{(0)}_{r2\ell 1} = B_{r2\ell 1; 021} b^{(0)}_{021}$$
$$b^{(i)}_{r2\ell 1} = B^{(i)}_{r2\ell 1; 121} b^{(i)}_{121} .$$

In the same sense as in Ref. (1), eq. (12) is a set of inhomogeneous linear equations with the same number of equations as that of unknowns. Thus the problem is formally solved.

The complete solution involves infinite determinants. We are, as yet, unable to discuss the convergence of these determinants nor the convergence of the "breaking off" processes which we will use as in Ref. (1).

IV. THE MAXWELL MOLECULES

We will use the same "successive approximation" method as in Ref. (1). Furthermore, we will restrict ourselves to the Maxwell molecules so that

(1) $J(\psi_{r\ell m}) = \lambda_{r\ell}\psi_{r\ell m}$,

(2) $\left[\frac{1}{c_x}\psi^*_{r\ell m}, \psi_{r'\ell' m}\right] = \lambda_{r'\ell'}\int d\vec{c}\, e^{-c^2}\frac{1}{c_x}\psi^*_{r\ell m}\psi_{r'\ell' m}$,

and

(3) $\left[\frac{1}{c_x}\psi^*_{r2\ell+11}, \psi_{021}\right] = 0$ unless both r and ℓ are zero which has as a consequence that all $b^{(0)}_{r(2\ell)1}$ are zero except $b^{(0)}_{021}$.

Since we do not expect to get anything from the zeroth approximation, we start with the first approximation.

A. First Approximation

As in the heat-conduction and sound-propagation[2] problems, we will use in the first approximation the eigenfunctions for which $2r + \ell \leq 3$, and $\ell < 3$. since only the ψ's with m = 1 enter the problem, we therefore need to take only ψ_{011}, ψ_{111}, and ψ_{021}. We know that

$$a_{021} = \text{const.} = b^{(0)}_{021}. \qquad (13a)$$

[2]C. S. Wang Chang and G. E. Uhlenbeck, "On the Propagation of Sound in Monatomic Gases", Univ. of Mich., Eng. Res. Inst., Proj. M999, Oct., 1952.

The other two a's are

$$a_{011} = n b_{021}\, x \left[\frac{1}{c_x} \psi^*_{011}, \psi_{021} \right], \tag{13b}$$

$$a_{111} = n b_{021}\, x \left[\frac{1}{c_x} \psi^*_{111}, \psi_{021} \right]. \tag{13c}$$

Since p = 0 in this case, the boundary condition (12) reduces to

$$w S_{021} = \frac{2-\alpha}{2\alpha} b^{(0)}_{021} - \frac{nd}{2} T^+_{021,\,021} b^{(0)}_{021}$$

where

$$S_{r2l1} = \int d\vec{c}\, e^{-c^2} \psi^*_{r2l1} c_z \frac{1+\mathrm{sign}\, c_x}{2} = (-)^l \frac{i\sqrt{\pi}}{2} N_{r2l1} \frac{l(l+\frac{1}{2})(l+r-\frac{3}{2})!}{r!}$$

Using the values of S and T^+ as given in tables in Appendix II, one finds

$$b^{(0)}_{021} = -2i\pi^{3/2} N_{021}\, w \frac{\lambda}{d} \frac{1}{1 + \frac{2-\alpha}{2\alpha} \frac{8\sqrt{\pi}}{3} \frac{\lambda}{d}}$$

where $\lambda \equiv \left(n A_2 \sqrt{\frac{\kappa}{kT}}\right)^{-1}$. Using eqs. (6) and (13), one obtains the following results:

$$p^{(1)}_{xz} = nkT \frac{4}{3} w \frac{\lambda}{d} \frac{1}{1 + \frac{2-\alpha}{2\alpha} \frac{8\sqrt{\pi}}{3} \frac{\lambda}{d}}, \tag{14a}$$

$$\bar{c}_z = \frac{w}{2} \left[1 - \frac{2x}{d} \frac{1}{1 + \frac{2-\alpha}{2\alpha} \frac{8\sqrt{\pi}}{3} \frac{\lambda}{d}} \right], \tag{14b}$$

and

$$q_z = -\frac{5}{2} nkT \frac{wx}{d} \sqrt{\frac{2kT}{m}} \frac{1}{1 + \frac{2-\alpha}{2\alpha} \frac{8\sqrt{\pi}}{3} \frac{\lambda}{d}}, \tag{14c}$$

where w is still in units $\sqrt{2kT/m}$. Therefore the velocity slip at the upper plate $x = d/2$ is

$$(\overline{c_z})_{x=\frac{d}{2}} = \Delta W = \frac{W}{2} \frac{2-\alpha}{2\alpha} \frac{8\sqrt{\pi}}{3} \frac{\lambda}{d} \frac{1}{1 + \frac{2-\alpha}{2\alpha} \frac{8\sqrt{\pi}}{3} \frac{\lambda}{d}} .$$

For small λ/d and $\alpha = 1$, this is in agreement with the result of Maxwell.

