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ON THE PROPAGATION OF SOUND IN MONATOMIC GASES

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ON THE PROPAGATION OF SOUND IN MONATOMIC GASES

I. INTRODUCTION

Among the problems connected with the study of the transport phenomena in gases, the question of the sound propagation in monatomic gases is of special interest. The reason for this lies in the fact that by varying the ratio of the wavelength of sound, Δ , to the mean free path of the gas molecules, λ , one can study in the most direct way the transition from the so-called Clausius-gas to the Knudsen-gas regimes, which correspond to the limiting cases $\Delta \gg \lambda$ and $\Delta \ll \lambda$.

The classical theory of sound propagation is based on the Stokes-Navier equations, which lead to the dispersion law:

$$\begin{aligned} \omega^3 - \omega V_0^2 \sigma^2 - \frac{4i}{3} \frac{\mu}{\rho} \omega^2 \sigma^2 \\ - \frac{2i}{3} \frac{m\gamma}{k\rho} \omega^2 \sigma^2 + \frac{2i}{5} V_0^2 \frac{m\gamma}{k\rho} \sigma^4 \\ - \frac{8}{9} \frac{m\mu\gamma}{k\rho^2} \omega \sigma^4 = 0 \end{aligned} \quad (1)$$

where ω is the frequency of sound, $\sigma = \sigma_1 - i\sigma_2$ is the complex wave number, V_0 is the velocity of sound at zero frequency, k is the Boltzmann constant, and m , ρ , μ , and γ are the molecular mass, the density, the viscosity coefficient, and the heat conduction coefficient of the gas respectively. One can solve Eq (1) exactly for the real and imaginary parts of σ , and thus obtain the "exact" expressions for the dispersion and absorption of sound. But for gases, it is in

the spirit of the derivation of the Stokes-Navier equations from the Boltzmann equation to develop Eq (1) in powers of λ/Λ , and keep only terms up to the first order in this ratio. Doing this, one finds that there is no dispersion, the velocity of sound, V , is

$$V = V_0,$$

and the absorption coefficient σ_2 is given by:

$$\sigma_2 = -\frac{\omega^2}{\rho V_0^3} \left(\frac{2}{3} \mu + \frac{2}{15} \frac{\eta}{\kappa} \right)$$

For Maxwell molecules, i.e. molecules repelling with the $\mathcal{K}r^{-5}$ force law, $\eta = \frac{15}{4} \frac{k}{m} \mu$, the expression reduces to

$$\sigma_2 = \frac{7}{6} \frac{\mu \omega^2}{\rho V_0^3}$$

One can go to higher approximations by calculating from the Boltzmann equation higher-order corrections to the local Maxwell-Boltzmann distribution. From these one obtains the higher-order terms in the pressure tensor and heat conduction vector, leading to the corresponding higher-order hydrodynamic equations. From the second-order hydrodynamic equations (the Burnett equations) Primakoff¹ and later Tsien and Schamberg² have computed the dispersion and absorption of sound. Their results were in error on account of an error in the heat conduction vector given in Chapman's book³. This error was corrected by Chang and Uhlenbeck⁴, who also carried the calculation one step further, i.e. computed the next term of the absorption coefficient. For Maxwell molecules the results are:

1 H. Primakoff, J. Acous. Soc. Am. 14, 14, 1942.

2 H. S. Tsien and R. Schamberg, J. Acous. Soc. Am. 18, 334, 1946.

3 Chapman and Cowling, The Mathematical Theory of Non-uniform Gases, Chap. 15.

4 C. S. Wang Chang and G. E. Uhlenbeck, "On the Transport Phenomena in Rarified Gases, APL/JHU CM-443 UMH-3-F, 1948.

$$\sigma_1 = \frac{\omega}{V} = \frac{\omega}{V_0} \left\{ 1 - \frac{215}{72} \frac{\mu^2 \omega^2}{\rho^2 V_0^4} + \dots \right\} \quad (2a)$$

$$\sigma_2 = \frac{\mu \omega^2}{\rho V_0^3} \left\{ \frac{7}{6} - \frac{5155}{6 \cdot 72} \frac{\mu^2 \omega^2}{\rho^2 V_0^4} + \dots \right\} \quad (2b)$$

(2b)

Chang⁵ has also reported results from the Burnett approximation for helium using for the force law a form derived from the second virial coefficient by de Boer and Michels. For room temperatures, the numerical coefficient 7/6 in Eq (2b) is replaced by 7.01/6, while the coefficient in Eq (2a) is replaced by 230.9/72, which indicates that the dependence of these coefficients on the molecular model is not very strong.

Recently measurements of the absorption and the dispersion of sound in helium have been made by Greenspan⁶ using very low pressure and very high frequency. These experiments go beyond the range of validity of the Stokes-Navier and Burnett approximations, and probably they are beyond the range of convergence of any series expansion in λ/Λ . Thus, it seems desirable to make a further study of the theoretical aspect of the problem. To do this, it seems better to go back to the Boltzmann equation. In the Enskog-Chapman successive-approximation method one does not distinguish between the size of the disturbance from the equilibrium state and the scale (compared to λ) of the disturbance. One concentrates on the derivation of the macroscopic equations of motion, which as a result are not linear. But for the sound propagation the nonlinear terms can always be neglected, since one can always assume that the intensity of the sound is sufficiently small. Thus the Enskog-Chapman method requires too much unnecessary work.

It is more practical to assume from the beginning that the size of the disturbance from the equilibrium state is small, and that the dependence on time and space coordinates is like $\exp i(\omega t - \sigma z)$ with no restrictions on the scale in time and space as compared to the time between collisions and the mean free path λ . It turns out that in this way it is easy to derive and extend results

5 C. S. Wang Chang, "On the Dispersion of Sound in Helium", APL/JHU CM-467 UMH-3-F, 1948.

6 M. Greenspan, Phys. Rev. 75, 197, 1949; J. Acous. Soc. Am. 22, 568, 1951.

like Eq (2), and one may hope to obtain in this way the complete dispersion law for all values of λ/Λ . The same idea has occurred to Dr. Mott-Smith*, and his results are in substantial agreement with ours. In the following, we will give an account of our work. A joint publication with Dr. Mott-Smith is in preparation.

II. THE BOLTZMANN EQUATION FOR A SMALL DISTURBANCE FROM EQUILIBRIUM

The Boltzmann equation for a monatomic gas when there is no outside force is:

$$\begin{aligned} \frac{Df}{Dt} &\equiv \frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z} = \\ &= \iiint d\xi_1 d\eta_1 d\zeta_1 \iint d\mathcal{E} d\theta \sin \theta \\ &\quad \cdot g I(g, \theta) (f' f'_1 - f f_1) \end{aligned} \quad (3)$$

$f(\xi, \eta, \zeta; x, y, z; t)$ is the velocity distribution function at time t and at the point (x, y, z) ; f_1 is the distribution function corresponding to the velocity components $\xi_1, \eta_1,$ and ζ_1 ; and f' and f'_1 are the distribution functions corresponding to the velocities after collision. $I(g, \theta)$ is the differential collision cross section corresponding to a turning of the relative velocity g over the angle θ into the solid angle $\sin \theta d\theta d\mathcal{E}$. For the case of a small disturbance the "Ansatz" for f is:

$$f = f_0 [1 + h(\xi, \eta, \zeta; x, y, z; t)] \quad (4)$$

* We thank Dr. Mott-Smith for sending us his manuscript.

where f_0 is the complete equilibrium distribution function with no mass velocity* and $h \ll 1$, but no assumptions are made about the space and time variations of h . Substituting Eq (4) into (3) and introducing the dimensionless velocity

$$\vec{c} = \sqrt{\frac{m}{2kT}} \vec{\xi}$$

we obtain, in the first approximation:

$$\sqrt{\frac{m}{2kT}} \frac{\partial h}{\partial t} + c_x \frac{\partial h}{\partial x} + c_y \frac{\partial h}{\partial y} + c_z \frac{\partial h}{\partial z} = nJ(h), \quad (5)$$

where n is the number density and

$$J(h) \equiv \frac{1}{\pi^{3/2}} \iiint d\vec{c}' e^{-c'^2} \iint d\epsilon d\theta \sin \theta \cdot$$

$$\cdot g I(g, \theta) (h' + h'_1 - h - h_1)$$

where g is also dimensionless. The operator J will be called the collision operator**. It has the dimension of an area and the order of magnitude of the collision cross section. Eq (5) is a linear integral differential equation. For further discussion, it is convenient to consider first the eigenfunctions, ψ_i , of the collision operator J defined as

$$J(\psi_i) = \lambda_i \psi_i, \quad (7)$$

* The density and the temperature are therefore supposed to be constant.

**It is to be noted that our definition of $J(h)$ differs from that of Chapman by a factor $-(m/2\pi kT)^{3/2} \sqrt{2kT/m} e^{-c^2}$.

where λ_i is the eigenvalue. We will list some simple properties of the system of eigenfunctions and eigenvalues which are valid for any kind of interatomic force.

1) As a consequence of the conservation theorems of the number of particles, momentum, and energy during a collision, five of the eigenfunctions are known. They are, except for normalization factors:

$$\psi_{000} \sim 1$$

$$\psi_{010} \sim c_z \quad \psi_{01-1} \sim c_x + ic_y \quad \psi_{011} \sim c_x - ic_y$$

and

$$\psi_{100} \sim c^2 - \frac{3}{2}$$

The corresponding eigenvalues are all zero. The meaning of the labels of the eigenfunctions will be explained below.

