ENGINEERING RESEARCH INSTITUTE
THE UNIVERSITY OF MICHIGAN
ANN ARBOR

Technical Report

ON THE BEHAVIOR OF A GAS NEAR A WALL; A PROBLEM OF KRAMERS'

C. S. Wang Chang
G. E. Uhlenbeck

Project 2457

OFFICE OF NAVAL RESEARCH, U. S. NAVY DEPARTMENT
CONTRACT NO. Nonr-1224-(15)

August 1956

TABLE OF CONTENTS

	Page
ABSTRACT	iii
OBJECTIVE	iii
I. INTRODUCTION	1
II. THE COUETTE FLOW NEAR A FIXED PLATE	2
III. SUCCESSIVE APPROXIMATIONS AND NUMERICAL RESULTS	13
A. FIRST APPROXIMATION	13
B. SECOND APPROXIMATION	13
C. THIRD APPROXIMATION	15
D. FOURTH APPROXIMATION	16
IV. TEMPERATURE DISTRIBUTION NEAR A FIXED PLATE	17
A. FIRST APPROXIMATION	22
B. SECOND APPROXIMATION	23
C. THIRD APPROXIMATION	25
V. CONCLUDING REMARKS	26
APPENDIX I	27
APPENDIX II	28
APPENDIX III	30
APPENDIX IV	31
REFERENCES	32

ABSTRACT

The deviations from the originally linear velocity or temperature distribution due to the introduction of an infinite wall in the plane $x = 0$ and with a velocity or temperature equal to that of the gas at $x = 0$ before its introduction have been studied by the same method employed in our previous reports. Formal solutions involving infinite determinants have been obtained. The general features of the solutions agree with the prediction of Kramers. Some successive approximation calculations were carried out. The first approximation results for both problems are in agreement with the classical results of Maxwell and others.

OBJECTIVE

A theoretical study of the behavior of a gas near a wall.

I. INTRODUCTION

The problem of the behavior of a gas near a solid wall is of long standing. The temperature and velocity jumps in the immediate vicinity of a wall were first observed by Knudt and Warburg[1] in their experimental investigation of the heat conduction and friction in rarefied gases. Approximate expressions for the temperature and velocity jumps have also been derived by several authors.[2] In two of our previous reports[3] we derived these expressions for the cases where there are two solid parallel plates and where, in the case of heat conduction, the ratio of the temperature difference between the plates to the mean temperature of the plates and, in the case of Couette flow, the ratio of the velocity difference between the plates to the mean molecular velocity are small. Kramers[4] in 1949 attempted to derive a more exact expression of the velocity jump by studying the modification of the distribution function due to the presence of a stationary plate in a gas moving in the z-direction with a constant velocity gradient dw/dx. Though he was unable to give a complete solution, he did suggest that by writing for the distribution function an "Ansatz" of the form

$$f = f_o \left\{ 1 - \frac{8}{3} \lambda \frac{dw}{dx} C_x C_z + \epsilon C_z + X \right\} ,$$

where f_o is the Maxwell distribution with a mass velocity in the z-direction given by $x(dw/dx)$, λ is a "free path," and ϵ a constant to be determined, and by a superposition of solutions of the type

$$X = \chi(\vec{c})e^{-\gamma x}$$

with constant γ, perhaps both the Boltzmann equation and the boundary conditions could be satisfied. We will confirm this expectation completely. Most recently, Welander[5] also derived the temperature distribution, the velocity distribution, and the temperature and velocity jumps in the neighborhood of a plate. Welander made the main assumption that the rate of change of the molecular distribution function due to collision is proportional to the deviation from the Maxwellian state and took for the proportionality factor the expression given by the Stokes-Navier approximation for Maxwell molecules. For perfect accommodation the following expressions for the temperature and velocity jumps were found:

$$(\Delta T)_w = \frac{15}{8}(1.152)\,\bar{l}\,\frac{dT}{dx}$$

$$(\Delta V)_w = (1.21)\,\bar{l}\,\frac{dw}{dx} \quad ,$$

where dT/dx and dw/dx are the temperature and velocity gradients, respectively, just beyond the transition region (i.e., a few mean free paths from the wall), and $\bar{l}$ is the mean free path related to the viscosity coefficient μ by the relation

$$l = \frac{2\mu}{nm\sqrt{\frac{8kT}{\pi m}}} \quad .$$

The numbers in the parenthesis are the correction factors for the expressions derived from the simple theory.[2]

The problems considered in Reference 3 are closely connected with Kramers' problem. The treatment presented there, with proper modifications, can be adapted to solve Kramers' problem. In fact, the velocity jump Kramers was seeking can be deduced from the results of those reports. However, the apparent differences between the problems, such as the nonexistence of parameters like $\Delta T/T$ and d/λ, where d is the distance between the plates, seem to warrant a separate treatment. For simplicity, we will limit ourselves to Maxwell molecules, i.e., molecules interacting with the force law, κr^{-5}, κ being the force constant.

II. THE COUETTE FLOW NEAR A FIXED PLATE

We consider an infinite space where the gas is flowing in the z-direction with a mass velocity $\bar{c}_z(x)$ proportional to x. A plate, with accommodation coefficient α, is introduced in the plane x = 0. The problem is to find in the stationary state the distribution function of the gas in the upper half infinite space (x > 0). The Boltzmann equation is*

$$c_x \frac{\partial f}{\partial x} = nJ(ff_1) \qquad (1)$$

where f is the distribution function, $\vec{c}$ is the molecular velocity, and J is the collision operator. We write again

$$f = f_0(1 + h) \ . \qquad (2)$$

*We will always use dimensionless velocity variables by measuring all velocities in units $(2kT/m)^{1/2}$.

Differing from Reference 3, we take for the zeroth-order distribution the Maxwellian distribution with a mass flow $\overline{c}_z = x(dw/dx)$ where dw/dx is the velocity gradient in the x-direction, i.e., we write

$$f_o = n \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-C^2} , \qquad (3)$$

where

$$C^2 = c_x^2 + c_y^2 + (c_z - x \frac{dw}{dx})^2$$

or $\vec{C} = \vec{c} - (\vec{\overline{c}})$. It is assumed that dw/dx is a constant, and that $\overline{l}(dw/dx)$ is small compared to the mean molecular speed $[\sim(kT/m)^{1/2}]$. Substituting Equations 2 and 3 into Equation 1, and dropping the term $h(\partial f_o/\partial x)$, one finds

$$2C_xC_z\frac{dw}{dx} + C_x \frac{\partial h}{\partial x} = nJ(h) , \qquad (4)$$

where now $\vec{C}$ may be considered to be identical with $\vec{c}$. Equation 4 is the integro-differential equation to be solved. The boundary conditions can be written as follows:

1. $x = \infty$ $\qquad h = -\frac{8}{3} \lambda \frac{dw}{dx} C_xC_z + \epsilon C_z$ (5a)

2. $x = 0$ and $C_x > 0$ $\qquad h(0,C_x, C_y, C_z) = (1-\alpha)h(0,-C_x,C_y,C_z)$. (5b)

The first term on the right-hand side of Equation 5a is the viscous effect as calculated by Chapman,[6] where λ is the mean free path defined by

$$\lambda = (nA_2\sqrt{\kappa/kT})^{-1} ,$$

κ being the force constant and

$$A_2 = 2\pi \int d\theta \sin^3\theta \, F(\theta)$$

and $F(\theta) = \sqrt{m/2\kappa}\, gI(g,\theta)$. The mean free path λ is related to the mean free path $\overline{l}$ defined before by the relation

$$\lambda = \frac{3}{2\sqrt{\pi}} \overline{l} . \qquad (6)$$

The second term is the effect at infinity due to the presence of the plate at x = 0. A consideration of the symmetry of the problem indicates that this effect should be proportional to C_z. The proportionality factor ε is independent of the velocity components and it is expected that it will have the form λ(dw/dx), except for a multiplicative constant which is to be determined. In Equation 5b the quantity α is the accommodation coefficient; it means that at the plate a fraction α of the molecules striking the plate will be reemitted with the Maxwellian distribution in equilibrium with the plate, while the fraction $(1 - \alpha)$ will be reflected specularly. Equation 5b is the mathematical statement of this fact. The problem is, thus, completely defined. To solve it, we first make a change of the dependent variable from $h(x, \vec{C})$ to $Y(x, \vec{C})$ where

$$h(x, \vec{C}) = Y(x, \vec{C}) - \frac{8}{3} \lambda \frac{dw}{dx} C_x C_z + \epsilon C_z \tag{7}$$

so as to make the integro-differential equation homogeneous, namely,

$$\frac{\partial Y}{\partial x} = \frac{n}{C_x} J(Y) . \tag{8}$$

The boundary conditions can now be written in the form

$$x = \infty \qquad Y = 0 \tag{9a}$$

$$x = 0 \qquad Y(0,\vec{C}) = [(1-\alpha)Y(0,-C_x, C_y, C_z) + (2-\alpha) \frac{8}{3} \lambda \frac{dw}{dx} C_x C_z - \alpha \epsilon C_z] \frac{1 + \text{sign } C_x}{2} + Y(0, \vec{C}) \frac{1 - \text{sign } C_x}{2} . \tag{9b}$$

It is to be noted that Equation 9b for $C_x < 0$ is just an identity; only for $C_x > 0$ does Equation 9b give a condition for $Y(0, \vec{C})$.

A natural way of solving Equation 8 with the boundary conditions Equations 9a and 9b is to use the Laplace transform. Introducing the Laplace transform $Z(s, \vec{C})$ of the function $Y(x, \vec{C})$ defined by

$$Z(s, \vec{C}) = \int_0^\infty dx e^{-sx} Y(x, \vec{C}) , \tag{10}$$

we have

$$sZ - Y(0, \vec{C}) = \frac{n}{C_x} J(Z) . \tag{11}$$

With $Y(0, \vec{C})$ assumed to be known, we will solve Equation 10 in the same way as the problems in References 3 were solved, namely, we will expand $Y(x, \vec{C})$, and consequently $Z(x, \vec{C})$ in terms of $\psi_{r,\ell,m}(\vec{C})$, the eigenfunctions of the collision operator J. As discussed in Reference 3, only terms with $m = \pm 1$ will enter, hence we write

$$Y = \sum_{r,\ell} [a_{r,\ell,1}(x)\psi_{r,\ell,1} + a_{r,\ell,-1}\psi_{r,\ell,-1}] \tag{11a}$$

$$Z = \sum_{r,\ell} [\alpha_{r,\ell,1}(s)\psi_{r,\ell,1} + \alpha_{r,\ell,-1}(s)\psi_{r,\ell,-1}] \ . \qquad (11b)$$

