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POSTBUCKLING BEHAVIOR OF RECTANGULAR ELASTIC PLATES

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PART I. PLATES WITH FOUR EDGES SIMPLY SUPPORTED

1. INTRODUCTION

The buckling behavior of plates with lateral edge support is quite different from that of a column. In the case of a column, when the applied thrust reaches the critical buckling load (Euler load), if there is no relative displacement of the two ends prescribed, the column will collapse. Hence the Euler load is the upper limit of the buckling strength of a column. It is the upper limit since for an ordinary column to a greater of lesser degree there are physical imperfections which make its buckling strength lower than that of an Euler load. For plates with lateral edge support, however, after the thrust reaches its critical buckling load, the plate buckles into a primary buckling mode. Nevertheless due to the lateral support from the undeflectable edges, the buckling mode can stand, and its amplitude is uniquely determined by the amount of the thrust applied. For some cases, (for instance, a very thin plate of high strength material), if the thrust increases further, secondary buckling occurs before it collapses. Hence, there is more buckling strength than the critical buckling load that can be developed from such a plate.

The need of utilizing this postbuckling strength is very urgent for engineers, particularly for those in aeronautical engineering. This is because a plate is a primary element in aircraft structures which is highly likely to buckle.

Because of the relatively large deflections (compared to the thickness of plate), the classic linear equations are no longer effective in governing the behavior of plates in the buckled state. A system of nonlinear equations governing the performance of them has been derived by Kármán(1)*. Due to the nonlinearity, the difficulty in solving them is tremendous, especially when secondary buckling is included.

A number of solutions, almost all of them approximate, have been produced in recent decades(2) to (12). In most of them, attention is directed to primary buckling. Among them there are two outstanding works. One is given by Friedrichs and Stoker(8) representing a set of complete solutions (in the sense that they are applicable to the whole range of the loading parameter) for a circular plate with axially symmetric buckling mode. The other one is for rectangular plates done by Levy(4), a formally exact solution of Kármán's pair of equations by Fourier series approach. It might be noteworthy that due to the convergence of Fourier series, the more terms in the series that are taken the closer the sum will approach the primary buckling

*Numbers in parentheses refer to the bibliography at the end of the thesis.

pattern. But no solution in connection with secondary buckling can be expected by this approach. However secondary buckling has been observed by experiments(13) to (15).

From an analytical point of view, the instability of the axially symmetric mode in the case of circular plate has been pointed out by several authors(8) (9) (10).

For rectangular plates, in Marguerre's(2) as well as Timoshenko's(3) work, in addition to the primary buckling mode, another coordinate function was included to accomplish the speculation about the secondary buckling. The evidence of the change of buckling patterns in the deep post buckling domain has also been noticed by Koiter(6), Alexeev(11), and Stein(12) through their analytic results. However in their studies, the primary buckling (for n = 1, n being the number of one-half waves in the loaded direction) and secondary buckling (for n = 2) were treated separately. Hence no interactions between these two buckling patterns were to be seen and no criterion for the stability for each of these buckling configurations could be investigated.

In this part of this thesis, an attempt is made to study the behavior of plates with four edges simply supported in the postbuckling domain. Both primary and secondary buckling modes are included. The investigation will include the modes themselves, the stabilities of each of the two modes and of the combination of them, and the transition from one mode to the other.

The problem is approached by two methods. One is similar to Marguerre's[2]. The other is a function space method. Since the former which will be referred to as the first method is more familiar, it is given first in the next section. Then a complete solution for a square plate follows in the third section. In the fourth section, an extension of the function space method to problems including secondary buckling is established. A comparison of this method to Marguerre's is given. The application of this method to the square plate is also demonstrated there.

A comparison of the present result with other works and discussion of it conclude this study.

2. OUTLINE OF THE FIRST METHOD

Consider an elastic isotropic plate of a thickness h which is assumed to be small compared with the other dimensions of the plate, subjected to a prescribed edge thrust λT_i on portion B' of boundary B and to a displacement λU_i on portion B", the remainder of B. (The index notation is used in the general discussion. The subscript letters i, j etc. standing for 1 or 2 refer to the Cartesian coordinates on the middle plane of the plate). When the positive increasing parameter λ stays below a certain critical value, say λ_o, the plate remains in its plane. This is a plane problem in elasticity. The membrane stresses λt^o_{ij} in the region of plate satisfy

$$t^o_{ij,j} = 0 \qquad \text{in } R \qquad (2.1)$$

and on boundary

$$t^o_{ij} n_j = T_i \qquad \text{on } B' \qquad (2.2)$$

where n_j is the direction cosines of the outward normal of the boundary. The displacement field λu^o_i is related to the strain field λe^o_{ij} by the equations

$$e^o_{ij} = \frac{1}{2}(u^o_{i,j} + u^o_{j,i}) \qquad (2.3)$$

and on boundary it satisfies

$$u^o_i = U_i \qquad \text{on } B'' \qquad (2.4)$$

The stresses for homogeneous and isotropic materials are related to the corresponding strains by Hook's Law, i.e.

$$t_{ij}^{\circ} = \frac{E}{1+\nu}\left[e_{ij}^{\circ} + \frac{\nu}{1-\nu} e_{kk}^{\circ} \delta_{ij}\right] \tag{2.5}$$

in which δ_{ij} is the Kronecker delta; E, Young's modulus of elasticity; ν, Poisson's ratio.

As the parameter λ increases and exceeds λ_o, the equilibrium state described in this manner becomes unstable. Then the plate buckles into a configuration $w(x_1, x_2)$ which is governed by

$$D w_{,iijj} = h\, t_{ij}\, w_{,ij} \qquad \text{in } R \tag{2.6}$$

and appropriate boundary conditions. In the present problem the edges are simply supported, hence both the lateral deflection of the plate and the normal resisting moment from the boundary vanish on boundary B. In Eq. (2.6), D is the flexural rigidity of the plate. Due to this configuration, in general, the membrane stresses t_{ij}, the associated strains e_{ij} and the displacements u_i are different from those in the unbuckled state. Let t_{ij}', e_{ij}' and u_i' be respectively their differences between the two states. Then one would have in the buckled state

$$t_{ij} = \lambda t_{ij}^{\circ} + t_{ij}' \tag{2.7}$$

$$e_{ij} = \lambda e_{ij}^{\circ} + e_{ij}' \tag{2.8}$$

$$u_i = \lambda u_i^{\circ} + u_i' . \tag{2.9}$$

These additional elements, on account of equations (2.1), (2.2) and (2.4) obey

$$t_{ij,j}' = 0 \qquad \text{in } R , \tag{2.10}$$

$$t_{ij}' n_j = 0 \qquad \text{on } B' \tag{2.11}$$

and

$$u_i' = 0 \qquad \text{on } B'' \tag{2.12}$$

The same stress-strain relations given by Eq. (2.5) hold for them, i.e.

$$t_{ij}' = \frac{E}{1+\nu}\left[e_{ij}' + \frac{\nu}{1-\nu} e_{kk}' \delta_{ij} \right] . \tag{2.13}$$

But the strain-displacement relations now become

$$e_{ij}' = \frac{1}{2}\left[u_{i,j}' + u_{j,i}' + w_i w_j \right] . \tag{2.14}$$

The preceding formulation of the problem which is equivalent to that done by Kármán was given in reference (9).

The difference between the present method and Marguerre's is in the first steps. In the present method an arbitrary but reasonable membrane stress field is assumed first, such that

$$T_{ij} = \lambda t_{ij}^{\circ} + T_{ij}' \tag{2.15}$$

in which $\bar{\tau}_{ij}$ is a "statically homogeneous" stress field. The term "statically homogeneous" means that $\bar{\tau}_{ij}$ satisfy the homogeneous equilibrium equations (2.10) and the homogeneous boundary conditions (2.11).* Replacing t_{ij} (which may be known as actual membrane stress field in contrast with the artificial $\bar{\tau}_{ij}$) by $\bar{\tau}_{ij}$, equation (2.6) may presumably be solved for the lateral deflection function $W(x_1, x_2)$, (not w which is actual one). This W is supposed to satisfy the appropriate boundary conditions as mentioned before and a characteristic equation arising from boundary conditions. Hence this is a linear eigenvalue problem. For the purpose of studying the buckling behavior of plates in the deep postbuckling domain, presumably such a particular eigenvalue can be chosen that with it both primary and secondary buckling modes exist. Let w^1 and w^2 be the respective configuration function of these modes and A_1 and A_2 be their corresponding arbitrary amplitude coefficients, then the solution of W can be expressed by

$$W = A_1 w^1 + A_2 w^2 \tag{2.16}$$

With this set of functions at hand, Marguerre's method(2) is readily followed up to determine the coefficients A_1 and A_2.

*Eqs. (2.12) and (2.14) need not be satisfied, i.e. the stress field need not be "compatible" or "geometrically homogeneous."

Now let u'_i , ε'_{ij} and τ'_{ij} be respectively the additional membrane displacement, strain and stress fields associated with the configuration $W(x_1, x_2)$. All of them shall satisfy Eqs. (2.10) to (2.14). Hence one has

$$\tau'_{ij} = \frac{E}{1+\nu}\left[\varepsilon'_{ij} + \frac{\nu}{1-\nu}\varepsilon'_{kk}\delta_{ij}\right] \tag{2.17}$$

and

$$\varepsilon'_{ij} = \frac{1}{2}\left[u'_{i,j} + u'_{j,i} + W_{,i}W_{,j}\right], \tag{2.18}$$

thus

$$\tau'_{ij} = \frac{E}{1-\nu^2}\left[\frac{1}{2}(1-\nu)(u'_{i,j} + u'_{j,i} + W_{,i}W_{,j}) + \nu(u'_{k,k} + \frac{1}{2}W_{,k}W_{,k})\delta_{ij}\right] \tag{2.19}$$

Substituting this expression of τ'_{ij} into Eqs. (2.10) and rearranging the result, one gets

$$(1-\nu)u'_{i,jj} + (1+\nu)u'_{j,ji} = -\left[(1-\nu)W_{,i}W_{,jj} + (1+\nu)W_{,j}W_{,ij}\right] \tag{2.20}$$

These two simultaneous, second order, non-homogeneous partial differential equations combined with boundary conditions (2.11) and (2.12) characterize the additional membrane displacements u'_i. With this set of equations solved, ε'_{ij} and τ'_{ij} can be computed accordingly. Hence a new membrane stress field denoted by τ_{ij} may be obtained as

$$\tau_{ij} = \lambda t^0_{ij} + \tau'_{ij} \tag{2.21}$$

Because of the undetermined amplitude coefficients of W, the magnitude of this improved stress field is also indefinite; otherwise it is uniquely determined. The principle of minimum potential energy will be employed for the determination of these coefficients.

It can be shown that the minimization of the potential energy of this system is equivalent to the minimization of its additional potential energy due to the lateral deflection. The latter is composed of three parts. They are the bending strain energy, U_b, work done by the edge thrust, W_e, and membrane strain energy, U_m, i.e.

$$V(W, \tau_{ij}') = U_b(W) + W_e(W, \lambda t_{ij}^{\circ}) + U_m(W, \tau_{ij}') \tag{2.22}$$

in which

$$U_b = \frac{D}{2}\int_R [(1-\nu) W_{,ij} W_{,ij} + \nu W_{,ii} W_{,jj}]\, dA$$

$$W_e = \frac{h}{2}\int_R \lambda t_{ij}^{\circ} W_{,i} W_{,j}\, dA \tag{2.23}$$

$$U_m = \frac{h}{4}\int_R \tau_{ij}' W_{,i} W_{,j}\, dA$$

The first two in the above expressions are well known.[19] The third one is derived from

$$U_m = \frac{h}{2}\int_R \tau_{ij}' \varepsilon_{ij}'\, dA \tag{2.24}$$

By relationship (2.19) and in view of the symmetry of the stress-strain matrix:

$$U_m = \frac{h}{2}\int_R \tau_{ij}\, u_{i,j}\, dA + \frac{h}{4}\int_R \tau_{ij}\, W_{,i}\, W_{,j}\, dA \,. \tag{2.25}$$

The first integral in the above equation vanishes. This can be seen if one applies Green's Theorem to it. Then all the resulting terms vanish based on the consideration of equations (2.10) to (2.12).

An equilibrium state is characterized by the vanishing of the first variation of the potential energy V , that is,

$$\delta V = 0 \,. \tag{2.26}$$

In the present case, the function W is in the form given by (2.16). Hence the potential energy is a function of A_1 and A_2, and consequently the variation of V with respect to W shall be performed independently with respect to A_1 and A_2. Thus condition (2.26) becomes

$$\frac{\partial V}{\partial A_1}\delta A_1 + \frac{\partial V}{\partial A_2}\delta A_2 = 0 \,.$$

Since δA_1 and δA_2 are arbitrary, two equations

$$\frac{\partial V}{\partial A_1} = 0 \quad \text{and} \quad \frac{\partial V}{\partial A_2} = 0 \tag{2.26a}$$

are available for the determination of A_1 and A_2.

The stability of the configurations so determined requires the second variation of the potential energy to be positive definite[20], i.e.

$$\delta(\delta V) > 0$$

For the present problem, this leads to

$$\frac{\partial^2 V}{\partial A_1^2}(\delta A_1)^2 + 2\frac{\partial^2 V}{\partial A_1 \partial A_2}(\delta A_1)(\delta A_2) + \frac{\partial^2 V}{\partial A_2^2}(\delta A_2)^2 > 0 . \qquad (2.27)$$

Considering the independence between δA_1 and δA_2, the necessary and sufficient conditions for the above inequality to be held are

$$\frac{\partial^2 V}{\partial A_1^2} > 0 \qquad \text{and} \qquad \left[\frac{\partial^2 V}{\partial A_1 \partial A_2}\right]^2 < \frac{\partial^2 V}{\partial A_1^2}\,\frac{\partial^2 V}{\partial A_2^2} \qquad (2.27a)$$

or

$$\frac{\partial^2 V}{\partial A_2^2} > 0 \qquad \text{and} \qquad \left[\frac{\partial^2 V}{\partial A_1 \partial A_2}\right]^2 < \frac{\partial^2 V}{\partial A_1^2}\,\frac{\partial^2 V}{\partial A_2^2} \qquad (2.27b)$$

The stability so characterized may be called relative stability. The counterpart, absolute stability, will be discussed in Section 4.

The criterion for the change of buckling patterns of plates from one to the other has rarely been discussed. Koiter's, Alexeev's and Stein's conjectures (see p. 3) were based on their computations that for certain loading parameter, the antisymmetric buckling mode resulted in a smaller thrust than that resulting from a symmetric mode. This is similar to what was proposed by Kármán and Tsien(21) for buckling of circular shells. However, such kind of criterion for shells has been rejected and a criterion based on energy consideration has been established.(22) According to this, the buckling pattern will change from one configuration to another when the potential energy associated with the latter is lower than that associated with the former.

For the present problem, when the plate has undergone large buckling deformations, the configuration resembles that of a shallow shell. It is therefore reasonable to apply the shell criterion to the buckled plate.

Hence, the primary buckling mode may change to the secondary one when

$$V_1 = V_2 \tag{2.28}$$

where V_1 is the potential energy associated with the primary mode and V_2 is the one associated with the secondary mode.

3. SOLUTION FOR A SQUARE PLATE

The approach to postbuckling problems outlined in the previous section can be applied to plates with any aspect ratio. In what follows, there is no particular aspect ratio specified up to the computation for eigenvalues. From there on, for simplicity, an aspect ratio equal to one will be assumed.

Let the plate considered have a length a in the loaded direction, a width $2b$, and a thickness h. It is simply supported in the lateral direction of the plate. A set of rectangular reference axes has its origin at one corner of the plate with x_1 in the loaded direction, x_2 in the other. Prior to buckling, the middle plane of the plate coincides with the $x_1 x_2$ - plane.

Let a relative displacement of two loaded edges be prescribed in proportion to a positive increasing value $\bar{\lambda}$ such that $\bar{\lambda} = K\lambda$ where $K = \frac{1-\nu^2}{hE}a$ and let it be free from shearing stress along these loaded edges. Let the edges, along $x_2 = 0$ and $2b$, be immovable in the x_2 direction and let the displacement along these edges in the x_1- direction be linear in x_1. Hence the boundary conditions for the membrane displacement field u_i° are

$$\left.\begin{aligned}
&u_1^\circ(0, x_2) = 0 \\
&u_1^\circ(a, x_2) = -k \\
&u_{2,1}^\circ(0, x_2) = u_{2,1}^\circ(a, x_2) = 0 \\
&u_2^\circ(x_1, 0) = 0 \\
&u_1^\circ(x_1, 0) = -k\frac{x_1}{a}
\end{aligned}\right\} \quad (3.1a)$$

Along the center line, the symmetric condition must be satisfied, namely

$$\left.\begin{aligned}
&u_2^\circ(x_1, b) = 0 \\
&u_{1,2}^\circ(x_1, b) = 0
\end{aligned}\right\} \quad (3.1b)$$

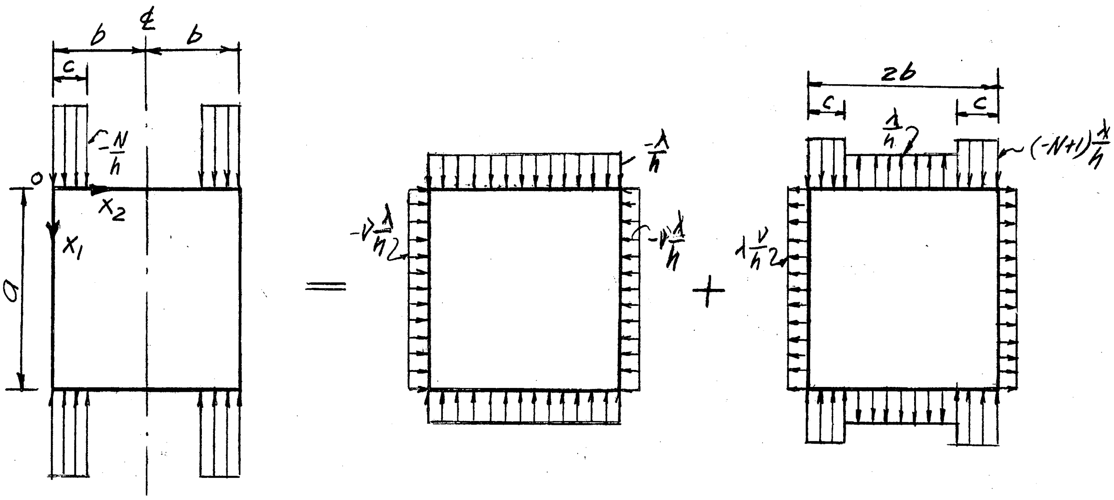
All of these boundary conditions are satisfied by assuming

$$\left.\begin{aligned}
&u_1^\circ(x_1, x_2) = -k\frac{x_1}{a} \\
&u_2^\circ(x_1, x_2) = 0
\end{aligned}\right\} \quad (3.1c)$$

From this displacement field one easily gets its corresponding stress field t_{ij}° . It is

$$t_{11}^\circ = -\frac{1}{h}, \quad t_{22}^\circ = -\nu\frac{1}{h} \quad \text{and} \quad t_{12}^\circ = 0 . \qquad (3.2)$$

For a start of the present approach to the problem, a membrane stress field T_{ij} shall be selected. In order to study the behavior of the plate in a deep postbuckling domain, it is reasonable to choose the stress distribution for ultimate strength of buckling of thin plates suggested by Kármán, Sechler and Donnell(18). It is shown schematically in Figure 1.



$$T_{ij} \quad = \quad \lambda t^{\circ}_{ij} \quad + \quad T'_{ij}$$

Figure 1. An Assumed Membrane Stress Field

The width of stress strip, c, shown in Figure 1 is known as the effective width in the literature. This is expressed then by the equations:

$$\left.\begin{array}{llll}
T_{11} = -\frac{N}{h} & 0 \leq x_2 < c, & T_{22} = T_{12} = 0 & \text{in } R \\
T_{11} = 0 & c \leq x_2 \leq b, & & \\
 & & & \\
t^{\circ}_{11} = -\frac{1}{h} & 0 \leq x_2 \leq b, & t^{\circ}_{12} = 0. & \text{in } R \\
t^{\circ}_{22} = -\nu\frac{1}{h} & 0 \leq x_1 \leq a, & & \\
 & & & \\
T'_{11} = -(N-1)\frac{1}{h} & 0 \leq x_2 < c, & T'_{22} = \nu\frac{\lambda}{h} & 0 \leq x_1 \leq a \\
T'_{11} = \frac{1}{h} & c < x_2 \leq b, & T'_{12} = 0 & \text{in } R.
\end{array}\right\} \quad (3.3)$$

As before, positive stress represents tension and the negative, compression. The "statically homogeneous" additional membrane stress field T'_{ij} so selected satisfies the equilibrium equation (2.10) as well as the boundary conditions (2.11). The satisfaction for the former one is quite obvious; while for the latter, because no normal stresses are prescribed along the edges, it is satisfied automatically.