B. Second Approximation ($2r + \ell \leq 4$ and $\ell < 4$)

The eigenfunctions entering the problem are ψ_{011}, ψ_{021}, ψ_{111}, ψ_{031}, and ψ_{121}. The ψ's with odd values of $\ell - m$ are ψ_{021} and ψ_{121}. The only nonzero value of p is

$$p = \pm \sqrt{\frac{5}{8}}\, A_2 \sqrt{\frac{\kappa}{kT}} .$$

The constants to be determined by the boundary conditions are $b_{021}^{(0)}$ and $b_{121}^{(0)}$. Equations (12) become a set of two equations:

$$w S_{021} = \left(\frac{2-\alpha}{2\alpha} - \frac{nd}{2} \overset{+}{T}_{021;021} \right) b_{021}^{(0)} - \frac{1}{p} \overset{+}{T}_{021;121} \sinh \frac{npd}{2}\, b_{121}$$

and

$$w S_{121} = -\frac{nd}{2} \overset{+}{T}_{121;021}\, b_{021}^{(0)} + \left[\frac{2-\alpha}{2\alpha} \cosh \frac{npd}{2} - \frac{1}{p} \overset{+}{T}_{121;121} \sinh \frac{npd}{2} \right] b_{121} .$$

Solving these equations, one then can calculate the quantities of physical interest. It turns out that in all approximations the drag, the velocity distribution, and the velocity jump have the form

$$p_{xz} = nkT \frac{4W}{3} \frac{1}{\frac{d}{\lambda} + \frac{2-\alpha}{2\alpha} \frac{8\sqrt{\pi}}{3} \frac{A}{B}} ,$$

$$\overline{c_z} = \frac{W}{2} \left[1 - \frac{2x}{d} \frac{\frac{d}{\lambda}}{\frac{d}{\lambda} + \frac{2-\alpha}{2\alpha} \frac{8\sqrt{\pi}}{3} \frac{A}{B}} - \frac{2-\alpha}{2\alpha} \frac{\frac{1}{B} \sum_i E_i \sinh np_i x}{\frac{d}{\lambda} + \frac{2-\alpha}{2\alpha} \frac{8\sqrt{\pi}}{3} \frac{A}{B}} \right] , \qquad (15)$$

$$\Delta W = \frac{W}{2} \frac{2-\alpha}{2\alpha} \frac{8\sqrt{\pi}}{3} \frac{\frac{A}{B} - \frac{3}{8\sqrt{\pi}} \frac{1}{B} \sum_i E_i S_i}{\frac{d}{\lambda} + \frac{2-\alpha}{2\alpha} \frac{8\sqrt{\pi}}{3} \frac{A}{B}} ,$$

where the sum is over all the roots p_i, and s_i = sinh $(np_i d/2)$. The functions A, B, and E_i are all functions of $K = d/\lambda$.

In this approximation, one finds

$$A = \frac{2-\alpha}{2\alpha} + \frac{13}{40}\sqrt{\frac{10}{\pi}}\, t \;,$$

$$B = \frac{2-\alpha}{2\alpha} + \frac{12}{40}\sqrt{\frac{10}{\pi}}\, t \;,$$

$$E = \frac{2\sqrt{10}}{15}\,\frac{1}{c} \;,$$

where t = tanh (npd/2), c = cosh (npd/2).

C. Third Approximation ($2r + \ell \leq 5$, and $\ell \leq 5$)

The eigenfunctions to be taken in this case are ψ_{011}, ψ_{021}, ψ_{111}, ψ_{031}, ψ_{121}, ψ_{041}, ψ_{211}, and ψ_{131}. There are four ψ's with odd values of ℓ - m. Hence there are two pairs of p differing from zero. They are

$$p_1 = 0.7151\, A_2' \quad \text{and} \quad p_2 = 1.342\, A_2' \;,$$

where $A_2' = A_2 \sqrt{\kappa/kT}$. We find, further,

$$b^{(1)}_{041} = -0.2456\, b^{(1)}_{121} \quad , \qquad b^{(2)}_{041} = 2.539\, b^{(2)}_{121}.$$

Solving the set of three equations for the boundary conditions, one finds then,

$$A = \left(\frac{2-\alpha}{2\alpha}\right)^2 + \frac{2-\alpha}{2\alpha}\left(0.6318\, t_1 + 0.5384\, t_2\right) + 0.3392\, t_1 t_2 \;,$$

$$B = \left(\frac{2-\alpha}{2\alpha}\right)^2 + \frac{2-\alpha}{2\alpha}\left(0.5367\, t_1 + 0.5067\, t_2\right) + 0.2714\, t_1 t_2 \;,$$

$$E_1 = \frac{1}{c_1}\left(\frac{2-\alpha}{2\alpha}\cdot 0.8992 + 0.4680\, t_2\right),$$

$$E_2 = \frac{1}{c_2}\left(\frac{2-\alpha}{2\alpha}\cdot 0.3000 + 0.1734\, t_1\right)$$

D. Fourth Approximation ($2r + \ell \leq 6$, and $\ell < 6$)

We proceed as before, with the four values of p's and the corresponding ratios of $b^{(i)}_{r(2\ell)1} / b^{(i)}_{121}$ given in the following table.

i	1	2	3	4
$p^{(i)}$	$0.4604A^{\frac{1}{2}}$	$1.065A^{\frac{1}{2}}$	$1.261A^{\frac{1}{2}}$	$1.610A^{\frac{1}{2}}$
$b^{(i)}_{041}/b^{(i)}_{121}$	$\frac{1}{2\sqrt{3}}(-0.711)$	$\frac{1}{2\sqrt{3}}(-5.525)$	$\frac{1}{2\sqrt{3}}(3.322)$	$\frac{1}{2\sqrt{3}}(15.58)$
$b^{(i)}_{221}/b^{(i)}_{121}$	$\frac{5}{12}(-1.218)$	$\frac{5}{12}(-6.204)$	$\frac{5}{12}(1.213)$	$\frac{5}{12}(8.442)$
$b^{(i)}_{141}/b^{(i)}_{121}$	$\frac{\sqrt{6}}{4\sqrt{11}}(1.362)$	$\frac{1}{4}\sqrt{\frac{6}{11}}(0.320)$	$\frac{1}{4}\sqrt{\frac{6}{11}}(-5.349)$	$\frac{1}{4}\sqrt{\frac{6}{11}}(32.51)$