Further, since $\psi_{r,\ell,-m}$ is the complex conjugate of $\psi_{r,\ell,m}$, the reality condition of h, and hence that of the functions Y and Z, requires that $a_{r,\ell,-1} = a^*_{r,\ell,1}$. Thus we need only to solve for $a_{r,\ell,1}$ or $\alpha_{r,\ell,1}$. In terms of the a's, Equation 8 becomes

$$\frac{da_{r,2\ell,1}}{dx} = n \sum_{r'\ell'} R_{r,2\ell,1;\ r';2\ell'+1,1}\ a_{r';2\ell'+1,1}(x) \qquad (12a)$$

$$\frac{da_{r,2\ell+1,1}}{dx} = n \sum_{r';\ell'} R_{r,2\ell+1,1;\ r';2\ell';1}\ a_{r';2\ell';1}(x) \qquad (12b)$$

and Equation 10 becomes

$$s\alpha_{r,\ell,1} - a_{r,\ell,1}(0) = n \sum_{r';\ell'} R_{r,\ell,1;\ r';\ell';1}\ \alpha_{r';\ell';1} \ , \qquad (13)$$

where the $a_{r,\ell,1}(0)$'s will be assumed to be known for the time being, and

$$R_{r,\ell,1;\ r';\ell';1} = [\frac{1}{C_x}\psi^*_{r,\ell,1},\ \psi_{r';\ell';1}]$$

$$= \lambda_{r'\ell'} \int d\vec{C} e^{-C^2} \frac{1}{C_x} \psi^*_{r,\ell,1}\ \psi_{r,\ell';1} \ .$$

The expression $\lambda_{r';\ell'}$ is the eigenvalue of the collision operator J, and it has the dimension of an area. As before, we separate Equation 13 into two sets, a set of even ℓ and a set of odd ℓ,

$$s\alpha_{r,2\ell,1} - a_{r,2\ell,1}(0) = n \sum_{r';\ell'} R_{r,2\ell,1;r';2\ell'+1,1}\ \alpha_{r';2\ell'+1,1} \qquad (13a)$$

$$s\alpha_{r,2\ell+1,1} - a_{r,2\ell+1,1}(0) = n \sum_{r';\ell'} R_{r,2\ell+1,1;\ r';2\ell';1}\ \alpha_{r';2\ell';1} \ . \qquad (13b)$$

Eliminating $\alpha_{r,2\ell+1,1}$, we find

$$s^2\alpha_{r,2\ell,1} - n^2 \sum_{r',\ell'} \sum_{r'',\ell''} R_{r,2\ell,1;\, r',2\ell'+1,1}\, R_{r',2\ell'+1,1;\, r'',2\ell'',1}\, \alpha_{r'',2\ell'',1}$$

$$= s a_{r,2\ell,1}(0) + n \sum_{r',\ell'} R_{r,2\ell,1;\, r',2\ell'+1,1}\, a_{r',2\ell'+1,1}(0) , \qquad (14)$$

a set of linear inhomogeneous equations for all $\alpha_{r,2\ell,1}$. Since

$$R_{0,2,1;\, r,2\ell+1,1} = 0 ,$$

a consequence of the conservation of linear momentum, and since for Maxwell molecules

$$R_{r,2\ell+1,1;\, r'',2\ell'',1} = 0 ,$$

unless $r = 0$ and $\ell = 0$, but $R_{r,2\ell,1;0,1,1} = 0$ because $\lambda_{01} = 0$, the set of Equations 14, can be written as

$$s\alpha_{0,2,1} = a_{0,2,1}(0) \qquad (14a)$$

and

$$\sum_{r'',\ell''} \left(s^2\delta_{rr''}\delta_{\ell\ell''} - n^2 \sum_{r',\ell'} R_{r,2\ell,1;r',2\ell'+1,1}\, R_{r',2\ell'+1,1;r'',2\ell'',1}\right) \alpha_{r'',2\ell'',1}$$

$$= s a_{r,2\ell,1}(0) + n \sum R_{r,2\ell,1;r',2\ell'+1,1}\, a_{r',2\ell'+1,1}(0) \quad (r,\ell),\ (r'',\ell'') \neq (0,1)$$

$$(14b)$$

The solutions of Equation 14b can be easily written down. We write,

$$\Delta = \left| s^2\delta_{rr''}\delta_{\ell\ell''} - n^2 \sum_{r',\ell'} R_{r,2\ell,1;r',2\ell'+1,1}\, R_{r',2\ell'+1,1;r'',2\ell'',1} \right|$$

$$= \prod_i (s - p_i)(s + p_i) .$$

The p_i's are the roots of the equation $\Delta(s) = 0$; they are of the form of a constant divided by the mean free path. Let $D_{r,2\ell,1}$ be the determinant obtained from Δ by replacing the $(r,2\ell,1)$ column by the inhomogeneous part of Equation 14b. By splitting in partial fractions, one then can write, except for $r = 0$ and $\ell = 1$,

$$\alpha_{r,2\ell,1} = \sum_i \frac{D_{r,2\ell,1}(s = p_i)}{\left(\frac{d\Delta}{ds}\right)_{s = p_i}} \frac{1}{s - p_i} + \sum_i \frac{D_{r,2\ell,1}(s = -p_i)}{\left(\frac{d\Delta}{ds}\right)_{s = -p_i}} \frac{1}{s + p_i} , \qquad (15a)$$

where the determinants $D_{r,2\ell,1}(s = \pm p_i)$ still contain the $a_{r,\ell,1}(0)$'s. Since the $D_{r,2\ell,1}$ depend linearly on the inhomogeneous part of Equation 14b, only one of the sets $D_{r,2\ell,1}$ $(s = \pm p_i)$ is independent. Choosing this set to be $D_{1,2,1}$, the others are given by*

$$D_{r,2\ell,1}(s = \pm p_i) = \frac{\Delta^{(i)}_{1,2,1;r,2\ell,1}}{\Delta^{(i)}_{1,2,1;1,2,1}} D_{1,2,1}(s = \pm p_i) ,$$

where $\Delta_{1,2,1;r,2\ell,1}$ is the algebraic complement of the $(1,2,1;r,2\ell,1)$ element of the determinant Δ. Substituting the expression for $\alpha_{r,2\ell,1}$, Equation 15a, into Equation 13b one finds

$$\alpha_{r,2\ell+1,1}(s) = \frac{a_{r,2\ell+1,1}(0)}{s} + n \sum_{r';\ell'} R_{r,2\ell+1,1;r';2\ell';1} \times$$

$$\left\{ \sum_i \frac{D_{r';2\ell';1}(s = p_i)}{\left(\frac{d\Delta}{ds}\right)_{s = p_i}} \frac{1}{s(s - p_i)} + \sum_i \frac{D_{r';2\ell';1}(s = -p_i)}{\left(\frac{d\Delta}{ds}\right)_{s = -p_i}} \frac{1}{s(s + p_i)} \right\} . \qquad (15b)$$

By taking the inverse Laplace transform one obtains for the solutions of Equations 12,

$$a_{o,2,1}(x) = a_{o,2,1}(0) = \text{const.} \qquad (16a)$$

$$a_{r,2\ell,1}(x) = \sum_i \frac{D_{r,2\ell,1}(s = p_i)}{\left(\frac{d\Delta}{ds}\right)_{s = p_i}} e^{p_i x} + \sum_i \frac{D_{r,2\ell,1}(s = -p_i)}{\left(\frac{d\Delta}{ds}\right)_{s = -p_i}} e^{-p_i x} \qquad (16b)$$

$$a_{r,2\ell+1,1}(x) = a_{r,2\ell+1,1}(0) + n \sum_{r';\ell'} R_{r,2\ell+1,1;r';2\ell';1} \left\{ \sum_i \frac{1}{p_i} \frac{D_{r,2\ell,1}(s=p_i)}{\left(\frac{d\Delta}{ds}\right)_{s = p_i}} (e^{p_i x} - 1) \right.$$

$$\left. - \sum \frac{1}{p_i} \frac{D_{r,2\ell,1}(s = -p_i)}{\left(\frac{d\Delta}{ds}\right)_{s = -p_i}} (e^{-p_i x} - 1) \right\} . \qquad (16c)$$

*See Appendix I.

Equations 16 are the general solutions of Equation 8. So far, we have not really taken into account the boundary conditions. The conditions, Equations 9a and 9b, in terms of the a's are, for all r and ℓ,

$$a_{r,\ell,1}(\infty) = 0 \tag{17a}$$

$$a_{r,\ell,1}(0) = \left\{(1-\alpha) - (-)^{\ell}\right\}\sum_{r'\ell'} (-)^{\ell'-1} M_{r,\ell,1;r',\ell',1}\, a_{r',\ell',1}(0)$$

$$+ \frac{4(2-\alpha)}{9iN_{021}} M_{r,\ell,1;0,2,1}\, \lambda \frac{dw}{dx} - \frac{\alpha\epsilon}{2iN_{011}} M_{r,\ell,1;0,1,1}\,, \tag{17b}$$

where the N's are the normalization constants for the ψ's and

$$M_{r,\ell,m;r',\ell',m} \equiv \int d\vec{C} e^{-C^2} \frac{1 + \text{sign } C_x}{2} \psi^*_{r,\ell,m}\, \psi_{r',\ell',m} \,.$$

The elements $M_{r,\ell,m;r',\ell',m}$ have the following properties:*

1. $M_{r,\ell,m;r',\ell',m} = M_{r',\ell',m;r,\ell,m}$ (18a)

2. $M_{r,2\ell,m;r',2\ell',m} = 1/2\, \delta_{rr'}\delta_{\ell\ell'}$ (18b)

   $M_{r,2\ell+1,m;r',2\ell'+1,m} = 1/2\, \delta_{rr'}\delta_{\ell\ell'}$ (18c)

3. $\sum_{r',\ell'} M_{r,2\ell,m;r',2\ell'+1,m}\, M_{r',2\ell'+1,m;r'',2\ell'',m} = 1/4\, \delta_{rr''}\delta_{\ell\ell''}$ (18d)

   $\sum_{r',\ell'} M_{r,2\ell+1,m;r',2\ell',m}\, M_{r',2\ell',m;r'',2\ell''+1,m} = 1/4\, \delta_{rr''}\delta_{\ell\ell''}$ (18e)

The conditions at infinity require that

$$a_{0,2,1}(x) = 0 \tag{19a}$$

$$D_{r,2\ell,1}(s = p_i) = 0, \quad \text{or} \quad D_{1,2,1}(s = p_i) = 0 \quad \text{for all } i, \tag{19b}$$

$$a_{r,2\ell+1,1}(0) + n \sum_{r',\ell'} R_{r,2\ell+1,1;r',2\ell',1} \sum_i \frac{1}{p_i} \frac{D_{r',2\ell',1}(s = -p_i)}{\left(\frac{d\Delta}{ds}\right)_{s=-p_i}} = 0, \tag{19c}$$

*See Appendix II.

where in arriving at Equation 19c, we have put Equation 19b into Equation 16c. Each of the Equations 19 is a linear relationship between the a(0)'s. However, these are not all independent because of the differential equation 12b. Let us consider the case where we take N eigenfunctions, among which there are N_e even ones (ℓ = even) and N_O odd ones, such that $N_e + N_O = N$. Since as in Reference 3 we always will cut off the series eigenfunctions for various values of $M = 2r + \ell$ with $\ell \leq M - 1$, one easily sees that $N_O \geq N_e$. It can be shown that Equations 19b follow automatically from Equations 19c, 11b, 18a, and 18b. Thus the condition at infinity furnishes us with $N_O + 1$ linear homogeneous relations for the a(o)'s.