Transforming the coordinate variables x_1 and x_2 into dimensionless quantities ξ and η such that $\xi = \frac{x_1}{c}$ and $\eta = \frac{x_2}{c}$ and replacing t_{ij} by T_{ij} , the equilibrium equation (2.6) takes the form

$$D[W_{,\xi\xi\xi\xi} + 2W_{,\xi\xi\eta\eta} + W_{,\eta\eta\eta\eta}] - c^2 N W_{,\xi\xi} = 0 \tag{3.4}$$

for $0 \leq \xi \leq \frac{a}{c}$, $0 \leq \eta \leq 1$ where c^2 comes from the transformation of the variables. For $0 \leq \xi \leq \frac{a}{c}$ and $0 \leq \eta \leq \frac{b}{c}$ one has

$$D[\overline{W}_{,\xi\xi\xi\xi} + 2\overline{W}_{,\xi\xi\eta\eta} + \overline{W}_{,\eta\eta\eta\eta}] = 0 \tag{3.5}$$

The bar over W in the above equation is used to distinguish between the coordinate functions in the different regions. This notation will be employed in similar occasions below.

The simply supported boundary conditions in the lateral direction of the plate may be stated as follows:

$$\left.\begin{aligned} &W(0,\eta)=W(\tfrac{a}{c},\eta)=W_{,\xi\xi}(0,\eta)=W_{,\xi\xi}(\tfrac{a}{c},\eta)=0 \\ &W(\xi,0)=W_{,\eta\eta}(\xi,0)=0 \end{aligned}\right\} \quad (3.6)$$

$$\overline{W}(0,\eta)=\overline{W}(\tfrac{a}{c},\eta)=\overline{W}_{,\xi\xi}(0,\eta)=\overline{W}_{,\xi\xi}(\tfrac{a}{c},\eta)=0. \quad (3.7)$$

Along the central line, on account of symmetry, one has

$$\overline{W}_{,\eta}(\xi,\tfrac{b}{c})=\overline{W}_{,\eta\eta\eta}(\xi,\tfrac{b}{c})=0. \quad (3.7a)$$

Along $\eta = 1$ the deflection, slope, moment, and shear must be continuous from one region to the other, that is,

$$\left.\begin{aligned} &W(\xi,1)=\overline{W}(\xi,1), \qquad W_{,\eta}(\xi,1)=\overline{W}_{,\eta}(\xi,1) \\ &W_{,\eta\eta}(\xi,1)=\overline{W}_{,\eta\eta}(\xi,1), \qquad W_{,\eta\eta\eta}(\xi,1)=\overline{W}_{,\eta\eta\eta}(\xi,1). \end{aligned}\right\} \quad (3.8)$$

To solve this system of equations for W, first let

$$W=f(\eta)\sin\gamma\xi, \qquad \text{where } \gamma=\frac{n\pi c}{a} \quad (3.9)$$

then the first of the boundary condition (3.6) is satisfied automatically. Substituting Eq. (3.9) into Eq. (3.4) and denoting the derivative with respect to η by prime "$'$", the partial differential equation (3.4) becomes

$$f''''-2\gamma^2 f''-\gamma^4\left[\frac{\rho^2}{\eta^2}-1\right]f=0 \qquad 0\leq\eta\leq 1 \quad (3.10)$$

in which

$$\rho^2 = \frac{Na^2}{\pi^2 D} \tag{3.10a}$$

The solution of the above equation with the satisfaction of the second of the boundary conditions (3.6) is in the following form:*

$$f(\eta) = A\left[\sin\alpha\eta + B\sinh\beta\eta\right] \tag{3.11}$$

where

$$\alpha = r\sqrt{\frac{\rho}{n} - 1} \qquad \beta = r\sqrt{\frac{\rho}{n} + 1} \tag{3.11a}$$

and where A and B are undetermined coefficients.

Similarly by letting

$$\bar{W} = \bar{f}(\eta)\sin r\xi \tag{3.12}$$

the condition (3.7) is satisfied. From Eq. (3.5), one has

$$\bar{f}'''' - 2r^2\bar{f}'' + r^4\bar{f} = 0 \tag{3.13}$$

The solution of this equation with the fulfillment of the boundary condition (3.7a) is found to be

$$\bar{f}(\eta) = A\Big\{[G + Hr\eta]\,e^{-r\eta} + \left[G + Hr\left(\tfrac{2b}{c} - \eta\right)\right]e^{-r\left(\frac{2b}{c} - \eta\right)}\Big\} \qquad 1 \le \eta \le \frac{b}{c} \tag{3.14}$$

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*It can be shown that $\rho > n$, that is, α is real.

where G and H are again undetermined coefficients.

The conditions of continuity (3.8) now become

$$f=\bar{f},\quad f'=\bar{f}',\quad f'=\bar{f}'' \quad \text{and} \quad f'''=\bar{f}''' \qquad \text{on } \eta=1 \tag{3.15}$$

The satisfaction of the above four equations leads to four linear homogeneous equations in the undetermined coefficients. In order for this system to have a non-trivial solution it is necessary and sufficient that the determinant vanish. The result is a characteristic equation:

$$\begin{aligned}
&\cos\alpha\Big\{\Big[\tanh^2\gamma\left(\tfrac{b}{c}-1\right)+\tfrac{1}{\gamma}\tanh\gamma\left(\tfrac{b}{c}-1\right)-1 \\
&+\frac{n\pi}{\gamma}\,\frac{e^{-2\gamma\left(\frac{b}{c}-1\right)}}{1+e^{-2\gamma\left(\frac{b}{c}-1\right)}}\left(1+\tanh\gamma\left(\tfrac{b}{c}\;\right)\right)\Big]\left(\beta\tan\alpha-\alpha\tanh\beta\right) \\
&-\frac{4}{\gamma}\,\frac{n}{\rho}\Big[\tan\gamma\left(\tfrac{b}{c}-1\right)\Big(\beta\tan\alpha+\alpha\tanh\beta \\
&+\frac{1}{\gamma}\tan\alpha\tanh\beta\tanh\gamma\left(\tfrac{b}{c}-1\right)\Big)+\frac{1}{\gamma}\alpha\beta\Big]\Big\}=0
\end{aligned} \tag{3.16}$$

This characteristic equation gives two groups of eigenvalues. One group is from

$$\cos\alpha = 0 \tag{3.16a}$$

The other is from

$$\begin{aligned}
&\Big\{\tanh^2\gamma\left(\tfrac{b}{c}-1\right)+\tfrac{1}{\gamma}\tanh\gamma\left(\tfrac{b}{c}-1\right)-1 \\
&+\frac{n\pi}{\gamma}\,\frac{e^{-2\gamma\left(\frac{b}{c}-1\right)}}{1+e^{-2\gamma\left(\frac{b}{c}-1\right)}}\left[1+\tanh\gamma\left(\tfrac{b}{c}-1\right)\right]\Big\}\left[\beta\tan\alpha-\alpha\tanh\beta\right]
\end{aligned} \tag{3.16b}$$

$$-\frac{4}{\gamma}\frac{n}{\rho}\Big\{\tan\gamma\left(\frac{b}{c}-1\right)\left[\beta\tan\alpha+\alpha\tanh\beta\right]$$

$$+\frac{1}{\gamma}\tan\alpha\tanh\beta\tanh\gamma\left(\frac{b}{c}-1\right)+\frac{1}{\gamma}\alpha\beta\Big\}=0. \tag{3.16b}$$

From (3.16a), one has

$$\alpha=\left(\frac{2n-1}{2}\right)\pi, \qquad n=1,2,3---, \tag{3.17}$$

Substituting α and γ given by Eqs. (3.11a) and (3.9) respectively into the above equation, one will find

$$\rho=\frac{a^2}{nc^2}\left[n^2-n+\frac{1}{4}\right]+n \tag{3.18}$$

It can be verified that when c is less than b, this mode leads to a higher eigenvalue than the mode to be investigated in the following paragraphs. It is, therefore not considered further beyond noting that, after minimizing it with respect to n, it corresponds essentially to the Euler load (except for a correction associated with the particular type of boundary conditions assumed in the present problem.)

Eq. (3.16b) is so complex that it is impossible to give an explicit solution. However, the eigenvalue ρ can be computed for a corresponding "effective width" c, if the dimensions of the plate, a and b, and the number of half waves, n, are specified.

Once an eigenvalue ρ is determined, and by going through equations (3.15), a set of corresponding coefficients, B, H and G can be calculated as follows:

$$B_n = \frac{(1-e^{-\delta_n})\gamma_n \sin\alpha_n + (1+e^{-\delta_n})\alpha_n \cos\alpha_n}{(1-e^{-\delta_n})\gamma_n \sinh\beta_n + (1+e^{-\delta_n})\beta_n \cosh\beta_n}$$

where $\qquad \delta_n = n\pi + 2\gamma_n$

$$H_n = \frac{\rho e^{\gamma_n}}{2(1+e^{-\delta_n})}\left(\sin\alpha_n - B_n \sinh\beta_n\right) \tag{3.19}$$

$$G_n = \frac{e^{\gamma_n}}{1+e^{-\delta_n}}\left\{\sin\alpha_n + B_n \sinh\beta_n - H_n e^{-\gamma_n}\left[\gamma_n + (n\pi-\gamma_n)e^{-\delta_n}\right]\right\}$$

in which the subscript n is attached to all the values except the eigenvalue ρ, in order to indicate their dependence on the parameter n. The only remaining amplitude coefficient A_n will be determined by energy consideration.

It is well known that the primary buckling mode is a multiple of half square waves, if the aspect ratio of the plate is an integer. Hence, without significant loss of generality, one may assume the aspect ratio of the present plate to be one, i.e. $\frac{a}{2b} = 1$. Then, by designating the parameter n to be 1, 2, 3 and 4*, a group of eigenvalues ρ with corresponding $\frac{c}{b}$ are computed from the characteristic equation (3.16b) for $\nu = 0.3$.

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*A limit case for n approaching infinity was studied by Koiter[6], who used the same initial stress field, but found his conclusions to be seriously at variance with experimental and other considerations.

Now one half of the total load P from Figure (1) and Eqs. (3.10a) is

$$P = NC = \rho^2 \frac{D\pi^2}{a^2} C$$

The one-half of the critical load prior to buckling for the plate can be found to be

$$P_o = \lambda_o b = \frac{4}{1+\nu} \frac{D\pi^2}{a^2} b^* \qquad (3.20)$$

Hence the ratio of the load P to P_o is

$$\frac{P}{P_o} = \frac{1+\nu}{4} \rho^2 \frac{c}{b}$$

With the computed values of ρ and $\frac{c}{b}$, a set of curves with $\frac{c}{b}$ as ordinate and $\frac{P}{P_o}$ as abscissa are depicted as shown by Figure 2.

These curves show that as the ratio $\frac{c}{b}$ decreases (due to the increases of the load), the lowest value of $\frac{P}{P_o}$ may change from one curve to the other. The intersecting point of the curves of n = 1 and 2, at which

$$\frac{c}{b} = .358, \quad \frac{P}{P_o} = 5.26 \quad or \quad \rho^2 = 45.35 \qquad (3.21)$$

gives the possibility that with this eigenvalue, either the first mode of buckling or the second one or the both

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*Put

$$t_{11} = -\frac{\lambda_o}{h}, \quad t_{22} = -\nu\frac{\lambda_o}{h}, \quad t_{12} = 0 \quad and \quad w = A \sin\frac{\pi x_1}{a} \sin\frac{\pi x_2}{a}$$

into Eq. (2.6) one will find the eigenvalue $\lambda_o = \frac{4}{1+\nu} \frac{D\pi^2}{a^2}$

of them can exist. This particular point is selected for the further investigation. Then, in general, the configuration $W(\xi,\eta)$ can be expressed as

$$W(\xi,\eta) = A_1 W^1 + A_2 W^2 \qquad (3.22)$$

where

$$W^n = f_n(\eta) \sin \gamma_n \xi \qquad n = 1, 2 \qquad (3.22a)$$

and

$$f_n(\eta) = \sin \alpha_n \eta + B_n \sin \beta_n \eta \qquad 0 \leq \eta \leq 1 . \qquad (3.22b)$$

For the interior region

$$\bar{W}(\xi,\eta) = A_1 \bar{W}^1 + A_2 \bar{W}^2 \qquad (3.23)$$

where

$$\bar{W}^n = \bar{f}_n(\eta) \sin \gamma_n \xi \qquad n = 1, 2 \qquad (3.23a)$$

and

$$\bar{f}_n = [G_n + H_n \gamma_n \eta] e^{-\gamma_n \eta} + [C_n + D_n \gamma_n \eta] e^{\gamma_n \eta}, \quad 1 \leq \eta \leq \frac{b}{c} \qquad (3.23b)$$

in which

$$\left. \begin{aligned} C_n &= (n\pi H_n + G_n) e^{-n\pi} \\ D_n &= (-H_n) e^{-n\pi} \end{aligned} \right\} \qquad (3.24)$$

obtained from Eq. (3.14).

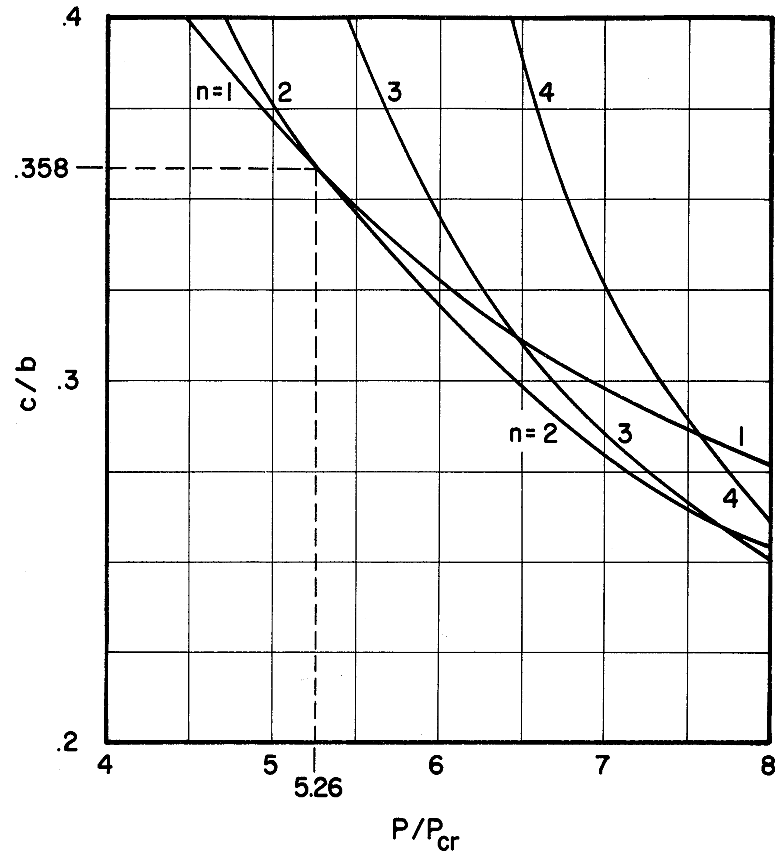

Figure 2. Solutions of the Characteristic Equation (3.16b)

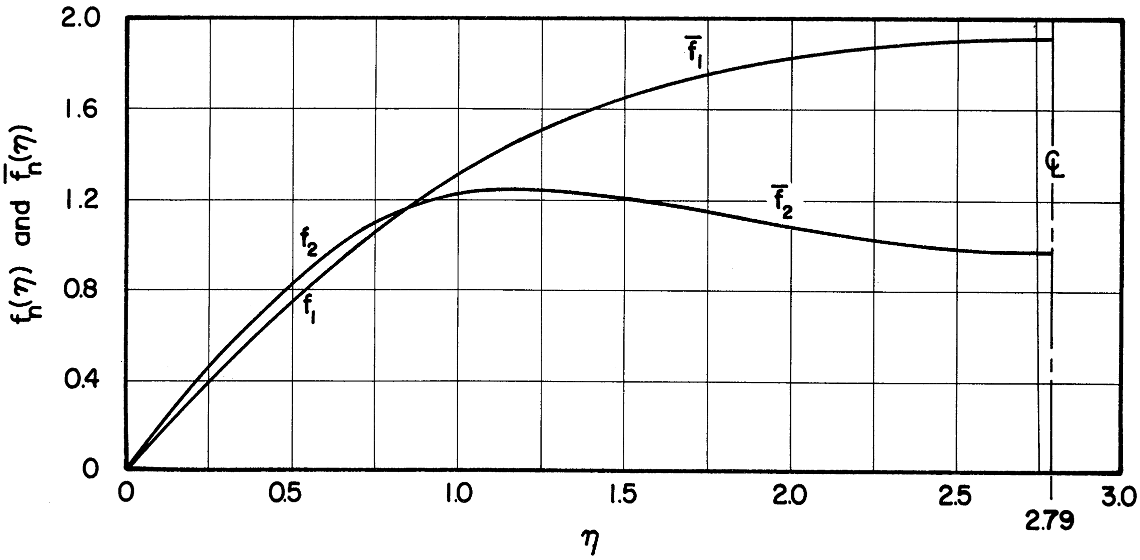

Figure 3. Buckling Modes in -Direction

With the eigenvalue given by (3.21) the parameters α_n , β_n , γ_n and the coefficients involved in Eqs. (3.22) and (3.23) are readily computed by their appropriate definitions from Eqs. (3.11a), (3.19) and (3.24). The results are given in Table I, with $\nu = 0.3$.

TABLE 1

CONSTANTS CONTAINED IN DEFLECTION FUNCTIONS

n	γ_n	α_n	β_n	B_n	G_n	H_n	C_n	D_n
1	0.56301	1.34822	1.5675	.14656	-.63988	3.33867	0.42561	-0.14428
2	1.12602	1.73243	2.35311	0.043735	-.95092	14.9877	.043648	-0.0072295

The configuration functions $f_n(\eta)$ and $\bar{f}_n(\eta)$ in η -direction are shown in Figure 3.

So far, one has found a particular deflection function $W(\xi, \eta)$ satisfying the equilibrium equation (2.6) under the assumed membrane stress field T_{ij} and including both symmetrical (n = 1) and antisymmetrical (n = 2) modes. The next step is to calculate its associate membrane stress field T_{ij}' . To do this, one has to solve for the membrane displacement u_i' from the set of Eqs. (2.20).

On substitution of Eq. (3.22) to the right side of Eq. (2.20), one has

$$(1-\nu)W_{,i}W_{,jj}+(1+\nu)W_{,j}W_{,ij}$$

$$=A_mA_n\Big[(1-\nu)\tfrac{1}{2}(W_{,i}^{m}W_{,jj}^{n}+W_{,i}^{n}W_{,jj}^{m}) \qquad (3.25)$$

$$+(1+\nu)\tfrac{1}{2}(W_{,j}^{m}W_{,ij}^{n}+W_{,j}^{n}W_{,ij}^{m})\Big]$$

in which m and n take the values 1 and 2. The summation convention of index notation applies to the repeated indices of m and n only when one is a subscript and the other is a superscript. For instance, $A_mW^m=A_1W^1+A_2W^2$ But $A_m\phi_m$ means $A_1\phi_1$ or $A_2\phi_2$. Such a convention will be used henceforth. Also note that the expression in the square bracket is symmetric with respect to m and n.

Let the U_i' in equation (2.20) be

$$U_i'=A_mA_nu_i^{mn} \qquad (3.26)$$

Then the Eq. (2.20) becomes

$$(1-\nu)u_{i,jj}^{mn}+(1+\nu)u_{j,ji}^{mn}$$

$$=-\Big[(1-\nu)\tfrac{1}{2}(W_{,i}^{m}W_{,jj}^{n}+W_{,i}^{n}W_{,jj}^{m}) \qquad (3.27)$$

$$+(1+\nu)\tfrac{1}{2}(W_{,j}^{m}W_{,ij}^{n}+W_{,j}^{n}W_{,ij}^{m})\Big]$$

The dropping of the amplitude coefficients A_mA_n in the above equations is based on the independence of these coefficients and the homogeneity of the boundary conditions. Hence in Eq. (3.27) there are, due to the symmetry of n and m, three sets of simultaneous, second order and non-homogeneous partial differential equations for U_i^{mn}, for the region of $0 \leq \xi \leq \frac{a}{c}$ and $0 \leq \eta \leq 1$.