Taking $\alpha = 1$ to simplify the computation we arrived at the following results:

$$A = 2.38 + 2.63\,t_1 + 2.73\,t_2 + 2.50\,t_3 + 2.48\,t_4 + 2.95\,t_1t_2 + 2.70\,t_1t_3 + 2.73\,t_1t_4 + 2.86\,t_2t_3 + 2.81\,t_2t_4 + 2.61\,t_3t_4 + 3.03\,t_1t_2t_3 + 3.03\,t_1t_2t_4 + 2.80\,t_1t_3t_4 + 2.94\,t_2t_3t_4 + 3.10\,t_1t_2t_3t_4\,,$$

$$B = 2.38 + 2.49\,t_1 + 2.37\,t_2 + 2.49\,t_3 + 2.33\,t_4 + 2.46\,t_1t_2 + 2.55\,t_1t_3 + 2.42\,t_1t_4 + 2.47\,t_2t_3 + 2.31\,t_2t_4 + 2.43\,t_3t_4 + 2.52\,t_1t_2t_3 + 2.40\,t_1t_2t_4 + 2.48\,t_1t_3t_4 + 2.41\,t_2t_3t_4 + 2.46\,t_1t_2t_3t_4\,,$$

$$E_1 = \frac{2.069}{c_1}\left(0.502 + 0.457\,t_2 + 0.502\,t_3 + 0.488\,t_4 + 0.453\,t_2t_3 + 0.437\,t_2t_4 + 0.482\,t_3t_4 + 0.433\,t_2t_3t_4\right),$$

$$E_2 = \frac{1.145}{c_2}\left(3.227 + 3.216\,t_1 + 3.338\,t_3 + 3.081\,t_4 + 3.276\,t_1t_3 + 3.071\,t_1t_4 + 3.183\,t_1t_3t_4 + 3.125\,t_1t_3t_4\right),$$

$$E_3 = \frac{0.331}{c_3}\left(0.805 + 0.517 t_1 + 0.668 t_2 + 0.833 t_4 + 0.409 t_1 t_2 + 0.554 t_1 t_4 + 0.700 t_2 t_4 + 0.454 t_1 t_2 t_4\right),$$

$$E_4 = \frac{0.658}{c_4}\left(2.563 + 2.604 t_1 + 2.455 t_2 + 2.705 t_3 + 2.495 t_1 t_2 + 2.694 t_1 t_3 + 2.603 t_2 t_3 + 2.589 t_1 t_2 t_3\right).$$

For the discussion of these results we have plotted three sets of curves. Figure 1 contains a set of four curves for $p_{xz}/(p_{xz})_{\text{Knudsen}}$ against the Knudsen number, d/λ. Curve I is for the Stokes-Navier approximation. Curves II, III, and IV are results from our first, second, and third approximations respectively. We have not plotted the curve for the fourth approximation because we do not expect anything new and the numerical computation is very laborious. The straight line on the left is the initial slope for the exact solution. All the curves from the different approximations have the same value (unity) for $K = 0$, and for K equal to infinity they all approach the same limit, zero. The initial slope for the four approximations are listed below.

	Initial slope
Exact	-1.242
First approximation	-0.423
Second "	-0.458
Third "	-0.534
Fourth "	-0.598

Figure 2 consists of a set of three curves for the first three approximations for the velocity slip at the upper plate as a function of K. The initial slope is given by:

	Initial slope
First approximation	-0.423
Second "	-0.494
Third "	-0.644
Fourth "	-0.755

We have drawn Fig. 3 to show the velocity distributions as functions of x for $K = 10$. Since the velocity distributions for the different approximations deviate very slightly from the straight-line distribution of the first