The conditions at x = 0, i.e., Equation 17b, can be separated into two sets, namely,

$$a_{r,2\ell,1}(0) + \frac{2\alpha}{2-\alpha} \sum_{r',\ell'} M_{r,2\ell,1;r',2\ell'+1,1}\, a_{r',2\ell'+1,1}(0)$$

$$= \frac{4}{9iN_{021}} \lambda \frac{dw}{dx} \delta_{ro}\delta_{\ell 1} - \frac{2\alpha}{2-\alpha} \frac{\epsilon}{2iN_{011}} M_{r,2\ell,1;0,1,1} \qquad (20a)$$

$$a_{r,2\ell+1,1}(0) + \frac{2(2-\alpha)}{\alpha} \sum_{r',\ell'} M_{r,2\ell+1,1;r',2\ell',1}\, a_{r',2\ell',1}(0)$$

$$= \frac{8(2-\alpha)}{9i\alpha N_{021}} M_{r,2\ell+1,1;0,2,1}\, \lambda \frac{dw}{dx} - \frac{\epsilon}{2iN_{011}} \delta_{ro}\delta_{\ell o} \; . \qquad (20b)$$

For N = finite, Equations (20a) and Equations (20b), N_e and N_O in number, respectively, are independent. Thus, for an approximate solution where we limit ourselves to a finite number of eigenfunctions, we have $2N_O + N_e + 1$ linear inhomogeneous relations for the N + 1 (N $a_{r,\ell,1}$'s and ϵ) unknowns. If we choose to have the condition at infinity (Equations 19a and 19c) and Equations 20a satisfied, then the Equations 20b can not be satisfied. Presumably by taking N bigger, the Equations 20b will be better satisfied. The last statement is confirmed by observing that for the exact solution, where $N \to \infty$, Equations 20a and 20b follow from each other. For instance, if we multiply Equations 20a by $M_{r',2\ell'+1,1;r,2\ell,1}$ and sum over all r and ℓ, by using Equations 18d, one finds

$$\sum_{r,\ell} M_{r',2\ell'+1,1;r,2\ell,1}(0) + \frac{\alpha}{2(2-\alpha)} a_{r',2\ell'+1,1}(0) =$$

$$= \frac{4}{9iN_{021}} \lambda \frac{dw}{dx} M_{r',2\ell'+1,1;0,2,1} - \frac{\alpha}{2(2-\alpha)} \frac{\epsilon}{2iN_{011}} \delta_{r'o}\delta_{\ell' o} \; ,$$

which is $\alpha/2(2-\alpha)$ times Equations 20b.

Making use of Equations 19, one finds

$$a_{r,2\ell,1}(x) = \sum_i \frac{D_{r,2\ell,1}(s = -p_i)}{\left(\frac{d\Delta}{ds}\right)_{s=-p_i}} e^{-p_i x} \qquad (21a)$$

$$a_{r,2\ell+1,1}(x) = -n \sum_{r'\ell'} R_{r,2\ell+1,1;r',2\ell',1} \sum_i \frac{1}{p_i} \frac{D_{r',2\ell',1}(s = -p_i)}{\left(\frac{d\Delta}{ds}\right)_{s=-p_i}} e^{-p_i x}. \qquad (21b)$$

For the actual solution for the constants ϵ and the $a_{r,\ell,1}(0)$'s, it is most convenient to call

$$b^{(i)}_{r,2\ell,1} = \frac{D_{r,2\ell,1}(s = -p_i)}{\left(\frac{d\Delta}{ds}\right)_{s=-p_i}} \quad . \qquad (22)$$

By the relationship between $D_{r,2\ell,1}$ and $D_{1,2,1}$, it follows that

$$b^{(i)}_{r,2\ell,1} = \frac{\Delta^{(i)}_{1,2,1;r,2\ell,1}}{\Delta^{(i)}_{1,2,1;1,2,1}} b^{(i)}_{1,2,1} \quad , \qquad (23)$$

where $b^{(i)}_{1,2,1}$ is the set of independent variables to be solved. Eliminating the odd a's between Equations 21b and Equations 20a, using 21a, 22, and 23, one arrives at the following set of equations for the determination of ϵ and $b^{(i)}_{1,2,1}$ for all i,

$$\frac{M_{r,2\ell,1;0,1,1}}{2iN_{011}} \epsilon + \sum_i Q^{(i)}_{r,2\ell,1} b^{(i)}_{1,2,1} = \frac{2-\alpha}{2\alpha} \frac{4}{9iN_{021}} \lambda \frac{dw}{dx} \delta_{r0}\delta_{\ell 1} \quad , \qquad (24)$$

with

$$Q^{(i)}_{0,2,1} = -n \sum_{r',\ell'} \sum_{r'',\ell''} M_{0,2,1;r',2\ell'+1,1} R_{r',2\ell'+1,1;r'',2\ell'',1} \left(\frac{1}{p} \frac{\Delta_{1,2,1;r'',2\ell'',1}}{\Delta_{1,2,1;1,2,1}}\right)_i \qquad (25a)$$

$$Q^{(i)}_{r,2\ell,1} = -n \sum_{r'\ell'} \sum_{r'',\ell''} M_{r,2\ell,1;r',2\ell'+1,1} R_{r',2\ell'+1,1;r'',2\ell'',1} \left(\frac{1}{p} \frac{\Delta_{1,2,1;r'',2\ell'',1}}{\Delta_{1,2,1;1,2,1}}\right)_i$$

$$+ \frac{2-\alpha}{2\alpha} \left(\frac{\Delta_{1,2,1;r,2\ell,1}}{\Delta_{1,2,1;1,2,1}}\right)_i \qquad r \neq 0 \quad \text{and} \quad \ell \neq 1 \; . \qquad (25b)$$

Equation 24 is a set of equations N_e in number for the unknowns ϵ and $b^{(i)}_{1,2,1}$, where i goes from 1 to $N_e - 1$. It is seen from Equation 24 that both ϵ and all the b's are proportional to the parameter $\lambda(dw/dx)$.

Formally our problem is solved. The deviation from the Maxwellian distribution with a streaming velocity in the z-direction equal to x(dw/dx) is

$$h = -\frac{8}{3}\lambda \frac{dw}{dx} C_x C_z + \epsilon C_z + \sum_{r,\ell} (a_{r,\ell,1}\psi_{r,\ell,1} + a_{r,\ell,-1}\psi_{r,\ell,-1}) , \qquad (26)$$

where all the $a_{r,\ell,\pm 1}$'s are linear combinations of the form e^{-px_i}. This is in agreement with the contention of Kramers as mentioned in the introduction. For distances much larger than the free path, $x \gg \lambda$, h is of the form

$$h = -\frac{8}{3}\lambda \frac{dw}{dx} C_x C_z + \epsilon C_z .$$

The general expression for the velocity in the z-direction is

$$\bar{c}_z = \frac{1}{n}\int d\vec{c}\; c_z f$$

$$= x\frac{dw}{dx} + \frac{\epsilon}{2} - \frac{a_{0,1,1}}{i\pi^{3/2} N_{011}} ; \qquad (27)$$

$a_{0,1,1}$ is purely imaginary, as can be easily deduced from the symmetry property. Since $a_{0,1,1}$ is a linear combination of negative exponential functions in x, the velocity gradient is therefore no longer constant. However, at points more than a few free paths from the plate (i.e., outside the transition region), the velocity gradient will still be a constant, and, in fact, the same as if the plate were not present. The velocity at a large distance from the wall is

$$(\bar{c}_z)_{x \gg \lambda} = x\frac{dw}{dx} + \frac{\epsilon}{2} , \qquad (28)$$

which can be written in Kramers' form, by putting $\epsilon = 2k\bar{\ell}(dw/dx)$, so that

$$(\bar{c}_z)_{x \gg \lambda} = (x + k\bar{\ell})\frac{dw}{dx} ,$$

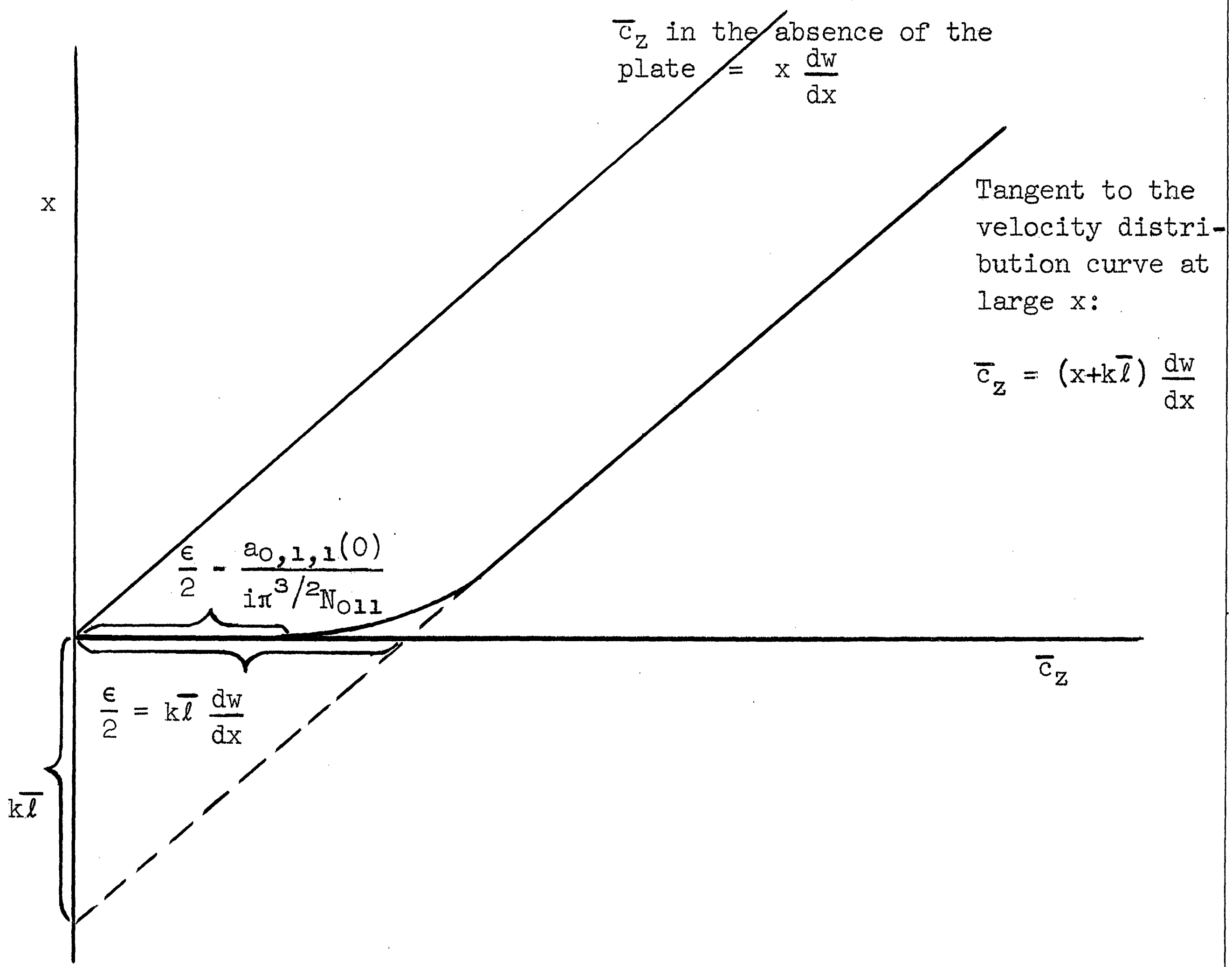
where k is called the slip coefficient, a numerical constant to be determined, and $\bar{\ell}$ is the free path defined in the introduction. Thus the velocity is still a linear function of x with the same gradient, but the plane of zero velocity is at $x = -k\bar{\ell}$ instead of at x = 0. The velocity at the wall, which is equal to the velocity jump defined in Reference 3, is

$$\Delta w = (\overline{c}_z)_{x=0} = \frac{\epsilon}{2} - \frac{a_{0,1,1}(0)}{i\pi^{3/2}N_{011}} \quad . \tag{29}$$

The velocity jump, $(\Delta w)_W$, used by Welander is defined as the difference between the velocity at the wall obtained from a linear extrapolation of the velocity distribution curve beyond the transition region and the velocity of the wall, i.e.,

$$(\Delta w)_W = \frac{\epsilon}{2} \quad . \tag{30}$$

The figure below gives a sketch of the velocity distribution.