Similarly there are another three sets of simultaneous equations for the interior region. They are

$$(1-\nu)\bar{U}_{i,jj}^{mn} + (1+\nu)\bar{U}_{j,ij}^{mn}$$

$$= -\Big[(1-\nu)\tfrac{1}{2}(\bar{W}_{,i}^{m}\bar{W}_{,jj}^{n} + \bar{W}_{,i}^{n}\bar{W}_{,jj}^{m}) \qquad (3.28)$$

$$+ (1+\nu)\tfrac{1}{2}(\bar{W}_{,j}^{m}\bar{W}_{,ij}^{n} + \bar{W}_{,j}^{n}\bar{W}_{,ij}^{m})\Big]$$

for $0 \leq \xi \leq \frac{a}{c}$ and $1 \leq \eta \leq \frac{b}{c}$. In the above equations it was assumed that

$$\bar{U}_i' = A_m A_n \bar{U}_i^{mn} \qquad (3.29)$$

and Eq. (3.23) was used.

Since the membrane displacements are prescribed all around the plate (see Eqs. (3.1)), the boundary conditions on the additional membrane displacements according to Eq. (2.12) are homogeneous. From Eqs. (3.1a)

$$\left.\begin{array}{l} U_1^{mn}(0,\eta) = U_1^{mn}(\frac{a}{c},\eta) = U_{2,1}^{mn}(0,\eta) = U_{2,1}^{mn}(\frac{a}{c},\eta) = 0 \\ U_2^{mn}(\xi,0) = U_1^{mn}(\xi,0) = 0 \\ \bar{U}_1^{mn}(0,\eta) = \bar{U}_1^{mn}(\frac{a}{c},\eta) = \bar{U}_{2,1}^{mn}(0,\eta) = \bar{U}_{2,1}^{mn}(\frac{a}{c},\eta) = 0 \\ \text{from Eqs. (3.1b)} \\ \bar{U}_2^{mn}(\xi,\frac{b}{c}) = \bar{U}_{1,2}^{mn}(\xi,\frac{b}{c}) = 0 \end{array}\right\} \qquad (3.30)$$

Along $\eta = 1$, the displacements on each side must match, i.e.,

$$u_1^{mn}(\xi,1) = \bar{u}_1^{mn}(\xi,1), \quad u_2^{mn}(\xi,1) = \bar{u}_2^{mn}(\xi,1). \tag{3.31a}$$

Due to the continuity of the additional membrane stress field τ_{ij}', the normal stress in η-direction, τ_{22}' and shearing stress, τ_{12}' must equal their correspondents $\bar{\tau}_{22}'$ and $\bar{\tau}_{12}'$, respectively. In turn these conditions may be expressed on the consideration of the stress-strain, strain-displacement relationships and condition (3.31a) as

$$u_{2,2}^{mn}(\xi,1) = \bar{u}_{2,2}^{mn}(\xi,1), \quad u_{1,2}^{mn}(\xi,1) = \bar{u}_{1,2}^{mn}(\xi,1) \tag{3.31b}$$

Hence the additional membrane displacement fields u_i^{mn} and $\bar{u}_i^{mn}$ have been characterized by the sets of differential equations (3.27) and (3.28), boundary conditions (3.30), and conditions of continuity (3.31a) and (3.31b).

The mathematical computations in solving these systems of equations are quite involved. The details of doing so will be shown in the Appendix. The resulting solutions of u_i^{mn} are in the following forms:

$$\left.\begin{aligned} u_1^{mn} &= \frac{1}{c}\left[G_{mn}(\eta)\sin\phi_{mn}\xi + I_{mn}(\eta)\sin\psi_{mn}\xi\right] \\ u_2^{mn} &= \frac{1}{c}\left[H_{mn}(\eta)\cos\phi_{mn}\xi + J_{mn}(\eta)\cos\psi_{mn}\xi\right] \end{aligned}\right\} \tag{3.32a}$$

where

$$\phi_{mn} = \gamma_m + \gamma_n \,, \qquad \psi_{mn} = \gamma_m - \gamma_n \,.$$

Note that when m = n, the second part in u_1^{mn} vanishes and $\cos \psi_{mn} \xi = 1$

Let $G^o_{mn}(\eta)$, $H^o_{mn}(\eta)$, $I^o_{mn}(\eta)$ and $J^o_{mn}(\eta)$ be considered as the complimentary parts of their corresponding solutions and $G^p_{mn}(\eta)$, $I^p_{mn}(\eta)$, $I^o_{mn}(\eta)$ and $J^p_{mn}(\eta)$ as their particular solutions, then they are in the following forms:

$$\left.\begin{aligned}
&\begin{bmatrix} G^o_{mn} \\ H^o_{mn} \end{bmatrix}^* = (a_1 + a_2 \phi_{mn}\eta) \cosh \phi_{mn}\eta + (a_3 + a_4 \phi_{mn}\eta) \sinh \phi_{mn}\eta \\
&\begin{bmatrix} I^o_{mn} \\ J^o_{mn} \end{bmatrix} = (b_1 + b_2 \psi_{mn}\eta) \cosh \psi_{mn}\eta + (b_3 + b_4 \psi_{mn}\eta) \sinh \psi_{mn}\eta \,. \\
&\text{The last set of solutions is for } m \neq n, \text{ while } m = n \\
&J^o_{mn} = b_1 + b_2 \phi_{mn}\eta
\end{aligned}\right\} \quad (3.32b)$$

and I_{mn} does not exist. And

---

*Solutions of similar forms are collected together. The differences among them are only in constant coefficients. This equation should be strictly written as

$$\begin{bmatrix} G^o_{mn} \\ H^o_{mn} \end{bmatrix} = \left(\begin{bmatrix} (a^1_{mn})_G \\ (a^1_{mn})_H \end{bmatrix} + \begin{bmatrix} (a^2_{mn})_G \\ (a^2_{mn})_H \end{bmatrix} \phi_{mn}\eta \right) \cosh \phi_{mn}\eta$$

$$+ \left(\begin{bmatrix} (a^3_{mn})_G \\ (a^3_{mn})_H \end{bmatrix} + \begin{bmatrix} (a^4_{mn})_G \\ (a^4_{mn})_H \end{bmatrix} \phi_{mn}\eta \right) \sinh \phi_{mn}\eta$$

For simplicity, all subscripts m, n, and G (or H) are omitted. This understanding will be used for similar occasions without notification.

$$\left.\begin{aligned}
\begin{Bmatrix} G^p_{mn} \\ I^p_{mn} \end{Bmatrix} &= C_1 \cos(\alpha_m - \alpha_n)\eta + C_2 \cos(\alpha_m + \alpha_n)\eta \\
&\quad + C_3 \cosh(\beta_m - \beta_n)\eta + C_4 \cosh(\beta_m + \beta_n)\eta \\
&\quad + C_5 \sinh\beta_m\eta \sin\alpha_n\eta + C_6 \sin\alpha_m\eta \sinh\beta_n\eta \\
&\quad + C_7 \cosh\beta_m\eta \cos\alpha_n\eta + C_8 \cos\alpha_m\eta \cosh\beta_n\eta \\
\begin{Bmatrix} H^p_{mn} \\ J^p_{mn} \end{Bmatrix} &= d_1 \sin(\alpha_m - \alpha_n)\eta + d_2 \sin(\alpha_m + \alpha_n)\eta \\
&\quad + d_3 \sinh(\beta_m - \beta_n)\eta + d_4 \sinh(\beta_m + \beta_n)\eta \\
&\quad + d_5 \cosh\beta_m\eta \sin\alpha_n\eta + d_6 \sin\alpha_m\eta \cosh\beta_n\eta \\
&\quad + d_7 \sinh\beta_m\eta \cos\alpha_n\eta + d_8 \cos\alpha_m\eta \sinh\beta_n\eta
\end{aligned}\right\} \quad (3.32c)$$

All the above solutions are valid for $0 \leq \eta \leq 1$. For the interior region, $1 \leq \eta \leq b/c$, one has the solution of $\bar{U}_i^{mn}$ as

$$\left.\begin{aligned}
\bar{U}_1^{mn} &= \frac{1}{c}\left[\bar{G}_{mn}(\eta) \sin\phi_{mn}\xi + \bar{I}_{mn}(\eta) \sin\psi_{mn}\xi\right] \\
\bar{U}_2^{mn} &= \frac{1}{c}\left[\bar{H}_{mn}(\eta) \cos\phi_{mn}\xi + \bar{J}_{mn}(\eta) \cos\psi_{mn}\xi\right]
\end{aligned}\right\} \quad (3.33a)$$

in which

$$\begin{bmatrix} \bar{G}_{mn} & \bar{I}_{mn} \\ \bar{H}_{mn} & \bar{J}_{mn} \end{bmatrix} = (p_1 + p_2 \phi_{mn}\eta + p_3 \phi_{mn}^2 \eta^2) e^{\phi_{mn}\eta} + (p_4 + p_5 \phi_{mn}\eta + p_6 \phi_{mn}^2 \eta^2) e^{-\phi_{mn}\eta}$$

$$+ (q_1 + q_2 \psi_{mn}\eta + q_3 \psi_{mn}^2 \eta^2) e^{\psi_{mn}\eta} + (q_4 + q_5 \psi_{mn}\eta + q_6 \psi_{mn}^2 \eta^2) e^{-\psi_{mn}\eta} \tag{3.33b}$$

for $m \neq n$;

$$\begin{bmatrix} G_{mn} \\ H_{mn} \\ J_{mn} \end{bmatrix} = (p_1 + p_2 \phi_{mn}\eta + p_3 \phi_{mn}^2 \eta^2) e^{\phi_{mn}\eta} + (p_4 + p_5 \phi_{mn}\eta + p_6 \phi_{mn}^2 \eta^2) e^{-\phi_{mn}\eta}$$

$$+ p_7 + p_8 \phi_{mn}\eta + p_9 \phi_{mn}^2 \eta^2 + p_{10} \phi_{mn}^3 \eta^3 \tag{3.33c}$$

for $m = n$. Both the complimentary and particular solutions are included in the above solutions.

The numerical results for all the foregoing solutions are given in Table II to IV.

With the obtained u_i^{mn} and $\bar{u}_i^{mn}$ combining with w^n and $\bar{w}^n$ given by Eqs. (3.22) and (3.23), the additional membrane stress field τ_{ij}' is readily computed from Eq. (2.19); this leads to:

$$\tau_{ij}' = \frac{E}{1-\nu^2} A_m A_n \Big[\frac{1}{2} (u_{i,j}^{mn} + u_{j,i}^{mn} + w_{,i}^{m} w_{,j}^{n})$$

$$+ \nu (u_{k,k}^{mn} + \frac{1}{2} w_{,k}^{m} w_{,k}^{n}) \delta_{ij} \Big] \tag{3.34}$$

and

$$\bar{\tau}_{ij} = \frac{E}{1-\nu^2} A_m A_n \Big[\frac{1}{2} \left(\bar{U}_{i,j}^{mn} + \bar{U}_{j,i}^{mn} + \bar{W}_{,i}^{m} \bar{W}_{,j}^{n} \right) + \nu \left(\bar{U}_{k,k}^{mn} + \frac{1}{2} \bar{W}_{,k}^{m} \bar{W}_{,k}^{n} \right) \delta_{ij} \Big] \tag{3.34}$$

Let

$$\left.\begin{aligned} \tau_{ij}' &= \frac{E}{1-\nu^2} \frac{1}{c^2} \left[A_m A_n \tau_{ij}^{mn} \right] \\ \bar{\tau}_{ij}' &= \frac{E}{1-\nu^2} \frac{1}{c^2} \left[A_m A_n \bar{\tau}_{ij}^{mn} \right] \end{aligned}\right\} \tag{3.35a}$$

in which

$$\left.\begin{aligned} \tau_{ij}^{mn} &= P_{ij}^{mn}(\eta) \cos \psi_{mn} \xi + Q_{ij}^{mn}(\eta) \cos \phi_{mn} \xi \\ \bar{\tau}_{ij}^{mn} &= \bar{P}_{ij}^{mn}(\eta) \cos \psi_{mn} \xi + \bar{Q}_{ij}^{mn}(\eta) \cos \phi_{mn} \xi \\ &\text{for } i = j; \\ \tau_{ij}^{mn} &= P_{ij}^{mn}(\eta) \sin \psi_{mn} \xi + Q_{ij}^{mn}(\eta) \sin \phi_{mn} \xi \\ \bar{\tau}_{ij}^{mn} &= \bar{P}_{ij}^{mn}(\eta) \sin \psi_{mn} \xi + \bar{Q}_{ij}^{mn}(\eta) \sin \phi_{mn} \xi \end{aligned}\right\} \tag{3.35b}$$

for $i \neq j$, and where

$$\begin{bmatrix} P_{ij}^{mn} \\ Q_{ij}^{mn} \end{bmatrix} = K_1 \cos(\alpha_m - \alpha_n)\eta + K_2 \cos(\alpha_m + \alpha_n)\eta + K_3 \cosh(\beta_m - \beta_n)\eta + K_4 \cosh(\beta_m + \beta_n)\eta + K_5 \sinh \beta_m \eta \sin \alpha_n \eta + K_6 \sin \alpha_m \eta \sinh \beta_n \eta \tag{3.35c*}$$

---

*It should be noted that strictly writing $K_s = \begin{bmatrix} (K_s^P)_{ij}^{mn} \\ (K_s^Q)_{ij}^{mn} \end{bmatrix}$, s = 1 to 12. So for ℓ's in (3.35d) and S's, t's in (3.35e).

$$+K_7 \cosh\beta_m\eta \cos\alpha_n\eta + K_8 \cos\alpha_m\eta \cosh\beta_n\eta$$

$$+\left(K_9 + K_{10}\left[\frac{\psi_{mn}}{\phi_{mn}}\right]\eta\right)\sinh\left[\frac{\psi_{mn}}{\phi_{mn}}\right]\eta + \left(K_{11}+K_{12}\left[\frac{\psi_{mn}}{\phi_{mn}}\right]\eta\right)\cosh\left[\frac{\psi_{mn}}{\phi_{mn}}\right]\eta$$

for $i = j$,

$$\begin{bmatrix} P_{ij}^{mn} \\ Q_{ij}^{mn} \end{bmatrix} = \ell_1 \sin(\alpha_m - \alpha_n)\eta + \ell_2 \sin(\alpha_m + \alpha_n)$$

$$+\ell_3 \sinh(\beta_m - \beta_n)\eta + \ell_4 \sinh(\beta_m + \beta_n)\eta$$

$$+\ell_5 \cosh\beta_m\eta \sin\alpha_n\eta + \ell_6 \cos\alpha_m\eta \sinh\beta_n\eta \qquad (3.35d)$$

$$+\ell_7 \sin\alpha_m\eta \cosh\beta_n\eta + \ell_8 \sinh\beta_m\eta \cos\alpha_n\eta$$

$$+\left(\ell_9 + \ell_{10}\left[\frac{\psi_{mn}}{\phi_{mn}}\right]\eta\right)\sinh\left[\frac{\psi_{mn}}{\phi_{mn}}\right]\eta + \left(\ell_{11} + \ell_{12}\left[\frac{\psi_{mn}}{\phi_{mn}}\right]\eta\right)\cosh\left[\frac{\psi_{mn}}{\phi_{mn}}\right]\eta$$

for $i \neq j$; and

$$\begin{bmatrix} \bar{P}_{ij}^{mn} \\ \bar{Q}_{ij}^{mn} \end{bmatrix} = (s_1 + s_2\phi_{mn}\eta + s_3\phi_{mn}^2\eta^2)e^{\phi_{mn}\eta}$$

$$+(s_4 + s_5\phi_{mn}\eta + s_6\phi_{mn}^2\eta^2)e^{-\phi_{mn}\eta}$$

$$+(t_1 + t_2\psi_{mn}\eta + t_3\psi_{mn}^2\eta^2)e^{\psi_{mn}\eta} \qquad (3.35e)$$

$$+(t_4 + t_5\psi_{mn}\eta + t_6\psi_{mn}^2\eta^2)e^{-\psi_{mn}\eta}$$

$$+s_7 + s_8\phi_{mn}\eta + s_9\phi_{mn}^2\eta^2$$

The computed numerical values of these constant coefficients are given in Tables V to VIII. The results of functions P_{ij}^{mn}, Q_{ij}^{mn} and $\bar{P}_{ij}^{mn}$, $\bar{Q}_{ij}^{mn}$ are shown by Figures 4 and 5.

TABLE II

CONSTANT COEFFICIENTS IN EQUATIONS (3.32 b)

	$ch\,\phi_{mn}\eta$	$\phi_{mn}\eta\,ch\,\phi_{mn}\eta$	$sh\,\phi_{mn}\eta$	$\phi_{mn}\eta\,sh\,\phi_{mn}\eta$	$ch\,\psi_{mn}\eta$	$\psi_{mn}\eta\,ch\,\psi_{mn}\eta$	$sh\,\psi_{mn}\eta$	$\psi_{mn}\eta\,sh\,\psi_{mn}\eta$	$\phi_{mn}\eta$
	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4	b_2
$\mathring{G}_{11}$	.05442	-.01398	-.02903	.02520	--	--	--	--	--
$\mathring{G}_{22}$	.04108	-.01194	-.02479	.01245	--	--	--	--	--
$\mathring{G}_{21}$	.05932	-.01713	-.03557	.02644	--	--	--	--	--
$\mathring{I}_{21}$	--	--	--	--	-.29587	.43479	.90301	-.29445	--
$\mathring{H}_{11}$	--	-.02520	-.002085	.01398	--	--	--	--	--
$\mathring{H}_{22}$	--	-.01245	-.01523	.01194	--	--	--	--	--
$\mathring{H}_{21}$	--	-.02644	-.004417	.01713	--	--	--	--	--
$\mathring{J}_{11}$	--	--	--	--	--	--	--	--	-.00567
$\mathring{J}_{22}$	--	--	--	--	--	--	--	--	.07288
$\mathring{J}_{21}$	--	--	--	--	--	.29445	-.31567	-.43479	--

TABLE III

CONSTANT COEFFICIENT IN EQUATIONS (3.32 c)

	$\cos(\alpha_m-\alpha_n)\eta$	$\cos(\alpha_m+\alpha_n)\eta$	$ch(\beta_m-\beta_n)\eta$	$ch(\beta_m+\beta_n)\eta$	$sh\beta_m\eta\sin\alpha_n\eta$	$\sin\alpha_m\eta\, sh\beta_n\eta$	$ch\beta_m\eta\cos\alpha_n\eta$	$\cos\alpha_m\eta\, ch\beta_n\eta$
	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
G^P_{11}	-.02786	-.03519	--	.000756	.001185	--	.007875	--
G^P_{22}	.02009	-.07038	--	.0001346	.002825	--	.009075	--
G^P_{21}	.02144	-.09631	-.003279	.0006205	-.0006425	.01350	.004710	.008465
I^P_{21}	.24968	-.01183	.002451	-.00004943	.00577	.06793	-.01232	.01566

	$\sin(\alpha_m-\alpha_n)\eta$	$\sin(\alpha_m+\alpha_n)\eta$	$sh(\beta_m-\beta_n)\eta$	$sh(\beta_m+\beta_n)\eta$	$ch\beta_m\eta\sin\alpha_n\eta$	$\sin\alpha_m\eta\, ch\beta_n\eta$	$sh\beta_m\eta\cos\alpha_n\eta$	$\cos\alpha_m\eta\, sh\beta_n\eta$
	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8
H^P_{11}	--	-.16853	--	-.004204	-.09918	--	-.12353	--
H^P_{22}	--	-.21656	--	-.0005626	-.04339	--	-.06120	--
H^P_{21}	-.25769	-.18955	.004329	-.001511	-.01250	-.07050	-.02570	-.06351
J^P_{11}	--	.07986	--	.002184	.05141	--	.05454	--
J^P_{22}	--	.09456	--	.0003007	.02038	--	.02288	--
J^P_{21}	1.2777	.17628	.005355	.001592	.01422	.06155	.02213	.05037

TABLE IV

CONSTANT COEFFICIENTS IN EQUATIONS (3.33 b) AND (3.33 c)

