TRANSPORT PHENOMENA IN VERY DILUTE GASES, II

BY

C. S. WANG CHANG

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INTRODUCTION

In a previous report[1] the method of successive collisions was used to calculate the heat flux between two parallel plates one step better than in the usual Knudsen approximation, in which only collisions between the gas molecules and the walls containing the gas are considered. The method is a general and a systematic successive approximation method. In order to use it in other cases -- and we will consider especially the problem of the drag on a sphere moving in a dilute gas -- it is necessary to find first the zeroth order volocity distribution function in each case. This involves a variation of the usual kinetic methods used for the Knudsen gas, and, although it leads, of course, to the same results for observable quantities like the drag or the average velocity distribution, the method is sufficiently different that it seems worth while to record the main steps. We will give the calculations in detail for the case of a sphere moving through a Knudsen gas and give only the results for the cases of (1) a small circular plate moving through a Knudsen gas and (2) the flow of a Knudsen gas through a narrow circular tube.

ZEROTH ORDER DISTRIBUTION FUNCTION FOR A SPHERE OF RADIUS R IN A GAS WITH STREAMING VELOCITY V

Our aim is to find the distribution function of a steady gas stream as modified by the presence of a small sphere fixed in space. The zeroth approximation

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1 C. S. Wang Chang and G. E. Uhlenbeck. Transport Phenomena in Very Dilute Gases. CM579, Nov. 15, 1949, UMH-3-F.

means that the gas is assumed to be so dilute and the size of the sphere so small that the collision between the molecules of the main stream and those reflected from the sphere can be neglected. We assume also that there are no outside forces For the calculation we will introduce the following notations:

$\vec{V}$, the velocity of the main stream

R, the radius of the sphere

$\vec{r}$, the point under consideration, or the point at which the velocity distribution is being calculated

$\vec{n}$, the outward normal to the sphere which passes through the point r. $\vec{r} = r\vec{n}$ if the center of the sphere is taken as the origin of the coordinate system.

$\vec{c}$, the velocity vector

α, the fraction of the molecules that is diffusely reflected from the sphere

$1-\alpha$, the fraction of the molecules that is specularly reflected from the sphere

$\vec{n}'$, the normal to the sphere at the point from which the molecules arriving at r with velocity $\vec{c}$ originated.

The geometry of the system is shown in Fig. 1. From this figure one deduces easily that

$$\vec{n}' = \frac{\vec{r}}{R} - \frac{\vec{r}\cdot\vec{c}\ \vec{c}}{Rc^2} + \sqrt{R^2 - r^2 + \left(\frac{r\cdot c}{c}\right)^2}\ \frac{\vec{c}}{Rc} \quad . \tag{1}$$

The zeroth order distribution $f^{(0)}(x,y,z;\ c_x,\ c_y,\ c_z)$ has to fulfill the Boltzmann equation without the collision term

$$c_\alpha \frac{\partial f^{(0)}}{\partial x_\alpha} = 0 \quad ,$$

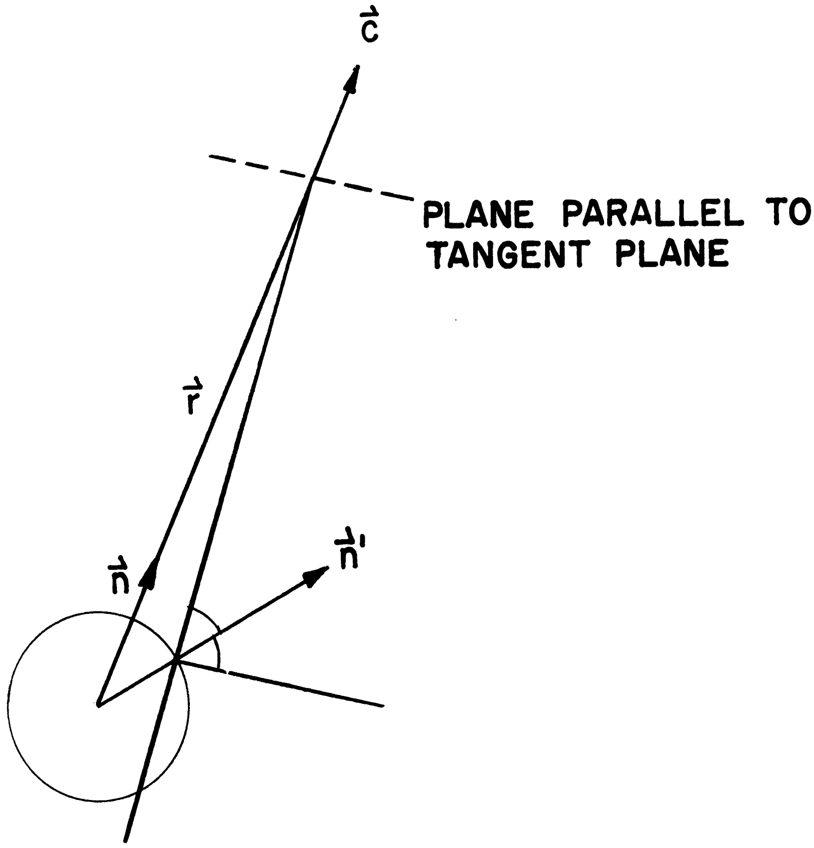

FIG. 1

and it has to satisfy the following boundary conditions:

(a) At infinity

$$f^{(0)} = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m}{2kT}(\vec{c}-\vec{V})^2} , \tag{2}$$

where the scalar n is the number density in the main stream or the number density at infinity.