2) All the other eigenvalues are negative. This can be shown as follows: Multiplying both sides of Eq (7) by $\psi_i e^{-c^2}$ and integrating over $d\vec{c}$,

$$\lambda_i = \frac{\iiint d\vec{c} e^{-c^2} \psi_i J(\psi_i)}{\iiint d\vec{c} e^{-c^2} (\psi_i)^2}$$

The denominator is positive, while the numerator* can be transformed into

* It is essentially the negative of the bracket symbol $[\psi, \psi]$ used by Chapman.

$$-\frac{1}{4} \int d\vec{c} \int d\vec{c}_1 e^{-c^2 - c_1^2} \int d\epsilon \int d\theta \sin\theta g I(g, \theta) \cdot$$

$$\cdot (\psi_i' + \psi_{i1}' - \psi_i - \psi_{i1})^2$$

and is therefore negative or zero, and zero only for the five values of ψ_i 's mentioned above. Hence all the nonzero eigenvalues are negative.

3) Both Enskog⁷ and Chapman have given the expression for J in the standard form, namely:

$$J(\psi) = -\sigma(c)\psi - \int d\vec{c}_1 e^{-c_1^2} K(\vec{c}, \vec{c}_1)\psi(\vec{c}_1), \quad (8)$$

where

$$\sigma(c) \equiv \frac{2}{\sqrt{\pi}} \int d\vec{c}_1 e^{-c_1^2} \int d\theta \sin\theta g I(g, \theta) \quad (9)$$

and

$$K(\vec{c}, \vec{c}_1) = \frac{2}{\sqrt{\pi}} \int_0^\pi d\theta \sin\theta \left\{ g I(g, \theta) - \right.$$

$$\left. - g c c_1 \frac{\theta}{2} \left[I(g \csc \frac{\theta}{2}, \theta) + I(g \csc \frac{\theta}{2}, \pi - \theta) \right] \cdot \quad (9a) \right.$$

$$\left. e^{-(c^2 + c_1^2 - 2\vec{c} \cdot \vec{c}_1) \cot^2 \frac{\theta}{2}} J_0(-2ic c_1 \sin \frac{\theta}{2}, \cot \frac{\theta}{2}) \right\}$$

⁷ D. Enskog: Kinetische Theorie der Vorgänge in Mässig Verdünnten Gasen. Dissertation, Uppsala 1917, p. 140.

One sees that $K(\vec{c}, \vec{c}_1)$ is symmetric in \vec{c} and \vec{c}_1 and is isotropic, which means that K depends only on the magnitudes of c and c_1 and on the angle, ϕ' , between them.

4) From the isotropy of the kernel $K(\vec{c}, \vec{c}_1)$ it follows that the eigenfunctions ψ will have the form:

$$\psi_e(\vec{c}) = f_e(c^2) Y_e(\phi, \chi), \quad (10)$$

where ϕ and χ are the polar angles of \vec{c} with respect to a set of fixed axes, and where the radial part $f_e(c^2)$ of the eigenfunction fulfills the equation:

$$-\sigma(c) f_e - \frac{4\pi}{2\ell+1} \int d\Omega c_1^2 e^{-c_1^2} f_e(c_1^2) G_\ell(c, c_1) = \lambda_e f_e \quad (11)$$

$G_\ell(c, c_1)$ is defined by the development of $K(c, c_1, \cos\phi')$ in zonal harmonics:

$$K(c, c_1, \cos\phi') = \sum_{\ell'=0}^{\infty} G_{\ell'}(c, c_1) P_{\ell'}(\cos\phi'). \quad (12)$$

The proof is as follows: Introducing Eq (10) and Eq (12) into the integral in Eq (8), one gets:

$$\sum_{\ell'=0}^{\infty} \int d\vec{c}_1 e^{-c_1^2} G_{\ell'}(c, c_1) P_{\ell'}(\cos\phi') Y_e(\phi, \chi) f_e(c^2),$$

where ϕ_1 and χ_1 are the polar angles of \vec{c}_1 with respect to the same set of axes used for \vec{c} . For the integration over \vec{c}_1 , we take the direction of \vec{c} as the polar axis, so that the polar angles are ϕ' and χ' , and $d\vec{c}_1 = c_1^2 dc_1 \sin\phi' d\phi' d\chi'$. The integration over χ' can then be performed, using

$$\int_0^{2\pi} d\chi' Y_{\ell}(\phi, \chi') = 2\pi P_{\ell}(\cos \phi) Y_{\ell}(\phi, \chi),$$

which is a consequence of the addition theorem for spherical harmonics, and one obtains:

$$2\pi \sum_{\ell'=0}^{\infty} Y_{\ell}(\phi, \chi) \int_0^{\infty} dc, c^2 e^{-c^2} f_{\ell}(c^2) G_{\ell'}(c, c).$$

$$\cdot \int_0^{\pi} d\phi' \sin \phi' P_{\ell'}(\cos \phi') P_{\ell}(\cos \phi')$$

$$= \frac{4\pi}{2\ell+1} Y_{\ell}(\phi, \chi) \int_0^{\infty} dc, c^2 e^{-c^2} f_{\ell}(c^2) G_{\ell}(c, c).$$

Therefore if $f_{\ell}(c^2)$ fulfills Eq (11), $\psi_{\ell}(\vec{c})$ as given by Eq (10) will be an eigenfunction of $J(\psi)$.

5) The eigenvalues λ_{ℓ} in Eq (11) will clearly depend on ℓ , and on a "radial quantum number" r . We will write the eigenvalues as $\lambda_{r\ell}$; they are $(2\ell + 1)$ -fold degenerate, corresponding to the $2\ell + 1$ different spherical harmonics $Y_{\ell}(\phi, \chi)$ of order ℓ . The radial eigenfunctions $f_{r\ell}(c^2)$ form an orthogonal set of functions with a weight factor $c^2 e^{-c^2}$. The complete eigenfunctions will be written as

$$\psi_{r\ell m}(\vec{c}) = N_{r\ell m} Y_{\ell}^m(\phi, \chi) f_{r\ell}(c^2),$$

where N_{rlm} are appropriate normalization constants, so that:

$$\int d\vec{c} e^{-c^2} \psi_{rlm} \psi_{r'l'm'} = \delta_{rr'} \delta_{ll'} \delta_{mm'} .$$

The simplest problem to consider is the case where the disturbance h depends only on t , but not on x , y , and z . Eq (5) becomes, then:

$$\frac{\partial h}{\partial t} = n \sqrt{\frac{2kT}{m}} J(h). \quad (13)$$

Developing h in terms of the set of normalized eigenfunctions of J ,

$$h = \sum_l^\infty \alpha_{r'l'm'} \psi_{r'l'm'}(\vec{c}).$$

Substituting in Eq (13), multiplying by $\psi_{rlm} e^{-c^2}$, and integrating over $d\vec{c}$, one obtains the set of simple differential equations;

$$\frac{d\alpha_{rlm}}{dt} = n \sqrt{\frac{2kT}{m}} \lambda_{rlm} \alpha_{rlm} \quad (14)$$

for all values of n , l , and m . The solutions are

$$\alpha_{rlm}(t) = \alpha_{rlm}(0) e^{n \sqrt{\frac{2kT}{m}} \lambda_{rlm} t}$$

(because $\lambda < 0$)

For all nonzero eigenvalues, the coefficients α_{rlm} approach zero exponentially. For the five zero eigenvalues, the α 's remain constant. They are determined by the initial values of the density, mass velocity and temperature. The solution

is clearly a special case of the general result of Boltzmann (proved by means of the H-theorem) that the equilibrium distribution, as described by a Maxwell distribution with a mass velocity, is reached monotonically in time.

For the problem of sound propagation we will assume h to be of the form:

$$h(x, y, z) e^{i(\omega t - \sigma z)}$$

with ω real and $\sigma = \sigma_1 - i\sigma_2$. Eq (5) now becomes:

$$i\left(\sqrt{\frac{m}{2kT}} \omega - \sigma c_z\right) h = nJ(h). \quad (15)$$

Developing h as before, one obtains an infinite set of homogeneous linear equations with coefficients α_{rlm} . In order that they have a solution, the infinite determinant must be zero. This gives a relation between ω and σ , which is an expression of the exact dispersion law for the gas.

For the further development we will limit ourselves to the case of Maxwell molecules. For this case we have found an explicit expression for the "radial" eigenfunctions f_{rl} , which leads to a straightforward evaluation of the coefficients in the series expansions in Eq (2).

III. PROPAGATION OF SOUND IN A GAS OF MAXWELL MOLECULES

For Maxwell molecules the collision probability per second is independent of the relative velocity g . It is convenient to write:

$$gI(g, \theta) = \sqrt{\frac{2kT}{m}} F(\theta) \quad (16)$$

and redefine the collision operator as:

$$J(h) = \frac{1}{\pi^{3/2}} \int d\vec{c}_1 e^{-c_1^2} \int d\epsilon \int d\theta \sin\theta F(\theta) \cdot (h' + h_1' - h - h_1), \quad (17)$$

where κ is the force constant and $F(\theta)$ is dimensionless and a function of θ only, so that the collision operator J is also dimensionless and the eigenvalues of J will be pure numbers. In Appendix II, the definition and some of the properties of $F(\theta)$ will be given.

Adopting these changes, Eq (15) becomes

$$i(\omega - \sqrt{\frac{2kT}{m}} \sigma c_z) h = \eta \sqrt{\frac{2\kappa}{m}} J(h), \quad (18)$$

where

$$J(h) = -A_0 h - \int d\vec{c}_1 e^{-c_1^2} h(\vec{c}_1) K(c, c_1, \cos\phi') \quad (19)$$

with

$$A_0 \equiv 2\pi \int_0^\pi d\theta \sin\theta F(\theta)$$

and

$$\begin{aligned}
 K(c, c_1, \cos \phi') &= \frac{1}{\pi^{3/2}} \left\{ A_0 - \right. \\
 &- 4\pi \int_0^\pi d\theta \cos \frac{\theta}{2} \operatorname{csc}^2 \frac{\theta}{2} [F(\theta) + F(\pi - \theta)] \cdot \\
 &\left. e^{-\frac{(c^2 + c_1^2 - 2c \cdot c_1) \cot^2 \frac{\theta}{2}}{2}} J_0(\omega i c c_1 \sin \phi' \cot \frac{\theta}{2}) \right\}
 \end{aligned} \tag{20}$$

Since for small θ , $F(\theta)$ becomes infinite as $\theta^{-5/2}$ (see Appendix II), both terms in $J(h)$ are really infinite. However, the combination of the two terms is finite.