	$10^{-3} \times e^{\phi_{mn}\eta}$			$e^{-\phi_{mn}\eta}$			$10^{-2} \times e^{\psi_{mn}\eta}$			$e^{-\psi_{mn}\eta}$		
	1	$\phi_{mn}\eta$	$\phi_{mn}^2\eta^2$	1	$\phi_{mn}\eta$	$\phi_{mn}^2\eta^2$	1	$\psi_{mn}\eta$	$\psi_{mn}^2\eta^2$	1	$\psi_{mn}\eta$	$\psi_{mn}^2\eta^2$
	p_1	p_2	p_3	p_4	p_5	p_6	q_1	q_2	q_3	q_4	q_5	q_6
$\bar{G}_{21}$	2.7525	-0.5653	0.02941	0.43465	0.14399	0.36386	-1.0883	3.5912	-1.3250	-0.6672	1.0955	-0.30662
$\bar{I}_{21}$	-2.6975	0.47535	-0.02121	-1.2761	-0.9346	-0.26229	0.544	3.9017	-1.8341	1.0890	1.7637	-0.43454
$\bar{H}_{21}$	-1.3250	0.3980	-0.02941	2.3006	-1.8313	0.36386	-2.376	1.0451	0.4416	-1.2189	.8841	-0.10221
$\bar{J}_{21}$	4.9750	-1.1025	0.12720	-2.9201	1.1985	-0.78835	-4.1034	-7.026	1.8341	1.8687	1.0404	-0.43454
	$10^{-3} e^{\phi_{mn}\eta}$			$e^{-\phi_{mn}\eta}$			1					
	1	$\phi_{mn}\eta$	$\phi_{mn}^2\eta^2$	1	$\phi_{mn}\eta$	$\phi_{mn}^2\eta^2$	1	$\phi_{mn}\eta$	$\phi_{mn}^2\eta^2$	$\phi_{mn}^3\eta^3$		
	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}		
$\bar{G}_{11}$	9.9050	-6.2010	-0.9888	0.10298	-0.01081	-0.52951	.15001	- .13845	.02204	--		
$\bar{G}_{22}$	0.19483	-0.0620	0.004925	-0.32465	0.014816	1.4240	-.03400	.03218	-.002562	--		
$\bar{H}_{11}$	-3.0070	3.7470	-0.9888	-0.53926	-1.3253	0.52951	-.35149	.11187	--	--		
$\bar{H}_{22}$	-0.12312	-0.07462	-0.004925	1.2446	-3.0187	1.4240	-.08118	.01300	--	--		
$\bar{J}_{11}$	8.1635	-5.7485	1.0722	-0.36742	0.5243	-0.5742	-.27192	-.05857	-.02099	.004454		
$\bar{J}_{22}$	0.32235	-0.11034	0.009064	-1.8993	2.8239	-2.7428	.66409	-.06941	-.017327	.0004546		

TABLE V

CONSTANT COEFFICIENT 2N EQUATIONS (3.35 C)

	$\cos(\alpha_m-\alpha_n)\eta$	$\cos(\alpha_m+\alpha_n)\eta$	$ch(\beta_m-\beta_n)\eta$	$ch(\beta_m+\beta_n)$	$sh\beta_m\eta\sin\alpha_n\eta$	$\sin\alpha_m\eta\, sh\beta_n\eta$	$ch\beta_m\eta\cos\alpha_n\eta$	$\cos\alpha_m\eta\, ch\beta_n\eta$
	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8
P_{11}^{11}	.10038	-.036056	--	.0007744	.021139	--	--	--
Q_{11}^{11}	--	--	--	--	.020206	--	-.006201	--
P_{22}^{11}	.25181	--	--	--	--	--	--	--
Q_{22}^{11}	-.21276	--	--	--	.000938	--	.00620	--
P_{11}^{22}	.22708	-.14448	--	.0002718	.025231	--	--	--
Q_{11}^{22}	--	--	--	--	.020777	--	-.014309	--
P_{22}^{22}	.25980	--	--	--	--	--	--	--
Q_{22}^{22}	-.34260	--	--	--	.004447	.014310	-.01013	.01960
P_{11}^{21}	.045828	-.13957	-.0018922	.0009440	.009525	.034908	.006379	-.034259
Q_{11}^{21}	-.023195	.004175	.0017288	-.00005519	.013376	.032269	-.005570	-.015954
P_{22}^{21}	.09842	-.004661	.0009676	-.00001934	-.0004452	.002187	.0002156	.0018147
Q_{22}^{21}	-.044823	.001254	.007956	-.00000938	-.0007595	.010009	.005570	.01594

TABLE VI

CONSTANT COEFFICIENTS IN EQUATIONS (3.35 C) (Continued)

	$sh\,\psi_{mn}\eta$	$\psi_{mn}\eta\, sh\,\psi_{mn}\eta$	$ch\,\psi_{mn}\eta$	$\psi_{mn}\eta\, ch\,\psi_{mn}\eta$	$sh\,\phi_{mn}\eta$	$\phi_{mn}\eta\, sh\,\phi_{mn}\eta$	$ch\,\phi_{mn}\eta$	$\phi_{mn}\eta\, ch\,\phi_{mn}\eta$
	K_9	K_{10}	K_{11}	K_{12}	K_9	K_{10}	K_{11}	K_{12}
P^{11}_{11}	--	--	--	--	--	--	--	--
Q^{11}_{11}	--	--	--	--	.027963	-.019859	-.05206	.011017
P^{11}_{22}	--	--	--	--	--	--	--	--
Q^{11}_{22}	--	--	--	--	-.005932	.019859	.012336	-.011017
P^{22}_{11}	--	--	--	--	--	--	--	--
Q^{22}_{11}	--	--	--	--	.04776	-.01960	.07380	.01882
P^{22}_{22}	--	--	--	--	--	--	--	--
Q^{22}_{22}	--	--	--	--	-.01013	.01960	.03460	-.01882
P^{21}_{11}	-.43497	.11604	.42016	-.17135	--	--	--	--
Q^{21}_{11}	--	--	--	--	.05141	-.031257	-.08457	.02025
P^{21}_{22}	.09227	-.11604	.061925	.17135	--	--	--	--
Q^{21}_{22}	--	--	--	--	-.010905	.031257	.022053	-.02025

TABLE VII

CONSTANT COEFFICIENTS IN EQUATIONS (3.35 d)

	$\sin(\alpha_m-\alpha_n)\eta$	$\sin(\alpha_m+\alpha_n)\eta$	$sh(\beta_m-\beta_n)\eta$	$sh(\beta_m+\beta_n)\eta$	$ch\beta_m\eta\sin\alpha_n\eta$	$\cos\alpha_m\eta\, sh\beta_n\eta$	$\sin\alpha_m\eta\, ch\beta_n\eta$	$sh\beta_m\eta\cos\alpha_n\eta$
	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8
Q_{12}^{11}	--	--	--	--	.006136	-.009750	--	--
Q_{12}^{22}	--	--	--	--	.006359	-.018375	--	--
P_{12}^{21}	.06716	-.025508	-.0013531	.0001357	.0023742	-.011782	-.0004850	.0001660
Q_{12}^{21}	-.10196	.002289	.003697	-.0000169	.005504	-.025054	.007088	-.007151

	$sh\psi_{mn}\eta$	$\psi_{mn}\eta\, sh\psi_{mn}\eta$	$ch\psi_{mn}\eta$	$\psi_{mn}\eta\, ch\psi_{mn}\eta$	$sh\phi_{mn}\eta$	$\phi_{mn}\eta\, sh\phi_{mn}\eta$	$ch\phi_{mn}\eta$	$\phi_{mn}\eta\, ch\phi_{mn}\eta$
	ℓ_9	ℓ_{10}	ℓ_{11}	ℓ_{12}	ℓ_9	ℓ_{10}	ℓ_{11}	ℓ_{12}
Q_{12}^{11}	--	--	--	--	-.032197	.011017	.016949	-.019859
Q_{12}^{22}	--	--	--	--	-.05420	.018815	.028945	-.019600
P_{12}^{21}	.05412	-.17135	-.26362	.11604	--	--	--	--
Q_{12}^{21}	--	--	--	--	-.05331	.02025	.031156	-.031256

TABLE VIII

CONSTANT COEFFICIENTS IN EQUATION (3.35 e) for M=N

	$10^{-3} \times e^{\phi_{mn}\eta}$			$e^{-\phi_{mn}\eta}$			1		
	1	$\phi_{mn}\eta$	$\phi_{mn}^2\eta^2$	1	$\phi_{mn}\eta$	$\phi_{mn}^2\eta^2$	1	$\phi_{mn}\eta$	$\phi_{mn}^2\eta^2$
	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
$\bar{P}^{11}_{11}$	13.03	-8.856	1.501	.02961	-.30812	.80383	.025812	.21826	-.06948
$\bar{Q}^{11}_{11}$	1.757	-1.752	.3753	-.05338	-.32128	.20096	.034737	--	--
$\bar{P}^{11}_{21}$	--	--	--	--	--	--	.25181	--	--
$\bar{Q}^{11}_{22}$	-4.022	2.502	-.3753	.07370	-.08068	-.20096	-.26564	.21826	-.06948
$\bar{Q}^{11}_{12}$	2.770	-2.177	.3753	.11403	.12030	-.20096	-.10913	.06948	--
$\bar{P}^{22}_{11}$	0.5496	-.18206	.01508	.26095	-2.1239	4.3233	.05916	.10146	-.01615
$\bar{Q}^{22}_{11}$	.10497	-.04021	.00384	.8360	-2.1559	1.0809	.00805	--	--
$\bar{P}^{22}_{22}$	--	--	--	--	--	--	.2598	--	--
$\bar{Q}^{22}_{22}$	-.15090	.04788	.00384	-.30167	-.00582	-1.0809	-.12508	.10146	-.01615
$\bar{Q}^{22}_{12}$	.12696	-.04404	.00384	-.29831	1.0752	-1.0809	-.05072	.01615	--

TABLE IX

CONSTANT COEFFICIENTS IN EQUATIONS (3.35 e) FOR M≠N

	$10^{-3} \times e^{\phi_{mn}\eta}$			$e^{-\phi_{mn}\eta}$			$10^{-2} \times e^{\psi_{mn}\eta}$			$e^{-\psi_{mn}\eta}$		
	1	$\phi_{mn}\eta$	$\phi_{mn}^2\eta^2$	1	$\phi_{mn}\eta$	$\phi_{mn}^2\eta^2$	1	$\psi_{mn}\eta$	$\psi_{mn}^2\eta^2$	1	$\psi_{mn}\eta$	$\psi_{mn}^2\eta^2$
	s_1	s_2	s_3	s_4	s_5	s_6	t_1	t_2	t_3	t_4	t_5	t_6
$\bar{P}^{21}_{11}$	-6.4195	-1.3846	.0752	.64883	- .41331	.93209	-3.064	1.5798	- .3483	- .47250	.14095	- .08057
$\bar{Q}^{21}_{11}$	.7355	- .2372	.01666	- .22135	- .96270	.20713	- .20016	.22442	.17407	.51442	- .30499	.04028
$\bar{P}^{21}_{22}$	-1.0708	.18729	.008359	- .60163	- .36886	- .10356	8.8125	- .29729	.3483	.67396	.18124	.08057
$\bar{Q}^{21}_{22}$	- .1310	.3040	- .01666	.90396	- .13418	- .20713	-3.0578	4.2461	-1.5665	-1.31442	1.29518	- .36252
$\bar{P}^{21}_{12}$	2.1515	- .51165	.02508	- .69990	- .48363	- .31070	-5.8405	2.2763	- .3483	.49268	.02017	.08057
$\bar{Q}^{21}_{12}$	1.0055	- .27000	.01666	.76979	.54844	- .20713	- .22946	- .37101	.5222	- .86981	.67339	- .12084

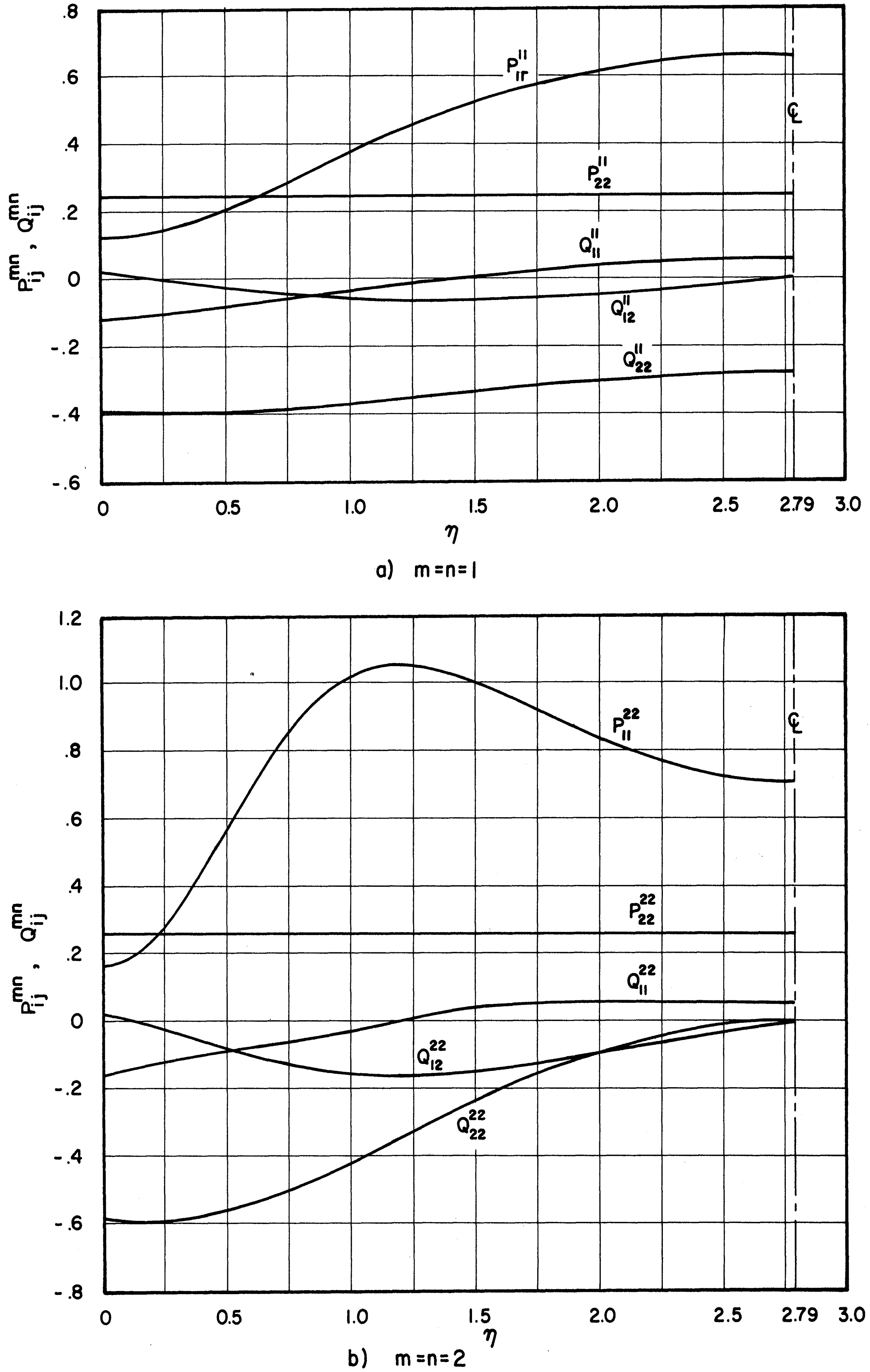

Figure 4. Functions P_{ij}^{mn} and Q_{ij}^{mn} , for m = n

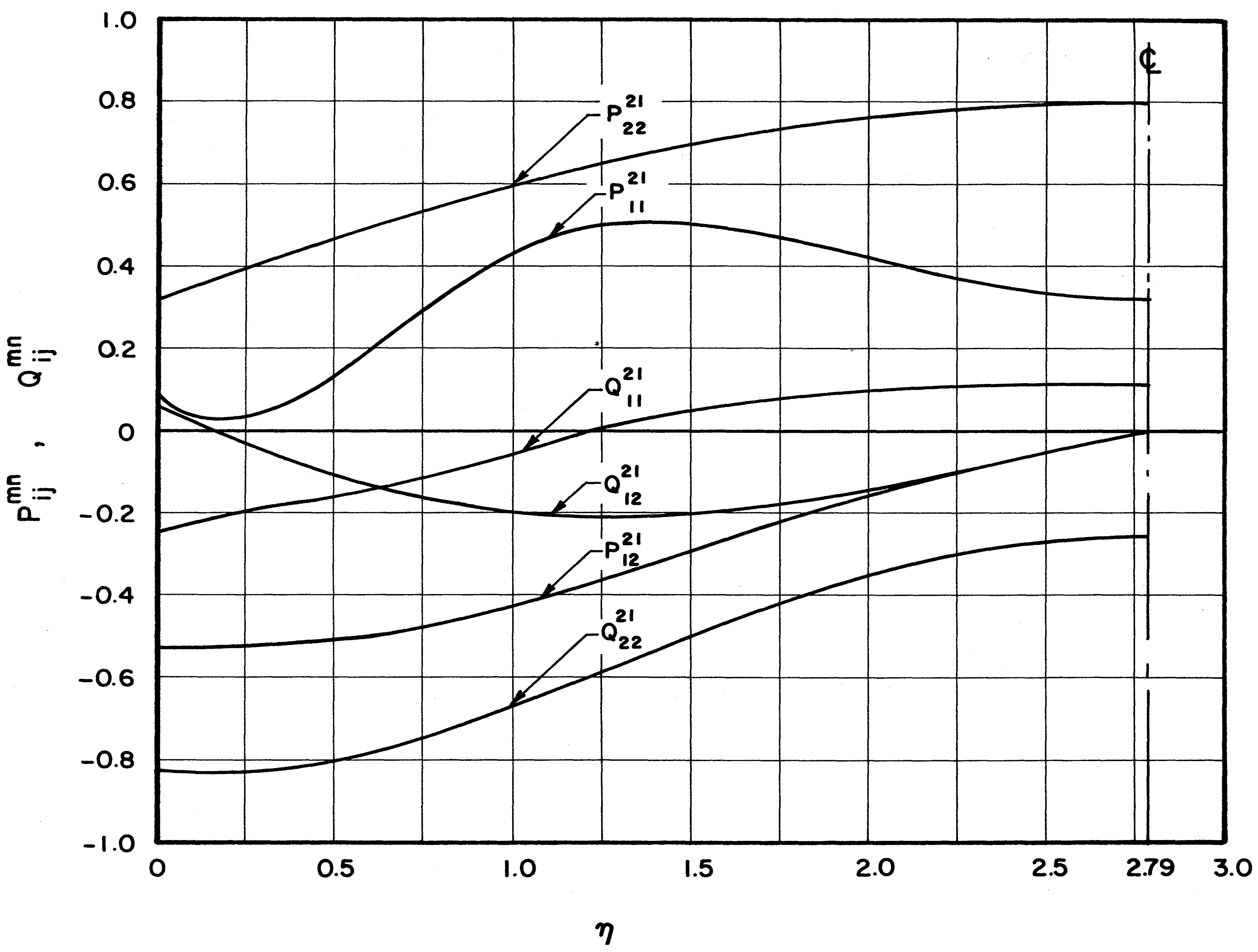

Figure 5. Functions P_{ij}^{mn} and Q_{ij}^{mn} for m ≠ n

In order to determine the amplitude coefficients A_m and to examine the stability of the corresponding configuration, the potential energy as defined by Eqs. (2.22) and (2.23) needs to be calculated. For the present case, because of the simply supported boundary conditions in the perpendicular direction to the plane of plate, the bending energy term may be simplified to

$$U_b = \frac{D}{2}\int_R [(A_m W^m)_{,ii}]^2 dA \tag{3.36}$$

where $dA = dx_1 dx_2$. After transforming it to $d\xi d\eta$, the integration is performed by taking ξ from o to $\frac{a}{c}$, η from o to 1 and then from 1 to $\frac{b}{c}$ so that only one half of the area is covered.