III. SUCCESSIVE APPROXIMATIONS AND NUMERICAL RESULTS

In the previous section, the formal exact solution of the problem is given. The actual solution consists first in finding the roots (p_i) of the infinite determinant Δ, and the solution of the set of infinite linear in-homogeneous equations, Equation 21, for ϵ and b. We will in this section find some approximate solutions by using the same cut-off process as employed in Reference 3.

A. FIRST APPROXIMATION

We take into account $\psi_{0,1,1}, \psi_{1,1,1}, \psi_{0,2,1}$. The solutions are clearly

$$a_{0,2,1} = 0$$

$$a_{0,1,1} = a_{1,1,1} = 0 \, .$$

The set of Equations 15 becomes one equation for ϵ, namely,

$$\frac{M_{0,2,1;0,1,1}}{2iN_{011}} \epsilon = \frac{2-\alpha}{2\alpha} \frac{4}{9iN_{021}} \lambda \frac{dw}{dx} ,$$

giving

$$\epsilon = \frac{2-\alpha}{\alpha} \frac{4\sqrt{\pi}}{3} \lambda \frac{dw}{dx} = \frac{2-\alpha}{\alpha} 2\overline{l} \frac{dw}{dx} .$$

The average velocity in the z-direction remains linear, and is given by

$$\overline{c}_z = (x + \frac{2-\alpha}{\alpha} \overline{l}) \frac{dw}{dx} .$$

Hence the slip coefficient is $(2-\alpha)/\alpha$ in agreement with the results of the older theory; $\overline{c}_z$ is linear in x throughout the whole space, $\Delta w = (\Delta w)_W = 2-\alpha/\alpha \, \overline{l} \, dw/dx$.

B. SECOND APPROXIMATION

For this approximation we use five eigenfunctions: $\psi_{0,2,1}, \psi_{1,2,1}; \psi_{0,1,1}, \psi_{1,1,1}, \psi_{0,3,1}$. The only nonzero value of p, as solved in Reference 3 is

$$p = \pm \sqrt{\frac{5}{8}} \frac{1}{\lambda} .$$

The constants to be determined are ϵ and $b_{1,2,1}$, given by the two equations

$$\frac{M_{0,2,1;0,1,1}}{2iN_{011}}\epsilon + Q_{0,2,1}b_{1,2,1} = \frac{2-\alpha}{2\alpha}\frac{4}{9iN_{021}}\lambda\frac{dw}{dx} ,$$

$$\frac{M_{1,2,1;0,1,1}}{2iN_{011}}\epsilon + Q_{1,2,1}b_{1,2,1} = 0 ,$$

where

$$Q_{0,2,1} = -\frac{n}{p}\sum M_{0,2,1;r;2\ell'+1,1}R_{r;2\ell'+1,1;1,2,1} = \frac{7}{4\sqrt{35\pi}}$$

$$Q_{1,2,1} = -\frac{n}{p}\sum M_{1,2,1;r;2\ell'+1,1}R_{r;2\ell'+1,1;1,2,1} + \frac{2-\alpha}{2\alpha} = \frac{13}{4\sqrt{10\pi}} + \frac{2-\alpha}{2\alpha} .$$

The solutions are

$$\epsilon = \frac{2-\alpha}{\alpha}\frac{4\sqrt{\pi}}{3}\lambda\frac{dw}{dx}\frac{\frac{2-\alpha}{2\alpha}+\frac{13}{4\sqrt{10\pi}}}{\frac{2-\alpha}{2\alpha}+\frac{12}{4\sqrt{10\pi}}} ,$$

where we have written in such a way that the last factor is the correction to the simple theory, and

$$b_{1,2,1} = -\frac{2-\alpha}{2\alpha}\frac{4}{9\sqrt{14}iN_{021}}\lambda\frac{dw}{dx}\frac{1}{\frac{2-\alpha}{2\alpha}+\frac{3}{\sqrt{10\pi}}} .$$

Using Equations 27 and 21b, the velocity distribution is given by

$$\overline{c}_z = \left(x + \frac{2-\alpha}{\alpha}\frac{2\sqrt{\pi}}{3}\lambda\frac{\frac{2-\alpha}{2\alpha}+\frac{13}{4\sqrt{10\pi}}}{\frac{2-\alpha}{2\alpha}+\frac{12}{4\sqrt{10\pi}}}\right)\frac{dw}{dx} - \frac{2-\alpha}{2\alpha}\frac{2}{3\sqrt{10}}\lambda\frac{dw}{dx}\frac{1}{\frac{2-\alpha}{2\alpha}+\frac{3}{\sqrt{10\pi}}}e^{-\sqrt{\frac{5}{8}}\frac{x}{\lambda}} .$$

The slope of the velocity at the wall is bigger than that at infinity by a factor

$$1 + \frac{2-\alpha}{2\alpha}\frac{1}{6}\frac{1}{\frac{2-\alpha}{2\alpha}+\frac{3}{\sqrt{10\pi}}} ,$$

which is independent of the density and is equal to 1.08 for $\alpha = 1$. The velocity at large x is

$$(\bar{c}_z)_{x \gg \lambda} = \left(x + \frac{2-\alpha}{\alpha} \bar{\ell} \frac{\frac{2-\alpha}{2\alpha} + \frac{13}{4\sqrt{10\pi}}}{\frac{2-\alpha}{2\alpha} + \frac{12}{4\sqrt{10\pi}}} \right) \frac{dw}{dx} ,$$

giving for the "velocity jump"

$$(\Delta w)_W = \frac{2-\alpha}{\alpha} \bar{\ell} \frac{\frac{2-\alpha}{2\alpha} + \frac{13}{4\sqrt{10\pi}}}{\frac{2-\alpha}{2\alpha} + \frac{12}{4\sqrt{10\pi}}} \frac{dw}{dx} .$$

For $\alpha = 1$, this gives the value 1.043 for the slip coefficient. The velocity at the wall is smaller than $(\Delta w)_W$; it is

$$(\bar{c}_z)_{x=0} = \Delta w = \frac{2-\alpha}{\alpha} \bar{\ell} \frac{\frac{2-\alpha}{2\alpha} + \frac{11}{4\sqrt{10\pi}}}{\frac{2-\alpha}{2\alpha} + \frac{12}{4\sqrt{10\pi}}} \frac{dw}{dx}$$

$$= 0.957 \, \bar{\ell} \frac{dw}{dx} \quad \text{for } \alpha = 1 .$$

C. THIRD APPROXIMATION

In this approximation, the functions $\psi_{0,4,1}$, $\psi_{2,1,1}$, and $\psi_{1,2,1}$ are added. The determinant Δ has two roots

$$p_1 = \frac{0.7151}{\lambda} \qquad p_2 = \frac{1.342}{\lambda} .$$

After solving the set of three equations of Equation 24, one finds

$$\epsilon = \frac{2-\alpha}{\alpha} \frac{4\sqrt{\pi}}{3} \lambda \frac{dw}{dx} \frac{0.945 + 3.259 \frac{2-\alpha}{2\alpha} + 2.785 \left(\frac{2-\alpha}{2\alpha}\right)^2}{0.756 + 2.906 \frac{2-\alpha}{2\alpha} + 2.785 \left(\frac{2-\alpha}{2\alpha}\right)^2}$$

and

$$\bar{c}_z = (x + k\bar{\ell}) \frac{dw}{dx}$$

$$- \frac{2-\alpha}{\alpha} \frac{2\sqrt{\pi}}{3} \lambda \frac{dw}{dx} \frac{1}{8\sqrt{\pi}} \frac{(3.909 + 7.514 \frac{2-\alpha}{2\alpha}) e^{-0.7151 \frac{x}{\lambda}} + (1.447 + 2.532 \frac{2-\alpha}{2\alpha}) e^{-1.342 \frac{x}{\lambda}}}{0.756 + 2.906 \frac{2-\alpha}{2\alpha} + 2.785 \left(\frac{2-\alpha}{2\alpha}\right)^2} .$$

Thus the slope of the velocity at the wall is

$$\frac{dw}{dx}\left\{1 + \frac{2-\alpha}{\alpha}\,\frac{1}{12}\,\frac{4.737 + 8.771\,\frac{2-\alpha}{2\alpha}}{0.756 + 2.906\,\frac{2-\alpha}{2\alpha} + 2.785\,\left(\frac{2-\alpha}{2\alpha}\right)^2}\right\} .$$

For $\alpha = 1$, this slope is $1.262\ dw/dx$. The velocity at the wall and the "velocity jump" are easily deduced. For $\alpha = 1$, they are

$$(\bar{c}_z)_{x=0} = 0.874\ \bar{\ell}\,\frac{dw}{dx}$$

$$(\Delta w)_w = 1.126\ \bar{\ell}\,\frac{dw}{dx} .$$

D. FOURTH APPROXIMATION

In addition to the ψ's in the third approximation we include also $\psi_{2,2,1}$, $\psi_{1,4,1}$, $\psi_{0,5,1}$. The four values of p's are

$$p_1 = \frac{0.4604}{\lambda}\ ,\quad p_2 = \frac{1.065}{\lambda}\ ,\quad p_3 = \frac{1.261}{\lambda}\ ,\quad p_4 = \frac{1.610}{\lambda} .$$