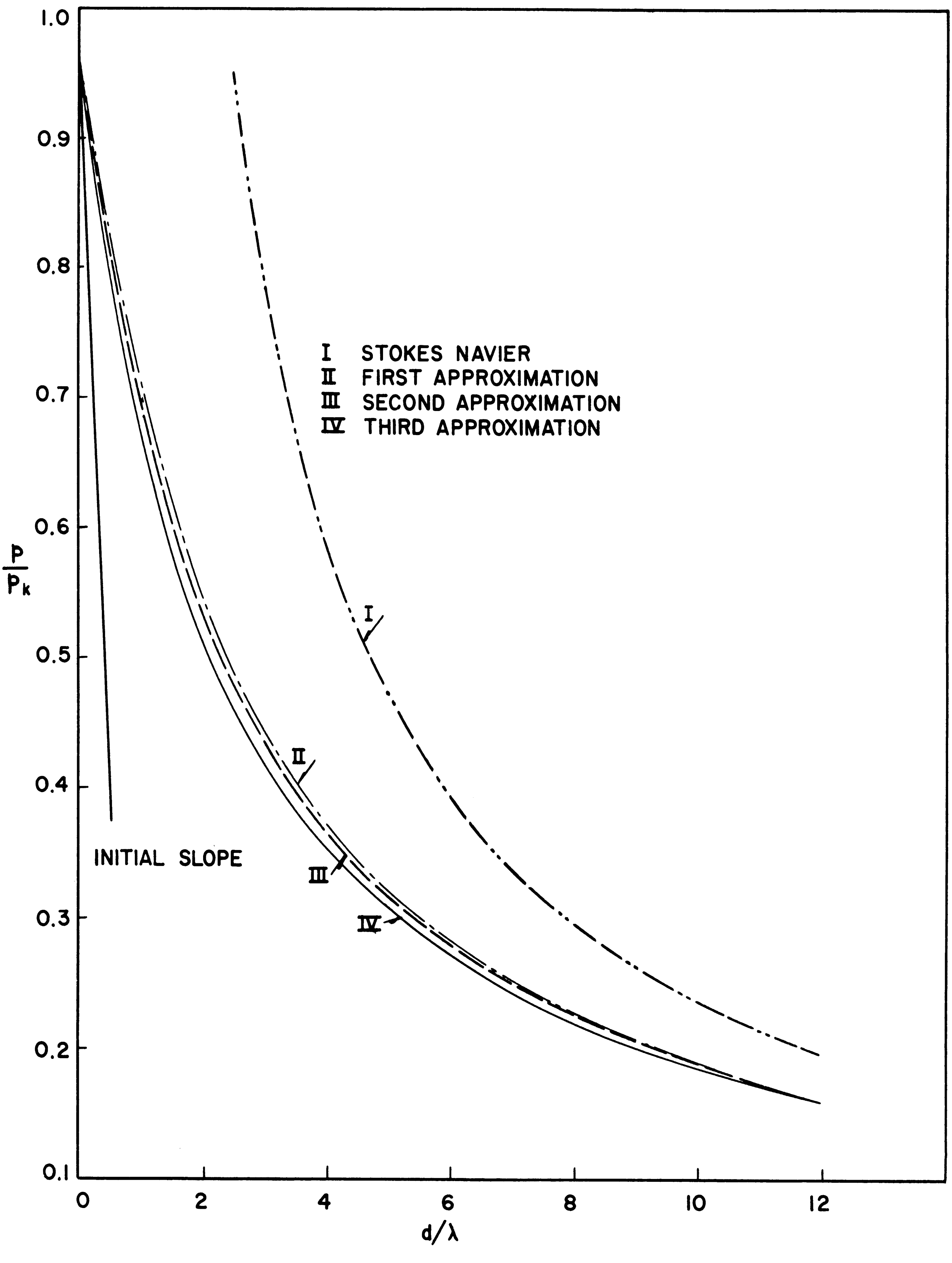

Fig. 1 The Drag as a Function of the Knudsen Number

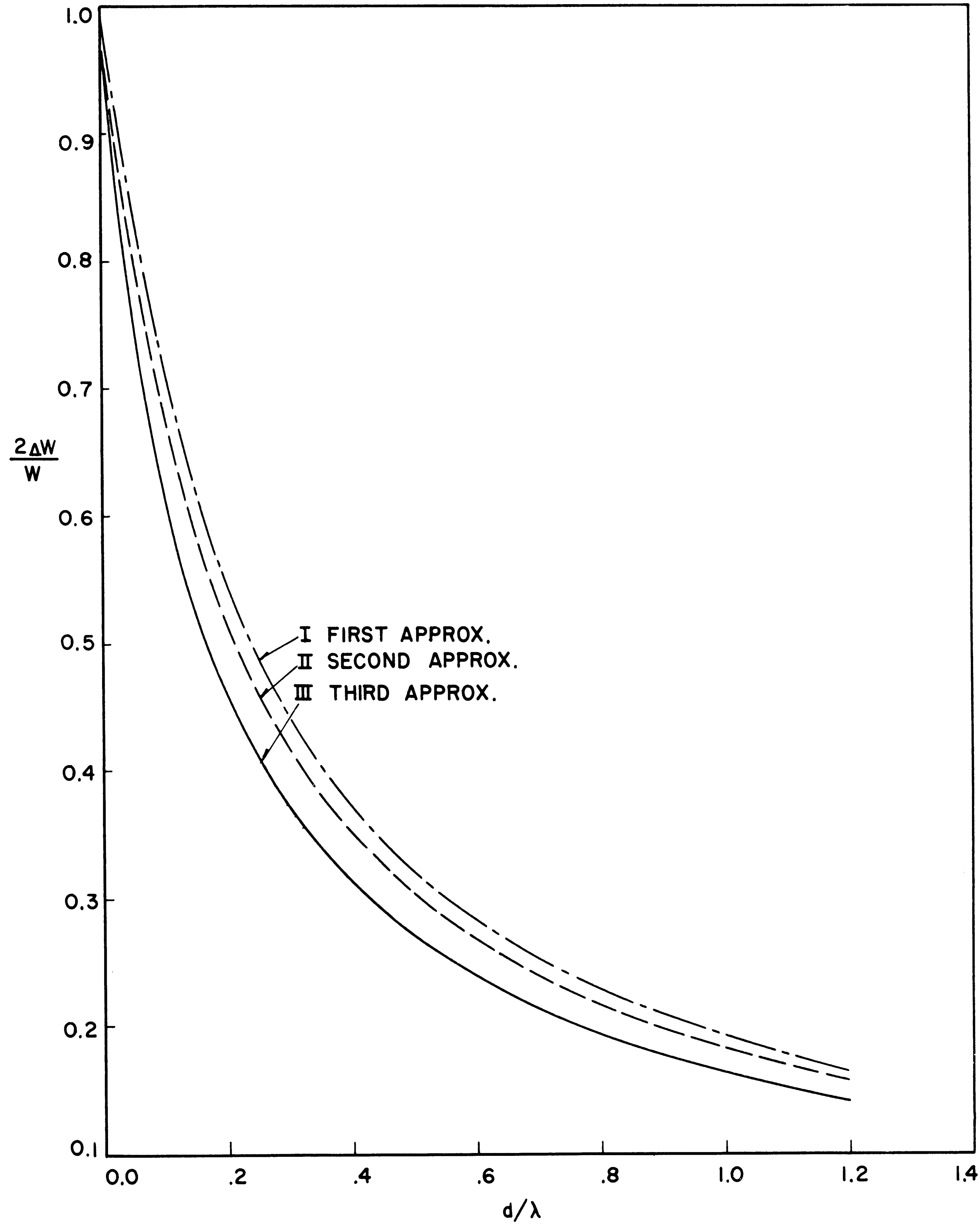

Fig. 2 Velocity Jump, $\frac{2\Delta W}{W}$ as a Function of the Knudsen Number

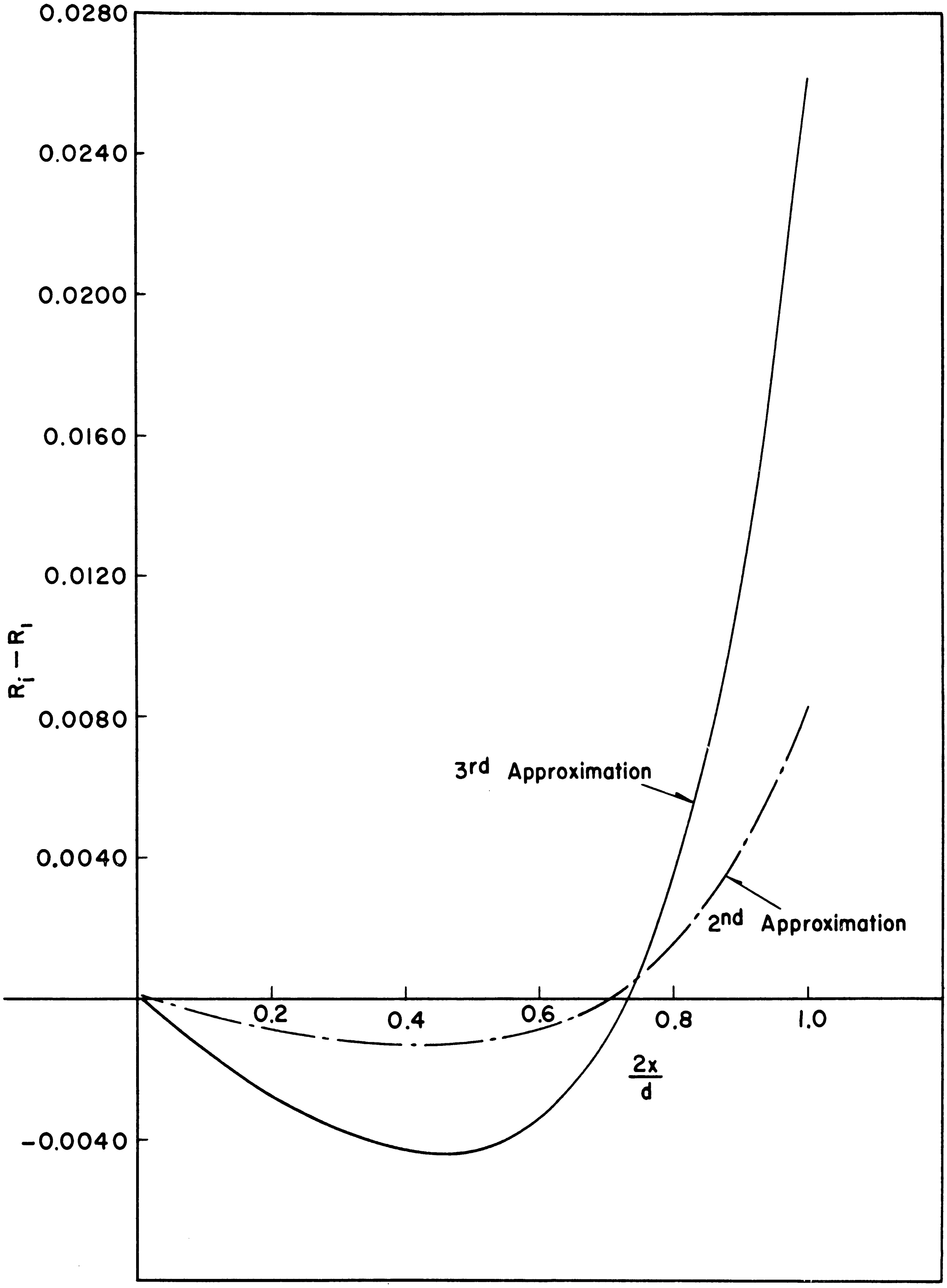

Fig. 3 Velocity Distribution in the Second and Third Approximation as Compared with the Linear Velocity Distribution R_1 (x) From the Stokes-Navier Approximation with slip for $d/\lambda = 0$

approximation we have drawn instead of $\overline{c}_z$ against x, the curves $R_1 - R$, where $R = 1 - 2\overline{c}_z(x)/w$ and R_1 is the value of R from the first approximation, namely,

$$R_1 = \frac{2x}{d} \frac{1}{1 + \frac{4\sqrt{\pi}}{3}\frac{\lambda}{d}} \; .$$

As in the case of heat conduction, the boundary effect is shown by the sharp rise of the curve near the wall.