With the t_{ij}^{o} given by Eqs. (3.2) the work done term (see Eq. (2.23)) may also be reduced somewhat to the form

$$W_e = -\frac{\lambda}{2}\int_R [(A_m W^m)_{,1}(A_n W^n)_{,1} + \nu(A_m W^m)_{,2}(A_n W^n)_{,2}] dA \tag{3.37}$$

The membrane strain energy term, with τ_{ij}' given by Eqs. (3.35a) may be expressed as

$$U_m = \frac{Eh}{1-\nu^2}\frac{1}{C^2}\frac{1}{4}\int_R [(A_p A_q \tau_{ij}^{pq})(A_m W^m)_{,i}(A_n W^n)_{,j}] dA \tag{3.38}$$

Now, let the integrals involved in U_b, and W_e be denoted by

$$\left.\begin{aligned} J_1 &= \frac{c^3}{a}\frac{1}{2}\int_R \left[(w_{,11}^1)^2 + 2(w_{,11}^1 w_{,22}^1) + (w_{,22}^1)^2\right] dA \\ J_2 &= \frac{c^3}{a}\frac{1}{2}\int_R \left[(w_{,11}^2)^2 + 2(w_{,11}^2 w_{,22}^2) + (w_{,22}^2)^2\right] dA \end{aligned}\right\} \quad (3.39)$$

and

$$\left.\begin{aligned} K_1 &= \frac{2}{1+\nu}\left(\frac{\pi c}{a}\right)^2 \int_R \left[w_{,1}^1 w_{,1}^1 + \nu w_{,2}^1 w_{,2}^1\right] dA \\ K_2 &= \frac{2}{1+\nu}\left(\frac{\pi c}{a}\right)^2 \int_R \left[w_{,1}^2 w_{,1}^2 + \nu w_{,2}^2 w_{,2}^2\right] dA \end{aligned}\right\} \quad (3.40)$$

respectively, and those in U_m be denoted by

$$\left.\begin{aligned} L &= 3\frac{c}{a}\int_R \left[\tau_{ij}^{11} w_{,i}^1 w_{,j}^1\right] dA \\ N &= 3\frac{c}{a}\int_R \left[\tau_{ij}^{22} w_{,i}^2 w_{,j}^2\right] dA \\ M' &= 3\frac{c}{a}\int_R \left[\tau_{ij}^{11} w_{,i}^2 w_{,j}^2\right] dA \\ M'' &= 3\frac{2c}{a}\int_R \left[\tau_{ij}^{12} w_{,i}^1 w_{,j}^2\right] dA \end{aligned}\right\} \quad (3.41)$$

and

$$M = M' + M''.$$

(The computed numerical values of these constants are given in Table IX.) Then by summing the three energy terms, the potential energy assumes the form

$$V = \frac{Da}{c^3}\left\{(J_1 - K_1\lambda')A_1^2 + (J_2 - K_2\lambda')A_2^2 + \frac{1}{h^2}(LA_1^4 + 2MA_1^2A_2^2 + NA_2^4)\right\} \quad (3.42)$$

where $\lambda' = \frac{\lambda}{\lambda_o}$ and λ_o is given in Eq. (3.20).

Note that the equality

$$\int_R \tau_{ij}^{11} w_{,i}^{2} w_{,j}^{2}\, dA = \int_R \tau_{ij}^{22} w_{,i}^{1} w_{,j}^{1}\, dA \tag{3.43a}$$

has been taken into consideration in the constant 2M'. It can be seen as follows. In view of the symmetry of the stress-strain matrix

$$\int_R \tau_{ij}^{11} \varepsilon_{ij}^{22}\, dA - \int_R \tau_{ij}^{22} \varepsilon_{ij}^{11}\, dA \tag{3.43b}$$

Replace the strain field ε_{ij}^{mn} on the both sides of the above equation, by its corresponding displacement field u_i^{mn} and W^n according to Eq. (2.19), then apply Green's Theorem to the integral involving u_i^{mn}. This integral will vanish because on the boundary, $u_i^{nn} = 0$ (for the present case; otherwise $\tau_{ij}' n_j = 0$), and in this region, the equations of equilibrium are satisfied by $\tau_{ij}'^{n}$. The remaining integral is the equality (3.43a).

Now the two equations for the determination of A_1 and A_2 based on Eqs (2.26a) are readily to be obtained from Eq. (3.42). They are

$$\left.\begin{aligned} &\frac{\partial V}{\partial A_1} = 0, \quad A_1\left[(J_1 - k_1\lambda') + 2L\left(\frac{A_1}{h}\right)^2 + 2M\left(\frac{A_2}{h}\right)\right] = 0 \\ &\frac{\partial V}{\partial A_2} = 0, \quad A_2\left[(J_2 - k_2\lambda') + 2M\left(\frac{A_1}{h}\right) + 2N\left(\frac{A_2}{h}\right)^2\right] = 0 \end{aligned}\right\} \tag{3.44}$$

Since A_1 and A_2 are independent there are three possible cases. Namely, case (1) with $A_2 = 0$, case (2) with $A_1 = 0$, and case (3) neither of them vanish.

Case (1): Let $A_2 = 0$, from Eqs. (3.44)

$$\left(\frac{A_1}{h}\right)_1^2 = \frac{1}{2L}(K_1\lambda' - J_1) > o \qquad \text{if } \lambda' > \frac{J_1}{K_1} \qquad (3.45)$$

The subscript on $\left(\frac{A_1}{h}\right)$ indicates the number of case.

Case (2): Let $A_1 = 0$ in Eqs. (3.44)

$$\left(\frac{A_2}{h}\right)_2^2 = \frac{1}{2N}(K_2\lambda' - J_2) > o \qquad \text{if } \lambda' > \frac{J_2}{K_2} \qquad (3.46)$$

Case (3): Solving Eqs. (3.44) simultaneously for A_1 and A_2

$$\left(\frac{A_1}{h}\right)_3^2 = \frac{1}{2}\frac{1}{M^2-LN}\left[(MK_2 - LK_1)\lambda' - (MJ_2 - LJ_1)\right]$$

$$> o \qquad \text{if } \lambda' > \frac{MJ_2 - LJ_1}{MK_2 - LK_1} \qquad (3.47a)$$

Note that $M^2 - LN > o$ for the present case (see Table IX). And

$$\left(\frac{A_2}{h}\right)^2 = \frac{1}{2}\frac{1}{M^2-LN}\left[(MK_1 - LK_2)\lambda' - (MJ_1 - LJ_2)\right] \qquad (3.47b)$$

$$> o \qquad \text{if } \lambda' > \frac{MJ_1 \cdot LJ_2}{MK_1 - LK_2}$$

Now the stabilities of these modes are going to be examined. From Eq. (3.42) or Eqs. (3.44)

$$\frac{\partial^2 V}{\partial A_1^2} = 2\left[(J_1 - k_1\lambda') + 6L\left(\frac{A_1}{h}\right)^2 + 2M\left(\frac{A_2}{h}\right)^2\right]$$

$$\frac{\partial^2 V}{\partial A_2^2} = 2\left[(J_2 - k_2\lambda') + 2M\left(\frac{A_1}{h}\right)^2 + 6N\left(\frac{A_2}{h}\right)^2\right] \qquad (3.48)$$

and

$$\frac{\partial^2 V}{\partial A_1 \partial A_2} = 8M\left(\frac{A_1}{h}\right)\left(\frac{A_2}{h}\right)$$

For case (1), with $A_2 = 0$, and $\left(\frac{A_1}{h}\right)^2$ given by Eq. (3.45)

$$\frac{\partial^2 V}{\partial A_1^2} = 8L\left(\frac{A_1}{h}\right)_1^2$$

$$\frac{\partial^2 V}{\partial A_2^2} = \frac{4(M^2 - LN)}{L}\left(\frac{A_1}{h}\right)_3^2 \qquad (3.49)$$

$$\frac{\partial^2 V}{\partial A_1 \partial A_2} = 0$$

Hence, according to condition (2.27a), the primary buckling mode is table if and only if

$$\left(\frac{A_1}{h}\right)_1^2 > 0 \quad \text{and} \quad \left(\frac{A_1}{h}\right)_3^2 > 0 \qquad (3.50)$$

For case (2), with $A_1 = 0$ and $\left(\frac{A_2}{h}\right)^2$ given by Eq. (3.46),

$$
\left.
\begin{aligned}
\frac{\partial^2 V}{\partial A_1^2} &= \frac{4(M^2-LN)}{L}\left(\frac{A_1}{h}\right)_2^2 \\
\frac{\partial^2 V}{\partial A_2^2} &= 8N\left(\frac{A_2}{h}\right)_2^2 \\
\frac{\partial^2 V}{\partial A_1 \partial A_2} &= 0
\end{aligned}
\right\} \tag{3.51}
$$

Thus again, according to condition (2.27a), the secondary buckling mode is stable if and only if

$$
\left(\frac{A_1}{h}\right)_3^2 > 0 \quad \text{and} \quad \left(\frac{A_2}{h}\right)_2^2 > 0 \tag{3.52}
$$

Now for case (3), with $\left(\frac{A_1}{h}\right)^2$ and $\left(\frac{A_2}{h}\right)^2$ obtained in Eqs. (3.47a) and (3.47b)

$$
\left.
\begin{aligned}
\frac{\partial^2 V}{\partial A_1^2} &= 8L\left(\frac{A_1}{h}\right)_3^2 \\
\frac{\partial^2 V}{\partial A_2^2} &= 8N\left(\frac{A_2}{h}\right)_3^2 \\
\frac{\partial^2 V}{\partial A_1 \partial A_2} &= 8M\left(\frac{A_1}{h}\right)_3\left(\frac{A_2}{h}\right)_3
\end{aligned}
\right\} \tag{3.53}
$$

It follows that

$$
\left[\frac{\partial^2 V}{\partial A_1 \partial A_2}\right]^2 - \frac{\partial^2 V}{\partial A_1^2}\frac{\partial^2 V}{\partial A_2^2} = 64(M^2-LN)\left(\frac{A_1}{h}\right)_3^2\left(\frac{A_2}{h}\right)_3^2 > 0 \tag{3.54}
$$

This fact shows that the configuration of the combination of the two modes is unstable in the whole postbuckling domain.

The numerical results of the amplitude coefficients A_1 and A_2 and the stability of each case are given in Table X.

TABLE X

INTEGRATED CONSTANTS IN EQUATION (3.42)

J_1	J_2	K_1	K_2	L	M	M'	N
.7453	3.0650	.6244	1.0293	1.8680	3.342	2.034	3.464

TABLE XI

AMPLITUDE COEFFICIENTS AND STABILITIES

Case	$(\frac{A_1}{h})^2$	Min. λ'	$(\frac{A_2}{h})^2$	Min. λ'	Stability
1	$.1671\,\lambda' - .1995$	1.19	0		Stable if $\lambda' > 1.19$
2	0		$.1486\,\lambda' - .4423$	2.98	Unstable if $\lambda' < 6.00$ Stable if $\lambda' > 6.00$
3	$.1359\,\lambda' - .8152$	6.00	$.01746\,\lambda' + .4104$		Unstable

It is seen from Table XI that the symmetric configuration starts at $\lambda' = 1.19$ and it is stable from there on. The antisymmetric configuration could exist after $\lambda' = 2.98$, but it is stable only after λ' exceeds 6.00.

Note that an actual symmetric buckling mode starts at $\lambda' = 1$. The difference between this value and the present one (1.19) is due the approximation involved in the configuration function.

The primary buckling mode may change to the secondary one when the potential energies associated with each mode are equal. (See Eq. (2.28)). On substitution of Eqs. (3.45) and (3.46) into Eq. (3.42) separately, one obtains

$$\left.\begin{aligned} V_1 &= -\frac{1}{4}\frac{1}{L}(K_1\lambda' - J_1)^2 \\ &\text{and} \\ V_2 &= -\frac{1}{4}\frac{1}{N}(K_2\lambda' - J_2)^2 \end{aligned}\right\} \quad (3.54)$$

respectively. Equating these two equations and solving for λ' results in

$$\lambda' = \frac{\sqrt{N}\,J_1 \mp \sqrt{L}\,J_2}{\sqrt{N}\,K_1 \mp \sqrt{L}\,K_2} \quad (3.55)$$

Using the constants given in Table X, one finds

$$\lambda' = 2.17 \text{ and } 11.42 \quad (3.55a)$$

The first value in the above results is in a range in which $\left(\frac{A_2}{h}\right)_2$ does not exist. Hence, at $\lambda' = 11.42$, the transition may occur.

With the determined lateral buckling amplitude coefficients $\left(\frac{A_1}{h}\right)_1^2$ and $\left(\frac{A_2}{h}\right)_2^2$, the additional membrane stress field τ_{ij}' as given by Eqs. (3.35) is uniquely determined. In turn one may find the new membrane stress field τ_{ij} as defined by Eq. (2.21) with the $\overset{\circ}{t}_{ij}$ given by Eqs. (3.2). For instance, in case (1), using $\lambda_0 = \frac{4}{1+\nu}\frac{\pi^2 D}{a^2}$

$$\left.\begin{aligned}\frac{\tau_{11}}{\lambda_0} &= -\lambda' + \frac{3(1+\nu)}{\ell_1^2}\left(\frac{A_1}{h}\right)_1^2 \tau_{11}'' \\ \frac{\tau_{22}}{\lambda_0} &= -\nu\lambda' + \frac{3(1+\nu)}{\ell_1^2}\left(\frac{A_2}{h}\right)_1^2 \tau_{22}'' \\ \frac{\tau_{12}}{\lambda_0} &= \frac{3(1+\nu)}{\ell_1^2}\left(\frac{A_1}{h}\right)_1^2 \tau_{12}''.\end{aligned}\right\} \quad (3.56)$$

By changing $\left(\frac{A_1}{h}\right)_1^2 \tau_{ij}''$ to $\left(\frac{A_2}{h}\right)_2^2 \tau_{ij}^{22}$ in the above equations the stress field of case (2) may be obtained. The distributions of the normal stress along the edges of $\xi = 0$ are depicted in Figure 6 for both cases.

A relation between the total thrust P and the parameter λ' can easily be found by integrating the normal stress τ_{11} along $\xi = 0$. The resulting equation for the case (1), is

$$\frac{P_1}{P_0} = 0.5530\lambda' + 0.5337 \qquad \text{for } \lambda' > 1.19 \quad (3.55a)^*$$

for the case (2),

$$\frac{P_2}{P_0} = 0.3075\lambda' + 2.0613 \qquad \text{for } \lambda' > 2.98 \quad (3.55b)$$

where P_1 and P_2 are the one-half thrust in their respective cases, while P_0 is the prior buckling load equal to $\lambda_0 b$. Equations (3.55) are shown graphically in Figure 7.

*The conventional coefficient of λ' is 0.5; the discrepancy is due again to the choice of the deflection function.

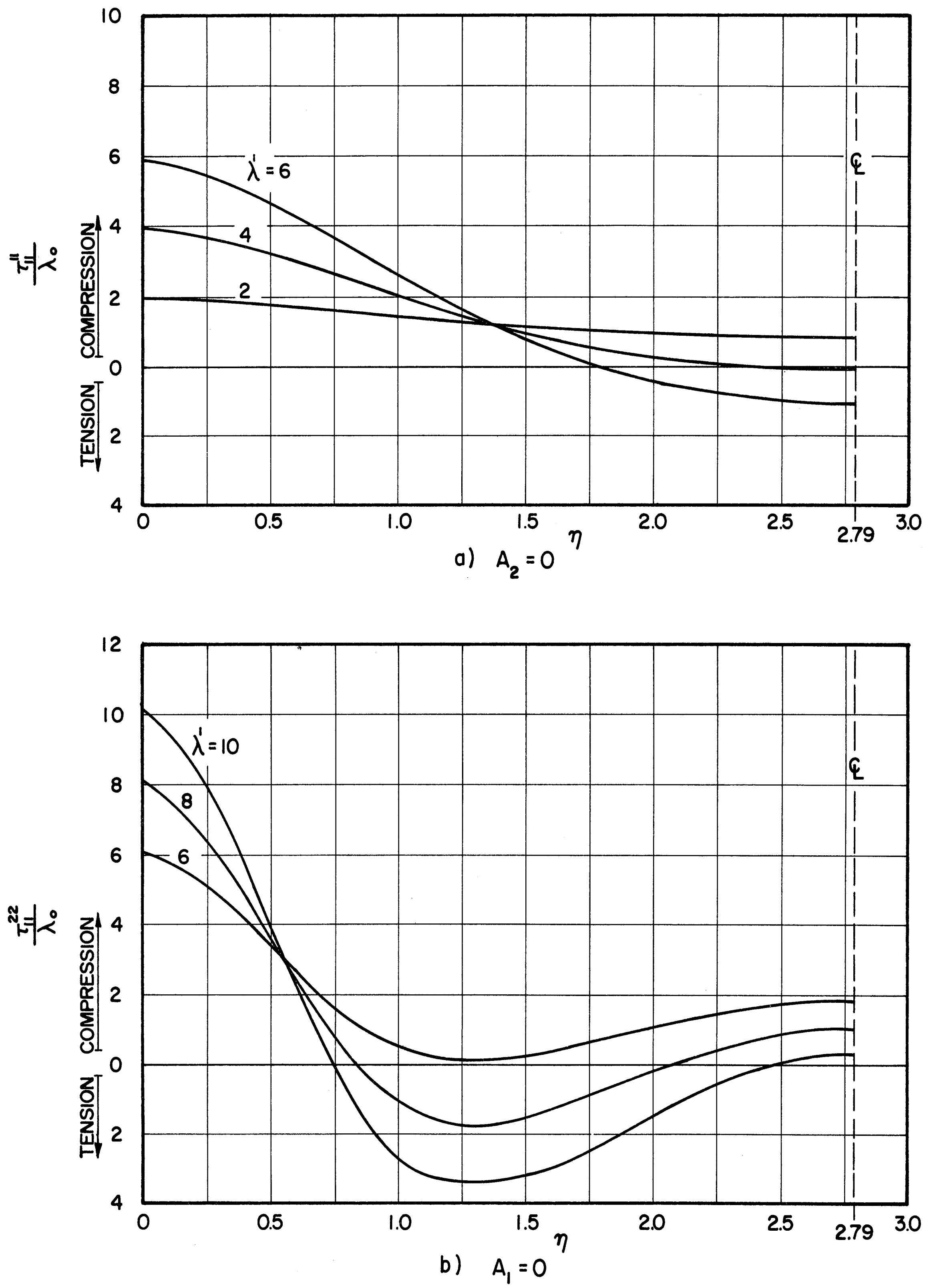

Figure 6. Normal Stress in -Direction
Along the Loaded Edge

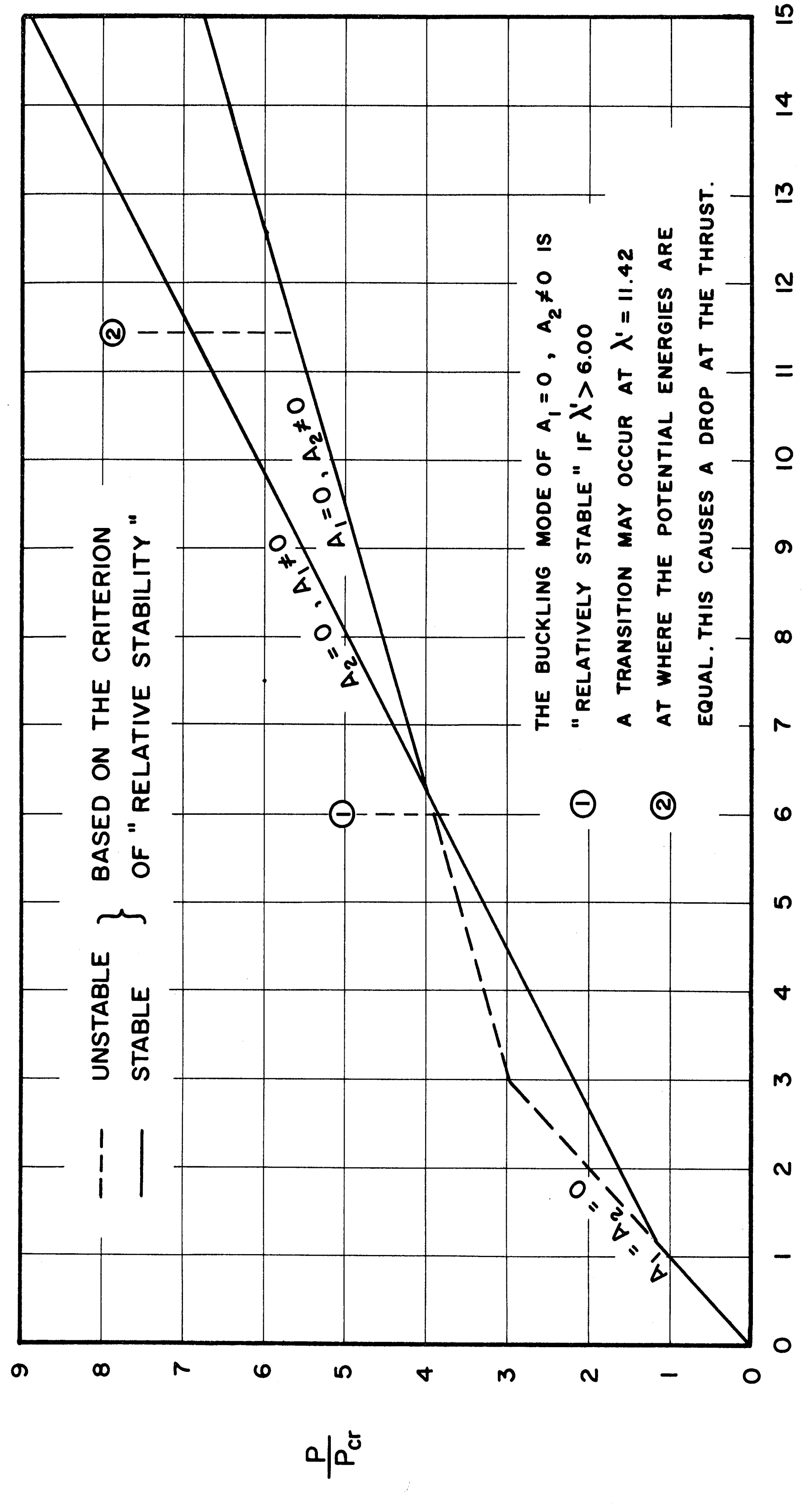

Figure 7. Thrust vs. Shortening Curves

4. APPLICATION OF FUNCTION SPACE METHOD TO THE SQUARE PLATE

The function space method was initially suggested by Prager and Synge* as an approximate method for problems in elasticity. The application of the concept of this method to buckling problems of plates and the basic principle involved was first given in a previous paper(9) on which the present study leans heavily. However, the method suggested there was primary for problems without including secondary buckling. An extension of this method in this direction is made in the present study. The illustration is given to the square plate discussed in Section 3.