The following results are obtained:

$$\epsilon = \frac{2-\alpha}{\alpha}\,\frac{4\sqrt{\pi}}{3}\,\lambda\,\frac{dw}{dx}\,\frac{4.37 + 33.24\,\frac{2-\alpha}{2\alpha} + 93.98\left(\frac{2-\alpha}{2\alpha}\right)^2 + 116.80\left(\frac{2-\alpha}{2\alpha}\right)^3 + 53.76\left(\frac{2-\alpha}{2\alpha}\right)^4}{3.48 + 27.73\,\frac{2-\alpha}{2\alpha} + 82.71\left(\frac{2-\alpha}{2\alpha}\right)^2 + 109.41\left(\frac{2-\alpha}{2\alpha}\right)^3 + 53.76\left(\frac{2-\alpha}{2\alpha}\right)^4}$$

$$\bar{c}_z = (x + k\bar{\ell})\,\frac{dw}{dx}$$

$$- \frac{2-\alpha}{\alpha}\,\frac{2\sqrt{\pi}}{3}\,\lambda\,\frac{dw}{dx}\,\frac{1}{14\sqrt{\pi}}\ \cdot\ \frac{1}{3.48 + 27.73\,\frac{2-\alpha}{2\alpha} + 82.71\left(\frac{2-\alpha}{2\alpha}\right)^2 + 109.41\left(\frac{2-\alpha}{2\alpha}\right)^3 + 53.76\left(\frac{2-\alpha}{2\alpha}\right)^4}$$

$$\times \left\{\left[6.61 + 41.94\,\frac{2-\alpha}{2\alpha} + 88.55\left(\frac{2-\alpha}{2\alpha}\right)^2 + 62.22\left(\frac{2-\alpha}{2\alpha}\right)^3\right] e^{-0.4604\,\frac{x}{\lambda}}\right.$$

$$+ \left[26.24 + 161.72\,\frac{2-\alpha}{2\alpha} + 326.90\left(\frac{2-\alpha}{2\alpha}\right)^2 + 218.89\left(\frac{2-\alpha}{2\alpha}\right)^3\right] e^{-1.065\,\frac{x}{\lambda}}$$

$$+ \left[1.11 + 8.17\,\frac{2-\alpha}{2\alpha} + 19.78\left(\frac{2-\alpha}{2\alpha}\right)^2 + 15.76\left(\frac{2-\alpha}{2\alpha}\right)^3\right] e^{-1.261\,\frac{x}{\lambda}}$$

$$\left. + \left[11.98 + 75.61\,\frac{2-\alpha}{2\alpha} + 151.21\left(\frac{2-\alpha}{2\alpha}\right)^2 + 100.17\left(\frac{2-\alpha}{2\alpha}\right)^3\right] e^{-1.610\,\frac{x}{\lambda}}\right\} .$$

The slope of the velocity at the wall is

$$\frac{dw}{dx}\left\{1+\frac{2-\alpha}{\alpha}\frac{1}{21}\,\frac{51.68+323.57\,\frac{2-\alpha}{2\alpha}+657.31\left(\frac{2-\alpha}{2\alpha}\right)^2+442.91\left(\frac{2-\alpha}{2\alpha}\right)^3}{3.48+27.73\,\frac{2-\alpha}{2\alpha}+82.71\left(\frac{2-\alpha}{2\alpha}\right)^2+109.41\left(\frac{2-\alpha}{2\alpha}\right)^3+53.76\left(\frac{2-\alpha}{2\alpha}\right)^4}\right\}.$$

For $\alpha = 1$, this is equal to 1.37 dw/dx. The corresponding velocity jumps are

$$\Delta w = (\bar{c}_z)_{x=0} = 0.852\,\bar{l}\,\frac{dw}{dx}$$

$$(\Delta w)_W = 1.134\,\bar{l}\,\frac{dw}{dx} \quad .$$

It is of interest to compare our result with Welander's value of $(\Delta w)_W$ = 1.21 $\bar{l}$ dw/dx.

IV. TEMPERATURE DISTRIBUTION NEAR A FIXED PLATE

We consider the infinite space filled with a gas which has no mass flow but the temperature distribution of which is

$$T_o(x) = T_o + tx ,$$

T_o being the temperature at x = 0 and t is a constant. It is assumed that the change of temperature over a mean free path is small, i.e., $\lambda t/T_o$ is small. An infinite plate at the temperature T_o and having the accommodation coefficient α is introduced in the plane x = 0. To find the new stationary state velocity distribution function f, we write again $f = f_o(1 + h)$, where for the present purpose we choose for f_o,

$$f_o = n_o(x)\left(\frac{m}{2\pi k T_o(x)}\right)^{3/2} e^{-\frac{mv^2}{2kT_o(x)}} ,$$

where $\vec{v}$ is the molecular velocity. The number density $n_o(x)$ and the temperature $T_o(x)$ are related by the equation $n_o(x)kT_o(x)$ = constant. To the first order of approximation in $\lambda t/T_o$, the Boltzmann integro-differential equation reads

$$(c^2 - \frac{5}{2})\frac{t}{T_o} + \frac{\partial h}{\partial x} = \frac{n_o}{c_x} J(h) ,$$

where, now both n and T can be treated as constant and replaced by n_o and T_o. The velocity $\vec{c}$ is the dimensionless velocity defined by

$$\vec{c} = \sqrt{\frac{m}{2kT_o}}\,\vec{v} \;.$$

The boundary conditions are

$$x = \infty \qquad h = 2\,\frac{\lambda t}{T_o}\left(\frac{5}{2} - c^2\right)c_x + \epsilon_1 + \epsilon_2\left(\frac{3}{2} - c^2\right) \tag{32a}$$

$$x = 0,\; c_x > 0 \qquad h(0;c_x,c_y,c_z) = \alpha B + (1 - \alpha)h(0, -c_x,c_y,c_z) \;. \tag{32b}$$

Equation 32a states that at distances far from the plate, h is composed of three terms, the ordinary heat-flux term of Chapman and two other terms responsible for whatever change in density and temperature is brought about by the presence of the plate. One must expect that both ϵ_1 and ϵ_2 are proportional to $\lambda t/T_o$. At large distances, the constancy of the scalar pressure, i.e., nkT = constant, should again hold true, even though the presence of the plate might disturb this relation in the immediate neighborhood of the plate. As a consequence, ϵ_1 and ϵ_2 are related, namely,

$$\epsilon_1 = \epsilon_2 \;. \tag{33}$$

Equation 32b is the boundary condition at the wall. The difference in form from Equation 5b of Section II is due to the fact that the density is now also a function of x. The unknown constant B is related to the density of the gas near the plate. As in the previous section, we bring the integro-differential equation, Equation 31, to the homogeneous form

$$\frac{\partial Y}{\partial x} = \frac{n}{c_x}\,J(Y) \;, \tag{34}$$

by introducing $Y(x,\vec{c})$ defined by

$$h = Y(x,\vec{c}) + 2\,\frac{\lambda t}{T_o}\left(\frac{5}{2} - c^2\right)c_x + \epsilon_1 + \epsilon_2\left(\frac{3}{2} - c^2\right) \;. \tag{35}$$

The corresponding boundary conditions become

$$x = \infty:$$

$$Y = 0 \tag{36a}$$

$$x = 0:$$

$$Y(0,c_x,c_y,c_z) = \Big\{\alpha B + (1 - \alpha)Y(0, -c_x,c_y,c_z) - (2-\alpha)\frac{2\lambda t}{T_o}\left(\frac{5}{2} - c^2\right)c_x$$

$$- \alpha\epsilon_1 - \alpha\epsilon_2\left(\frac{3}{2} - c^2\right)\Big\}\frac{1 + \text{sign}\, c_x}{2} + Y(0,\vec{c})\,\frac{1 - \text{sign}\, c_x}{2} . \qquad (36b)$$

Introducing the Laplace transforms, making expansion in the eigenfunctions $\psi_{r,\ell}$,[≠] and finally eliminating $\alpha_{r,2\ell}$, as in the previous section, one is led to the set of equations

$$\sum_{r'',\ell''}\left(s^2\delta_{rr''}\delta_{\ell\ell''} - n^2\sum_{r',\ell'} R_{r,2\ell+1;r',2\ell'}\, R_{r',2\ell';r'',2\ell''+1}\right)\alpha_{r'',2\ell''+1}$$

$$= s a_{r,2\ell+1}(0) + n\sum_{r',\ell'} R_{r,2\ell+1;r',2\ell'}\, a_{r',2\ell'}(0) , \qquad (37)$$

corresponding to Equation 14 in Section II. As consequences of the conservation laws,

$$R_{0,1;r,2\ell} = R_{1,1;r,2\ell} = R_{r,2\ell;0,1} = 0 .$$

Furthermore, for Maxwell molecules

$$R_{r,2\ell;1,1} = 0 ,$$

except for $(r,2\ell) = (0,0)$ or $(1,0)$. One finds then easily the solutions

$$a_{0,1} = \text{const.} \qquad (38a)$$

$$a_{1,1} = \text{const.} \qquad (38b)$$

$$a_{r,2\ell+1}(x) = \sum_i \frac{D_{r,2\ell+1}(s = p_i)}{\left(\frac{d\Delta}{ds}\right)_{s = p_i}} e^{p_i x} + \sum_i \frac{D_{r,2\ell+1}(s = -p_i)}{\left(\frac{d\Delta}{ds}\right)_{s = -p_i}} e^{-p_i x} \qquad (38c)$$

[≠] It is clear in this problem only m = 0 is of interest; hence, in this section we will drop the index m altogether.

$$a_{r,2\ell}(x) = a_{r,2\ell}(0)$$

$$+ n \sum_{r',\ell'} R_{r,2\ell;r',2\ell'+1} \left\{ \sum_i \frac{1}{p_i} \frac{D_{r,2\ell+1}(s = p_i)}{\left(\frac{d\Delta}{ds}\right)_{s = p_i}} (e^{p_i x} - 1)\right.$$

$$\left. - \sum_i \frac{1}{p_i} \frac{D_{r',2\ell'+1}(s = - p_i)}{\left(\frac{d\Delta}{ds}\right)_{s = -p_i}} (e^{-p_i x} - 1) \right\} ,$$

where the notations have the same meanings as before, and the basic determinant Δ is

$$\Delta = / s^2 \delta_{rr''} \delta_{\ell\ell''} - n^2 \sum_{r',\ell'} R_{r,2\ell+1;r',2\ell'} R_{r',2\ell';r'',2\ell''+1} / .$$

The boundary condition at ∞ requires that

$$a_{0,1} = 0 \tag{39a}$$

$$a_{1,1} = 0 \tag{39b}$$

$$D_{r,2\ell+1}(s = p_i) = 0 \tag{39c}$$

$$a_{r,2\ell}(0) + n \sum_{r',\ell'} R_{r,2\ell;r',2\ell'+1} \sum_i \frac{1}{p_i} \frac{D_{r',2\ell'+1}(s = - p_i)}{\left(\frac{d\Delta}{ds}\right)_{s = - p_i}} = 0 . \tag{39d}$$