V. THE APPROACH TO THE KNUDSEN LIMIT; A DIVERGENCE DIFFICULTY

In Ref. (1) we pointed out that the behavior of the exact solution for the heat-conduction problem is for large K quite different from the behavior at small Knudsen number. The approach to the Clausius gas limit ($K \gg 1$) is complicated by the occurrence of the hyperbolic functions, so that a development in inverse powers of K is not possible. We see from section III that the same is true for the Couette flow. On the other hand, it seems that the approach to the Knudsen gas regime ($K \ll 1$) is quite regular so that a development of all relevant physical quantities in powers of K should be possible.

In fact, in Ref. (1) we gave the first two terms in such a development for the heat flux. Analogous results can be found for the drag*. In the zeroth approximation one finds

$$p_{xz}^{(0)} = nkT \frac{\alpha}{2-\alpha} \frac{w}{\sqrt{\pi}} \tag{16}$$

with w still in units of $\sqrt{2kT/m}$. In the first approximation

$$p_{xz}^{(1)} = p_{xz}^{(0)} \frac{\alpha}{2-\alpha} \frac{nd}{\pi} \left[c_z \operatorname{sign} c_x \, , \, c_z \operatorname{sign} c_x \right] . \tag{17}$$

For $\alpha = 1$ this reduces to the expression given in Ref. (3).[3] There the square bracket which is always negative has been evaluated for elastic spheres and Maxwell molecules. The results are as follows.

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*These results can be found either by the method described in Chap. III of Ref. (1), or from the general solution.

[3]C. S. Wang and G. E. Uhlenbeck, "Transport Phenomena in Very Dilute Gases", CM 579, UMH-3-F, Univ. of Mich., Eng. Res. Inst., Proj. M604-6, Nov. 15, 1949.

For elastic spheres of diameter σ,

$$\frac{p_{xz}^{(1)}}{p_{xz}^{(0)}} = -\frac{\sqrt{2}\, n d \pi \sigma^2}{48}\left(5 + 26\sqrt{2}\right), \tag{18a}$$

For Maxwell molecules,

$$\frac{p_{xz}^{(1)}}{p_{xz}^{(0)}} = -nd\sqrt{\frac{\pi\kappa}{kT}}\int_0^{\pi} d\theta \sin\theta\, F(\theta)\Big\{\pi - (1-\cos\theta)\sin^{-1}\frac{1-\cos\theta}{2} - (1+\cos\theta)\sin^{-1}\frac{1+\cos\theta}{2} + \frac{1}{2}\frac{\sin^2\theta}{\sqrt{4-(1-\cos\theta)^2}} + \frac{1}{2}\frac{\sin^2\theta}{\sqrt{4-(1+\cos\theta)^2}}\Big\}. \tag{18b}$$

In this case the result obtained from the general solution is expressed in the eigenvalues $\lambda_{r\ell}$ of the collision operator in the form

$$\frac{p_{xz}^{(1)}}{p_{xz}^{(0)}} = -\frac{\alpha}{2-\alpha}\,\frac{nd}{4\pi}\sum_{r,\ell}\lambda_{r2\ell}\,(4\ell+1)\,\frac{\ell(\ell+\frac{1}{2})\left[(\ell+r-\frac{3}{2})!\right]^2}{r!\,(2\ell+r+\frac{1}{2})!}. \tag{19}$$

The summation can be carried out (for details see Appendix I) and confirms eq. (18b). The integral was evaluated numerically in Ref. (3), with the result

$$\frac{p_{xz}^{(1)}}{p_{xz}^{(0)}} = -\frac{\alpha}{2-\alpha}\,\frac{\lambda}{d}\,(1.242)$$

where $\lambda^{-1} = nA_2\sqrt{\kappa/kT}$. Thus for $\alpha = 1$, the exact value of the initial slope in Fig. 1 is -1.242.

A difficulty appears when one wants to calculate the second approximation of the drag, or when one wants to find analogous expansions for the average velocity distribution or for the velocity jump at the plates. Formally, one finds

$$p_{xz}^{(2)} = p_{xz}^{(0)}\left(\frac{\alpha}{2-\alpha}\right)^2\frac{n^2d^2}{8}\left[\frac{1}{c_x}\,\mathrm{sign}\,c_x\, J(c_z\,\mathrm{sign}\,c_x),\; c_z\,\mathrm{sign}\,c_x\right], \tag{20}$$

and in the first approximation,

$$\overline{c_z}^{(1)} = \frac{\alpha}{2-\alpha}\,\frac{nxw}{\pi^{3/2}}\left[\frac{c_z}{c_x},\; c_z\,\mathrm{sign}\,c_x\right].$$

The bracket expressions are both divergent integrals because of the factor $1/c_x$. The same difficulty occurs in the heat-conduction problem, although we did not notice it at that time. The second approximation to the heat flux given in

Ref. (1), p. 8, eq. (19) is also divergent, and the temperature distribution in the first approximation for which one can derive the formal expression

$$T^{(1)}(x) = \frac{\alpha}{2-\alpha}\,\frac{2}{3}\,\frac{\Delta T\, n x}{\pi^{3/2}}\left[\left(c^2-\frac{3}{2}\right)\frac{1}{c_x}\,,\quad (c^2-2)\,\mathrm{sign}\,c_x\right],$$

is divergent just as $\overline{c}_z^{(1)}$

The divergence of all these integrals is of a logarithmic nature and is always due to the factor $1/c_x$. The origin of these divergence difficulties is therefore clearly the parallel-plate geometry. Molecules which are emitted nearly parallel to a plate will have to transverse a very long path before reaching the other plate, so that the Knudsen approximation will not be valid for these molecules. Or one can say, the average Knudsen number for molecules emitted in all directions is logarithmically infinite even if $d/\lambda \ll 1$. We have found that by taking two concentric spheres (radii a and b) instead of two parallel plates all divergences disappear. For $d = a - b \ll (1/2)(a + b) = R$, quantities such as the temperature distribution will in first approximation in $K = d/\lambda$ contain terms proportional to $\ln (R/d)$, which blows up for the parallel plate geometry.