To find the radial eigenfunctions one first has to determine $G_\rho(c, c_1)$. From Eq (12), we have:

$$G_\rho = \frac{2l+1}{2} \int_0^\pi d\phi' \sin \phi' P_\rho(\cos \phi') K(c, c_1, \cos \phi').$$

With the help of the integral formula⁸

$$\begin{aligned}
 \int_0^\pi d\theta \sin^{\gamma+\frac{1}{2}} \theta e^{iz \cos \theta \cos \psi} J_{\gamma-\frac{1}{2}}(z \sin \theta \sin \psi) C_r^\nu(\cos \theta) &= \\
 &= \sqrt{\frac{2\pi}{z}} i^\gamma \sin^{\nu-\frac{1}{2}} \psi C_r^\nu(\cos \psi) J_{\nu+r}(z).
 \end{aligned}$$

⁸ Watson, Bessel Functions, p 379.

the ϕ' integration can be performed, leading to:

$$\begin{aligned}
 G_\ell(c, c_1) &= \frac{1}{\pi^{3/2}} A_0 S_{20} - \\
 &- \frac{2(2\ell+1)}{\sqrt{cc_1}} (i)^{\ell+\frac{1}{2}} \int_0^\pi d\theta \sqrt{\cos \frac{\theta}{2}} \csc \frac{\theta}{2} P_\ell(\cos \frac{\theta}{2}) \cdot \quad (21) \\
 &\cdot [F(\theta) + F(\pi-\theta)] \cdot e^{-(c^2+c_1^2)\cot^2 \frac{\theta}{2}} J_{\ell+\frac{1}{2}}(-2ic_1 \cot \frac{\theta}{2} \csc \frac{\theta}{2})
 \end{aligned}$$

and the "radial" integral equation:

$$\begin{aligned}
 -A_0 f_\ell(c^2) - \frac{4}{\sqrt{\pi}} A_0 S_{20} \int_0^\infty dc_1 c_1^2 e^{-c_1^2} f_\ell(c_1^2) + \\
 + \frac{8\pi}{\sqrt{cc_1}} (i)^{\ell+\frac{1}{2}} \int_0^\infty dc_1 c_1^2 e^{-c_1^2} f_\ell(c_1^2) \int_0^\pi d\theta \sqrt{\cos \frac{\theta}{2}} \csc \frac{\theta}{2} \cdot \\
 \cdot P_\ell(\cos \frac{\theta}{2}) [F(\theta) + F(\pi-\theta)] e^{-(c^2+c_1^2)\cot^2 \frac{\theta}{2}} \quad (22) \\
 \cdot J_{\ell+\frac{1}{2}}(-2ic_1 \cot \frac{\theta}{2} \csc \frac{\theta}{2}) = \lambda e f_\ell(c^2).
 \end{aligned}$$

We assert that the eigenfunctions are:

$$f_{rl}(c^2) = c^l S_{l+\frac{1}{2}}^{(r)}(c^2) \equiv c^l \sum_p (-)^p \frac{c^{2p} (l+\frac{1}{2}+r)!}{p!(r-p)!(l+\frac{1}{2}+p)!}, \quad (23)$$

where $S_{l+\frac{1}{2}}^{(r)}(c^2)$ are the Sonine polynomials of degree r and order $l + 1/2$, which are defined by the last part of Eq (23). For some of the properties of these polynomials, see Appendix III. To verify the above statement and to find the eigenvalues we substitute Eq (23) into Eq (22). The second term of Eq (22) is zero unless l is zero, and because of the orthogonality of the Sonine polynomials r must also be zero. Since f_{00} is 1, the second term can be written as $-A_0 \delta_{r0} \delta_{l0} f_{rl}$. Hence, interchanging the order of integration in the last term, one finds:

$$-A_0 f_{rl}(c^2) - A_0 \delta_{r0} \delta_{l0} f_{rl}(c^2) +$$

$$+ 8\pi (i)^{l+\frac{1}{2}} \frac{1}{\sqrt{c}} \sum_p (-)^p \int_0^\pi d\theta \sqrt{\cos \frac{\theta}{2}} \csc \frac{\theta}{2} P_l(\cos \frac{\theta}{2}) e^{-c^2 \cot^2 \frac{\theta}{2}}$$

$$\cdot [F(\theta) + F(\pi - \theta)] \frac{(l+\frac{1}{2}+r)!}{p!(r-p)!(l+\frac{1}{2}+p)!}$$

$$\cdot \int_0^\infty d\zeta \zeta^{l+\frac{1}{2}+2p} e^{-\zeta^2 \csc^2 \frac{\theta}{2}} J_{l+\frac{1}{2}}(-2i c \zeta \cot \frac{\theta}{2} \csc \frac{\theta}{2})$$

The c_1 integral is evaluated by using the formula⁹:

⁹ Watson, Watson, Bessel Functions, Eq (3), p 394.

$$\int_0^{\infty} dt t^{\mu-1} e^{-pt^2} J_{\nu}(at) =$$

$$= \frac{\Gamma(\frac{\mu+\nu}{2}) (\frac{a}{2p})^{\nu}}{2p^{\mu} \Gamma(\nu+1)} e^{-\frac{a^2}{4p^2}} {}_1F_1(\frac{\nu-\mu}{2}+1, \nu+1, \frac{a^2}{4p^2}),$$

and one obtains:

$$-A_0 f_{re}(c^2) (1 + \delta_{r0} \delta_{l0}) + \frac{2\pi}{(l+\frac{1}{2})!} c^l \int_0^{\pi} d\theta \sin \theta \cos^l \frac{\theta}{2}.$$

$$\cdot P_l(\cos \frac{\theta}{2}) [F(\theta) + F(\pi-\theta)] \cdot \sum_{p=0}^l (-)^p \frac{(l+\frac{1}{2}+r)!}{p!(l-p)!}.$$

$$\cdot \sin^{2p} \frac{\theta}{2} {}_1F_1(-p, l+\frac{3}{2}, -c^2 \cot^2 \frac{\theta}{2}) =$$

$$= -A_0 f_{re}(c^2) (1 + \delta_{r0} \delta_{l0}) + 2\pi c^l \int_0^{\pi} d\theta \sin \theta \cos^l \frac{\theta}{2}.$$

$$\cdot P_l(\cos \frac{\theta}{2}) [F(\theta) + F(\pi-\theta)] \sum_{p=0}^l (-)^p \frac{(l+\frac{1}{2}+r)!}{p!(l-p)!} \sin^{2p} \frac{\theta}{2}.$$

$$\cdot \sum_{j=0}^p \frac{p!}{j!(p-j)!(l+\frac{1}{2}+j)!} c^{2j} \cos^{2j} \frac{\theta}{2} \sin^{-2j} \frac{\theta}{2}$$

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Interchanging the order of summations, the sum over p is seen to be just $(-)^j \cos^{2(r-j)}\theta/2$; the sum over j is then:

$$\sum_{j=0}^r (-)^j c^{2j} \frac{(l+\frac{1}{2}+r)!}{j!(r-j)!(l+\frac{1}{2}+j)!} = S_{l+\frac{1}{2}}^{(r)}(c^2).$$

We thus arrive at the desired result; the eigenvalues are given by:

$$\lambda_{rl} = 2\pi \int_0^\pi d\theta \sin^2 \theta F(\theta).$$

(24)

$$\cdot \left[\cos^{2r+l} \frac{\theta}{2} P_l(\cos \frac{\theta}{2}) + \sin^{2r+l} \frac{\theta}{2} P_l(\sin \frac{\theta}{2}) - (1 + \delta_{r0} \delta_{l0}) \right].$$

All eigenvalues are negative except λ_{00} , λ_{10} , and λ_{01} , which are zero as in the general case. For the same value of l the eigenvalues decrease with increasing values of r . It can be proved that there is no lower bound (see Appendix IV). A few of the eigenvalues are listed in Table I. Here $A_{2k} = \int_0^\pi d\theta \sin^{2k+1} \theta F(\theta)$.