In the present case, with the same cut-off procedure, it is easily seen that $N_e > N_o$; hence, Equation 39c follows Equation 39d and the differential equations for the a's. The boundary condition at x = 0 gives the following relations:

$$a_{r,2\ell+1}(0) + \frac{2\alpha}{2-\alpha} \sum_{r',\ell'} M_{r,2\ell+1;r,2\ell'} a_{r',2\ell'}(0)$$

$$= - \frac{1}{N_{11}} \frac{2\lambda t}{T_o} \delta_{r1}\delta_{\ell o} + \frac{2\alpha}{2-\alpha} \frac{B-\epsilon_1}{N_{oo}} M_{r,2\ell+1;o,o} - \frac{2\alpha}{2-\alpha} \frac{\epsilon_2}{N_{1o}} M_{r,2\ell+1;1,o} \tag{40a}$$

$$a_{r,2\ell}(0) + \frac{2(2-\alpha)}{\alpha} \sum_{r',\ell'} M_{r,2\ell;r',2\ell'+1}\, a_{r',2\ell'+1}(0)$$

$$= -\frac{2(2-\alpha)}{\alpha} \frac{1}{N_{11}} \frac{2\lambda t}{T_o} M_{r,2\ell;1,1} + \frac{B-\epsilon_1}{N_{oo}} \delta_{ro}\delta_{\ell o} - \frac{\epsilon_2}{N_{1o}} \delta_{r1}\delta_{\ell o} \quad . \qquad (40b)$$

As remarked in the previous section, Equations 40a and 40b follow from each other when all the eigenfunctions $\psi_{r\ell}$ are taken into account. Otherwise we have to make a choice as to which set of equations we want to be satisfied. In the present case, it seems natural to take Equation 40a. Introducing

$$b_{o,3}^{(i)} = \frac{D_{o,3}(s = -p_i)}{\left(\frac{d\Delta}{ds}\right)_{s=-p_i}} ,$$

making use of the relations between the different D's and substituting the expressions of $a_{r,2\ell+1}(0)$ and $a_{r,2\ell}(0)$ in terms of the b's into Equations 40a, we arrive at the following set of equations

$$\frac{B-\epsilon_1}{N_{oo}} M_{r,2\ell+1;o,o} - \frac{\epsilon_2}{N_{1o}} M_{r,2\ell+1;1,o} + \sum_i Q_{r,2\ell+1}^{(i)}\, b_{o,3}^{(i)} = \frac{2-\alpha}{2\alpha} \frac{1}{N_{11}} \frac{2\lambda t}{T_o} \delta_{r1}\delta_{\ell o} \qquad (41)$$

where

$$Q_{o,1}^{(i)} = n_o \sum_{r',\ell'} \sum_{r'',\ell''} M_{o,1;r',2\ell'};\, R_{r',2\ell';r'',2\ell''+1} \frac{1}{p_i} \left(\frac{\Delta_{o,3;r'',2\ell''+1}}{\Delta_{o,3;o,3}}\right)_i$$

$$Q_{1,1}^{(i)} = n_o \sum_{r',\ell'} \sum_{r'',\ell''} M_{1,1;r',2\ell'}\, R_{r',2\ell';r'',2\ell''+1} \frac{1}{p_i} \left(\frac{\Delta_{o,3;r'',2\ell''+1}}{\Delta_{o,3;o,3}}\right)_i$$

$$Q_{r,2\ell+1}^{(i)} = n_o \sum_{r',\ell'} \sum_{r'',\ell''} M_{r,2\ell+1;r',2\ell'}\, R_{r',2\ell';r'',2\ell''+1} \frac{1}{p_i} \left(\frac{\Delta_{o,3;r'',2\ell''+1}}{\Delta_{o,3;o,3}}\right)_i - \frac{2-\alpha}{2\alpha} \left(\frac{\Delta_{o,3;r,2\ell+1}}{\Delta_{o,3;o,3}}\right)_i \quad .$$

The Equations 11 are a set of N_o linear inhomogeneous equations for the N_o unknowns $b_{o,3}^{(i)}$ (N_o - 2 in number), B and ϵ_2 (and hence ϵ_1, because of 33b). Upon the solution of these equations, the complete distribution function is known:

$$h = \epsilon_1 + \epsilon_2 \left(\frac{3}{2} - c^2\right) + \frac{2\lambda t}{T_o} \left(\frac{5}{2} - c^2\right) c_x + \sum_{r,\ell} a_{r,\ell}(x)\psi_{r,\ell} \quad ,$$

where all the coefficients are of the form e^{-px}. The physical quantities of interest are given by

$$n = n_o(x)\left\{1 + \frac{1}{\pi^{3/2}N_{oo}}\left(\frac{\epsilon_1}{N_{oo}} + a_{oo}(x)\right)\right\}$$

$$T = T_o(x)\left\{1 - \frac{2}{3\pi^{3/2}N_{1o}}\left(\frac{\epsilon_2}{N_{1o}} + a_{1o}(x)\right)\right\} .$$

For large x, the a's all vanish, the temperature becomes a linear function of x, and the relation nkT = constant is satisfied. It will be seen that this is not the case for $x \sim \lambda$. The temperature jump, ΔT, is given by

$$\Delta T = T(x = 0) - T_o(x = 0) = -\left(\epsilon_2 + \frac{2}{3\pi^{3/2}N_{1o}}\, a_{1o}(0)\right) .$$

The corresponding "temperature jump," $(\Delta T)_w$ as defined by Welander, namely, the difference between the temperature at the wall obtained from a linear extrapolation of the temperature curve beyond the transition region and the temperature of the wall T_o is

$$(\Delta T)_w = T_o\,[1 - \epsilon_2] - T_o = -\,\epsilon_2\, T_o .$$

To get some idea of the magnitude of these quantities, we give numerical results for some "successive approximation" calculations.

A. FIRST APPROXIMATION: $\psi_{o,1}, \psi_{1,1}; \psi_{o,o}, \psi_{1,o}, \psi_{o,2}$

The obvious solutions are

$$a_{o,1} = a_{1,1} = 0$$

$$a_{o,o} = a_{1,o} = a_{o,2} = 0 .$$

Equations 37 become

$$\alpha\left(\frac{B}{N_{oo}} - \frac{\epsilon_1}{N_{oo}}\right) M_{o,1;o,o} - \alpha\,\frac{\epsilon_2}{N_{1,o}}\, M_{o,1;1,o} = 0$$

$$\alpha\left(\frac{B}{N_{oo}} - \frac{\epsilon_1}{N_{oo}}\right) M_{1,1;o,o} - \alpha\,\frac{\epsilon_2}{N_{1,o}}\, M_{1,1;1o} = \frac{2-\alpha}{2}\,\frac{2\lambda t}{T_o} ,$$

giving

$$\epsilon_1 = \epsilon_2 = -\,\frac{2-\alpha}{\alpha}\,\frac{5\sqrt{\pi}}{4}\,\frac{\lambda t}{T_o} .$$

The temperature distribution is linear throughout the entire space. The temperature jump is

$$\Delta T = (\Delta T)_w = \frac{2-\alpha}{\alpha}\frac{5\sqrt{\pi}}{4}\lambda t = \frac{2-\alpha}{\alpha}\frac{15}{8}\bar{l}\frac{dT_o(x)}{dx} ,$$

in agreement with the results of Maxwell and others.

B. SECOND APPROXIMATION

To the functions used in the first approximation, we now add $\psi_{o,3}$, $\psi_{1,2}$, and $\psi_{2,o}$. The determinant Δ has one pair of roots, namely,

$$p = \pm \frac{\sqrt{15}}{4}\frac{1}{\lambda} .$$

Equations 37 now consist of three equations:

$$\frac{B-\epsilon_1}{N_{oo}} M_{o,1;o,o} - \frac{\epsilon_2}{N_{1o}} M_{o,1;1,o} + \frac{n_o}{p}\sum_{r;l'} M_{o,1;r;2l'} R_{r;2l;o,3}\, b_{o,3} = 0$$

$$\frac{B-\epsilon_1}{N_{oo}} M_{1,1;o,o} - \frac{\epsilon_2}{N_{1o}} M_{1,1;1,o} + \frac{n_o}{p}\sum_{r;l'} M_{1,1;r;2l'} R_{r;2l;o,3} b_{o,3} = \frac{2-\alpha}{\alpha}\frac{\lambda t}{T_o}$$

$$\frac{B-\epsilon_1}{N_{oo}} M_{o,3;o,o} - \frac{\epsilon_2}{N_{1o}} M_{o,3;1,o} + \left(\frac{n_o}{p}\sum_{r'l'} M_{o,3;r;2l'} R_{r;2l;o,3} - \frac{2-\alpha}{2\alpha}\right) b_{o,3} = 0.$$

Together with Equation 29b, we find the following for the complete solution:

$$\epsilon_1 = \epsilon_2 = -\frac{2-\alpha}{\alpha}\frac{5\sqrt{\pi}}{4}\frac{\lambda t}{T_o}\,\frac{\frac{2-\alpha}{2\alpha} + \frac{39}{10\sqrt{15\pi}}}{\frac{2-\alpha}{2\alpha} + \frac{15}{4\sqrt{15\pi}}}$$

$$a_{o,1} = a_{1,1} = a_{1,2} = a_{2,o} = 0$$

$$a_{o,3} = \frac{2-\alpha}{\alpha}\frac{1}{4}\sqrt{\frac{5}{6}}\,\pi^{3/4}\frac{\lambda t}{T_o}\,\frac{e^{-\frac{\sqrt{15}}{4}\frac{x}{\lambda}}}{\frac{2-\alpha}{2\alpha} + \frac{15}{4\sqrt{15\pi}}}$$

$$a_{o,o} = -\frac{2-\alpha}{\alpha}\,\frac{1}{4}\sqrt{\frac{3}{5}}\,\pi^{3/4}\,\frac{\lambda t}{T_o}\,\frac{e^{-\frac{\sqrt{15}}{4}\frac{x}{\lambda}}}{\frac{2-\alpha}{2\alpha}+\frac{15}{4\sqrt{15\pi}}}$$

$$a_{1,o} = \frac{2-\alpha}{\alpha}\,\frac{1}{4}\sqrt{\frac{2}{5}}\,\pi^{3/4}\,\frac{\lambda t}{T_o}\,\frac{e^{-\frac{\sqrt{15}}{4}\frac{x}{\lambda}}}{\frac{2-\alpha}{2\alpha}+\frac{15}{4\sqrt{15\pi}}}$$

$$a_{o,2} = \frac{2-\alpha}{\alpha}\,\frac{\sqrt{5}}{8}\,\pi^{3/4}\,\frac{\lambda t}{T_o}\,\frac{e^{-\frac{\sqrt{15}}{4}\frac{x}{\lambda}}}{\frac{2-\alpha}{2\alpha}+\frac{15}{4\sqrt{15\pi}}} .$$