One may conclude, therefore, that these divergence difficulties will not affect any real physical situation, and that the behavior of the solutions near the Knudsen limit can still be considered to be regular, so that a development in powers of K is possible.

APPENDIX I

DERIVATION OF EQUATION (18b) FROM EQUATION (19)

In terms of the eigenvalues $\lambda_{r\ell}$ of the collision operator, the first order correction to the pressure p_{xz} in the Knudsen limit is given by

$$\frac{p_{xz}^{(1)}}{p_{xz}^{(0)}} = -\frac{\alpha}{2-\alpha}\,\frac{nd}{4\pi}\sum_{r,\ell}\lambda_{r2\ell}\,(4\ell+1)\,\frac{\ell(\ell+\frac{1}{2})\left[(\ell+r-\frac{3}{2})!\right]^2}{r!\,(2\ell+r+\frac{1}{2})!} \tag{23}$$

The expression for $\lambda_{r2\ell}$ is

$$\lambda_{r2\ell} = 2\pi\sqrt{\frac{\kappa}{kT}}\int_0^\pi d\theta\,\sin\theta\,F(\theta)\left\{\cos^{2r+2\ell}\frac{\theta}{2}\,P_{2\ell}\left(\cos\frac{\theta}{2}\right) + \right.$$
$$\left. + \sin^{2r+2\ell}\frac{\theta}{2}\,P_{2\ell}\left(\sin\frac{\theta}{2}\right) - (1+\delta_{r0}\delta_{\ell 0})\right\}. \tag{24}$$

In the braces, we see that the second, third, and fourth terms are obtainable from the first term by replacing θ by $\pi - \theta$, 0, and $\pi/2$. Thus, substituting eq. (24) into eq. (23), we see that we need only to calculate the sum

$$S = \sum_{r,\ell}(4\ell+1)\,\frac{\ell(\ell+\frac{1}{2})\left[(\ell+r-\frac{3}{2})!\right]^2}{r!\,(2\ell+r+\frac{1}{2})!}\cos^{2r+2\ell}\frac{\theta}{2}\,P_{2\ell}\left(\cos\frac{\theta}{2}\right).$$

Putting $r + \ell = i$, we have

$$S = \sum_i\left[(i-\tfrac{3}{2})!\right]^2\cos^{2i}\frac{\theta}{2}\sum_\ell(4\ell+1)\frac{\ell(\ell+\frac{1}{2})}{(i-\ell)!\,(i+\ell+\frac{1}{2})!}\,P_{2\ell}\left(\cos\frac{\theta}{2}\right).$$

The last sum can be done with the help of the relation

$$(2n+1)\cos^{2n}x = \sum_\ell(4\ell+1)\,P_{2\ell}(\cos x)\,\frac{n!\,(n+\frac{1}{2})!}{(n-\ell)!\,(n+\ell+\frac{1}{2})!},$$

leading to

$$S = -2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\sum_i \frac{\Gamma(i+\frac{1}{2})}{i!}\cos^{4i}\frac{\theta}{2} + 2\cos^4\frac{\theta}{2}\sum_i \frac{[\Gamma(i+\frac{1}{2})]^2}{i!\,\Gamma(i+\frac{3}{2})}\cos^{4i}\frac{\theta}{2}$$

$$= -2\Gamma(\tfrac{1}{2})\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\,F(\tfrac{1}{2};\alpha,\alpha;\cos^4\tfrac{\theta}{2}) + 2\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})}\cos^4\frac{\theta}{2}\,F(\tfrac{1}{2};\tfrac{1}{2},\tfrac{3}{2};\cos^4\tfrac{\theta}{2})$$

$$= 2\sqrt{\pi}\left\{(1+\cos\theta)\sin^{-1}\frac{1+\cos\theta}{2} - \frac{1}{2}\frac{\sin^2\theta}{\sqrt{4-(1+\cos\theta)^2}}\right\},$$

and thus to eq. (18b).

APPENDIX II

TABLE USED FOR NUMERICAL COMPUTATIONS

1. Table of Eigenvalues. $\left(A_2' = A_2\sqrt{\frac{\kappa}{kT}}\right.$, $\left.\frac{A_4}{A_2} = R = 0.636\right)$

r \ ℓ	0	1	2	3	4
0	0	0	$-\frac{3}{4}A_2'$	$-\frac{9}{8}A_2'$	$-\frac{7}{4}A_2'(1-\frac{5R}{16})$
1	0	$-\frac{A_2'}{2}$	$-\frac{7A_2'}{8}$	$-\frac{11}{8}A_2'(1-\frac{5R}{22})$	$-2A_2'(1-\frac{115R}{256})$
2	$-\frac{A_2'}{2}$	$-\frac{3A_2'}{4}$	$-\frac{9}{8}A_2'(1-\frac{R}{6})$		