A principle of minimum strain energy of the additional membrane stress field was established in reference (9). The inequality, in present notation,

$$\mathcal{U}(T_{ij}, W) \equiv \mathcal{U}_b(w,w) + W_e(\Delta t_{ij}; w,w) + W_e(T_{ij}; w,w) \geq 0 \quad (4.1)$$

is used to characterize a surface which divides the space into two parts. Any stress field T_{ij} satisfying (4.1) is on one side and is stable. Those of the other side are unstable. In the inequality (4.1), $W(x_1, x_2)$ is any non-trivial configuration which has piecewise continuous second

* See REFERENCES > in reference (9)

derivatives and satisfies the geometric boundary conditions relating W on B. Since the actual w also satisfies these conditions, therefore, $U(T_{ij}', w) \geq 0$. The equality holds when the membrane stress field is also an actual one, i.e. $U(t_{ij}', w) = 0$. This can be shown by multiplying Eq. (2.6) by w and integrating by parts. Hence,

$$W_e(T_{ij}' - t_{ij}', w.w) \geq 0$$

i.e.

$$h\int_R (T_{ij}' - t_{ij}')\, w_{,i}\, w_{,j}\, dA \geq 0 \tag{4.2}$$

which, in view of Eq. (2.14), can be written as

$$h\int_R (T_{ij}' - t_{ij}')\, e_{ij}'\, dA - h\int_R (T_{ij}' - t_{ij}')\, u_{i,j}'\, dA \geq 0. \tag{4.3}$$

Applying Green's Theorem to the second integral in (4.3), the result vanishes because $u_i' = 0$ on B", $(T_{ij}' - t_{ij}')\eta_j = 0$ on B' and $(T_{ij}' - t_{ij}')_{,j} = 0$ in R.

In view of the symmetry of the stress-strain matrix

$$2T_{ij}' e_{ij}' = t_{ij}' e_{ij}' + T_{ij}' E_{ij}' - (T_{ij}' - t_{ij}')(E_{ij}' - e_{ij}')$$

in which E_{ij}' is the strain field associated with T_{ij}'. Then the remaining integral in (4.3) may be written as

$$\begin{aligned} h\int_R T_{ij}' E_{ij}'\, dA &\geq h\int_R t_{ij}'\, e_{ij}'\, dA + h\int_R (T_{ij}' - t_{ij}')(E_{ij}' - e_{ij}')\, dA \\ &\geq h\int_R t_{ij}'\, e_{ij}\, dA \end{aligned} \tag{4.4}$$

This is the principle of minimum additional membrane strain energy. State in words, "the strain energy associated with the additional membrane stress is as small as possible subject to the requirement of static admissibility."

Some basic aspects about the function space are these. A vector in function space represents a stress field. The inner product between two vectors is defined by

$$\bar{t}^1 \cdot \bar{t}^2 \equiv L(\bar{t}^1)\, L(\bar{t}^2) \cos\theta = h\int_R t^1_{ij}\, e^2_{ij}\, dA \tag{4.6a}$$

from which the length of a vector is given by

$$L(\bar{t}) = +\left[h\int_R t_{ij}\, e_{ij}\, dA\right]^{\frac{1}{2}} \tag{4.6b}$$

while the angle θ between two vectors can be found from (4.6a).

Applying these concepts to the inequality (4.4), one now may write it as

$$\bar{T}'\cdot\bar{T}' \geq \bar{t}'\cdot\bar{t}' + (\bar{T}'-\bar{t}')(\bar{T}'-\bar{t}') \geq \bar{t}'\cdot\bar{t}' \tag{4.7}$$

This indicates that the surfaces S defined by (4.1) is concave to the opposite side of the origin O as shown in Figure 8. In this figure P is a generic point associated with $\bar{T}'$ and p is at a point such that vector $\overline{op}$ indicates the actual additional membrane stress field.

It also has been proved that the vector $\bar{\tau}'$, representing the additional membrane stress field τ'_{ij}, which is the statically homogeneous part of a stress field

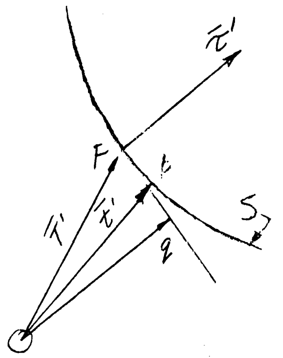

Figure 8. Diagram in Function Space
for Primary Buckling

associated with a strain field derived from

$$\varepsilon_{ij} = \frac{1}{2} w_{,i} w_{,j}$$

is normal to the surface at the corresponding generic point as shown in Figure 8.

In fact, the stress fields T'_{ij} and τ'_{ij} are defined in the formulation for the first method by Eqs. (2.15) and (2.21) respectively.

The stress field τ'_{ij} , due to the undetermined amplitude coefficient in W (in the present discussion, W includes primary buckling only), is also indefinite; otherwise it is uniquely determined.

The undetermined coefficient can be determined in the following way. Let (Figure 8)

$$\overline{oq} = \overline{T}' + \overline{pq} = \overline{\tau}' \tag{4.8}$$

Taking inner product of the above vector with $\overline{\tau}'$ and considering the fact that

$$\bar{\tau}'\cdot\overline{pq}=0$$

one has

$$\bar{T}\cdot\bar{\tau}'=\bar{\tau}:\bar{\tau}' \tag{4.9}$$

This equation enables one to compute the coefficient.

Note that referring to the inequality (4.7), the length of vector $\bar{T}'$ represents an upper bound to the actual one, the length of $\bar{t}'$. The length of vector $\overline{oq}$ is the lower bound to it, since this length does not exceed the length of $\bar{t}'$ due to the convexity of the surface S. Hence, the length of $\bar{\tau}'$ so determined is the lower bound of the actual one.

For different buckling modes which are characterized by different eigenvalues, there are different surfaces in function space. Each one represents one mode. Usually they are independent and separate from one to the other.

However, in the case of the problem discussed in Section 3, there are two surfaces, say S_1 and S_2 for the buckling modes of $n = 1$ and $n = 2$ respectively, passing through the point P which is located by the vector $\bar{T}'$ given by Eq. (3.3). This is because of the particularly selected eigenvalue (see (3.21)).

The additional membrane stress vector, $\bar{\tau}'$, is in the form of

$$\bar{\tau}' = \frac{E}{1-\nu^2}\frac{1}{c^2}\left[A_1^2\bar{\tau}^{11} + 2A_1A_2\bar{\tau}^{12} + A_2^2\bar{\tau}^{22}\right] \qquad (4.10)$$

where $\bar{\tau}^{mn}$ are the vectors representing the stress fields of τ_{ij}^{mn} given by Eqs. (3.35). It is normal to the surface S_1 when $A_2 = 0$. It is normal to the surface S_2 when $A_1 = 0$. When neither A_1 nor A_2 vanish, and by varying these two coefficients, the locus of the vector $\bar{\tau}'$ is an elliptic cone. This can be shown in the following way. Write

$$\bar{\tau}' = \alpha^2\bar{N}^{11} + 2\alpha\beta\bar{N}^{12} + \beta^2\bar{N}^{22} \qquad (4.11)$$

in which $\bar{N}^{11}$ and $\bar{N}^{22}$ are normalized stress vectors of $\bar{\tau}^{11}$ and $\bar{\tau}^{22}$ respectively. If one takes inner products of the vector $\bar{\tau}'$ given by Eq. (4.10) with $A_2 = 0$ and of the one given by Eq. (4.11) with $\beta_2 = 0$ to themselves separately, then comparing the results and since $\bar{N}^{11}\cdot\bar{N}^{11} = 1$, one obtains

$$\left.\begin{aligned} \alpha^2 &= \frac{E}{1-\nu^2}\frac{1}{c^2}A_1^2\left[\bar{\tau}^{11}\cdot\bar{\tau}^{11}\right]^{\frac{1}{2}} \\ \text{And} \qquad & \\ \beta^2 &= \frac{E}{1-\nu^2}\frac{1}{c^2}A_2^2\left[\bar{\tau}^{22}\cdot\bar{\tau}^{22}\right]^{\frac{1}{2}} \end{aligned}\right\} \qquad (4.12)$$

is obtained by a similar way.

It can be shown that by substituting the stress field τ_{ij}^{mn} given by Eqs. (3.35) into Eq. (4.6a)

$$\left.\begin{aligned} &\bar{\tau}^{11}\cdot\bar{\tau}^{12} = \bar{\tau}^{22}\cdot\bar{\tau}^{12} = 0 \\ \text{i.e.} \\ &\bar{N}^{11}\cdot\bar{N}^{12} = \bar{N}^{22}\cdot\bar{N}^{12} = 0\,. \end{aligned}\right\} \quad (4.13)$$

This means for the present problem, $\bar{N}^{12}$ is orthogonal to $\bar{N}^{11}$ and $\bar{N}^{22}$. Let

$$\bar{N}^{12}\cdot\bar{N}^{12} = \frac{\bar{\tau}^{12}\cdot\bar{\tau}^{12}}{(\bar{\tau}^{11}\cdot\bar{\tau}^{11})^{1/2}(\bar{\tau}^{22}\cdot\bar{\tau}^{22})^{1/2}} = \delta^2 \qquad (4.14a)$$

One also has

$$\left.\begin{aligned} &\bar{N}^{11}\cdot\bar{N}^{11} = 1, \qquad \bar{N}^{22}\cdot\bar{N}^{22} = 1 \\ \text{and} \\ &\bar{N}^{11}\cdot\bar{N}^{22} = \cos 2\theta\,. \end{aligned}\right\} \quad (4.14b)$$

The angle 2θ included by $\bar{N}^{11}$ and $\bar{N}^{22}$, or by their equivalences $\bar{\tau}^{11}$ and $\bar{\tau}^{22}$, can be computed by

$$\cos 2\theta = \left[\frac{\bar{\tau}^{11}\cdot\bar{\tau}^{22}}{(\bar{\tau}^{11}\cdot\bar{\tau}^{11})(\bar{\tau}^{22}\cdot\bar{\tau}^{22})}\right]^{1/2}. \qquad (4.15)$$

Referring to Eqs. (3.41) and Table X

$$\cos 2\theta = \left[\frac{M'}{LN}\right]^{1/2} = .7996, \text{ or } 2\theta = 36^\circ 54' \qquad (4.15a)$$

Let $\bar{i}$, $\bar{j}$ and $\bar{k}$ be three unit vectors along x, y, and z, the three Cartesian coordinate axes respectively as shown in Figure 9. Then one has

$$\left.\begin{aligned}\bar{N}^{11} &= \cos\theta\,\bar{i} + \sin\theta\,\bar{j}\\ \bar{N}^{22} &= \cos\theta\,\bar{i} - \sin\theta\,\bar{j}\\ \bar{N}^{12} &= \delta\,\bar{k}.\end{aligned}\right\} \qquad (4.16)$$

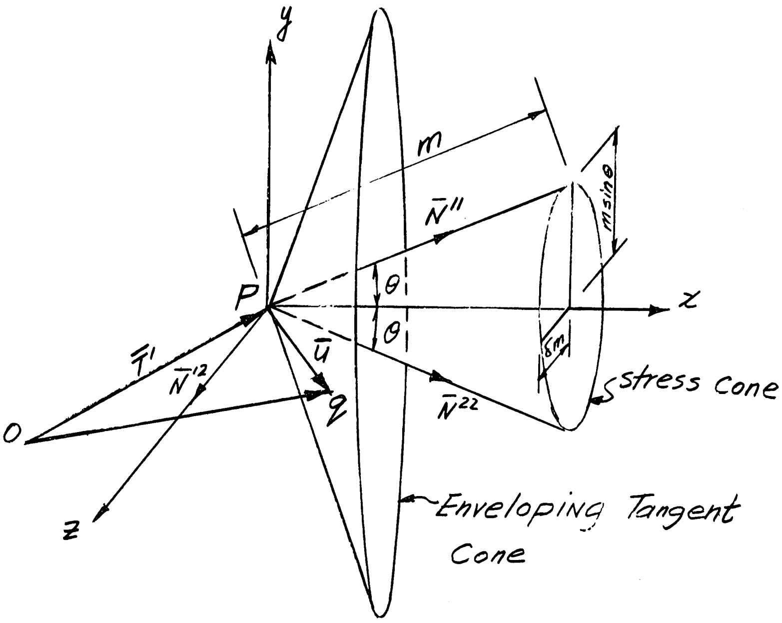

Figure 9. Stress and Enveloping Tangent Cones in Function Space

Hence, Eq. (4.11) can be converted to

$$\bar{\tau}' = \tau_x\,\bar{i} + \tau_y\,\bar{j} + \tau_z\,\bar{k}$$

where

$$\tau_x = (\alpha^2 + \beta^2)\cos\theta \qquad (4.16a)$$

$$\tau_y = (\alpha^2 - \beta^2)\sin\theta$$

and

$$\tau_z = 2\alpha\beta\delta.$$

The projection of the stress vectors $\tau_y \bar{j}$ and $\tau_z \bar{k}$ on the plane

$$\tau_x = m \cos\theta$$

where

$$m = \alpha^2 + \beta^2$$

can be found by eliminating α and β in τ_y and τ_z . The result is

$$\left(\frac{\tau_y}{m \sin\theta}\right)^2 + \left(\frac{\tau_z}{\delta m}\right)^2 = 1. \qquad (4.17)$$

This is an ellipse with axes

$$a_y = m \sin\theta, \qquad a_z = \delta m.$$

It is a circle if $\delta = \sin\theta$.

Let this cone be called stress cone.

A tangent plane $\bar{U}$ which is normal to $\bar{\tau}'$ is given by the equation

$$\bar{U} \cdot \bar{\tau}' = 0 \qquad (4.18)$$

The totality of all vectors $\bar{\tau}'$ as it has been shown, lie on a cone. Hence, the envelope of the tangent planes also forms a cone. The equations of this enveloping cone for which α and β are varying parameters, may be derived from

$$\frac{\partial}{\partial\alpha}(\bar{U}\cdot\bar{\tau}')=0 \quad \text{and} \quad \frac{\partial}{\partial\beta}(\bar{U}\cdot\bar{\tau}')=0$$

which, on account of Eq. (4.11) become

$$\left.\begin{aligned} &\alpha\bar{U}\cdot\bar{N}^{11}+\beta\bar{U}\cdot\bar{N}^{12}=0 \\ \text{and} \qquad & \\ &\alpha\bar{U}\cdot\bar{N}^{12}+\beta\bar{U}\cdot\bar{N}^{22}=0 \end{aligned}\right\} \qquad (4.19)$$

respectively. Eqs. (4.19) have non-trivial solutions if and only if

$$(\bar{U}\cdot\bar{N}^{11})(\bar{U}\cdot\bar{N}^{22})-(\bar{U}\cdot\bar{N}^{12})^2=0 \qquad (4.20)$$

which is the equation for the enveloping tangent cone as shown in Fig. 9. From Eqs. (4.19)

$$\frac{\beta}{\alpha}=-\frac{\bar{U}\cdot\bar{N}^{11}}{\bar{U}\cdot\bar{N}^{12}}=-\frac{\bar{U}\cdot\bar{N}^{12}}{\bar{U}\cdot\bar{N}^{22}}. \qquad (4.21)$$

Substituting it into (4.11)

$$\bar{\tau}'=\frac{1}{p}\left[(\bar{U}\cdot\bar{N}^{22})\bar{N}^{11}-2(\bar{U}\cdot\bar{N}^{12})\bar{N}^{12}+(\bar{U}\cdot\bar{N}^{11})\bar{N}^{22}\right] \qquad (4.22)$$

where p is a proportionality constant such that

$$\bar{U}\cdot\bar{N}^{22}=\alpha^2p,\quad \bar{U}\cdot\bar{N}^{12}=-\alpha\beta p \quad \text{and} \quad \bar{U}\cdot\bar{N}^{11}=\beta^2p \qquad (4.23)$$

Now in order to determine the coefficients α and β in a similar way as shown by Eq. (4.8), put

$$\overline{oq} = \overline{T}' + \overline{U} = \overline{\tau}'$$

By Eq. (4.11)

$$\overline{T}' + \overline{U} = \alpha^2 \overline{N}^{11} + 2\alpha\beta \overline{N}^{12} + \beta^2 \overline{N}^{22} \qquad (4.24)$$

where $\overline{T}'$ is known, given by Eqs. (3.3) and $\overline{U}$ is on the enveloping tangent cone. Taking inner products of the vector given by (4.24) with $\overline{N}^{11}$, $\overline{N}^{12}$ and $\overline{N}^{22}$ consecutively and considering the relations given in (4.13), (4.14b) and (4.23), one obtains respectively

$$\left.\begin{aligned} \alpha^2 + \beta^2 \cos 2\theta - p\beta^2 &= F_{11} \\ (2\delta^2 + p)\alpha\beta &= F_{12} \\ \alpha^2 \cos 2\theta + \beta^2 - p\alpha^2 &= F_{11} \end{aligned}\right\} \qquad (4.25)$$

where

$$F_{mn} = \overline{T}' \cdot \overline{N}^{mn} \qquad (4.25a)$$

are known quantities.

The three equations given by (4.25) serve to determine the three unknown parameters α, β and p. Solving for p from the second equation and substituting it into the other two equations, a pair of equations for α and β are obtained as

$$\left.\begin{aligned} &\alpha\left[\alpha^2+\beta^2(\cos 2\theta+2\delta^2)-F_{11}\right]-\beta F_{12}=0 \\ &\beta\left[\beta^2+\alpha^2(\cos 2\theta+2\delta^2)-F_{22}\right]-\alpha F_{12}=0 \end{aligned}\right\} \quad (4.26)$$

To show explicitly that this way of determining α and β is identical to Marguerre's method as used in Section 3, let the first two terms included in the potential energy V in Eq. (2.22) and defined by Eqs. (2.23) be converted into such form that

$$\begin{aligned} U_b + W_e &= \frac{h}{2}\int_R \{D[(1-\nu)W_{,ij}W_{,ij}+\nu W_{,ii}W_{,jj}] \\ &\quad +(\lambda t^{\circ}_{ij}+T'_{ij})W_{,i}W_{,j}\}dA - \frac{h}{2}\int_R T'_{ij}W_{,i}W_{,j}\,dA \qquad (4.27) \\ &= -\frac{h}{2}\int_R T'_{ij}W_{,i}W_{,j}\,dA. \end{aligned}$$

The vanishing of the first integral can be shown by multiplying the equilibrium equation (2.6) by W , by integrating it over the region, and by applying Green's Theorem and boundary conditions to the integration. It vanishes because the equilibrium equation has been satisfied by $T_{ij}\ (=\lambda t^{\circ}_{ij}+T'_{ij})$ and W as shown by Eqs. (3.4) and (3.5). It can be shown by a similar way as was done for Eqs. (2.24) and (2.25)

$$\frac{h}{2}\int_R T'_{ij}W_{,i}W_{,j}\,dA = h\int_R T'_{ij}\varepsilon'_{ij}\,dA = \bar{T}'\cdot\bar{\tau}'. \qquad (4.28)$$

The second equality in the above equation is based on (4.6a).