(b) On the surface of the sphere

$$f^{(0)} = n\left(\frac{m}{2\pi kT}\right)^{3/2}\left\{\frac{1-\operatorname{sign}\vec{n}\cdot\vec{c}}{2}\, e^{-\frac{m}{2kT}(\vec{c}-\vec{V})^2} + \frac{\alpha}{2}(1+\operatorname{sign}\vec{n}\cdot\vec{c})A_R^{(0)} e^{-\frac{mc^2}{2kT}} + \frac{1-\alpha}{2}(1+\operatorname{sign}\vec{n}\cdot\vec{c})B_R^{(0)} e^{-\frac{m}{2kT}(\vec{c}-\vec{V}-2\vec{n}\,\vec{n}\cdot\vec{c})^2}\right\} , \tag{3}$$

where $A_R^{(o)}$ and $B_R^{(o)}$ are constants, independent of $\vec{c}$, to be determined by the third boundary condition, namely:

(c) On the surface

$$\overline{\vec{n}\cdot\vec{c}}^{\,c} = 0 \quad . \tag{4}$$

The meaning of these boundary conditions is clear. The first one (a) says simply that at infinity the influence due to the presence of the sphere at the origin is so small that the distribution function is not modified. The second condition (b) says that on the surface of the sphere, the molecules consist of three groups: (1) those incident (signified by $1/2(1\text{-sign } \vec{n}\cdot\vec{c})$) on the sphere from the main stream; (2) the molecules emitted (signified by $1/2\ (1 + \text{sign } \vec{n}\cdot\vec{c})$) by the sphere by way of "diffuse reflection" (second term); and (3) by way of specular reflection (third term). The third condition (c) states the fact that the sphere neither absorbs nor emits gas molecules by itself so that all those that hit the sphere are reemitted. This condition is to be true for all values of α, hence it is sufficient for the determination of both $A_R^{(o)}$ and $B_R^{(o)}$.

The distribution function we aim to find is a superposition of three distribution functions, namely:

$$n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m}{2kT}(\vec{c}-\vec{V})^2}$$ of the main stream

$$n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mc^2}{2kT}}$$ of the diffusely reflected molecules

$$n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m}{2kT}(\vec{c}-\vec{V}-2\vec{n}'\,\vec{n}'\cdot c)^2}$$ of the specularly reflected molecules.

Thus one can write the distribution function as a linear combination of the three distribution functions given above with coefficients determined solely by the geometry of the problem. We write:

$$f^{(0)} = n\left(\frac{m}{2\pi kT}\right)^{3/2}\Big\{ S\, e^{-\frac{m}{2kT}(\vec{c}-\vec{V})^2}$$

$$+ \alpha A^{(0)}(V\cdot n')(1-S)\, e^{-\frac{mc^2}{2kT}} \tag{5}$$

$$+ (1-\alpha)B^{(0)}(V\cdot n')(1-S)\, e^{-\frac{m}{2kT}(\vec{c}-\vec{V}-2\vec{n}'\vec{n}'\cdot c)^2},$$

where $S(x,y,z;\ \vec{n}_c)$, $\vec{n}_c$ being the unit vector in the direction of $\vec{c}$, is a discontinuous function defined as follows: It is zero if the molecule with the velocity $\vec{c}$ at the point $\vec{r}(x,y,z)$ comes from the sphere, and one otherwise. In other words, S is equal to zero if the vector $\vec{n}_c$ points away from the sphere and lies within the solid angle subtended by the sphere at the point under consideration, and is one otherwise. Analytically speaking S is zero if:

$\vec{n}\cdot\vec{n}_c$ is positive

and $\quad r^2 - (\vec{r}\cdot\vec{n}_c)^2 < R^2.$

An analytical representation of 1-S is thus

$$1-S = \frac{1}{4}(1+\operatorname{sign}\vec{n}\cdot\vec{c})\left[1-\operatorname{sign}\left\{r^2-(\vec{r}\cdot\vec{c}_n)^2-R^2\right\}\right]$$

$$= \frac{1}{4}(1+\operatorname{sign}\vec{n}\cdot\vec{c})\left[1-\operatorname{sign}\left(\sqrt{1-\frac{R^2}{r^2}}-\vec{n}\cdot\vec{c}_n\right)\right]. \tag{6}$$

S can then be written as

$$S = \frac{1}{2}(1-\text{sign}\,\vec{n}\cdot\vec{c}) + \frac{1}{4}(1+\text{sign}\,\vec{n}\cdot\vec{c})\left[1+\text{sign}\left(\sqrt{1-\frac{R^2}{r^2}} - \frac{\vec{n}\cdot\vec{c}}{c}\right)\right],$$

where we have written it in a form so that the physical meaning is clear.

The $f^{(o)}$ so constructed (Eq 5) with S given by Eq 6 satisfies already the boundary conditions (a) and (b). For $r = \infty$, $\text{sign}\left(\sqrt{1-\frac{R^2}{r^2}} - \frac{\vec{n}\cdot\vec{c}}{c}\right)$ is positive (except for $\vec{c}\cdot\vec{n} = 1$, for which the sign function is not defined),

$$S = 1$$

$$1 - S = 0\,,$$

hence, $f^{(o)}$ has the form given by Eq 2. For $r \rightarrow R$,

$$S = (1/2)(1 - \text{sign}\,\vec{n}\cdot\vec{c})$$

$$1 - S = (1/2)(1 + \text{sign}\,\vec{n}\cdot\vec{c}),$$

and n' = n, Eq 5 reduces to Eq 3.