For the problem of the propagation of sound we can take for the complete set of eigenfunctions:

$$\psi_{rl} = \sqrt{\frac{r!(l+\frac{1}{2})}{\pi(l+\frac{1}{2}+r)!}} c^l P_l(\cos \phi) S_{l+\frac{1}{2}}^{(r)}(c^2)$$

The ψ_{rl} are normalized to unity with the weight factor e^{-c^2} . They are polynomials in c_x , c_y , and c_z of degree $2r + l$. And the first eight are listed in Table II. Developing the disturbance h in the ψ_{rl} , one gets for the coefficients α_{rl} the set of equations:

TABLE I

A Partial List of the Eigenvalues of the Collision Operator for the Maxwell Molecules

$\begin{array}{c} r \\ \ell \end{array}$	0	1	2
0	0	0	$-\frac{A_2}{2}$
1	0	$-\frac{A_2}{2}$	$-\frac{3A_2}{4}$
2	$-\frac{3A_2}{4}$	$-\frac{7A_2}{8}$	$-\frac{9A_2}{8} + \frac{3A_4}{16}$
3	$-\frac{9A_2}{8}$	$-\frac{11A_2}{8} + \frac{5A_4}{16}$	$-\frac{13A_2}{8} + \frac{19A_4}{32}$
4	$-\frac{7A_2}{4} + \frac{35A_4}{64}$	$-\frac{16A_2}{8} + \frac{115A_4}{128}$	$-\frac{18A_2}{8} + \frac{171A_4}{128} - \frac{35A_6}{256}$

TABLE II

First Eight of the Eigenfunctions of the collision Operator for the Maxwell Molecules

ψ_{00}	$\pi^{-\frac{3}{4}}$
ψ_{01}	$\sqrt{2} \pi^{-\frac{3}{4}} c_z$
ψ_{10}	$\sqrt{\frac{2}{3}} \pi^{-\frac{3}{4}} (\frac{3}{2} - c^2)$
ψ_{02}	$\frac{1}{\sqrt{3}} \pi^{-\frac{3}{4}} (3c_z^2 - c^2)$
ψ_{11}	$\frac{2}{\sqrt{15}} \pi^{-\frac{3}{4}} c_z (\frac{5}{2} - c^2)$
ψ_{03}	$\sqrt{\frac{2}{15}} \pi^{-\frac{3}{4}} (5c_z^3 - 3c^2 c_z)$
ψ_{12}	$\sqrt{\frac{2}{21}} \pi^{-\frac{3}{4}} (3c_z^2 - c^2) (\frac{7}{2} - c^2)$
ψ_{20}	$\sqrt{\frac{2}{15}} \pi^{-\frac{3}{4}} (\frac{15}{4} - 5c^2 + c^4)$

$$+ i\omega\alpha_{rl} - i\sqrt{\frac{2kT}{m}} \sigma \sum_{r'l'} M_{rl, r'l'} \alpha_{r'l'} = \eta \sqrt{\frac{2kT}{m}} \alpha_{rl} \lambda_{rl}, \quad (25)$$

where

$$M_{rl, r'l'} = \int d\vec{c} e^{-c^2} c_y \Psi_{rl} \Psi_{r'l'}$$

$M_{rl, r'l'}$ is symmetrical with respect to the pair of indices rl and $r'l'$.
Using the recurrence relation

$$\cos\theta P_l(\cos\theta) = \frac{1}{2l+1} [(l+1)P_{l+1} + lP_{l-1}]$$

and the integral properties of the Sonine polynomial given in Appendix III, one deduces easily the selection rules for the matrix elements $M_{rl, r'l'}$, namely

$$M_{rl, r'l'} = 0$$

unless $l' = l \pm 1$ and $2r' + l' = 2r + l \pm 1$. The complete expression for M is:

$$M_{rl, r'l'} = (l+1) \sqrt{\frac{(r+l+\frac{3}{2})}{(2l+1)(2l+3)}} \delta_{rr'} \delta_{l+1, l'} - (l+1) \sqrt{\frac{r}{(2l+1)(2l+3)}} \delta_{r-1, r'} \delta_{l+1, l'} +$$

$$\begin{aligned}
 & + l \sqrt{\frac{r+l+\frac{1}{2}}{(2l-1)(2l+1)}} \delta_{rr'} \delta_{l-l'} - \\
 & - l \sqrt{\frac{r+1}{(2l-1)(2l+1)}} \delta_{r+1r'} \delta_{l-l'}.
 \end{aligned}$$

The infinite set of linear equations (25) will have a solution if their determinant is zero. This will give a relation between ω and σ which is the general dispersion law of the gas.

We have not succeeded in developing a general discussion of the dispersion law, and we have therefore gone back to a successive-approximation method analogous to the Enskog-Chapman development. To do this, we have to choose a special ordering of the linear equations and consequently of the infinite determinant. The most natural ordering is according to the degree $2r + l$ of the eigenfunctions ψ_{rl} and for each group according to increasing values of l . The first eleven rows and columns of the determinant ordered in this way are shown in the page following. We have further introduced the dimensionless quantities

$$\omega_0 = \frac{\omega}{n \sqrt{\frac{2k}{m}}}, \quad \sigma_0 = \sqrt{\frac{kT}{\pi}} \frac{\sigma}{n},$$

so that

$$\frac{\sigma_0}{\omega_0} = \sqrt{\frac{2kT}{m}} \frac{\sigma}{\omega}.$$

The heavy lines indicate the successive stages of approximation, the choice of which will be explained in the next section.

2r+l	0		1		2		3		4			5	
	l	r	0	1	0	2	1	3	0	2	4	1	3
0	0	0	ω_0	$-\frac{1}{\sqrt{2}}\sigma_0$	0	0	0	0	0	0	0	0	0
1	1	0	$-\frac{1}{\sqrt{2}}\sigma_0$	ω_0	$\frac{1}{\sqrt{3}}\sigma_0$	$-\frac{\sqrt{2}}{\sqrt{3}}\sigma_0$	0	0	0	0	0	0	0
2	0	1	0	$\frac{1}{\sqrt{3}}\sigma_0$	ω_0	0	$-\frac{\sqrt{2}}{\sqrt{3}}\sigma_0$	0	0	0	0	0	0
	2	0	0	$-\frac{\sqrt{2}}{\sqrt{3}}\sigma_0$	0	$\omega_0 + i\lambda_{02}$	$\frac{2}{\sqrt{15}}\sigma_0$	$-\frac{3}{\sqrt{10}}\sigma_0$	0	0	0	0	0
3	1	1	0	0	$-\frac{\sqrt{2}}{\sqrt{3}}\sigma_0$	$\frac{2}{\sqrt{15}}\sigma_0$	$\omega_0 + i\lambda_{11}$	0	$\frac{\sqrt{2}}{\sqrt{3}}\sigma_0$	$-\frac{\sqrt{14}}{\sqrt{15}}\sigma_0$	0	0	0
	3	0	0	0	0	$-\frac{3}{\sqrt{10}}\sigma_0$	0	$\omega_0 + i\lambda_{03}$	0	$\frac{3}{\sqrt{35}}\sigma_0$	$-\frac{\sqrt{7}}{\sqrt{10}}\sigma_0$	0	0
4	0	2	0	0	0	0	$\frac{\sqrt{2}}{\sqrt{3}}\sigma_0$	0	$\omega_0 + i\lambda_{20}$	0	0	$-\frac{\sqrt{7}}{\sqrt{6}}\sigma_0$	0
	2	1	0	0	0	0	$-\frac{\sqrt{14}}{\sqrt{15}}\sigma_0$	$\frac{3}{\sqrt{35}}\sigma_0$	0	$\omega_0 + i\lambda_{12}$	0	$\frac{\sqrt{6}}{\sqrt{15}}\sigma_0$	$-\frac{2}{\sqrt{70}}\sigma_0$
	4	0	0	0	0	0	0	$-\frac{\sqrt{7}}{\sqrt{6}}\sigma_0$	0	0	$\omega_0 + i\lambda_{04}$	0	$\frac{4}{\sqrt{63}}\sigma_0$
5	1	2	0	0	0	0	0	0	$-\frac{\sqrt{7}}{\sqrt{6}}\sigma_0$	$\frac{\sqrt{6}}{\sqrt{15}}\sigma_0$	0	$\omega_0 + i\lambda_{21}$	0
	3	1	0	0	0	0	0	0	0	$-\frac{2}{\sqrt{70}}\sigma_0$	$\frac{4}{\sqrt{63}}\sigma_0$	0	$\omega_0 + i\lambda_{13}$

IV. SUCCESSIVE APPROXIMATIONS OF THE DISPERSION LAW FOR A MAXWELL GAS

As in the Enskog development, we will try to construct a successive-approximation method which will give the dispersion law as a series in λ/Λ , the ratio of the mean free path to the wavelength of the sound. It is to be noted that in the determinant the $\lambda_{r\ell}$ are of the order unity, while ω_0 and σ_0 are small, of the order λ/Λ .

1. Zeroth Approximation ("Ideal Fluid")

Clearly, taking only the first three rows and columns of the determinant, one uses only the zero eigenvalues of the collision operator $J(h)$. These correspond to the basic conservation laws of number, momentum, and energy, and one must expect therefore that this approximation corresponds to the ideal fluid approximation in the usual theory. One gets in fact,

$$\Delta_3 = \begin{vmatrix} \omega_0 & -\frac{\sigma_0}{\sqrt{2}} & 0 \\ -\frac{\sigma_0}{\sqrt{2}} & \omega_0 & \frac{\sigma_0}{\sqrt{3}} \\ 0 & \frac{\sigma_0}{\sqrt{3}} & \omega_0 \end{vmatrix} = \omega_0 \left(\omega_0^2 - \frac{5}{6} \sigma_0^2 \right) = 0$$

of which the roots are

$$\omega_0 = 0$$

and

$$\omega_0^2 = \frac{\sqrt{5}}{6} \sigma_0^2 \quad \text{or} \quad \omega = \frac{1}{6} \sigma$$

with

$$V_0 = \sqrt{\frac{5kT}{3m}},$$

which is the well-known result. One can also determine the zeroth approximation of the first three expansion coefficients $\alpha_r l$. One of them is arbitrary, which corresponds clearly to the arbitrariness of the amplitude of the sound. One finds for the root $\omega_0 = \sqrt{5/6} \sigma_0$:

$$h = \alpha_{00} \left\{ 1 + \sqrt{\frac{5m}{3kT}} \beta - \frac{2}{3} \left(\frac{3}{2} - \frac{m\bar{v}^2}{2kT} \right) \dots \right\} \quad (26)$$

By computing the number density, the mass velocity, and the temperature, it can easily be verified that Eq (26) is of the form to be expected.