The membrane strain energy term defined in Eqs. (2.23) can also be expressed in the vectorial form as

$$U_m = \frac{h}{4}\int \tau'_{ij}\, W_{,i}\, W_{,j}\, dA = \frac{1}{2}\,\bar{\tau}'\cdot\bar{\tau}'. \tag{4.29}$$

Thus the potential energy obtained by combining Eqs. (4.28) and (4.29), written in present notation is

$$V = \frac{1}{2}(\bar{\tau}'\cdot\bar{\tau}') - (\bar{T}'\cdot\bar{\tau}'). \tag{4.30}$$

Using the $\bar{\tau}'$ given by Eqs. (4.11) and applying the relationships of inner products among the components (see Eqs. (4.13), (4.14a) and (4.14b)) and those between $\bar{T}'$ and $\bar{\tau}'$ (see Eqs. (4.25a)), to Eq. (4.30), one finds

$$V = \frac{1}{2}\left[\alpha^4 + 2\alpha^2\beta^2(2\delta^2 + \cos 2\theta) + \beta^4\right] - \alpha^2 F_{11} - 2\alpha\beta F_{12} - \beta^2 F_{22}. \tag{4.31}$$

Marguerre's method leads to

$$\left.\begin{aligned} \frac{\partial V}{\partial \alpha} = 0,\quad & \alpha\left[\alpha^2 + \beta^2(2\delta^2 + \cos 2\theta) - F_{11}\right] - \beta F_{12} = 0 \\ \frac{\partial V}{\partial \beta} = 0,\quad & \beta\left[\beta^2 + \alpha^2(2\delta^2 + \cos 2\theta) - F_{22}\right] - \alpha F_{12} = 0 \end{aligned}\right\} \tag{4.32}$$

This set of equations is identical to those given by Eqs. (4.26).

For the present problem it can be shown that

$$F_{12} = \bar{T}'\cdot\bar{N}^{12} = 0$$

Then the Eqs. (4.32) are reduced to be

$$\left.\begin{aligned} &\alpha\left[\alpha^2+\beta^2(2\delta^2+\cos 2\theta)-F_{11}\right]=0 \\ &\beta\left[\beta^2+\alpha^2(2\delta^2+\cos 2\theta)-F_{22}\right]=0 \end{aligned}\right\} \quad (4.32a)$$

These two equations differ from those given by Eqs. (3.44) only by a constant. As it has been discussed there, that due to the independence of α and β, there are three cases. Namely case (1) $\beta = 0$, case (2) $\alpha = 0$, and case (3) $\alpha \neq 0$, $\beta \neq 0$. Further, it also has been seen in Section 3, that the configuration in the last case is unstable. Hence hereafter the discussion will be confined to the first two cases only.

The fact that

$$\bar{T}'\cdot\bar{N}^{12} = \bar{N}^{11}\cdot\bar{N}^{12} = \bar{N}^{22}\cdot\bar{N}^{12} = 0$$

as mentioned before implies that the origin is on a "plane" normal to $\bar{N}^{12}$ and containing $\bar{T}'$, $\bar{N}^{11}$ and $\bar{N}^{22}$ as shown in Fig. 10. Consequently, the solution of case (1) with $\beta = 0$ is easily to be figured out from Figure 10, simply by setting

$$\overline{oq_1} = \bar{T}'\cdot\bar{N}^{11} \quad (4.33)$$

It can also be seen from Eqs. (4.32a) that for $\beta = 0$ but $\alpha \neq 0$

Hence

$$\left.\begin{aligned} \alpha^2 &= F_{11} = \bar{T}' \cdot \bar{N}^{11} \\ \alpha^2 &= \overline{oq}_1 . \end{aligned}\right\} \quad (4.34)$$

Let V_1 be denoted as the potential energy when $\beta = 0$ then applying the result of (4.34) to Eq. (4.31) with $\beta = 0$

$$V_1 = -\frac{1}{2}(F_{11})^2 = -\frac{1}{2}|\overline{oq}_1|^2 \qquad (4.35)$$

Knowing from Eq. (3.45) $\left(\frac{A_1}{h}\right)^2$ and hence α^2 or $\overline{oq}_1$ is linearly proportional to λ'. So as λ' increases, instead of moving the cones, one may consider the origin O being moved. When $|\overline{oq}_1| > 0$ if $\lambda' > 1.19$ as seen from Table **XI**. Hence the origin starts at $\lambda' = 1.19$.

By the same procedures one will find the solution for case (2) with $\alpha = 0$, as

$$\left.\begin{aligned} |\overline{oq}_2| &= \bar{T}' \cdot \bar{N}^{22} = F_{22} = \beta^2 \\ V_2 &= -\frac{1}{2}(F_{22})^2 = -\frac{1}{2}|\overline{oq}_2|^2 \\ \text{and} \quad & \\ |\overline{oq}_2| &> 0 \quad \text{if} \quad \lambda' > 2.98 \end{aligned}\right\} \quad (4.36)$$

As the origin moves up to and on the central line of the cones, then, (this time $\overline{oq}_1$ and $\overline{oq}_2$ are on opposite sides of the central line)

$$|\overline{oq}_1| = |\overline{oq}_2|$$

Hence

$$V_1 = V_2$$

It occurs when $\lambda' = 11.42$ as known from Eq. (3.55a). There is another value given there that when $\lambda' = 2.17$, V_1 is also equal to V_2. However, this point is in a region where $\overline{oq}_2$ is in a negative direction as shown in Fig. 10. Hence β is imaginary. This agrees with what was concluded there. With these particular values of λ' in mind and using a certain scale, one is able to draw a straight line as the path of the moving origin as shown in Fig. 10.

Let the variation of the stress reactor $\bar{\tau}'$ (see Eq. (4.11)) be

$$\Delta\bar{\tau}' = (\Delta\alpha)^2\bar{N}^{11} + 2(\Delta\alpha)(\Delta\beta)\bar{N}^{12} + (\Delta\beta)^2\bar{N}^{22} \qquad (4.37)$$

and that of the potential energy V (see Eq. (4.30)) be

$$\Delta V = V(\bar{T}', \bar{\tau}' + \Delta\bar{\tau}') - V(\bar{T}', \bar{\tau}').$$

Then one gets

$$\Delta V = (\bar{\tau}'\cdot\Delta\bar{\tau}') - (\bar{T}'\cdot\Delta\bar{\tau}') + \tfrac{1}{2}(\Delta\bar{\tau}'\cdot\Delta\bar{\tau}') \qquad (4.38)$$

If only the terms with the second order variation of the amplitude coefficients α and β are maintained, using the $\bar{\tau}'$ and $\Delta\bar{\tau}'$ given by Eqs. (4.11) and (4.37) respectively and also the relationships given by Eqs. (4.13), (4.14a), (4.14b) and (4.25a), one assumes

$$\Delta V = (\alpha^2+\beta^2\cos 2\theta - F_{11})(\Delta\alpha)^2 + 2(2\alpha\beta\delta^2 - F_{12})(\Delta\alpha)(\Delta\beta)$$
$$+(\alpha^2\cos 2\theta+\beta^2-F_{22})(\Delta\beta)^2 \tag{4.39}$$

Let

$$\left.\begin{aligned} V_{\alpha\alpha} &= \alpha^2+\beta^2\cos 2\theta - F_{11} \\ V_{\beta\beta} &= \alpha^2\cos 2\theta+\beta^2-F_{22} \\ V_{\alpha\beta} &= 2\alpha\beta\delta^2 - F_{12} \end{aligned}\right\} \tag{4.40}$$

be the coefficient of $(\Delta\alpha)^2$, $(\Delta\beta)^2$ and $2(\Delta\alpha)(\Delta\beta)$ respectively, then the sufficient conditions for ΔV to be positive is

$$\left.\begin{aligned} & V_{\alpha\alpha} > 0 \quad \text{and} \quad V_{\alpha\beta}^2 - V_{\alpha\alpha}V_{\beta\beta} < 0 \\ \text{or} \quad & \\ & V_{\beta\beta} > 0 \quad \text{and} \quad V_{\alpha\beta}^2 - V_{\alpha\alpha}V_{\beta\beta} < 0 \end{aligned}\right\} \tag{4.41}$$

These are the criteria that shall be used to test the stability of the configurations concerned. Note that these are different from those given by Eqs. (3.48) which expressed in present notation, derived from Eq. (4.31) are

$$\left.\begin{aligned} \frac{\partial^2 V}{\partial\alpha^2} &= 3\alpha^2+\beta^2(2\delta^2+\cos 2\theta)-F_{11} \\ \frac{\partial^2 V}{\partial\beta^2} &= 3\beta^2+\alpha^2(2\delta^2+\cos 2\theta)-F_{22} \end{aligned}\right\} \tag{4.42}$$

$$\frac{\partial^2 V}{\partial\alpha\,\partial\beta} = 2\alpha\beta\,(2\delta^2 + \cos 2\theta) - F_{12} \tag{4.42}$$

It can readily be shown that the stability criterion according to Eq. (4.41) is more severe than the one based on Eqs. (4.42); in other words, stability according to the former implies also stability according to the latter, although the inverse is not true. Moreover, if Eq. (4.41) is used as a stability criterion, it is easy to demonstrate that an absolute minimum of potential energy is attained, that is, that any other configuration, not necessarily in the neighborhood of the one being considered, exhibits a higher level of potential energy even if terms of order higher than the second are included. The term "absolute stability" is therefore applied to Eq. (4.41), although the latter represents only sufficiency, but not necessity conditions.

Now for the present problem, $F_{12} = 0$. Case (3) is unstable according to the criterion based on (4.42) or its equivalent (3.48). Thus it will be unstable by (4.41). However the absolute stability of cases (1) and (2) needs to be examined.

For case (1) with $\beta = 0$ and $\alpha^2 = F_{11}$, seen from Eq. (4.34), one has

$$\left.\begin{aligned} V_{\alpha\alpha} &= 0 \\ V_{\beta\beta} &= F_{11}\cos 2\theta - F_{22} \\ V_{\alpha\beta} &= 0 \end{aligned}\right\} \tag{4.43}$$

A comparison of the potential energy given by Eq. (3.42) with that given by Eq. (4.31), using the relationships between α^2 and A_1^2 , β^2 and A_2^2 given by Eqs. (4.12) and regardless of the common constant, one finds the following relationships among the constants

$$\left.\begin{aligned} F_{11} &= \frac{1}{\sqrt{L}}(K_1\lambda' - J_1) \\ F_{22} &= \frac{1}{\sqrt{N}}(K_2\lambda' - J_2) \\ \cos 2\theta &= \frac{M'}{\sqrt{L}\sqrt{N}}. \end{aligned}\right\} \qquad (4.44)$$

Thus

$$V_{\beta\beta} = \frac{1}{\sqrt{N}}\left[\left(J_2 - \frac{M'}{L}J_1\right) - \left(K_2 - \frac{M'}{L}K_1\right)\lambda'\right].$$

Taking the numerical values of the constants from Table X, the result is

$$V_{\beta\beta} = \frac{1}{\sqrt{N}}\left[2.253 - .3494\lambda'\right] > 0 \quad \text{if } \lambda' < 6.45. \qquad (4.45)$$

This means the configuration of case (1) with $\beta = 0$, $\alpha^2 = |\overline{og_1}|$ is stable only for $\lambda' < 6.45$. Since $\overline{og_1}$ starts at $\lambda' = 1.19$, one can conclude that this configuration exists and is absolutely stable only in the region of $1.19 < \lambda < 6.45$.

As for case (2) with $\alpha' = 0$ and $\beta^2 = |\overline{og_2}|$ (see Eqs. (4.36)), the coefficients defined by Eqs. (4.40), using the numerical values given by Eqs. (4.44) and Table X, assumes

$$
\begin{aligned}
V_{\alpha\alpha} &= F_{22}\cos 2\theta - F_{11} \\
&= \frac{1}{IL}\left[\left(J_1 - \frac{M'}{N}J_2\right) - \left(K_1 - \frac{M'}{N}K_2\right)\lambda'\right] \\
&= \frac{1}{IL}\left[-1.504 - .0200\lambda'\right] \quad < 0
\end{aligned}
\tag{4.46}
$$

$$V_{\beta\beta} = 0$$

$$V_{\alpha\beta} = 0$$

Thus the antisymmetrical mode given by case (2) is unstable in the whole postbuckling domain according to the present criterion of absolute stability.

To interpret the foregoing results geometrically, it would be convenient to give a name for every point as shown in Fig. 10 on the path line. The line starts from point a, where λ' = 1.19. When it is between a and b, $\overline{oq_1}$ is positive; while $\overline{oq_2}$ measured in the opposite direction, is negative. Since $\beta^2 = |\overline{oq_2}|$, the negative $\overline{oq_2}$ means β is imaginary. Hence there is no possibility of having the antisymmetrical mode of buckling exist in this region. After it passes the tangent surface or the tangent line in Fig. 10, marked by $\alpha = 0$, $\overline{oq_1}$ and $\overline{oq_2}$ both are positive, that is both α and β exist. However, in the region between b and d, $\overline{oq_1}$ is always greater than $\overline{oq_2}$. In view of Eqs. (4.35) and (4.36), this fact means in this region, $V_1 < V_2$. From the point of view that a configuration with lower potential energy is more stable, hence the symmetrical mode of buckling is maintained up to the point d. But this mode itself is absolutely stable only up to the point c. Right on this point, all the secondary variations given by (4.43) vanish. Then this is a neutral state. From this diagram when λ' reaches 6.45, the point q_1 coincides with P and which may be considered as a singularity. Hence before the moving origin hits the extension line of $\bar{N}^{11}$, the configuration with $\beta = 0$ is stable. This interpretation may also be applied to the case (2) with $\alpha = 0$. If the λ' - path line had gone through the extended stress cone, then beyond the extension line of $\bar{N}^{22}$, in that region $|o\bar{q}_1| < |o\bar{q}_2|$ hence $V_2 < V_1$, the antisymmetrical mode would be stable if it stayed out of the cone. However, for the present case, the λ'-path will never pass through the other side of the cone. Therefore it is unstable as it is indicated by Eqs. (4.46).

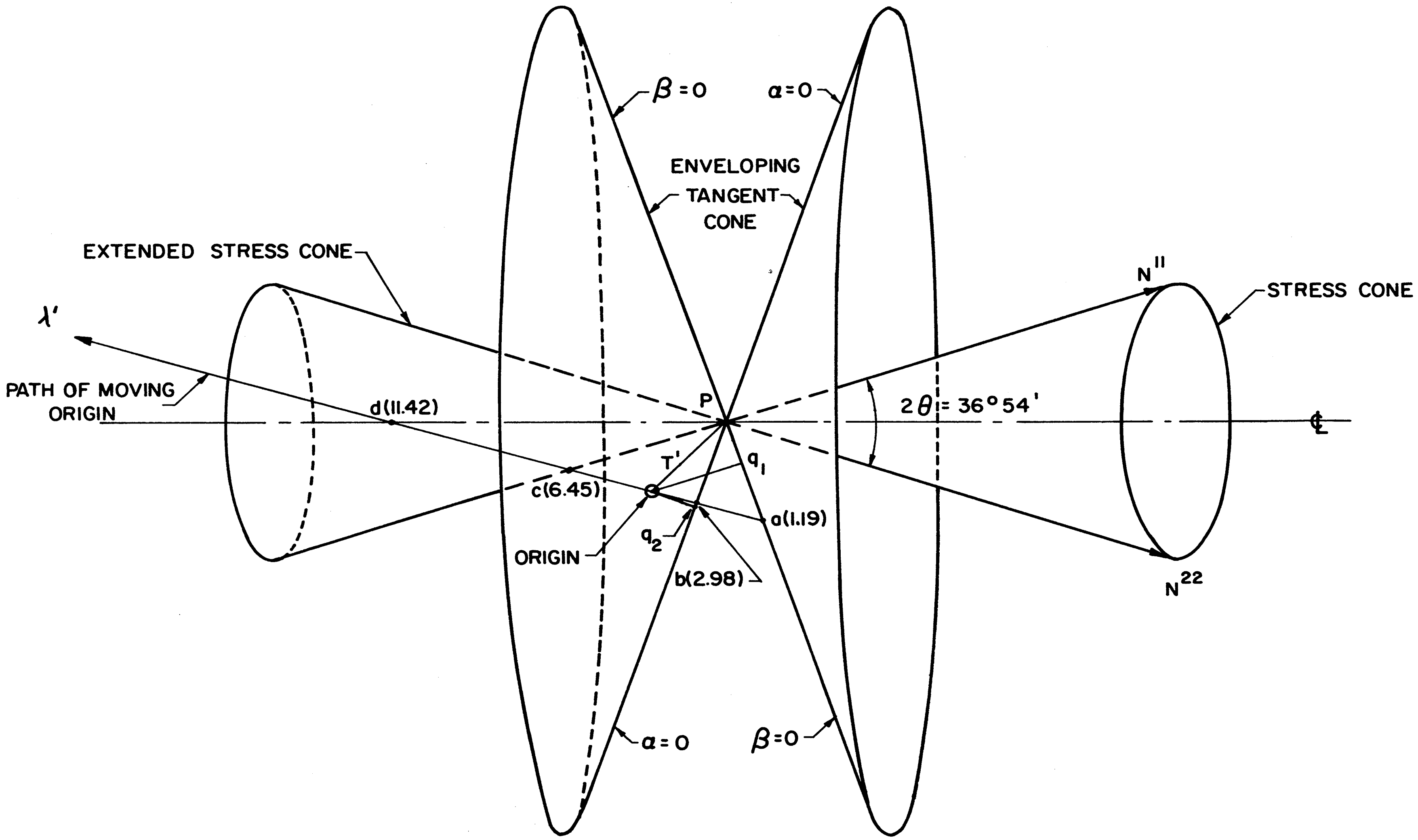

Figure 10. Result of the Buckling Problem of the Square Plate on a Function Space Diagram

5. DISCUSSION AND CONCLUSION

A difficulty arises when one intends to make a comparison of the present work with others. In most of the other works, the edges, paralleled to the thrust were allowed to move in their normal membrane direction. However, in this study the edges are immovable in this direction. Only the case with zero displacement along these edges in that direction was given by Marguerre.[2] But his solution for this case included the symmetric primary buckling mode only. No such case was investigated in his "refined solution". However, since the results are given in dimensionless quantities, it is still possible to make a rough comparison. For instance, the bifurcation point of two modes on load $(\frac{P}{P_0})$ versus shortening (λ') curves according to Stein's work occurs at $\frac{P}{P_0}$ about 2, λ' about 3. But Alexeev claimed for same type of plate, it would occur for $\lambda' > 10$ ($K = 40$ in his notation. No $\frac{P}{P_0}$ value was given). The present result seen from Fig. 7 it is at $\frac{P}{P_0} = 3.97$, $\lambda' = 6.22$. The experiments done by Yamaki[16] with four different boundary conditions in the normal to the plane of the plate direction show that up to $\frac{P}{P_c} = 3$ no change of buckling patterns were observed. This, of course, only tells that the present result is reasonable and the experimental results favor it.

The change of buckling patterns, in view of the present result would occur when $V_1 = V_2$ at $\lambda' = 11.42$. This value is much greater than that at the bifurcation point. Hence, if there is a change of buckling patterns, the external thrust will have a drop (shown in Fig. 7). This phenomenon has been observed in experimental works.(12)

It has been shown in the last section that the two methods used in solving the buckling problem of the square plate lead to the same results in the determination of the amplitude coefficients.

The conclusions about the stability of the two modes, based on different criteria agree with each other only for $\lambda' < 6.00$. From there on different conclusions are reached.

From the point of view of the absolute stability, the symmetrical mode is stable only for $\lambda' < 6.45$. Because of the singularity of the point P and the instability of the two modes for $\lambda' \geqslant 6.45$, no behavior of the buckling of the plate can be told in this region. A further study along this line of approach is desirable.

From the point of view of the relative stability, a transition from the symmetric mode to the unsymmetric one may occur at a relatively large value of λ' (= 11.42). This value is far from that an usual experiment can reach. To corroborate the analytic prediction, a special (say, for a plate of very thin and of high strength material) test program is highly desirable.

The solution obtained in this study is an approximate one. However, it shows some behavior of the buckled plate that has not been shown by other analytical works. Moreover, while quantitative corroboration is still lacking, the results show adequate qualitative agreement with available experimental data.