The coefficients $A^{(o)}$ and $B^{(o)}$ are to be determined by the third boundary condition, (c). Actually Eq 4 gives only the constants $A_R^{(o)}$ and $B_R^{(o)}$. But by simple physical arguments one can arrive at the more general expressions $A^{(o)}$ and $B^{(o)}$. Since Eq 4 is to be true for all values of α, we have from Eq 4, the following equations:

$$\iiint d\vec{c}\;\vec{n}\cdot\vec{c}\,(1-\text{sign}\,\vec{n}\cdot\vec{c})\,e^{-\frac{m}{2kT}(\vec{c}-\vec{V})^2} + \tag{7}$$

$$+ A_R^{(o)}\iiint d\vec{c}\;\vec{n}\cdot\vec{c}\,(1+\text{sign}\,\vec{n}\cdot\vec{c})\,e^{-\frac{mc^2}{2kT}} = 0\,,$$

$$\iiint d\vec{c}\ \vec{n}\cdot\vec{c}\ (1-\text{sign}\ \vec{n}\cdot\vec{c})\ e^{-\frac{m}{2kT}(\vec{c}-\vec{V})^2}$$

$$+B_R^{(0)}\iiint d\vec{c}\ \vec{n}\cdot\vec{c}\ (1+\text{sign}\ \vec{n}\cdot\vec{c})\ e^{-\frac{m}{2kT}(\vec{c}-\vec{V}-2\vec{n}\,\vec{n}\cdot\vec{c})^2} = 0 \qquad (8)$$

Introducing dimensionless velocities, using rectangular axes and taking $\vec{n}$ to be the x-axis, Eqs 7 and 8 become:

$$A_R^{(0)}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty dc_x\,dc_y\,dc_z\ c_x\,e^{-c^2}$$

$$= -\int_{-\infty}^0\int_{-\infty}^\infty\int_{-\infty}^\infty dc_x\,dc_y\,dc_z\ c_x\,e^{-(c^2+V^2-2\vec{c}\cdot\vec{V})}$$

$$B_R^{(0)}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty dc_x\,dc_y\,dc_z\ c_x\,e^{-(c^2+V^2-2\vec{c}\cdot\vec{V}+4c_xV_x)}$$

$$= -\int_{-\infty}^0\int_{-\infty}^\infty\int_{-\infty}^\infty dc_x\,dc_y\,dc_z\ c_x\,e^{-(c^2+V^2-2\vec{c}\cdot\vec{V})}$$

Integrating, we have:

$$A_R^{(0)} = 2\int_0^\infty dc_x\,c_x\,e^{-(c_x+V_x)^2}$$

$$B_R^{(0)}\int_0^\infty dc_x\,c_x\,e^{-(c_x^2+2c_xV_x+V_x^2)}$$

$$= \int_0^\infty dc_x\,c_x\,e^{-(c_x^2+2c_xV_x+V_x^2)}$$

or

$$A_R^{(0)} = e^{-V_x^2} - \sqrt{\pi} V_x + \sqrt{\pi} V_x \, \mathrm{erf}(V_x)$$

$$B_R^{(0)} = 1 .$$

The error function is defined as:

$$\mathrm{erf}(V_x) = \frac{2}{\sqrt{\pi}} \int_0^{V_x} dx \, e^{-x^2} .$$

Remembering that we have taken $\vec{n}$ as the x-axis, we conclude now that

$$A_R^{(0)} = e^{-(\vec{V}\cdot\vec{n})^2} - \sqrt{\pi} \vec{V}\cdot\vec{n} + \sqrt{\pi} \vec{V}\cdot\vec{n} \, \mathrm{erf}(\vec{V}\cdot\vec{n})$$

$$B_R^{(0)} = 1$$

We have so far obtained the values of $A^{(0)}$ and $B^{(0)}$ on the surface of the sphere. For any other point, since it is $\vec{n}'$ rather than $\vec{n}$ that plays the more important role, and since it is evident physically that both $A^{(0)}$ and $B^{(0)}$ can depend only on the scalar V and the scalar product $\vec{V}\cdot\vec{n}'$, we can write

$$A^{(0)} = e^{-(\vec{V}\cdot\vec{n}')^2} - \sqrt{\pi} \vec{V}\cdot\vec{n}' + \sqrt{\pi} \vec{V}\cdot\vec{n}' \, \mathrm{erf}(\vec{V}\cdot\vec{n}')$$

(9)

$$B^{(0)} = 1 .$$

The distribution function, Eq 5, with S and 1-S given by Eq 6 and the above expressions for $A^{(o)}$ and $B^{(o)}$, satisfies the Boltzmann equation except at the boundary, r = R, where we do not expect it to do so. This can be verified by straightforward differentiation.

For very small V, i.e., $V \ll \sqrt{2kT/m}$, one can make series developments of the exponentials and keep only terms up to the first power in V. Doing this, one finds

$$f^{(0)} = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mc^2}{2kT}} \Big\{ S\left(1-\frac{m}{kT}\vec{c}\cdot\vec{V}\right) + \alpha A^{(0)}(1-S) + (1-\alpha)(1-S)\left(1-\frac{m}{kT}\vec{c}\cdot\vec{V} + \frac{2m}{kT}\vec{n}'\cdot\vec{c}\;\vec{n}'\cdot\vec{V}\right)\Big\}, \tag{10}$$

where

$$A^{(0)} = 1-\sqrt{\frac{\pi m}{2kT}}\;\vec{n}'\cdot\vec{V} \tag{11}$$

in this approximation.

THE DRAG ON THE SPHERE

Given $f^{(o)}$ one can calculate the pressure tensor at any point in space. In particular one can calculate the different components of the pressure tensor on the surface of the sphere. This is given by

$$\begin{aligned}(p_{ij})_R &= m\iiint d\vec{c}\; C_i C_j f_R^{(0)} \\ &= m\iiint d\vec{c}\,(c-c_0)_i\,(c-c_0)_j\, f_R^{(0)}.\end{aligned}$$

$C_i = (c - c_o)_i$ is the ith component of the velocity of a molecule relative to the ith component of the streaming velocity c_{oi} of the system. The subscript R means that the quantities are evaluated at r = R. From this pressure tensor one can calculate the total force, F, in the direction $\vec{V}$ exerted by the gas on the sphere.