2. First Approximation ("Stokes-Navier")

For this approximation we add two rows and two columns to the determinant. A comparison with the Enskog development shows that this corresponds to the Stokes-Navier approximation. Calling the determinant Δ_5 and developing, one obtains

$$\begin{aligned} \Delta_5 = & (\omega_0 + i\lambda_{02})(\omega_0 + i\lambda_{11}) \Delta_3 - \frac{2}{3} \omega_0^2 \sigma_0^2 (\omega_0 + i\lambda_{11}) - \\ & - \frac{5}{8} \sigma_0^2 \left(\omega_0^2 - \frac{\sigma_0^2}{2} \right) (\omega_0 + i\lambda_{02}) - \frac{4}{15} \omega_0^3 \sigma_0^2 + \frac{1}{9} \omega_0 \sigma_0^4 \end{aligned} \quad (27)$$

where $\lambda_{02} = -3A_2/4$ and $\lambda_{11} = -A_2/2$ as given in Table I. We are mainly interested in the development around the zeroth-order root $\sigma_0^2 = 6/5 \omega_0^2$, so we put

$$\sigma_0^2 = \frac{6\omega_0^2}{5} + \frac{b_1}{A_2} + \frac{b_2}{A_2^2} + \dots$$

as a solution of $\Delta_5 = 0$ and determine b_1 and b_2 by equating coefficients of equal powers of A_2 . This yields

$$b_1 = -\frac{56i\omega_0^3}{25} \qquad b_2 = -\frac{64.11\omega_0^4}{125}$$

so that

$$\sigma_0^2 = \frac{6\omega_0^2}{5} \left(1 - \frac{28}{15} \frac{i\omega_0}{A_2} - \frac{25.11}{3.5^2} \frac{\omega_0^2}{A_2^2} + \dots \right). \quad (28)$$

Changing to ordinary units and making use of the connection between A_2 and the viscosity coefficient, μ ,

$$\frac{1}{\eta \sqrt{\frac{2\pi}{m}} A_2} = \frac{5\mu}{4\rho V_0^2};$$

it follows that

$$\sigma^2 = \frac{\omega^2}{V_0^2} - \frac{7i\mu\omega^3}{3\rho V_0^4} - \frac{22}{3} \frac{\mu^2\omega^4}{\rho^2 V_0^6} + \dots \quad (29)$$

Eq (29) gives the following expressions for the velocity and coefficient of absorption of sound

$$\frac{c}{V} = \frac{c}{V_0} \left\{ 1 - \frac{215}{72} \frac{\mu^2 \omega^2}{\rho^2 V_0^4} \right\} \quad (29a)$$

$$\alpha_2 = \frac{7\mu\omega^2}{6\rho V_0^3} \quad (29b)$$

The first five expansion coefficients can again be determined in terms of α_{00} which remains arbitrary. For the root given by Eq (29) and up to the second power of μ , one gets:

$$\alpha_{01} = \sqrt{\frac{5}{3}} \alpha_{00} \left\{ 1 + \frac{7}{6} \frac{i\mu\omega}{\rho V_0^2} + \frac{13}{8} \left(\frac{\mu\omega}{\rho V_0^2} \right)^2 + \dots \right\}$$

$$\alpha_{10} = -\sqrt{\frac{2}{3}} \alpha_{00} \left\{ 1 + \frac{5}{2} \frac{i\mu\omega}{\rho V_0^2} + \frac{5}{2} \left(\frac{\mu\omega}{\rho V_0^2} \right)^2 + \dots \right\}$$

(30)

$$\alpha_{02} = \frac{10\sqrt{3}}{9} \alpha_{00} \frac{i\mu\omega}{\rho V_0^2} \left\{ 1 - \frac{2}{3} \frac{i\mu\omega}{\rho V_0^2} \right\}$$

$$\alpha_{11} = -\frac{5\sqrt{6}}{6} \alpha_{00} \frac{i\mu\omega}{\rho V_0^2} \left\{ 1 - \frac{1}{6} \frac{i\mu\omega}{\rho V_0^2} \right\}$$

It is to be noted that the dispersion law, Eq (29) or Eqs (29a) and (29b), agrees with the result obtained from the Stokes-Navier equations (Section I) only up to the first order in μ . If the Stokes-Navier equations had been considered as exact and if they had been developed in powers of μ (specializing for the Maxwell gas for which $\nu = (15/4)(k/m)\mu$), the velocity would have been obtained as

$$\frac{\omega}{V} = \frac{\omega}{V_0} \left\{ 1 - \frac{141}{72} \left(\frac{\mu c_0}{\rho V_0^2} \right)^2 + \dots \right\}$$

Our first approximation is therefore not quite identical with the Stokes-Navier approximation in the Enskog development. That the coefficient 215/72 in Eq (29a) is the right one*, follows from the second approximation, discussed below.

3. Second Approximation ("Burnett")

Three rows and columns are added which is again suggested by the Enskog development, giving:

$$\begin{aligned} \Delta_8 = & \left\{ (a_0 + i\lambda_{03})(a_0 + i\lambda_{20})(a_0 + i\lambda_{12}) - \frac{\rho_0^2}{35} (a_0 + i\lambda_{20}) \right\} \Delta_5 - \\ & - \sigma_0^2 \left\{ \frac{14}{15} (a_0 + i\lambda_{02})(a_0 + i\lambda_{03})(a_0 + i\lambda_{20}) + \frac{2}{3} (a_0 + i\lambda_{02})(a_0 + i\lambda_{03})(a_0 + i\lambda_{12}) + \right. \\ & \left. + \frac{2}{10} (a_0 + i\lambda_{11})(a_0 + i\lambda_{20})(a_0 + i\lambda_{12}) \right\} \Delta_3 + \\ & + \sigma_0^4 \left\{ \frac{28}{45} \omega_0^2 (a_0 + i\lambda_{20})(a_0 + i\lambda_{03}) + \frac{4}{9} \omega_0^2 (a_0 + i\lambda_{03})(a_0 + i\lambda_{12}) + \right. \\ & \left. + \frac{3}{4} (c_0^2 - \frac{\sigma_0^2}{2}) (a_0 + i\lambda_{20})(a_0 + i\lambda_{12}) \right\} + \end{aligned}$$

* It follows also from the Enskog development but only from the next-order or the Burnett approximation, as noted in Reference 4.

$$\begin{aligned}
 & + \sigma_0^4 \left\{ \frac{6}{25} (\omega_0 + i\lambda_{02}) + \frac{9}{25} (\omega_0 + i\lambda_{20}) + \frac{3}{5} (\omega_0 + i\lambda_{12}) \right\} \Delta_3 - \\
 & - \frac{2}{5} \omega_0 \sigma_0^6 (\omega_0 + i\lambda_{20}) - \frac{4}{25} \omega_0^2 \sigma_0^6 = 0
 \end{aligned}$$

Using the values of λ_{r1} given in Table I, and making the same kind of expansion for σ_0^2 , but now up to A_2^{-4} , one gets:

$$\begin{aligned}
 \sigma_0^2 = \frac{6\omega_0^2}{5} \left\{ 1 - \frac{28}{15} \frac{i\omega_0}{A_2} - \frac{2^{5.11}}{3.5^2} \frac{\omega_0^2}{A_2^2} + \right. \\
 \left. + \frac{2^{5.37}}{3.5^2} \frac{i\omega_0^3}{A_2^3} + \frac{2^{7.56761}}{3^3.5^{4.7}} \frac{\omega_0^4}{A_2^4} \dots \right\}.
 \end{aligned} \tag{31}$$

The expressions for the velocity and the absorption coefficient, which follow from this equation, are

$$\begin{aligned}
 \frac{\omega}{V} &= \frac{\omega}{V_0} \left\{ 1 - \frac{215}{72} \frac{\mu^2 \omega^2}{\rho^2 V_0^4} + \frac{4,115,101}{27.3^{4.7}} \frac{\mu^4 \omega^4}{\rho^4 V_0^8} \dots \right\} \\
 &= \frac{\omega}{V_0} \left\{ 1 - 2.99 \frac{\mu^2 \omega^2}{\rho^2 V_0^4} + 56.70 \frac{\mu^4 \omega^4}{\rho^4 V_0^8} \dots \right\}
 \end{aligned} \tag{31a}$$

and

$$\sigma_2 = \frac{\mu\omega^2}{\rho V_0^3} \left\{ \frac{7}{6} - \frac{5155}{6.72} \frac{\mu\omega^2}{\rho^2 V_0^4} \dots \right\}$$

(31b)

$$= \frac{\mu\omega^2}{\rho V_0^3} \left\{ 1.17 - 11.93 \frac{\mu\omega^2}{\rho^2 V_0^4} \dots \right\}$$