2. Normalization constants:

$$N^2_{r\ell m} = \frac{r!\,(\ell+\frac{1}{2})}{\pi\,(r+\ell+\frac{1}{2})!}\,\frac{(\ell-m)!}{(\ell+m)!}\,.$$

r \ ℓm ($N^2_{r\ell m}$)	11	21	31	41	51
0	$\pi^{-\frac{3}{2}}$	$\frac{2}{9}\pi^{-\frac{3}{2}}$	$\frac{2}{45}\pi^{-\frac{3}{2}}$	$\frac{2^2}{3\cdot 5^2\cdot 7}\pi^{-\frac{3}{2}}$	$\frac{2^4}{3^4\cdot 5^2\cdot 7}\pi^{-\frac{3}{2}}$
1	$\frac{2}{5}\pi^{-\frac{3}{2}}$	$\frac{2^2}{3^2\cdot 7}\pi^{-\frac{3}{2}}$	$\frac{2^2}{3^4\cdot 5}\pi^{-\frac{3}{2}}$	$\frac{2^3}{3\cdot 5^2\cdot 7\cdot 11}\pi^{-\frac{3}{2}}$	
2	$\frac{2^3}{5\cdot 7}\pi^{-\frac{3}{2}}$	$\frac{2^4}{3^4\cdot 7}\pi^{-\frac{3}{2}}$			

3. $L_{r\ell m;\, r'\ell' m'} = \int d\vec{c}\; e^{-c^2} \frac{1}{c_x} \psi^*_{r\ell m} \psi_{r'\ell' m'}$

$r'\ell'm'$ \ $r\ell m$	011	111	031	211	131	051
021	$3\pi^{\frac{3}{2}} N_{011} N_{021}$	0	0	0	0	0
121	$3\pi^{\frac{3}{2}} N_{011} N_{121}$	$\frac{15}{2}\pi^{\frac{3}{2}} N_{111} N_{121}$	0	0	0	0
041	$-10\pi^{\frac{3}{2}} N_{011} N_{041}$	$10\pi^{\frac{3}{2}} N_{111} N_{041}$	$\frac{105}{2}\pi^{\frac{3}{2}} N_{031} N_{041}$	0	0	0
221	$3\pi^{\frac{3}{2}} N_{011} N_{221}$	$\frac{15}{2}\pi^{\frac{3}{2}} N_{111} N_{221}$	0	$\frac{105}{8}\pi^{\frac{3}{2}} N_{211} N_{221}$	0	0
141	$-20\pi^{\frac{3}{2}} N_{011} N_{141}$	$-15\pi^{\frac{3}{2}} N_{111} N_{141}$	$\frac{105}{2}\pi^{\frac{3}{2}} N_{031} N_{141}$	$35\pi^{\frac{3}{2}} N_{211} N_{141}$	$\frac{945}{4}\pi^{\frac{3}{2}} N_{131} N_{141}$	0

4. $L^+_{r2\ell 1,\, r'2\ell' 1} = \int d\vec{c}\; e^{-C^2} \frac{1}{c_x} \psi_{r2\ell 1,\, r'2\ell' 1} \frac{1 + \operatorname{sgn} c_x}{2}$

$\left(T^+_{r2\ell 1,\, r'2\ell' 1} = \lambda_{r'2\ell'} L^+_{r2\ell 1,\, r'2\ell' 1} \right)$

$r'2\ell'1$ \ $r2\ell 1$	021	121	041	221	141
021	$\frac{9\pi}{2} N_{021}^{\;2}$	$\frac{9\pi}{4} N_{021} N_{121}$	$-\frac{15\pi}{2} N_{041} N_{021}$	$\frac{27\pi}{16} N_{221} N_{021}$	$-\frac{3^2 \cdot 5}{2^2}\pi N_{141} N_{021}$
121	$\frac{9\pi}{4} N_{021} N_{121}$	$\frac{9 \cdot 13}{8}\pi N_{121}^{\;2}$	$\frac{15\pi}{4} N_{041} N_{121}$	$\frac{3^5}{3^5}\pi N_{221} N_{121}$	$\frac{3 \cdot 5 \cdot 13}{8}\pi N_{141} N_{121}$
041	$-\frac{15\pi}{2} N_{021} N_{041}$	$\frac{15\pi}{4} N_{121} N_{041}$	$\frac{5^2 \cdot 19}{4}\pi N_{041}^{\;2}$	$\frac{15}{16}\pi N_{221} N_{041}$	$\frac{5^2 \cdot 19}{2^3}\pi N_{141} N_{041}$
221	$\frac{27\pi}{16} N_{021} N_{221}$	$\frac{3^5}{2^5}\pi N_{121} N_{221}$	$\frac{15}{16}\pi N_{041} N_{221}$	$\frac{3^4 \cdot 7^2}{2^7}\pi N_{221}^{\;2}$	$\frac{3 \cdot 5^2 \cdot 7}{2^5}\pi N_{141} N_{221}$
141	$-\frac{45\pi}{4} N_{021} N_{141}$	$-\frac{3 \cdot 5 \cdot 13}{8}\pi N_{121} N_{141}$	$\frac{5^2 \cdot 19}{8}\pi N_{041} N_{141}$	$\frac{3 \cdot 5^2 \cdot 7}{2^5}\pi N_{221} N_{141}$	$\frac{3 \cdot 5^2 \cdot 7 \cdot 19}{2^4}\pi N_{141}^{\;2}$

4. $$S_{r2l1} = \int d\vec{c}\, e^{-c^2} \psi^{*}_{r2l1}\, c_z\, \frac{1+\operatorname{sign} c_x}{2}$$

$$= (-)^{l}\, \frac{i\sqrt{\pi}}{2}\, N_{r2l1}\, \frac{l(l+\frac{1}{2})(l+r-\frac{3}{2})!}{r!} .$$

021	121	041	221	141
$-\frac{3i\pi}{4} N_{021}$	$-\frac{3i\pi}{8} N_{121}$	$\frac{5i\pi}{4} N_{041}$	$-\frac{9i\pi}{2^5} N_{221}$	$\frac{15i\pi}{8} N_{141}$