$$F = -R^2 \iint d\vec{n} \left(\frac{p_{\alpha\beta}\, n_\alpha V_\beta}{V} \right)_{r=R} ,$$

where $\vec{n}$ is the outward normal to the sphere. Inserting the expression for p_{ij}

$$F = -\frac{mR^2}{V} \iint d\vec{n} \iiint d\vec{c}\, (c - c_o)_\alpha (c - c_o)_\beta\, n_\alpha V_\beta f_R^{(0)} .$$

Because of the third boundary condition, this is equal to

$$F = -\frac{mR^2}{V} \iint d\vec{n} \iiint d\vec{c}\, c_\alpha c_\beta\, n_\alpha V_\beta f_R^{(0)}$$

$$= -\frac{nmR^2}{\alpha V}\left(\frac{m}{2\pi kT}\right)^{3/2} \iint d\vec{n} \iiint d\vec{c}\, \vec{n}\cdot\vec{c}\; \vec{c}\cdot\vec{V} \Big\{ (1 - \operatorname{sign} \vec{n}\cdot\vec{c})\, e^{-\frac{m}{2kT}(\vec{c}-\vec{V})^2}$$

$$+ \alpha (1 + \operatorname{sign} \vec{n}\cdot\vec{c}) \Big(e^{-\frac{m}{2kT}(\vec{V}\cdot\vec{n})^2} - \sqrt{\frac{\pi m}{2kT}}\, \vec{V}\cdot\vec{n} + \sqrt{\frac{\pi m}{2kT}}\, \vec{V}\cdot\vec{n}\, \operatorname{erf}\Big(\sqrt{\frac{\pi m}{2kT}}\, V\Big)\Big) e^{-\frac{mc^2}{2kT}}$$

$$+ (1 - \alpha)(1 + \operatorname{sign} \vec{n}\cdot\vec{c})\, e^{-\frac{m}{2kT}(\vec{c} - \vec{V} - 2\vec{n}\, \vec{n}\cdot\vec{c})^2} \Big\} .$$

One can introduce integral representations for the sign functions, but the calculation is much simplified by letting the limits of integration take care of these functions. Introducing dimensionless velocities and choosing $\vec{c}$ as the polar axis for the $\vec{n}$ integration (θ,ϕ) and $\vec{V}$ as the polar axis for the $\vec{c}$ integration (θ_0, ϕ_0) for the first and the third terms, while $\vec{n}$ as the polar axis for the $\vec{c}$ integration (θ,ϕ) and $\vec{V}$ as the polar axis for the $\vec{n}$ integration for the second term, we get:

$$F = -\frac{nkTR^2}{\pi^{3/2}} \int_0^\infty dc\, c^4 \int_0^\pi d\theta \sin\theta \cos\theta \int_0^{2\pi} d\phi \int_0^\pi d\theta_0 \sin\theta_0 \cos\theta_0 \int_0^{2\pi} d\phi_0 \cdot$$

$$\Big\{ (1-\mathrm{sign}\cos\theta_0)\, e^{-(c^2+V^2-2cV\cos\theta)} +$$

$$+ (1-\alpha)(1+\mathrm{sign}\cos\theta_0)\, e^{-[c^2+V^2-2cV\cos\theta+4cV\cos\theta(\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos(\phi-\phi_0))]} \Big\} +$$

$$+ \frac{nkTR^2}{\pi^{3/2}} \int_0^\pi d\theta_0 \sin\theta_0 \int_0^{2\pi} d\phi_0 \int_0^\infty dc\, c^4 e^{-c^2} \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \cdot$$

$$\cdot \cos\theta \left[\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi-\phi_0)\right] \alpha (1+\mathrm{sign}\cos\theta) \cdot$$

$$\cdot \left[e^{-V^2\cos^2\theta_0} - \sqrt{\pi}\, V\cos\theta_0 + \sqrt{\pi}\, V\cos\theta_0\, \mathrm{erf}(V\cos\theta_0) \right]$$

The last term is simple. It gives $-4\sqrt{\pi}\, nkTR^2(\alpha\pi V/6)$. Integrating over ϕ and ϕ_0 in the first two integrals:

$$F = -4\sqrt{\pi}\, n k T R^2 \Big\{ -\frac{\alpha\pi V}{6} +$$

$$+ 2\int_0^\infty dc\, c^4 e^{-c^2-V^2} \int_0^\pi d\theta \sin\theta \cos\theta \int_{\frac{\pi}{2}}^{\pi} d\theta_0 \sin\theta_0 \cos\theta_0\, e^{2cV\cos\theta} +$$

$$+ 2\int_0^\infty dc\, c^4 e^{-c^2-V^2} \int_0^\pi d\theta \sin\theta \cos\theta \int_0^{\frac{\pi}{2}} d\theta_0 \sin\theta_0 \cos\theta_0 \,\cdot$$

$$\cdot\, e^{2cV\cos\theta(1-2\cos^2\theta_0)} J_0(4icV\cos\theta \sin\theta \sin\theta_0) \Big\}$$

$$= -4\sqrt{\pi}\, n k T R^2 \Big\{ -\frac{\alpha\pi V}{6} + \sqrt{\frac{\pi i}{V}} \int_0^\infty dc\, c^{7/2} e^{-c^2-V^2} J_{3/2}(2icV) +$$

$$+ 2\sqrt{2\pi}(2V) \int_0^\infty dc\, c^5 e^{-c^2-V^2} \frac{J_{3/2}(2icV)}{(2icV)^{3/2}} \int_0^{\pi/2} d\theta_0 \sin\theta_0 \cos\theta_0 (1-2\cos^2\theta_0) \Big\}, \tag{12}$$

where we have made use of the integral formulas

$$J_{n+\frac{1}{2}}(x) = (-i)^n \sqrt{\frac{x}{2\pi}} \int_0^\pi d\theta \sin\theta\, P_n(\cos\theta)\, e^{ix\cos\theta}$$

$$\int_0^\pi d\theta \sin\theta \cos\theta\, e^{ia\cos\theta} J_0(b\sin\theta) = i\sqrt{2\pi}\, a\, \frac{J_{3/2}(\sqrt{a^2+b^2})}{(a^2+b^2)^{3/2}} .$$