The corresponding expansion coefficients $\alpha_{r\ell}$ are given as follows:

$$\alpha_{01} = \sqrt{\frac{5}{3}} \alpha_{00} \left\{ 1 + \frac{7}{6} \frac{i\mu\omega}{\rho V_0^2} + \frac{13}{8} \left(\frac{\mu\omega}{\rho V_0^2} \right)^2 - \frac{2831}{24.33} i \left(\frac{\mu\omega}{\rho V_0^2} \right)^3 - \frac{5^2 \cdot 87907}{27.34.7} \left(\frac{\mu\omega}{\rho V_0^2} \right)^4 \dots \right\}$$

$$\alpha_{10} = -\sqrt{\frac{2}{3}} \alpha_{00} \left\{ 1 + \frac{5}{2} \frac{i\mu\omega}{\rho V_0^2} + \frac{5}{2} \left(\frac{\mu\omega}{\rho V_0^2} \right)^2 - \frac{5.13}{4} i \left(\frac{\mu\omega}{\rho V_0^2} \right)^3 - \frac{5.9.59}{2^2.7} \left(\frac{\mu\omega}{\rho V_0^2} \right)^4 \dots \right\}$$

$$\alpha_{02} = \frac{10\sqrt{3}}{9} \frac{i\mu\omega}{\rho V_0^2} \alpha_{00} \left\{ 1 - \frac{2}{3} \frac{i\mu\omega}{\rho V_0^2} - \frac{19}{9} \left(\frac{\mu\omega}{\rho V_0^2} \right)^2 + \frac{1333}{2.3^3.7} i \left(\frac{\mu\omega}{\rho V_0^2} \right)^3 \dots \right\}$$

$$\alpha_{11} = -\frac{5\sqrt{6}}{6} \frac{i\mu\omega}{\rho V_0^2} \alpha_{00} \left\{ 1 - \frac{1}{6} \frac{i\mu\omega}{\rho V_0^2} - \frac{89}{24} \left(\frac{\mu\omega}{\rho V_0^2} \right)^2 + \frac{67033}{2^4.3^3.7} i \left(\frac{\mu\omega}{\rho V_0^2} \right)^3 \dots \right\}$$

$$\alpha_{23} = -\frac{20}{7} \left(\frac{\mu\omega}{\rho V_0^2} \right)^2 \alpha_{00} \left\{ 1 - \frac{263}{2 \cdot 3^2 \cdot 7} \frac{i\mu\omega}{\rho V_0^2} \right\}$$

$$\alpha_{20} = -\frac{25}{130} \left(\frac{\mu\omega}{\rho V_0^2} \right)^2 \alpha_{00} \left\{ 1 - \frac{7}{2} \frac{i\mu\omega}{\rho V_0^2} \right\}$$

$$\alpha_{12} = \frac{10}{7} \sqrt{\frac{14}{3}} \left(\frac{\mu\omega}{\rho V_0^2} \right)^2 \alpha_{00} \left\{ 1 - \frac{13}{7} \frac{i\mu\omega}{\rho V_0^2} \right\}$$

One can go on in this way. In the third approximation eleven rows and columns must be taken, which permits development up to terms of the order μ^6 . The calculation becomes lengthy, and since the coefficients in the series increase rather rapidly so that the range of applicability of such a development seems rather limited, we have not gone any further. Several points should be noted in connection with the development outlined:

1) When the calculation is performed as outlined above, at no stage of the approximation are the results of the earlier approximations changed; each new stage simply adds two higher-order terms. This is seen by comparing Eqs (28) and (31), and it can also be proved in general*. It is due to the fact that in the n th approximation the added rows and columns have no nonvanishing elements extending farther than the blocks belonging to the $(n-1)$ th approximation. This in turn is due to the selection rule

$$2r' + l' = 2r + l \pm 1$$

in the matrix element $M_{rl, r'l'}$. This is also the reason why one gets at the n th stage of approximation a result for the dispersion law which is n steps better than the result obtained from the corresponding stage of approximation in

* The scheme for the higher approximations is that in the fourth and fifth approximation, one adds successively 4 rows and columns to the determinant; then in the next two orders of approximations one adds 5 rows and columns, and so on.

the Enskog development, as we noted previously, in the Stokes-Navier and the Burnett approximations.

2) For the expansion coefficients $\alpha_{r,p}$ of the distribution function, analogous theorems hold. At every stage of the approximation two further terms are obtained in the development in powers of μ , and these terms are unaltered by the higher approximations.

3) We know from Section III that for $\sigma_0 = 0$ the equation has three zero roots and all other roots are pure imaginary. For $\sigma_0 \neq 0$ all these roots will become functions of σ_0 , and may be called different modes of motion. Up to now we considered only the mode which in the zeroth order is given by $\omega_0^2 = 5\sigma_0^2/6$ and which therefore starts for $\sigma_0 = 0$ from two of the zero roots. For the mode starting from the third zero root one finds:

$$\omega^{(3)} = \frac{3}{2} \frac{i\mu\sigma^2}{\rho} \left\{ 1 - \frac{15}{4} \frac{\mu^2\sigma^2}{\rho^2 V_0^2} \right\} \quad (32)$$

For real σ all terms are pure imaginary, so that this mode may be called non-propagating, in contrast to the mode starting from the other zero roots, given by Eq (29), which can also be written:

$$\omega^{(1,2)} = \pm V_0 \sigma \left\{ 1 \pm \frac{7}{6} \frac{i\mu\sigma}{\rho V_0} + \frac{13}{8} \left(\frac{\mu\sigma}{\rho V_0} \right)^2 \pm \right\} \quad (33)$$

Here all powers of σ occur in the series expansion and the terms are alternately real and imaginary. For real σ , ω is complex, so the mode is propagating. The question arises whether there are other propagating modes.

4) The form of the series expansions of the different modes in powers of σ follows from the fact* that Δ is an even function of σ . Since in any approximation Δ is a homogeneous polynomial in ω_0 , σ_0 , and the iA_k , and since the A_k are proportional to $1/\mu$, the dispersion equation has in any approximation the form:

* This is physically obvious since σ and $-\sigma$ should always be equivalent; for a formal proof see Appendix IV.

$$P(x, y) = 0, \quad (34)$$

where P is a polynomial in $x = i\mu\omega_0$ and $y = i\mu\sigma_0$ with real coefficients. From the theory of the algebraic functions it is known that all the branches starting from the distinct roots of Eq (34) in x for $y = 0$ must be analytic functions of y . Since only even powers of y can occur in Eq (34), it is seen that in these cases the form of the development is:

$$i\mu\omega_0 = a_0 + a_1(\mu\sigma_0)^2 + a_2(\mu\sigma_0)^4 \dots$$

where the a_k 's are real. Therefore all distinct roots will not give rise to propagating modes.

5) Clearly the general situation will be that there are for $\sigma_0 = 0$ three zero roots and otherwise distinct and purely imaginary roots. One must expect, therefore, that in general there is only one propagating mode, corresponding to Eq (33). However, in the case of Maxwell molecules there are still "accidental" degeneracies. For instance, the eigenvalues λ_{11} and λ_{20} are equal and as a result the mode starting from these eigenvalues will have a form analogous to Eq (33). One finds in fact:

$$\omega = \frac{2i}{5} \frac{\rho V_0^2}{\mu} \left\{ 1 \pm \sqrt{5} \frac{\mu\sigma}{\rho V_0} \dots \right\}.$$

This should therefore also be called a propagating mode, although since the first term is purely imaginary the damping will be much stronger, so that it is unlikely that these modes have any physical significance.

V. EXTENSION TO OTHER MOLECULAR MODELS; FINAL REMARKS

In principle, the method of Section III can also be used for other force laws. However, there is then the practical difficulty that the eigenfunctions and eigenvalues of the collision operator are not known. In addition, the successive-approximation method described in Section IV is not so simple and straightforward.

The most obvious method for other molecular models is to use the same set of eigenfunctions in the expansion of the perturbation h as is used for Maxwell molecules*. Putting

$$h = \sum_{r'l'} \alpha_{r'l'} \psi_{r'l'}(\vec{c})$$

in Eq (15), multiplying by $\psi_{rl} e^{-c^2}$, and integrating over $d\vec{c}$, one finds:

$$i\omega \alpha_{rl} - i\sqrt{\frac{2kT}{m}} \sigma \sum_{r'l'} M_{rl,r'l'} \alpha_{r'l'} = \tag{35}$$

$$= \eta \sqrt{\frac{2kT}{m}} \sum_{r'l'} a_{rl,r'l'} \alpha_{r'l'},$$

where

* This is also essentially what is done in the Enskog-Chapman method.

$$\begin{aligned}
 a_{r'l, r'l'} &\equiv \int d\vec{c} e^{-c^2} \psi_{r'l} J(\psi_{r'l'}) = \\
 &= -\pi^{3/2} \sqrt{\frac{m}{2kT}} [\psi_{r'l}, \psi_{r'l'}]
 \end{aligned}
 \tag{36}$$

using the bracket symbol of Chapman. The $a_{r'l, r'l'}$ will be zero if $l' \neq l$ because of the isotropy of the collision operator J , but there will be no restriction on the "radial" numbers r and r' . If we now call:

$$\omega_i = \frac{\omega}{m \sqrt{\frac{2kT}{m}}} \qquad \sigma_i = \frac{\sigma}{m}
 \tag{37}$$

and arrange the determinant in the same manner as we did previously, then in the determinant Δ all elements proportional to σ are unchanged. The diagonal elements will be of the form $\omega_1 + ia_{r'l, r'l}$. The only difference from the form of the determinant in Section III is that there are new off-diagonal elements $ia_{r'l, r'l'}$.

Using the same successive-approximation method to derive the dispersion law for the propagating mode as before, one obtains in the zeroth approximation the same result as in the Maxwell case. This is, of course, to be expected, since the zeroth-order solution is governed solely by the conservation theorems. For the first approximation the expression for σ_1^2 is formally the same as Eq (27) in Section IV, except that in place of λ_{02} and λ_{11} we have $a_{02,02}$ and $a_{11,11}$ respectively. A comparison with the Enskog-Chapman development reveals that $a_{02,02}$ and $a_{11,11}$ are related to μ_1 and ν_1 , the "first approximation" of Chapman's expression for the viscosity and heat conduction coefficients, in exactly the same manner as the λ_{02} and λ_{11} are related to μ and ν in the Maxwell case. One finds, to the first order of small quantities:

$$\sigma_1^2 = \frac{6\omega_1^2}{5} \left\{ 1 + i\omega_1 \left[\frac{2}{5a_{11,11}} + \frac{4}{5a_{02,02}} \right] \dots \right\}$$

which, when transformed to ordinary units, reads:

$$\sigma^2 = \frac{\omega^2}{V_0^2} \left\{ 1 - \frac{i\omega}{\rho V_0^2} \left(\frac{4}{15} \frac{mV_1}{k} + \frac{4}{3} \mu_1 \right) \dots \right\}$$

This is the same expression as that derived from the Stokes-Navier equations except that μ_1 and ν_1 take the place of μ and ν .