The number density and temperature are given by

$$n = n_o(x)\left\{1 - \frac{2-\alpha}{\alpha}\,\frac{5\sqrt{\pi}}{4}\left[\frac{\frac{2-\alpha}{2\alpha}+\frac{39}{10\sqrt{15\pi}}}{\frac{2-\alpha}{2\alpha}+\frac{15}{4\sqrt{15\pi}}} + \frac{\frac{3}{5\sqrt{15\pi}}\,e^{-\frac{\sqrt{15}}{4}\frac{x}{\lambda}}}{\frac{2-\alpha}{2\alpha}+\frac{15}{4\sqrt{15\pi}}}\right]\frac{\lambda t}{T_o}\right\}$$

$$T = T_o(x)\left\{1 + \frac{2-\alpha}{\alpha}\,\frac{5\sqrt{\pi}}{4}\left[\frac{\frac{2-\alpha}{2\alpha}+\frac{39}{10\sqrt{15\pi}}}{\frac{2-\alpha}{2\alpha}+\frac{15}{4\sqrt{15\pi}}} - \frac{\frac{2}{5\sqrt{15\pi}}\,e^{-\frac{\sqrt{15}}{4}\frac{x}{\lambda}}}{\frac{2-\alpha}{2\alpha}+\frac{15}{4\sqrt{15\pi}}}\right]\frac{\lambda t}{T_o}\right\}$$

so that

$$nkT = n_o(x)kT_o(x)\left\{1 - \frac{2-\alpha}{\alpha}\,\frac{1}{4}\sqrt{\frac{5}{3}}\,\frac{e^{-\frac{\sqrt{15}}{4}\frac{x}{\lambda}}}{\frac{2-\alpha}{2\alpha}+\frac{15}{4\sqrt{15\pi}}}\,\frac{\lambda t}{T_o}\right\} .$$

Hence, only when $x \gg \lambda$ is the scalar pressure a constant. The temperature distribution is no longer linear in x; however, the deviation from the linearity is again appreciable only in a very small region where x is a few mean free paths. The quantity ΔT is given by

$$\Delta T = \frac{2-\alpha}{\alpha}\,\frac{15}{8}\,\bar{\ell}\,\frac{dT_o(x)}{dx}\,\frac{\frac{2-\alpha}{2\alpha}+\frac{14}{4\sqrt{15\pi}}}{\frac{2-\alpha}{2\alpha}+\frac{15}{4\sqrt{15\pi}}}$$

and

$$(\Delta T)_w = \frac{2-\alpha}{\alpha}\frac{15}{8}\bar{\ell}\frac{dT_0(x)}{dx}\frac{\frac{2-\alpha}{2\alpha}+\frac{39}{10\sqrt{15\pi}}}{\frac{2-\alpha}{2\alpha}+\frac{15}{4\sqrt{15\pi}}} .$$

For $\alpha = 1$, we have

$$\Delta T = (0.965)\frac{15}{8}\bar{\ell}\frac{dT_0(x)}{dx}$$

$$(\Delta T)_w = (1.021)\frac{15}{8}\bar{\ell}\frac{dT_0(x)}{dx} ,$$

giving a value of 1.021 x 15/8 for the slip coefficient defined by

$$(\Delta T)_w\Big/\bar{\ell}\frac{dT_0(x)}{dx} .$$

C. THIRD APPROXIMATION

Three more eigenfunctions are taken into account. They are $\psi_{0,4}$, $\psi_{2,1}$, and $\psi_{1,3}$. The following are the values of the p's:

$$p_1 = \frac{0.4798}{\lambda} , \quad p_2 = \frac{0.8344}{\lambda} , \quad p_3 = \frac{1.293}{\lambda} .$$

The temperature distribution is found to be

$$T = T_0(x)\left\{1 + \frac{\frac{2-\alpha}{\alpha}\frac{5\sqrt{\pi}}{4}\frac{\lambda t}{T_0}}{2.05 + 11.65\frac{2-\alpha}{2\alpha} + 22.09\left(\frac{2-\alpha}{2\alpha}\right)^2 + 13.96\left(\frac{2-\alpha}{2\alpha}\right)^3}\right.$$

$$\left[2.63 + 13.92\frac{2-\alpha}{2\alpha} + 24.29\left(\frac{2-\alpha}{2\alpha}\right)^2 + 13.96\left(\frac{2-\alpha}{2\alpha}\right)^3\right.$$

$$- \frac{1}{3\sqrt{5\pi}}\left([11.49 + 44.01\frac{2-\alpha}{2\alpha} + 42.04\left(\frac{2-\alpha}{2\alpha}\right)^2] e^{-\frac{0.4798x}{\lambda}}\right.$$

$$+ [0.71 + 2.73\frac{2-\alpha}{2\alpha} + 2.64\left(\frac{2-\alpha}{2\alpha}\right)^2] e^{-\frac{0.8344x}{\lambda}}$$

$$\left.\left.\left. + [-0.11 + 0.36\frac{2-\alpha}{2\alpha} + 1.09\left(\frac{2-\alpha}{2\alpha}\right)^2] e^{-1.293\frac{x}{\lambda}}\right)\right]\right\} ,$$

so that the temperature jumps are given by

$$\Delta T = \frac{15}{8} \bar{l} \frac{dT_0(x)}{dx} \frac{2-\alpha}{\alpha} \frac{1.61 + 9.96 \frac{2-\alpha}{2\alpha} + 20.45 \left(\frac{2-\alpha}{2\alpha}\right)^2 + 13.96 \left(\frac{2-\alpha}{2\alpha}\right)^3}{2.05 + 11.65 \frac{2-\alpha}{2\alpha} + 22.09 \left(\frac{2-\alpha}{2\alpha}\right)^2 + 13.96 \left(\frac{2-\alpha}{2\alpha}\right)^3}$$

and

$$(\Delta T)_w = \frac{15}{8} \bar{l} \frac{dT_0(x)}{dx} \frac{2-\alpha}{\alpha} \frac{2.63 + 13.92 \frac{2-\alpha}{2\alpha} + 24.29 \left(\frac{2-\alpha}{2\alpha}\right)^2 + 13.96 \left(\frac{2-\alpha}{2\alpha}\right)^3}{2.05 + 11.65 \frac{2-\alpha}{2\alpha} + 22.09 \left(\frac{2-\alpha}{2\alpha}\right)^2 + 13.96 \left(\frac{2-\alpha}{2\alpha}\right)^3} .$$

The corresponding values for $\alpha = 1$ are

$$\Delta T = (0.888) \frac{15}{8} \bar{l} \frac{dT_0(x)}{dx}$$

$$(\Delta T)_w = (1.150) \frac{15}{8} \bar{l} \frac{dT_0(x)}{dx} .$$

Welander obtained the result

$$(\Delta T)_w = (1.152) \frac{15}{8} \bar{l} \frac{dT_0(x)}{dx} .$$

V. CONCLUDING REMARKS

The outstanding question, which remains to be answered for this problem just as for the problems treated in References 2 and 3, is the question of the convergence of the successive approximation procedure. Practically speaking, the convergence seems quite rapid; only relatively small changes are produced by going beyond the first approximation which is identical with the classical results.

A second question is whether other ways of breaking off the infinite sets of equations, or other ways of selecting the boundary conditions to be satisfied, would lead to appreciably different results. If so, this might throw some light on the convergence question.*

Finally, the effect of the choice of molecular models needs further investigation.

*Dr. H. M. Mott-Smith in the report, "A New Approach in the Kinetic Theory of Gases," Mass. Inst. of Tech. Lincoln Lab. Group Report V-2, Dec., 1954, has investigated other selections of boundary conditions for the heat flow problem between parallel plates and for the Couette flow.

APPENDIX I

PROOF THAT $D_{r,2\ell,1}(s = \pm p_i) = \dfrac{\Delta^{(i)}_{1,2,1;r,2\ell,1}}{\Delta^{(i)}_{1,2,1;1,2,1}} D_{1,2,1}(s = \pm p_i)$

Let us consider the system of equations

$$\sum_k (s^2\delta_{jk} - b_{jk})\, \alpha_k = a_j \quad ,$$

and let $s = \pm p_i$ be the roots of the determinant $\Delta = /s^2\delta_{jk} - b_{jk}/ = 0$. Then

$$a_k = \sum_i \frac{D_k(s = p_i)}{\left(\frac{d\Delta}{ds}\right)_{s = p_i}} \frac{1}{s - p_i} + \sum_i \frac{D_k(s = - p_i)}{\left(\frac{d\Delta}{ds}\right)_{s = - p_i}} \frac{1}{s + p_i} \quad ,$$

where D_k is obtained from the determinant Δ by replacing the k^{th} column by the a's. Hence,

$$D_k = \sum_m a_m \Delta_{mk} \quad ,$$

where Δ_{mk} is the algebraic complement of the (mk) element of Δ. For $s = \pm p_i$, we have the relation

$$\Delta_{mk} : \Delta_{m\ell} = \Delta_{nk} : \Delta_{n\ell}$$

for all m, n, k, and ℓ. Making use of this,

$$D_k\,(s = \pm\, p_i) = \sum_m a_m\,(s = \pm\, p_i)\,\Delta^{(i)}_{mk} \frac{\Delta^{(i)}_{nk}}{\Delta^{(i)}_{n\ell}}$$

$$= \frac{\Delta^{(i)}_{nk}}{\Delta^{(i)}_{n\ell}} D_\ell\,(s = \pm\, p_i) \quad ,$$

where

$$\Delta^{(i)}_{nk} = \Delta_{nk}\,(s = \pm p_i) \; .$$

APPENDIX II

PROOF OF THE PROPERTIES OF $M_{r,\ell,m;r',\ell',m}$

$$M_{r,\ell,m;r',\ell',m} \equiv \int d\vec{c}\ e^{-c^2} \frac{1 + \text{sign } c_x}{2} \psi^*_{r,\ell,m} \psi_{r',\ell',m}$$

where

$$\psi_{r,\ell,m} = N_{r\ell m} S^{(r)}_{\ell+1/2}(c^2)\, c^\ell P^m_\ell(\cos\theta)\, e^{im\phi} \quad .$$

1. Since the same value of m appears in both ψ^* and ψ, it is clear that $M_{r,\ell,m;r',\ell',m}$ is symmetrical with respect to the two sets of indices.