The last integral in (12) vanishes on θ_0 integration, leaving

$$F = -4\sqrt{\pi}\, n k T R^2 \left\{ -\frac{\alpha\pi V}{6} + \sqrt{\frac{\pi}{V}} \int_0^\infty dc\, c^{7/2} e^{-c^2-V^2} J_{3/2}(2icV) \right\}$$

$$= +4\sqrt{\pi}\, n k T R^2 \left\{ \frac{\alpha\pi V}{6} + e^{-V^2}\left(\frac{V}{2} + \frac{1}{4V}\right) + \frac{\sqrt{\pi}}{2}\left(V^2 + 1 - \frac{1}{4V^2}\right) \operatorname{erf}(V) \right\} . \qquad (13)$$

In this expression V is the dimensionless streaming velocity of the main stream. This result, of course, agrees with the results of Heineman[2] and Ashley[3]. For $V \ll 1$, we have

$$F = +4\sqrt{\pi}\, n k T R^2 V \left(\frac{\alpha\pi}{6} + \frac{4}{3} \right) \qquad (14)$$

$$= \begin{cases} +\frac{16}{3}\sqrt{\pi}\, n k T R^2 V \left(1 + \frac{\pi}{8}\right) & \text{diffuse reflection} \\ +\frac{16}{3}\sqrt{\pi}\, n k T R^2 V \; . & \text{specular reflection} \end{cases}$$

The ratio of this force to the Stokes expression which is true also for $V \ll 1$ but for $R \gg \lambda_{tr}$ is of the order R/λ_{tr}, where λ_{tr} is the transport free path defined by Chang and Uhlenbeck[1]. Knudsen and Weber[4] have derived an empirical formula for the drag on a small sphere from their experimental results, with the pressure of the gas ranging from 0.14 dynes/cm^2 to 1.102 x 10^6 dynes/cm^2.

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2 M. Heineman. Comm. on Applied Math. 1, 259, 1948.

3 H. Ashley. Journ. Aero. Sc. 16, 95, 1949.

4 M. Knudsen and S. Weber. Ann. d. Phys., 36, 981, 1911.

Their formula is

$$F = +6\pi\mu R V \left(1 + 0.683\,\frac{\lambda}{R} + 0.354\,\frac{\lambda}{R}\,e^{-1.845\frac{R}{\lambda}}\right)^{-1},$$

where μ is the viscosity coefficient and λ is defined as:

$$\lambda \equiv \frac{1}{0.30967}\sqrt{\frac{\pi}{8}}\,\frac{\mu}{\sqrt{p\rho}} = \frac{1}{0.30967}\sqrt{\frac{\pi}{8}}\,\frac{\mu}{n\sqrt{mkT}}.$$

In order to compare with our expressions we prefer to use as definition for λ the relationship between λ and μ for elastic spheres as given by Chapman, namely:

$$\lambda = \frac{16}{1.016 \times 5 \times \sqrt{2\pi}}\,\frac{\mu}{n\sqrt{mkT}}.$$

This λ is really the Maxwell mean free path and is the same as λ_{tr} for elastic spheres. Introducing further the dimensionless velocity, the empirical formula of Knudsen and Weber becomes

$$F\ +6\pi\mu R V\sqrt{\frac{2kT}{m}}\left(1 + 1.100\,\frac{\lambda}{R} + 0.570\,\frac{\lambda}{R}\,e^{-1.146\frac{R}{\lambda}}\right)^{-1}$$

$$= \begin{cases} 1\ 6\pi\mu R V\sqrt{\frac{2kT}{m}} & R >> \lambda \text{ - Stokes Formula} \\ \\ 1\ 1.79 \cdot 4\sqrt{\pi}\,n k T R V\left(1 - 0.208\,\frac{R}{\lambda} \cdots\right) & R << \lambda \end{cases}$$

The factor 1.79 is to be compared with $4(1 + \pi/8)/3 = 1.857$ and $4/3 = 1.333$ for diffuse reflection and specular reflection respectively. Their result seems to indicate that a large portion of the molecules is diffusely reflected, i.e., a large portion of the molecules that hit the sphere stays on it long enough to forget their previous history. If one can carry out the calculation for the drag to the next approximation (taking one collision into account) one will get a check for the next term in R/λ in the formula of Knudsen and Weber.

THE FLOW PATTERN

With $f^{(o)}$ known, we can calculate other physical quantities pertaining to the system, for instance, the number density and the velocity of flow at any point in space. It is perhaps of some interest to find out the flow pattern. For this purpose we calculate both the number density and the components of the velocity of flow as functions of space coordinates. Since there is cylindrical symmetry around the main streaming velocity, it is sufficient to calculate the velocity components parallel and perpendicular to V. To simplify the calculation, we will assume that (1) $V \ll 1$, and (2) $\alpha = 1$. The distribution function is then given by

$$f^{(o)} = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mc^2}{2kT}} \left\{ S\left(1 + \frac{m}{kT}\vec{c}\cdot\vec{V}\right) + (1-S)\left(1 - \sqrt{\frac{\pi m}{2kT}}\, V\cdot n'\right) \right\}$$

$$= n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mc^2}{2kT}} \left\{ 1 + S\frac{m}{kT}\vec{c}\cdot\vec{V} - (1-S)\sqrt{\frac{\pi m}{2kT}}\, V\cdot n' \right\},$$

where n' is given by Eq 1 and is real only when $R^2 - r^2 + \left(\frac{c\cdot r}{c}\right)^2 \geq 0$. To find the average values, one integrates over all possible values of $\vec{c}$. This is done most simply by choosing $\vec{n}$ as the polar axis, since, then, one does not need to introduce integral representations for the sign functions but can let the limits of integrations take care of them. For $\vec{n}'$ real, $r^2 - R^2 - \left(\frac{\vec{c}\cdot\vec{r}}{c}\right)^2$ is negative; then