Going over to the second approximation, one finds, in contrast to the case of the Maxwell molecules, that the coefficient of the first approximation is changed. The result, to the first order of small quantities, is now

$$\sigma_1^2 = \frac{6\omega_1^2}{5} \left\{ 1 + i\omega_1 \left[\frac{2}{5a_{11,11}} + \frac{4}{5a_{02,02}} \frac{a_{02,02} a_{12,12}}{a_{02,02} a_{12,12} - (a_{02,12})^2} \right] \dots \right\}$$

The change is that $1/a_{02,02}$ is replaced by:

$$\frac{1}{a_{02,02} \frac{a_{02,02} a_{12,12}}{a_{02,02} a_{12,12} - (a_{02,12})^2}},$$

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which in Chapman's notation means that μ_1 is replaced by μ_2 , the "second approximation" of the viscosity coefficient according to the method of Chapman.

The development proceeds in this manner. At every stage of the approximation the coefficients of all the lower-order terms change, and especially in the first-order term one gets successively higher approximations of the viscosity and heat conduction coefficients according to the method of Chapman.

This complication of our successive-approximation method for the dispersion law cannot be avoided, as one might perhaps surmise, by the use of the exact eigenfunctions of the collision operator, which would make the matrix $a_{r'l, r'l}$ diagonal in both r and l . Even with these eigenfunctions (supposing that one knew them!), one will in general not have the same selection rules for the "radial" number r in the matrix $M_{r'l, r'l}$ as one has for the case of the Maxwell molecules. As a result it is easily shown that in the second approximation the coefficient of the first-order or Stokes-Navier correction is modified.

From a practical point of view, this complication of the method is not very serious, since one can expect from the work of Chapman and Enskog that for the usual force laws the successive changes of the coefficients in each order will converge rapidly*. Clearly the origin of the method's complication lies in the fact that the arrangement of the determinant adopted for the Maxwell molecules is not suitable for other molecular models. We will now present an arrangement which will lead to "exact" expressions for the viscosity and heat conduction coefficients, and which presumably will enable one to make a more consistent series-expansion development of the dispersion law. For this purpose we use the set of eigenfunctions belonging to the Maxwell model, so that the set of linear equations is the set given by Eq (25). In the arrangement of the determinant we leave the zeroth approximation unchanged but order the rest according to the "azimuthal number" l . In each block corresponding to a value of l , the arrangement is in the order of increasing values of r up to infinity. For the first approximation the blocks $l = 1$ and $l = 2$ are added to the zeroth-order determinant. The second approximation is obtained by the further addition of the blocks $l = 0$ and $l = 3$ and all values of r . For the m th approximation, where $m > 2$, one simply borders the determinant for the $(m - 1)$ th approximation by the rows and columns corresponding to $l = m + 1$, and again all values of r . Using the notations defined by Eqs (36) and (37), the determinant for the first or the Stokes-Navier approximation is given below. The zeroth-order solution is

$$\sigma_1^2 = \frac{6\omega_1^2}{5}$$

* Compare the values of the successive approximations μ_1, μ_2, \dots and ν_1, ν_2, \dots of the viscosity and heat conduction coefficients in Chapman and Cowling, Chap. 10.

while for the first approximation one easily obtains the equation:

$$\sigma_1^2 = \frac{6\omega_1^2}{5} \left\{ 1 + i\omega_1 \left(\frac{2}{5} \frac{A_{11}}{A} + \frac{4}{5} \frac{B_{11}}{B} \right) \dots \right\} \quad (38)$$

$\begin{matrix} l \\ r \end{matrix}$	$\begin{matrix} l \\ r \end{matrix}$	0	1	0	1	2
0	0	ω_1	$-\frac{\sigma_1}{\sqrt{2}}$	0	0	0
1	0	$-\frac{\sigma_1}{\sqrt{2}}$	ω_1	$\frac{\sigma_1}{\sqrt{3}}$	0	$-\frac{\sqrt{2}}{3}\sigma_1$
0	1	0	$\frac{\sigma_1}{\sqrt{3}}$	ω_1	$-\frac{\sqrt{5}}{6}\sigma_1$	0
1	1		$-\frac{\sqrt{5}}{6}\sigma_1$			
2	2					0
3	3					
1	1	0	0	0		$\alpha\sigma_1$
2	2					$\alpha\sigma_1$
2	2					
2	2	0	0	0		

Handwritten annotations in the table include:

- $\omega_1 + ia_{r1,s1}$ (diagonal)
- $ia_{r1,s1}$ (off-diagonal)
- $\alpha\sigma_1$ (off-diagonal)
- $\omega_1 + ia_{r2,s2}$ (diagonal)
- $ia_{r2,s2}$ (off-diagonal)
- $\alpha\sigma_1$ (off-diagonal)

where $A = |a_{r1,s1}|$, r and s taking all values from 1 to ∞ , and $B = |a_{r2,s2}|$ with r and s starting from 0. A_{11} and B_{11} are the first minors of A and B respectively when the corresponding first row and first column are struck out. The form of Eq (38) suggests the following identifications*:

* For justification of these identifications, see Appendix V.

$$\frac{\omega_1 A_{11}}{A} = -\frac{2}{3} \frac{\eta}{k} \frac{\gamma \omega}{\rho V_0^2} \quad (39a)$$

$$\frac{\omega_1 B_{11}}{B} = -\frac{5}{3} \frac{\mu \omega}{\rho V_0^2}, \quad (39b)$$

so that Eq (38) reduces in ordinary units to

$$\sigma^2 = \frac{\omega^2}{V_0^2} \left\{ 1 - \frac{i\omega}{\rho V_0^2} \left(\frac{4m}{15k} \gamma + \frac{4}{3} \mu \right) \right\}$$

the same as the Stokes-Navier expression with μ and γ given by Chapman's formulas.

We conclude with some remarks on questions which have to be investigated further.

1) Clearly the outstanding problem is the question of the convergence of the series development for the dispersion law of the propagating mode, say, for Maxwell molecules. At every stage of approximation one gets a convergent series, as follows from the theory of algebraic functions. However, it is not sure that the regions of convergence in the successive stages of approximation have a region in common.

2) It would be very desirable to obtain an expression for the dispersion law which could be used for larger values of λ/Λ , so that a comparison with the experiments of Greenspan could be made. A development in powers of Λ/λ seems indicated, but we have not succeeded in obtaining such a series. It is perhaps possible to cast the infinite determinant into other analytical forms for the case of Maxwell molecules, and perhaps in this way a better discussion of the dispersion laws for the different modes can be made.

3) For a proper discussion of the experiments it is perhaps necessary to include the effect of the solid walls through which the sound vibrations are

transmitted to the gas. In this connection it will be of interest to treat simple problems, like the Couette flow or the heat conduction between two parallel plates, where one has to take the effect of the walls into account. If the flow is slow, or the relative temperature difference small, then it is possible to formulate the problem for all values of λ/L (L = distance between plates) in a way analogous to the dispersion-of-sound problem.

4) These problems touch also the question of the calculation of the viscosity slip in gases as formulated recently by Kramers¹⁰ .

5) Finally, one should perhaps emphasize again that in the method proposed here it is in principle possible, for all types of flow problems, to separate sharply the effect of the (molecular) Mach number from the effect of the Knudsen number. For example, it will now perhaps be possible to discuss properly the drag of a sphere in a gas for small speeds, but for arbitrary ratio of the mean free path to the diameter of the sphere.

¹⁰ H. A. Kramers, Nuovo Cimento, 6, 297, 1949.

APPENDICES

APPENDIX I. DEFINITION AND PROPERTIES OF $F(\theta)$

From the relationship between the angle θ through which the relative velocity is turned in a collision, and the collision parameter b , one obtains for Maxwell molecules:

$$\frac{\pi - \theta}{2} = \int_0^{x'} \frac{dx}{\sqrt{1 - x^2 - \frac{1}{2} \left(\frac{x}{\alpha}\right)^4}}$$

where $x = b/r$, $\alpha = \sqrt[4]{\frac{mg^2}{2k}}$ b , and x' is the smallest positive root of the equation:

$$1 - x^2 - \frac{1}{2} \left(\frac{x}{\alpha}\right)^4 = 0.$$

Making the substitution

$$\alpha^4 = 2 \cot^2 2\phi,$$

so that the range $b = 0$ to $b = \infty$ corresponds to $\phi = \pi/4$ to $\phi = 0$, one finds

$$\frac{\pi - \theta}{2} = \sqrt{\cos 2\phi} K(\sin \phi),$$

where

$$K(\sin \phi) = \int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1 - \sin^2 \phi \sin^2 \psi}}$$

is the complete elliptical integral of the first kind. This relationship between θ and ϕ has been tabulated by Maxwell¹¹ and is reproduced in Table III.