2.

$$M_{r,2\ell,m;r',2\ell',m} = \int_0^{2\pi} d\phi \int_0^1 dx \int_0^\infty dc c^2\, e^{-c^2}$$

$$\cdot\ \psi^*_{r,2\ell,m}(c,x,\phi)\psi_{r',2\ell',m}(c,x,\phi)$$

where $x = \cos\theta$. Since $\psi^*_{r,2\ell,m}\psi_{r',2\ell',m}$ is an even function of x,

$$M_{r,2\ell,m;r',2\ell',m} = \frac{1}{2}\int d\vec{c} e^{-c^2} \psi^*_{r,2\ell,m}\psi_{r',2\ell',m}$$

$$= \frac{1}{2}\delta_{rr'}\delta_{\ell\ell'}$$

by the orthonormal property of the ψ's. Similarly,

$$M_{r,2\ell+1,m;r',2\ell'+1,m} = \frac{1}{2}\delta_{rr'}\delta_{\ell\ell'} \quad .$$

3. To find the sum

$$\sum_{r''\ell''} M_{r,2\ell+1,m;r'',2\ell'',m} M_{r'',2\ell'',m;r',2\ell'+1,m}$$

we first expand

$$\psi_{r,2\ell+1,m}\frac{1 \pm \text{sign } c_x}{2}$$

in terms of the complete set of eigenfunctions $\psi_{r,\ell,m}$

$$\psi_{r,2\ell+1,m} \frac{1 \pm \text{ sign } c_x}{2} = \sum_{r',\ell'} b_{r,2\ell+1,m;r',\ell',m} \psi_{r',\ell',m}$$

where

$$b_{r,2\ell+1,m;r',\ell',m} = \int d\vec{c} e^{-c^2} \psi_{r,2\ell+1,m} \psi^*_{r',\ell',m} \frac{1 \pm \text{ sign } c_x}{2} .$$

Thus,

$$\psi_{r,2\ell+1,m} \frac{1 \pm \text{ sign } c_x}{2} = \frac{1}{2} \psi_{r,2\ell+1,m} \pm \sum_{r',\ell'} M_{r,2\ell+1,m;r',2\ell',m} \psi_{r',2\ell',m} .$$

For all values of C_x except $C_x = 0$, one can write

$$\psi_{r,2\ell+1,m} \frac{1 \pm \text{sign} c_x}{2} = \frac{1}{2} \psi_{r,2\ell+1,m} \frac{1 \pm \text{sign } c_x}{2} \pm \sum_{r',\ell'} M_{r,2\ell+1,m;r',2\ell',m} \psi_{r',2\ell',m} \frac{1 \pm \text{sign} c_x}{2}$$

or

$$\psi_{r,2\ell+1,m} \frac{1 \pm \text{ sign } c_x}{2} = \pm 2 \sum_{r',\ell'} M_{r,2\ell+1,m;r',2\ell',m} \psi_{r',2\ell',m} \frac{1 \pm \text{ sign } c_x}{2} . \quad \text{(A)}$$

Similarly, one finds

$$\psi_{r,2\ell,m} \frac{1 \pm \text{ sign } c_x}{2} = \pm 2 \sum_{r',\ell'} M_{r,2\ell,m;r',2\ell'+1,m} \psi_{r',2\ell'+1,m} \frac{1 \pm \text{ sign } c_x}{2} . \quad \text{(B)}$$

By substituting Equation B into Equation A it follows that

$$\psi_{r,2\ell+1,m} = 4 \sum_{r',\ell'} M_{r,2\ell+1,m;r',2\ell',m} M_{r',2\ell',m;r'',2\ell''+1,m} \psi_{r'',2\ell''+1,m}$$

Hence,

$$\sum_{r',\ell'} M_{r,2\ell+1,m;r',2\ell',m} M_{r',2\ell',m;r'',2\ell+1'',m} = \frac{1}{4} \delta_{rr''} \delta_{\ell\ell''} ,$$

and similarly

$$\sum_{r',\ell'} M_{r,2\ell,m;r',2\ell'+1,m} M_{r',2\ell'+1,m;r'',2\ell'',m} = \frac{1}{4} \delta_{rr''} \delta_{\ell\ell''} .$$

APPENDIX III

$$R_{r,\ell,m;r',\ell',m} = \lambda_{r',\ell'} \int d\vec{c}\, e^{-c^2} \frac{1}{c_x} \psi^*_{r,\ell,m} \psi_{r',\ell',m}$$

$$= \lambda_{r',\ell'} L_{r',\ell,m;r',\ell',m}$$

where $L_{r,\ell,m;r',\ell',m}$ is symmetrical with the two sets of indices

A. m = 0. Table of $L_{r,\ell;r',\ell'}$

r',ℓ' \ r,ℓ	0,0	1,0	0,2	1,2	2,0	0,4
1,1	$\frac{2}{\sqrt{5}}$	$\sqrt{\frac{6}{5}}$	0	0	0	0
0,3	$-2\sqrt{\frac{2}{15}}$	$\frac{4}{3\sqrt{5}}$	$\frac{\sqrt{10}}{3}$	0	0	0
2,1	$\frac{4}{\sqrt{35}}$	$2\sqrt{\frac{6}{35}}$	0	0	$\sqrt{\frac{6}{7}}$	0
1,3	$-\frac{8}{3\sqrt{15}}$	$-\frac{2}{9}\sqrt{\frac{2}{5}}$	$\frac{2}{9}\sqrt{5}$	$\frac{1}{9}\sqrt{70}$	$\frac{4\sqrt{2}}{9}$	0

B. m = 1. Table of $L_{r,\ell,1;r',\ell',1}$

$r',\ell',1$ \ $r,\ell,1$	0,1,1	1,1,1	0,3,1	2,1,1	1,3,1	0,5,1
0,2,1	$\sqrt{2}$	0	0	0	0	0
1,2,1	$\frac{2}{\sqrt{7}}$	$\sqrt{\frac{10}{7}}$	0	0	0	0
0,4,1	$-\frac{4}{\sqrt{21}}$	$4\sqrt{\frac{2}{105}}$	$\sqrt{\frac{14}{15}}$	0	0	0
2,2,1	$\frac{4}{3\sqrt{7}}$	$\frac{2}{3}\sqrt{\frac{10}{7}}$	0	$\frac{\sqrt{10}}{3}$	0	0
1,4,1	$-8\sqrt{\frac{2}{231}}$	$-\frac{4\sqrt{3}}{\sqrt{5\cdot 7\cdot 11}}$	$\frac{2\sqrt{7}}{\sqrt{15\cdot 11}}$	$\frac{8}{\sqrt{3\cdot 5\cdot 11}}$	$\sqrt{\frac{42}{55}}$	0

APPENDIX IV

$$M_{r,\ell,m;r',\ell',m} = \int d\vec{c}\, e^{-c^2}\, \frac{1 + \text{sign } c_x}{2}\, \psi^*_{r,\ell,m}\, \psi_{r',\ell',m}$$

A. $M_{r,2\ell,m;r',2\ell',m} = \frac{1}{2}\, \delta_{rr'}\, \delta_{\ell\ell'}$

$$M_{r,2\ell+1,m;r',2\ell'+1,m} = \frac{1}{2}\, \delta_{rr'}\, \delta_{\ell\ell'}$$

B. Table of $M_{r,\ell;r',\ell'}$ (m = 0)

r,2ℓ+1 \ r',2ℓ'	0,0	1,0	0,2	1,2	2,0
0,1	$\frac{1}{\sqrt{2\pi}}$	$-\frac{1}{2\sqrt{3\pi}}$	$\frac{1}{\sqrt{6\pi}}$	$\frac{1}{2\sqrt{21\pi}}$	$-\frac{1}{4\sqrt{15\pi}}$
1,1	$\frac{1}{2\sqrt{5\pi}}$	$\frac{7}{4}\sqrt{\frac{2}{15\pi}}$	$-\frac{1}{2\sqrt{15\pi}}$	$\frac{11}{4}\sqrt{\frac{2}{105\pi}}$	$-\frac{17}{40}\sqrt{\frac{2}{3\pi}}$
0,3	$-\frac{1}{\sqrt{30\pi}}$	$\frac{1}{2\sqrt{5\pi}}$	$\sqrt{\frac{2}{5\pi}}$	$-\frac{1}{\sqrt{35\pi}}$	$-\frac{1}{20\sqrt{\pi}}$
2,1	$\frac{3}{4\sqrt{35\pi}}$	$\frac{13}{8}\sqrt{\frac{2}{105\pi}}$	$-\frac{1}{4\sqrt{105\pi}}$	$-\frac{25}{56}\sqrt{\frac{2}{15\pi}}$	$\frac{157}{80}\sqrt{\frac{2}{21\pi}}$
1,3	$-\frac{1}{2\sqrt{15\pi}}$	$-\frac{1}{12}\sqrt{\frac{2}{5\pi}}$	$\frac{1}{3\sqrt{5\pi}}$	$\frac{5}{2}\sqrt{\frac{2}{35\pi}}$	$\frac{7}{40}\sqrt{\frac{2}{\pi}}$

C. Table of $M_{r,\ell,1;r',\ell',1}$

r',ℓ',1 \ r,ℓ,1	0,1,1	1,1,1	0,3,1	2,1,1	1,3,1	0,5,1
0,2,1	$\frac{1}{\sqrt{2\pi}}$	$-\frac{1}{2\sqrt{5\pi}}$	$\frac{1}{\sqrt{5\pi}}$	$-\frac{1}{4\sqrt{35\pi}}$	$\frac{1}{3\sqrt{10\pi}}$	$-\frac{1}{3\sqrt{14\pi}}$
1,2,1	$\frac{1}{2\sqrt{7\pi}}$	$\frac{11}{2\sqrt{70\pi}}$	$-\frac{1}{\sqrt{70\pi}}$	$-\frac{25}{28\sqrt{10\pi}}$	$\frac{5}{2\sqrt{35\pi}}$	$\frac{1}{14\sqrt{\pi}}$
0,4,1	$-\frac{1}{\sqrt{21\pi}}$	$\frac{\sqrt{3}}{\sqrt{70\pi}}$	$\frac{3\sqrt{3}}{\sqrt{70\pi}}$	$-\frac{3}{14\sqrt{30\pi}}$	$-\frac{\sqrt{3}}{2\sqrt{35\pi}}$	$\frac{6}{7\sqrt{3\pi}}$
2,2,1	$\frac{1}{4\sqrt{7\pi}}$	$\frac{7}{4\sqrt{70\pi}}$	$-\frac{1}{6\sqrt{70\pi}}$	$\frac{111}{56\sqrt{10\pi}}$	$-\frac{11}{12\sqrt{35\pi}}$	$-\frac{1}{84\sqrt{\pi}}$
1,4,1	$-\frac{\sqrt{3}}{\sqrt{2\cdot 7\cdot 11\pi}}$	$-\frac{\sqrt{7}}{2\sqrt{165\pi}}$	$\frac{3\sqrt{3}}{2\sqrt{385\pi}}$	$\frac{87}{28\sqrt{165\pi}}$	$\frac{19\sqrt{3}}{2\sqrt{770\pi}}$	$-\frac{\sqrt{6}}{7\sqrt{11\pi}}$

REFERENCES

1. A. Kundt and E. Warburg, Ann. Phys. und Chemie, 155, 337, 525 (1875).

2. J. C. Maxwell, Scientific Papers, II. Cambridge Univ. Press, 1890, pp. 26-78, 681-712; M. von Smoluchowski, Akad. Wiss. Wien., 107, 304 (1898) and 108, 5 (1899); H. Grad, Commun. Pure Appl. Math., 2, 331 (1949).

3. C. S. Wang Chang and G. E. Uhlenbeck, "The Heat Transport Between Two Parallel Plates as a Function of the Knudsen Number," Project M999, Eng. Res. Inst., Univ. of Mich., Sept., 1953; "The Couette Flow Between Two Parallel Plates as a Function of the Knudsen Number," Project 1999, Eng. Res. Inst., Univ. of Mich., June, 1954.

4. H. A. Kramers, Nuovo Cimento, 6, suppl., 297 (1949).

5. P. Welander, Arkiv. för Fysik, 7, 507 (1954).

6. S. Chapman and T. W. Cowling, The Mathematical Theory of Non-uniform Gases. Cambridge Univ. Press, 1952.