$$S = (1/2)(1 - \text{sign}\ \vec{n}\cdot\vec{c})$$

$$1 - S = (1/2)(1 + \text{sign}\ \vec{n}\cdot\vec{c}).$$

Complex values of $\vec{n}'$ have, of course, no physical meaning but this is taken care of by the fact that, if $-R^2 + r^2 - \left(\frac{c\cdot r}{c}\right)^2$ is positive, 1-S = 0 and S = 1. Thus, instead of the sign functions one can separate the Θ integral, where Θ is the angle between $\vec{r}$ and $\vec{c}$, into two regions:

$$\cos^2\theta \gtrless 1 - \frac{R^2}{r^2}$$

$$\cos\theta = \begin{cases} \sqrt{1-\frac{R^2}{r^2}} \text{ to } 1 & S=0 \quad 1-S=1 \\ -\sqrt{1-\frac{R^2}{r^2}} \text{ to } -1 & S=1 \quad 1-S=0 \end{cases}$$

$$\cos\theta = -\sqrt{1-\frac{R^2}{r^2}} \text{ to } +\sqrt{1-\frac{R^2}{r^2}} \qquad S=1 \qquad 1-S=0.$$

Zeroth Order Number Density

$$n^{(0)} = n\left(\frac{m}{2\pi kT}\right)^{3/2} \iiint d\vec{c}\, e^{-\frac{mc^2}{2kT}} \left\{ 1 + S\frac{m}{kT}\vec{c}\cdot\vec{V} - \sqrt{\frac{\pi m}{2kT}}(1-S)\vec{V}\cdot\vec{n}' \right\}.$$

Choosing r as the polar axis and letting

$$\vec{c}: \quad c,\ \theta,\ \phi$$

$$\vec{V}: \quad V,\ \theta_o,\ \phi_o$$

$$n^{(0)} = \frac{n}{\pi^{3/2}} \int_0^\infty dc\, c^2 e^{-c^2} \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \,\cdot$$

$$\cdot \Big\{ 1 + 2ScV\left[\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi-\phi_0)\right] -$$

$$- (1-S)\frac{\sqrt{\pi}\,Vr}{R}\left[\cos\theta_0 - \left(\cos\theta - \sqrt{\cos^2\theta - \left(1-\frac{R^2}{r^2}\right)}\right)\left(\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi-\phi_0)\right)\right]\Big\}$$

$$n^{(0)} = n - \frac{nV\cos\theta_0}{\sqrt{\pi}}\left\{\frac{R^2}{r^2}\left(1+\frac{\pi}{6}\right) + \frac{\pi r}{2R}\left(1-\sqrt{1-\frac{R^2}{r^2}}\right) - \frac{\pi r}{6R}\left[1-\left(1-\frac{R^2}{r^2}\right)^{3/2}\right]\right\}. \tag{15}$$

Limiting cases:

1) $r \to \infty$

$n^{(o)} = n$, the number density of the main stream

2) $r = R$

$$n^{(0)} = n\left\{1 - \frac{V\cos\theta_0}{\sqrt{\pi}}\left(1+\frac{\pi}{2}\right)\right\}.$$

The number density is no longer uniform on account of the presence of the sphere. The correction term is largest on the surface of the sphere and falls off like r^{-2} at large distances. The number density is larger at the back than in the front. At $\theta_0 = \pi/2$ and $3\pi/2$, the correction term vanishes, which assures no discontinuity of the number density. All these conclusions are expected on physical grounds.

<u>Zeroth Order Mean Velocity</u>

If we denote the components of the average velocity in directions parallel and perpendicular to the direction of V by $c_{\parallel}^{(0)}$ and $c_{\perp}^{(0)}$ respectively, similar calculations lead to

$$n^{(0)} c_{\parallel}^{(0)} = nV\sqrt{\frac{2kT}{m}}\,(b + a\sin^2\theta_0)$$

$$n^{(0)} c_{\perp}^{(0)} = -nV\sqrt{\frac{2kT}{m}}\,a\sin\theta_0\cos\theta_0$$

where

$$b = \frac{3}{8} - \frac{R^2}{8r^2} - \frac{R^3}{4r^3} + \frac{r}{8R}\left(1-\frac{R^2}{r^2}\right)^2 \ln\frac{1+\frac{R}{r}}{\sqrt{1-\frac{R^2}{r^2}}} + \frac{1}{2}\left(1-\frac{R^2}{r^2}\right)^{3/2}$$

$$a = -\frac{1}{16} + \frac{3R^2}{16r^2} + \frac{3R^3}{8r^3} + \frac{r}{4R}\left(1-\frac{R^2}{r^2}\right)\ln\frac{1+\frac{R}{r}}{\sqrt{1-\frac{R^2}{r^2}}}$$

$$-\frac{3r}{16R}\left(1-\frac{R^2}{r^2}\right)^2 \ln\frac{1+\frac{R}{r}}{\sqrt{1-\frac{R^2}{r^2}}} + \frac{3R^2}{4r^2}\sqrt{1-\frac{R^2}{r^2}}\,.$$

Up to the order linear in V

$$c_{\parallel}^{(o)} = V\sqrt{\frac{2kT}{m}}\left(b + a\sin^2\theta_o\right) \tag{16a}$$

$$c_{\perp}^{(o)} = -V\sqrt{\frac{2kT}{m}}\, a\sin\theta_o\cos\theta_o \tag{16b}$$

The sign in front of $c_{\perp}^{(o)}$ is so chosen that the condition $\overline{c_\alpha n_\alpha} = 0$ on the surface of the sphere is fulfilled.