TABLE III
Relationship Between θ and ϕ for Maxwell molecules

ϕ		θ		ϕ		θ	
o	'	o	'	o	'	o	'
0	0	0	0	40	0	94	40
5	0	1	2	41	0	103	4
10	0	4	10	42	0	112	52
15	0	9	34	43	0	124	44
20	0	17	30	44	0	140	36
25	0	28	30	44	30	152	2
30	0	43	24	45	0	180	0
35	0	63	58				

In terms of the variable ϕ , $F(\theta)$, which is defined as

$$\begin{aligned} F(\theta) \sin \theta d\theta &= \sqrt{\frac{m}{2\pi}} g I(g, \theta) \sin \theta d\theta \\ &= \sqrt{\frac{m}{2\pi}} g b db, \end{aligned}$$

¹¹ Maxwell Collected Papers II, p 42.

is given by

$$F(\theta) = \frac{1}{4} \frac{\sqrt{\cos 2\phi}}{\sin \theta \sin 2\phi (\cos^2 \phi K - \cos 2\phi E)},$$

where

$$E = \int_0^{\frac{\pi}{2}} d\psi \sqrt{1 - \sin^2 \phi \sin^2 \psi}$$

is the complete elliptical integral of the second kind. $F(\theta)$ is a monotonically decreasing function of θ . For small θ

$$F(\theta) \approx \frac{\sqrt{5}\pi}{16} \theta^{-\frac{5}{2}} \left(1 + \frac{35}{24\pi} \theta + \right),$$

while for $\theta = \pi$

$$F(\pi) = \frac{1}{4 [K(\frac{\pi}{4})]^2} = 0.0727$$

APPENDIX II. PROPERTIES OF THE SONINE POLYNOMIALS

The Sonine polynomial $S_n^{(m)}(x)$ is defined as the coefficient of s^m in the expansion of

$$(1-s)^{-n-1} e^{-\frac{xs}{1-s}}$$

Its explicit expression is

$$S_n^{(m)}(x) = \sum_{p=0}^m (-x)^p \frac{(n+m)!}{p! (m-p)! (n+p)!},$$

so that

$$S_n^{(0)} = 1$$

$$S_n^{(1)} = n+1 - x$$

$$S_n^{(2)} = \frac{1}{2}(n+1)(n+2) - (n+2)x + \frac{1}{2}x^2,$$

etc. The following integrals can be easily derived from the generating function:

1) Orthogonality relation

$$\int_0^{\infty} dx x^n e^{-x} S_n^{(m)} S_n^{(m')} = \frac{\Gamma(n+m+1)}{\Gamma(m+1)} \delta_{mm'}$$

$$2) \int_0^{\infty} dx x^n e^{-x} S_{n-1}^{(m)} S_n^{(m)} = \frac{\Gamma(n+m+1)}{\Gamma(m+1)}$$

$$3) \int_0^{\infty} dx x^n e^{-x} S_{n-1}^{(m+1)} S_n^{(m)} = -\frac{\Gamma(n+m+1)}{\Gamma(m+1)}$$

APPENDIX III. THERE IS NO LOWER BOUND FOR THE SET OF EIGENVALUES

$$\lambda_{rl} = 2\pi \int_0^\pi d\theta \sin\theta F(\theta) \cdot$$

$$\cdot \left\{ \cos^{2r+l} \frac{\theta}{2} P_l(\cos \frac{\theta}{2}) + \sin^{2r+l} \frac{\theta}{2} P_l(\sin \frac{\theta}{2}) - \right. \\ \left. - (1 + \delta_{r0} \delta_{l0}) \right\}$$

We prove this statement in two steps:

1) $|\lambda_{rl}| > |\lambda_{r0}|.$

This is evident for $r = 0$. For $r \neq 0$

$$\lambda_{rl} - \lambda_{r0} = 2\pi \int_0^\pi d\theta \sin\theta F(\theta) \left\{ \cos^{2r} \frac{\theta}{2} \left[\cos^l \frac{\theta}{2} P_l(\cos \frac{\theta}{2}) - 1 \right] + \right. \\ \left. + \sin^{2r} \frac{\theta}{2} \left[\sin^l \frac{\theta}{2} P_l(\sin \frac{\theta}{2}) - 1 \right] \right\}$$

Since $F(\theta)$ is always positive, and in the interval $0 \leq x \leq 1$

$$x^l P_l(x) \leq 1,$$

it follows that

$$\lambda_{re} - \lambda_{ro} = \text{negative}$$

or

$$|\lambda_{re}| > |\lambda_{ro}|$$

because λ_{re} is negative. Hence if λ_{ro} has no lower bound, there will also be no lower bound for λ_{re} .

2) There is no lower bound for λ_{ro} .

For $r \neq 0$,

$$\lambda_{ro} = 2\pi \int_0^\pi d\theta \sin\theta F(\theta) \left(\cos^{2r} \frac{\theta}{2} + \sin^{2r} \frac{\theta}{2} - 1 \right)$$

Since $F(\theta)$ diverges at $\theta = 0$ and decreases monotonically with increasing θ ; and for large r , $(\cos^{2r} \theta/2 + \sin^{2r} \theta/2 - 1)$ is approximately -1 except for θ near 0 and π , we see that the main contribution to the integral will be from the neighborhood $\theta = 0$. Thus we can approximate $F(\theta)$ by

$$F(\theta) = \frac{1}{64} \sqrt{\frac{3\pi}{2}} \sin^{-\frac{5}{2}} \frac{\theta}{2}$$

which has the correct behavior for small θ , and lies always below the true $F(\theta)$ curve for $\theta > 0$. Further, we can drop the term $\sin^{2r} \theta/2$, since its contribution goes to zero as $r \rightarrow \infty$. Changing the integration variable from θ to $x = \theta/2$,

$$\lambda_{r0} \cong \frac{\pi}{8} \sqrt{\frac{3\pi}{2}} \int_0^{\frac{\pi}{2}} dx \sin x \cos x \frac{1}{\sin \frac{5}{2} x} (\cos^{2r} x - 1)$$

$$= -\sqrt{\frac{3}{2}} \frac{\pi}{16} \sum_{i=0}^{r-1} \frac{\Gamma(i+1) \Gamma(\frac{3}{4})}{\Gamma(i+\frac{7}{4})}$$

For large r, the ratio of the i + 1 to the ith term of the sum is:

$$\frac{i+1}{i+\frac{7}{4}} \cong 1 - \frac{3}{4i}$$

Thus:

$$\lim_{r \rightarrow \infty} \lambda_{r0} = -\sqrt{\frac{3}{2}} \frac{\pi}{16} \Gamma(\frac{3}{4}) \lim_{r \rightarrow \infty} \sum_{i=0}^{r-1} \frac{\Gamma(i+1)}{\Gamma(i+\frac{7}{4})}$$

$$= -\infty$$

APPENDIX IV. PROOF THAT THE DETERMINANT Δ IS AN EVEN FUNCTION OF σ_0

This property of the determinant is independent of the interatomic forces. It is a consequence of the selection rule

$$l' = l \pm 1.$$

We will assume h to be developed in the set of eigenfunctions ψ_{rl} of the linear operator J , and we will arrange the determinant according to the values of l . The determinant then takes the form:

l	0	1	2	3
l	0	0	0	0
0	0	$\sim \sigma_0$	0	0
1	$\sim \sigma_0$	0	$\sim \sigma_0$	0
2	0	$\sim \sigma_0$	0	$\sim \sigma_0$
3	0	0	$\sim \sigma_0$	0

Along the main diagonal we have $\omega_0 + i\lambda_{rl}$, the nondiagonal elements in the diagonal blocks are all zero, and σ_0 appears only in the off-diagonal blocks as indicated. These blocks may be square or not, and they may have any

number of rows and columns, depending on the atomic model and on the order of approximation. If we now change the signs of all the elements in the rows and the corresponding columns, all the signs of the σ_0 's change to their negatives, while the signs along the main diagonal remain unchanged. The determinant Δ is, however, not changed because we have made an even number of sign changes and therefore we have shown that:

$$\Delta(\sigma_0) = \Delta(-\sigma_0).$$

APPENDIX V. JUSTIFICATIONS FOR EQS (39a) AND (39b)

By comparing the definitions of $a_{rl,sl}$ with the quantities a_{rs} and b_{rs} used by Chapman and Cowling¹², it is easily seen that:

$$a_{r1, s1} = -\frac{\pi^{3/2}}{3} \sqrt{\frac{m}{2kT}} N_{r1} N_{s1} a_{rs}$$

$$a_{r2, s2} = -\frac{3\pi^{3/2}}{10} \sqrt{\frac{m}{2kT}} N_{r2} N_{s2} b_{rs},$$

where N_{rl} is the normalization factor for ψ_{rl} , namely,

$$N_{rl} = \sqrt{\frac{r!(l+\frac{1}{2})}{\pi(l+r+\frac{1}{2})!}}$$

If we call $(A)_C = |a_{rs}|$, then:

$$A^{(p)} = (-)^p \left(\frac{\pi^{3/2}}{3}\right)^p \left(\frac{m}{2kT}\right)^{\frac{p}{2}} (N_{11} N_{21} \dots N_{p1})^2 (A^{(p)})_C$$

$$A_{11}^{(p)} = (-)^{p-1} \left(\frac{\pi^{3/2}}{3}\right)^{p-1} \left(\frac{m}{2kT}\right)^{\frac{p-1}{2}} (N_{21} N_{31} \dots N_{p1})^2 (A_{11}^{(p)})_C$$

¹² Chapman and Cowling, The Mathematical Theory of Non-Uniform Gases, Chap. 7.

where the superscript p means that p rows and p columns are taken in A and $(A)_C$.
Hence

$$\frac{\omega_1 A_{11}^{(p)}}{A^{(p)}} = - \frac{3}{\pi \pi^{3/2}} \frac{\omega}{N_{11}^2} \frac{(A_{11}^{(p)})_C}{(A^{(p)})_C}$$

Since $N_{11} = \sqrt{\frac{4}{5}} \pi^{-3/4}$ and

$$\lim_{p \rightarrow \infty} \frac{(A_{11}^{(p)})_C}{(A^{(p)})_C} = \frac{16}{225} \cdot \frac{3m\gamma}{2k^2 T} \quad 12$$

it follows that

$$\frac{\omega_1 A_{11}}{A} = \lim_{p \rightarrow \infty} \frac{\omega_1 A_{11}^{(p)}}{A^{(p)}} = - \frac{2}{3} \frac{m \gamma \omega}{k \rho V_0^2}$$

Similarly, by making use of $N_{02} = \sqrt{\frac{4}{3}} \pi^{-3/4}$ and Chapman's result:

$$\lim_{p \rightarrow \infty} \frac{(B_{11}^{(p)})_C}{(B^{(p)})_C} = \frac{2\mu}{5kT},$$

one establishes Eq (39b).