Limiting cases:

1) $r = \infty$, $a = 0$, $b = 1$

$$c_{\parallel}^{(o)} = V\sqrt{\frac{2kT}{m}}$$

$$c_{\perp}^{(o)} = 0 \;,$$

2) $r = R$, $a = 1/2$, $b = 0$

$$c_{\parallel}^{(o)} = \frac{V}{2}\sqrt{\frac{2kT}{m}}\sin^2\theta_o$$

$$c_{\perp}^{(o)} = -\frac{V}{2}\sqrt{\frac{2kT}{m}}\sin\theta_o\cos\theta_o \;.$$

These results are also readily understandable.

For an ideal fluid flow, these two components of velocity are given by similar expressions, except that a and b are much simpler.

$$a = \frac{3R^3}{2r^3}$$

$$b = 1 - \frac{R^3}{r^3} \quad .$$

To compare the flow pattern, it is simplest to calculate $\cot \phi = c_{\parallel}^{(0)}/c_{\perp}^{(0)}$. In both cases

$$\cot \phi = - \frac{b}{a \sin\theta_0 \cos\theta_0} - \tan\theta_0 \ ,$$

except a and b have different r dependence in the two cases. It is found that the ratio b/a is larger in the case of the ideal fluid than in the case of the Knudsen gas for all values of $r \geq R$. Thus the lines of flow for the ideal, non-viscous gas are always flatter than those for the Knudsen gas.

ZEROTH ORDER DISTRIBUTION FUNCTION FOR A SMALL CIRCULAR PLATE IN A STREAMING GAS AND THE DRAG ON THE PLATE

The zeroth order distribution function for a system consisting of a main gas stream with velocity V and an infinitely thin circular plate of radius R situated with its center at the origin and its plane in the y-z plane has been constructed. The corresponding drag on the plate has also been calculated. The one simplification in this problem compared with the problem with the sphere is that there are only two directions of the normal exactly opposite to each other, and $\vec{n}'$ coincides with $\vec{n}$. However, the S-function seems to be more complicated except at the surface of the plate. We found

$$f^{(0)} = n\left(\frac{m}{2\pi kT}\right)^{3/2}\Big\{ S e^{-\frac{m}{2kT}(\vec{c}-\vec{V})^2} + \alpha A^{(0)}(\vec{V}\cdot\vec{n})(1-S)\, e^{-\frac{mc^2}{2kT}} +$$

$$+ (1-\alpha)(1-S)\, e^{-\frac{m}{2kT}(\vec{c}-\vec{V}-2\vec{n}\,\vec{n}\cdot\vec{c})^2}\Big\} \tag{17}$$

where

$$A^{(0)} = e^{-\frac{m}{2kT}(\vec{V}\cdot\vec{n})^2} + \sqrt{\frac{\pi m}{2kT}}\,\vec{V}\cdot\vec{n} + \sqrt{\frac{\pi m}{2kT}}\,\vec{V}\cdot\vec{n}\ \mathrm{erf}\left(\sqrt{\frac{m}{2kT}}\,\vec{V}\cdot\vec{n}\right) \tag{18}$$

$\vec{n}$ takes the values $\pm$ 1 depending on whether one is in the $\pm$ half space with respect to the y-z plane. We have not found a S that seems to be suitable for future calculation purposes. However, one can write down some representations for S, as for instance:

$$S = \tfrac{1}{2}(1 - \mathrm{sign}\,\vec{n}\cdot\vec{c}) +$$

$$+ \tfrac{1}{4}(1+\mathrm{sign}\,\vec{n}\cdot\vec{c})\Big\{1 + \mathrm{sign}\Big[(\vec{n}\times(\vec{c}\times\vec{r}))^2 - R^2(\vec{n}\cdot\vec{c})^2\Big]\Big\} .$$

This is similar to that for a sphere, except that it is much more complicated. An alternative form is

$$S = \tfrac{1}{2}(1 - \mathrm{sign}\,\vec{n}\cdot\vec{c}) + \tfrac{1}{2}(1+\mathrm{sign}\,\vec{n}\cdot\vec{c})(1-T)$$

$$= 1 - \tfrac{T}{2}(1+\mathrm{sign}\,\vec{n}\cdot\vec{c}) ,$$

where

$$T = \frac{R}{2\pi}\int_0^{2\pi} d\phi_1 \, \frac{\cos\phi_1 (R\cos\phi_1 - y + x\tan\theta\cos\phi) + \sin\phi_1 (R\sin\phi_1 - z + x\tan\theta\sin\phi)}{(R\cos\phi_1 - y + x\tan\theta\cos\phi)^2 + (R\sin\phi_1 - z + x\tan\theta\sin\phi)^2}$$

θ, ϕ are the polar angles for the $\vec{c}$-vector and x,y,z are the rectangular coordinates of $\vec{r}$. T is a discontinuous function. It is zero when the velocity vector at the point under consideration lies outside of the cone subtended at the point by the plate and T = 1 when the velocity vector lies within this cone. In problems with finite objects in a gas, there will always be discontinuous functions of this kind. In fact, the generalization to other more complicated geometry is straightforward in principle.

The drag calculated from this distribution function is:

$$F = -2\sqrt{\pi}\, nkTR^2 \left\{ \frac{\alpha\pi V}{2} + (2-\alpha) V e^{-V^2} + \frac{\sqrt{\pi}}{2}(2-\alpha)(1+2V^2)\,\mathrm{erf}(V) \right\} \qquad (19)$$

For V small, this goes into:

$$F = -2\sqrt{\pi}\, nkTR^2 V \left\{ 4 + \alpha\left(\frac{\pi}{2} - 2\right) \right\} \qquad (20)$$

$$= \begin{cases} -4\sqrt{\pi}\, nkTR^2 V \left(1 + \frac{\pi}{4}\right) \\ \\ -8\sqrt{\pi}\, nkTR^2 V \end{cases}$$

The force on a plate when R is much larger than λ_{tr} can be deduced from that on an oblate ellipsoid[5] whose eccentricity is $\varepsilon = (a^2-b^2)/a^2$

$$F = \frac{8\pi\mu b V}{\frac{1-\varepsilon^2}{\varepsilon^2} - (1-2\varepsilon^2)\frac{\sqrt{1-\varepsilon^2}}{\varepsilon^3}\sin^{-1}\varepsilon}$$

For an infinitely thin plate of radius R, a = R, b = 0 and   = 1

$$F = 16\mu R V$$

Thus the ratio of these two forces is again proportional to R/λ_{tr}.

FLOW OF GAS THROUGH A SMALL CIRCULAR TUBE

The present problem differs in one respect from all the other cases previously considered. There is no discontinuous function entering in the distribution function if the tube is taken as infinite in length. Let n(x) be the number density due to the emission by an element of area at the level x (x along the length of the tube). Assuming n(x) is slowly varying, one finds

$$f^{(0)} = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mc^2}{2kT}} \left\{ n(x) - \frac{dn}{dx}\cos\theta\left[\sqrt{a^2-(y\sin\alpha - z\cos\alpha)^2} - (y\cos\alpha + z\sin\alpha)\right]\right\} \qquad (21)$$

where θ is the angle the velocity vector makes with the x-axis, α = angle the projection of the velocity vector in the y-z plane makes with the y-axis and a is the radius of the tube.

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5 R. Gans. Ann. d. Physik, 86, 628, 1928.

With the distribution function (21) one can find the average values of functions of $\vec{c}$. It is found that the number density is uniform through a cross section and that it is equal to n(x). This result comes essentially from the assumption that n(x) is slowly varying, so that all x-derivatives of n(x) higher than the first are dropped in the Taylor development of $n(x + \Delta x)$ in deriving $f^{(o)}$. The mean streaming velocity is given by:

$$c_{oy} = c_{oz} = 0$$

$$c_{ox} = -\frac{1}{n}\frac{dn}{dx}\sqrt{\frac{2kT}{\pi m}}\; aE\left(\frac{\pi}{2}, \frac{r}{a}\right),$$

where

$$E\left(\frac{\pi}{2}, \frac{r}{a}\right) = \int_0^{\frac{\pi}{2}} d\phi \sqrt{1 - \frac{r^2}{a^2}\sin^2\phi},$$

r is the distance from the axis of the tube. The flow velocity has a maximum at the center of the tube. The total mass flowing through the tube per second is

$$G = \frac{4\pi}{3}\sqrt{\frac{2mkT}{\pi}}\;\frac{dn}{dx}\; a^3,$$

a formula derived also by Knudsen[6] and Lorentz[7].

In the case of Poiseuille flow

$$c_{ox} = \frac{a^2 - r^2}{4\mu} kT \frac{dn}{dx},$$

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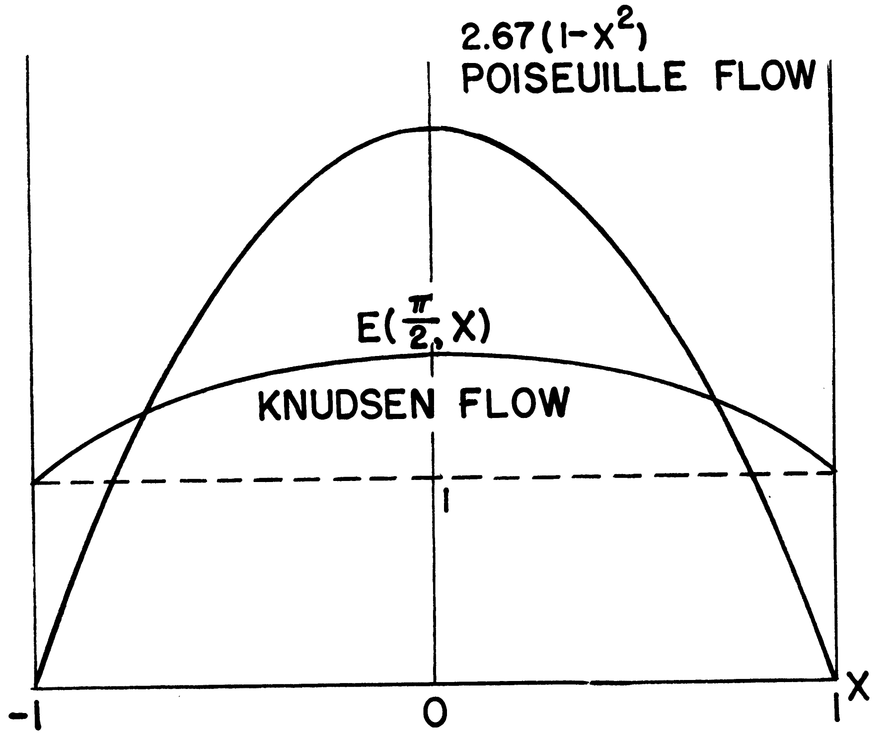
[6] M. Knudsen. Ann. d. Physik, 28, 75, 1909.

[7] H. A. Lorentz. Lectures on Theoretical Physics, Vol. I, Chap. III

where μ, the viscosity coefficient, is $K\rho\bar{c}\lambda$, K being a numerical constant of the order unity, and $\bar{c}$, the mean molecular velocity. The amount flowing through per second is

$$G = \frac{\pi}{8K}\sqrt{\frac{mkT}{3}}\,\frac{R^4}{\lambda}\,.$$

The ratio of the G's is again of the order R/λ. The velocity profile is quite different in the two cases. The velocity is zero in the Poiseuille flow at the



wall, while the velocity is not zero for the Knudsen gas. This is of the same nature as the velocity and temperature jumps in the report by Chang and Uhlenbeck[1]. The velocity profiles are shown in Fig. 2. For comparison we have drawn the curves for the same amount of flow per second.