PART II. PLATES WITH TWO FREE EDGES

I. INTRODUCTION

It has been seen in the last part that there is a considerable strength in the postbuckling domain of the plate with all edges simply supported, (see Fig. 7), and the intensity of the normal stress in the loading direction has the tendency of shifting toward the two edges parallel to the thrust (see Fig. 6). In the limit almost all the buckling strength is developed along those edges. There is even tension produced in the interior portion. On the contrary, for plates with two free edges which are parallel to the thrust, the buckling strength is limited to the critical buckling strength--Euler load. However, if the shortening, instead of force, is prescribed, the deflection surface is primarily a cylindrical type with a generator perpendicular to the free edges. But it will be seen later that the boundary conditions along the free edges can not be fully satisfied by the cylindrical deflection surface. A "boundary layer" along each of these edges is developed so that the remaining boundary condition is satisfied. Consequently a membrane stress field will be developed and a tensile normal stress may be produced along the trims of the free edges. Nevertheless, the total buckling strength of the plate remains bounded.

In this part, a general discussion on the post-buckling behavior of plates with two free edges and two loaded edges arbitrarily supported is given. An example of the two loaded edges simply supported follows.

2. FORMULATION AND GENERAL SOLUTIONS OF BOUNDARY LAYER EQUATIONS

Consider an isotropic and elastic plate of thickness h with a set of reference axes x_1 and x_2 as shown in Fig. 11. It has two free edges at $x_2 = 0$ and $2b$. It is arbitrarily supported at $x_1 = 0$ and a. However the supports are statically stable. Each of the supported edges is attached to a rigid bar. But there is no shearing stress produced between the plate and the rigid bar. A relative shortening of the bars is prescribed. Before buckling, the middle plane of the plate coincides with x_1x_2-plane.

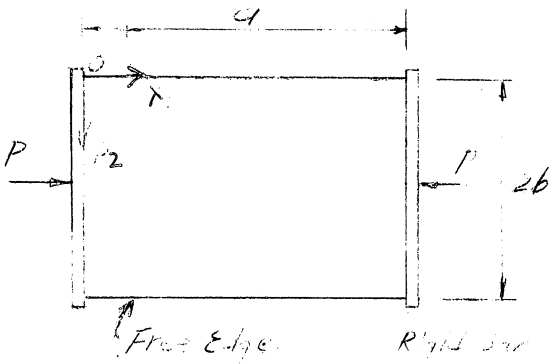

Figure 11. A Plate with Two Free Edges

The pair of basic equations governing the behavior of large deflection of thin plates given by Karman[1], are equations of equilibrium in the lateral direction of the plate

$$\frac{D}{h}\nabla^4 w - F_{,22} w_{,11} - F_{,11} w_{,22} + 2F_{,12} w_{,12} = 0 \quad \text{in } R \qquad (2.1)$$

and equation of compatibility

$$\nabla^4 F = -E\left(w_{,11} w_{,22} - w_{,12}^2\right) \qquad (2.2)$$

in which $w(x_1, x_2)$, D and E are defined in Part I. $F(x_1, x_2)$ is the Airy's stress function such that

$$F_{,22} = \tau_{11}, \quad F_{,11} = \tau_{22} \quad \text{and} \quad F_{,12} = -\tau_{12}. \qquad (2.3)$$

Along the free edges, both moment and shearing force in the lateral direction of the plate vanish. Expressing them in terms of w , one has

$$\left.\begin{aligned} & w_{,22} + \nu w_{,11} = 0 \\ & \left[w_{,22} + (2-\nu) w_{,11}\right]_{,2} = 0 \end{aligned}\right\} \text{on } x_2 = 0, 2b \qquad (2.4)$$

Also, there is neither normal nor shearing membrane stress. Hence,

$$\left.\begin{aligned} & F_{,11} = 0 \\ & F_{,12} = 0 \end{aligned}\right\} \text{on } x_2 = 0, 2b \qquad (2.5)$$

On the other two edges, the boundary conditions on w depend upon the conditions of supports. However each end supplies two. Totally there are four boundary conditions on w available. From statics one requires

$$h\int_0^{2b} F_{,22}\, dx_2 = P \qquad \text{on } x_1 = 0, a \qquad (2.6)$$

and with no shearing stress

$$F_{,12} = 0 \qquad \text{on } x_1 = 0,\ a \qquad (2.6a)$$

First, one may assume that the deflection surface is a cylindric type of function of x_1 . If we let

$$w = k\,W(x_1) \qquad (2.7)$$

$$F = -\frac{1}{2}\frac{N}{h}x_2^2 \qquad (2.8)$$

where k is the amplitude coefficient, then Eq. (2.1) becomes

$$W_{,1111} + \frac{N}{D}W_{,11} = 0 \qquad (2.9)$$

and its solution is

$$W(x_1) = \sin\sqrt{\frac{N}{D}}\,x_1 + B\cos\sqrt{\frac{N}{D}}\,x_1 + Cx_1 + E, \quad 0 \le x_1 \le a \qquad (2.10)$$

The three arbitrary constants B, C, E, and the eigenvalue N may be determined by the four conditions at x_1 = 0 and a. The Eq. (2.2) is satisfied identically. However, the reduced boundary condition from conditions (7.4) which is

$$W_{,11} = 0 \qquad \text{on } x_2 = 0,\ 2b \qquad (2.11)$$

cannot be satisfied by functions (2.10). Applying a technique similar to that used in connection with boundary layer phenomena in fluid mechanics[23] to the present problem let

$$
\begin{aligned}
w &= kW(x_1) + u(x_1, x_2) \\
F &= -\frac{1}{2}\frac{N}{h}x_2^2 + \frac{1}{h}\phi(x_1, x_2)
\end{aligned}
\qquad (2.12)
$$

Substituting these two equations into Eqs. (2.1) and (2.2) and on account of Eq. (2.9) they result in

$$
\begin{aligned}
&D\nabla^4 u + Nu_{,11} - \phi_{,22}(kW_{,11} + u_{,11}) - \phi_{,22}u_{,22} + \phi_{,12}u_{,12} = 0 \\
&\frac{1}{h}\nabla^4\phi = -E\left[(kW_{,11} + u_{,11})u_{,22} - u_{,12}^2\right]
\end{aligned}
\qquad (2.13)
$$

The boundary conditions on u and ϕ are

$$
\left.
\begin{aligned}
&u_{,22} + \nu u_{,11} = -\nu kW_{,11} \\
&\left[u_{,22} + (2-\nu)u_{,11}\right]_{,2} = 0
\end{aligned}
\right\} \text{ on } x_2 = 0,\ 2b \qquad (2.14)
$$

from Eq. (2.5), and from conditions (2.6)

$$
\phi_{,11} = \phi_{,12} = 0 \qquad \text{on } x_2 = 0,\ 2b \qquad (2.15)
$$

The system of equations defined by Eqs. (2.13) to (2.15) is no better than the original set if one intends to find an exact solution of this system. However, a set of approximate equations--so-called boundary layer equations--can be derived by considering the additional functions u and ϕ to be confined in a boundary layer along the free edges. This is accomplished by introducing a new variable

$$\eta = k^{\frac{1}{2}} x_2$$

so that x_2 -coordinate has been "blown up" by the parametric variable k. The set of Eqs. (2.13) now become

$$\left.\begin{aligned}
&D[u_{,1111} + k\, u_{,11\eta\eta} + k^2 u_{,\eta\eta\eta\eta}] + N u_{,11} \\
&-k\phi_{,\eta\eta}(k W_{,11} + u_{,11}) - k\phi_{,11} u_{,\eta\eta} + 2k\phi_{,1\eta} u_{,1\eta} = 0 \\
&[\phi_{,1111} + k\phi_{,11\eta\eta} + k^2 \phi_{,\eta\eta\eta\eta}] = -Eh[(kW_{,11} + u_{,11}) k u_{,\eta\eta} - k u_{,1\eta}^2],
\end{aligned}\right\} \quad (2.16)$$

The equations are divided by k^2 and k is permitted to approach to infinity, a set of boundary layer equations for u and ϕ are obtained. They are

$$\left.\begin{aligned}
&D u_{,\eta\eta\eta\eta} - \phi_{,\eta\eta} W_{,11} = 0 \\
&\phi_{,\eta\eta\eta\eta} + Eh\, u_{,\eta\eta} W_{,11} = 0
\end{aligned}\right\} \quad (2.17)$$

in which $W(x_1, x_2)$ is known from Eq. (2.10). By a similar procedure, the boundary conditions (2.14) and (2.15) are reduced to the following forms:

$$\left.\begin{aligned}
&u_{,\eta\eta} = -\nu W_{,11} \\
&u_{,\eta\eta\eta} = 0
\end{aligned}\right\} \text{ on } \eta = 0 \qquad (2.18)$$

and

$$\phi_{,11} = \phi_{,1\eta} = 0 \qquad \text{on } \eta = 0 \qquad (2.19)$$

respectively. The functions u, ϕ and their derivatives shall vanish as $\eta \to \infty$

Now let

$$u_{,\eta\eta} = Ae^{\eta f}$$
$$\phi_{,\eta\eta} = Be^{\eta f} \tag{2.20}$$

where A, B and F are arbitrary functions of x_1 .
Put them into Eqs. (2.17), which become

$$ADf^2 - BW_{,11} = 0$$

$$EhAW_{,11} + Bf^2 = 0$$

The necessary condition for A and B to be nontrivial is that the determinant formed by the coefficients of the above simultaneous equations must vanish. This results in

$$f^4 + 4\alpha^4 = 0$$

where

$$\alpha^4 = \frac{hE}{4D} W_{,11}^2$$

Hence

$$f = \pm\alpha(1 \pm i)$$

in which

$$\alpha \equiv \left| \frac{hE}{4D} W_{,11}^2 \right|^{1/4} \tag{2.21}$$

Thus

$$u_{,\eta\eta} = e^{-\alpha\eta}(A_1 \cos\alpha\eta + B_1 \sin\alpha\eta) + e^{\alpha\eta}(C_1 \cos\alpha\eta + D_1 \sin\alpha\eta) \tag{2.22}$$

$$\phi_{,\eta\eta} = e^{-\alpha\eta}(A_2\cos\alpha\eta + B_2\sin\alpha\eta) + e^{\alpha\eta}(C_2\cos\alpha\eta + D_2\sin\alpha\eta) \quad (2.22)$$

Since both the functions shall vanish at $\eta = \infty$

$$C_1 = D_1 = C_2 = D_2 = 0 \quad (2.23)$$

Through Eqs. (2.17), one will find

$$\left.\begin{aligned} B_2 &= \frac{1}{W_{,11}} 2\alpha^2 D A_1 \\ A_2 &= -\frac{1}{W_{,11}} 2\alpha^2 D B_1 \end{aligned}\right\} \quad (2.24)$$

The first of boundary conditions (2.18) gives

$$A_1 = -\nu W_{,11} \quad (2.25a)$$

and the second of the conditions yields

$$A_1 = B_1 \quad (2.25b)$$

With the coefficients given by the foregoing three sets of equations, the solutions (2.22) assumes the form:

$$\left.\begin{aligned} u_{,\eta\eta} &= -\nu W_{,11} e^{-\alpha\eta}[\cos\alpha\eta + \sin\alpha\eta] \\ \phi_{,\eta\eta} &= 2\alpha^2 D\nu e^{-\alpha\eta}[\cos\alpha\eta - \sin\alpha\eta] \end{aligned}\right\} \; 0 \le \eta < \infty \quad (2.26)$$

Integrating of these equations twice with respect to η results in

$$\left.\begin{aligned} u &= -\nu W_{,11}\frac{e^{-\alpha\eta}}{2\alpha^2}\left[\cos\alpha\eta - \sin\alpha\eta\right] + \eta f_1(x_1) + f_2(x_1) \\ \phi &= -\nu D e^{-\alpha\eta}\left[\cos\alpha\eta + \sin\alpha\eta\right] + \eta F_1(x_1) + F_2(x_1) \end{aligned}\right\} \quad (2.27)$$

The integration functions f_1 and f_2 must vanish because the function u will vanish when $\eta = \infty$. The satisfaction of boundary conditions (2.19) yields

$$\begin{aligned} \frac{dF_1}{dx_1} &= 0 \qquad \therefore F_1 = a_1 \\ \frac{d^2F_2}{dx_1^2} &= 0 \qquad \therefore F_2 = a_2 x_1 + a_3 \end{aligned} \quad (2.28)$$

These linear functions, however contribute nothing to stresses; both of them may be neglected. The final forms of the solutions (2.27) are

$$\left.\begin{aligned} u &= -\nu \frac{W_{,11}}{2\alpha^2} e^{-\alpha\eta}\left[\cos\alpha\eta - \sin\alpha\eta\right] \\ \phi &= -\nu D e^{-\alpha\eta}\left[\cos\alpha\eta + \sin\alpha\eta\right] \end{aligned}\right\} 0 \le \eta < \infty \quad (2.29)$$

Let τ_{ij}' be the additional stress field due to the membrane effect. Then

$$\tau_{11} = -\frac{N}{h} + \tau_{11}' \quad (2.30)$$

and from Eqs. (2.3), (2.12) and (2.30), one has

$$\tau_{11}' = \frac{k}{h}\phi_{\eta\eta} = \frac{k}{h}\nu(2\alpha^2)D e^{-\alpha\eta}\left[\cos\alpha\eta - \sin\alpha\eta\right] \quad (2.31)$$

3. AN EXAMPLE

Take a plate which is simply supported along the two loaded edges, as an example. The boundary conditions along these edges are that the lateral deflection and moment vanish. They are satisfied by setting the constants B, C and E in Eq. (2.10) equal to zero and $N = \frac{\pi^2 D}{a^2}$. Hence

$$W(x_1) = \sin \frac{\pi x_1}{a} \tag{3.1}$$

Substitution of the last equation into Eq. (2.21), gives

$$\alpha = \left| \frac{hE}{4D} \left(\frac{\pi}{a}\right)^2 \sin \frac{\pi x_1}{a} \right|^{\frac{1}{4}} \tag{3.2}$$

Then one has the solutions for the additional functions u and ϕ immediately by putting expressions (3.1) and (3.2) into Eqs. (2.29). They are

$$U = \nu \sqrt{\frac{D}{hE}}\, e^{-\alpha\eta} [\cos\alpha\eta - \sin\alpha\eta] \tag{3.3}$$

$$\phi = -\nu D e^{-\alpha\eta} [\cos\alpha\eta + \sin\alpha\eta] \tag{3.4}$$

These solutions are similar to what have been given by Ashwell(24), Fung and Wittrick(25) in the study of large deflection of the same plate by lateral bending.

The solution given by Eq. (3.3) shows the additional deflection due to the anticlastic curvature in the classic theory which now is confined along the two free edges with a maximum at $\eta = 0$. Since $D = \frac{Eh^3}{12(1-\nu^2)}$ and considering $\nu = 0.3$, one has the maximum

$$U(0, x_1) = \nu\sqrt{\frac{D}{hE}} \approx \frac{1}{10} h \qquad (3.5)$$

The additional normal stress by Eq. (2.31) and after simplification, is

$$\tau_{11}' = \frac{\nu k h E}{\sqrt{12(1-\nu^2)}} \left(\frac{\pi}{a}\right)^2 \sin\frac{\pi x_1}{a} e^{-\alpha\eta} \left[\cos\alpha\eta - \sin\alpha\eta\right] \qquad (3.6)$$

which means that a thin boundary layer of indefinitely increasing tensile stress is formed along the free edges. Although this result can be visualized with some effort, it is nevertheless considered startling in view of the overall compressive stress in the plate.

APPENDIX

SOLUTIONS OF THE DIFFERENTIAL EQUATIONS FOR THE MEMBRANE DISPLACEMENTS

The pair of differential equations (3.27) governing the displacement components u_i^{mn} are expanded as

for $i = 1$,

$$\left.\begin{aligned} &2u_{1,11}^{mn} + (1-\nu)u_{1,22}^{mn} + (1+\nu)u_{2,21}^{mn} = -\Phi_1^{mn} \\ &\text{for } i = 2, \\ &2u_{2,22}^{mn} + (1-\nu)u_{2,11}^{mn} + (1+\nu)u_{1,12}^{mn} = -\Phi_2^{mn} \end{aligned}\right\} \tag{A.1}$$

where

$$\Phi_i^{mn} = \frac{1}{2}\left[2W_{,i}^m W_{,11}^n + (1-\nu)W_{,i}^m W_{,22}^n + (1+\nu)W_{,2}^m W_{,2i}^n\right] \tag{A.2}$$
$$+\frac{1}{2}\left[2W_{,i}^n W_{,11}^m + (1-\nu)W_{,i}^n W_{,22}^m + (1+\nu)W_{,2}^n W_{,2i}^m\right]$$

On account of the functions W^n given by Eqs. (3.22)

$$\Phi_1^{mn} = \frac{1}{2}\left[M_{mn}(\eta) + M_{nm}(\eta)\right]\sin\phi_{mn}\xi$$
$$+\frac{1}{2}\left[N_{mn}(\eta) + N_{nm}(\eta)\right]\sin\psi_{mn}\xi \tag{A.3}$$

in which

$$M_{mn}(\eta) = -\gamma_m\gamma_n^2 f_m f_n + \frac{1-\nu}{2}\gamma_m f_m f_n'' + \frac{1+\nu}{2}\gamma_n f_m' f_n' \tag{A.3a}$$

and

$$N_{mn}(\eta) = \gamma_m\gamma_n^2 f_m f_n - \frac{1-\nu}{2}\gamma_m f_m f_n'' + \frac{1+\nu}{2}\gamma_n f_m' f_n' \tag{A.3b}$$

However, M_{nm} and N_{nm} are of the same forms as that of M_{mn} and N_{mn} respectively by changing the subscripts m and n accordingly. This kind of notation will be applied to similar occasions without further remarks.

And

$$\Phi_2^{mn} = \frac{1}{2}\left[K_{mn}(\eta) + K_{nm}(\eta)\right]\cos\phi_{mn}\xi \qquad (A.4)$$
$$+\frac{1}{2}\left[L_{mn}(\eta) + L_{nm}(\eta)\right]\cos\psi_{mn}\xi$$

in which

$$K_{mn}(\eta) = -f_m' f_n'' + \frac{1-\nu}{2}\gamma_n^2 f_m' f_n + \frac{1+\nu}{2}\gamma_m\gamma_n f_m f_n' \qquad (A.4a)$$

$$L_{mn}(\eta) = f_m' f_n'' - \frac{1-\nu}{2}\gamma_n^2 f_m' f_n + \frac{1+\nu}{2}\gamma_m\gamma_n f_m f_n' \qquad (A.4b)$$

Notice that when $m = n$, $\psi_{mn} = 0$, the second functional term in Φ_1^{mn} vanishes and that in Φ_2^{mn} is the function of η only.

It might be noteworthy that the components of the membrane displacement field, introduced by Eqs. (3.26) and (3.29), characterized by Eqs. (3.27) and (3.28) and boundary conditions (3.30) and (3.31) are symmetrical about m and n. Now for simplicity, but without loss of generality, let

$$U_1^{mn} = \frac{1}{c}\left[G_{mn}(\eta)\sin\phi_{mn}\xi + I_{mn}(\eta)\sin\psi_{mn}\xi\right]$$
$$U_2^{mn} = \frac{1}{c}\left[H_{mn}(\eta)\cos\phi_{mn}\xi + J_{mn}(\eta)\cos\psi_{mn}\xi\right] \qquad (A.5)$$

in which G_{mn}, H_{mn}, I_{mn} and J_{mn} are yet arbitrary functions of η. Due to the nonsymmetry of $\sin\psi_{mn}\xi$ one shall have

$$I_{mn}(\eta) = -I_{nm}(\eta)$$

However $u_i^{mn} = u_i^{nm}$ still hold and hence the computations will be carried out for u_i^{11}, u_i^{22} and u_i^{21} only. So for $\bar{u}_i^{mn}$

Substituting functions (A.2) and (A.5) into the first of Eqs. (A.1) and equating the functional coefficients of $\sin\phi_{mn}\xi$ and $\sin\psi_{mn}\xi$ separately, one gets two independent equations. Repeating this process with the second of Eqs. (A.1), one obtains two more equations. After rearrangement, they are

$$\left.\begin{aligned} -2\phi_{mn}^2 G_{mn} + (1-\nu)G_{mn}'' - (1+\nu)\phi_{mn}H_{mn}' &= -\frac{1}{2}[M_{mn}+M_{nm}] \\ (1+\nu)\phi_{mn}G_{mn}' + 2H_{mn}'' - (1-\nu)\phi_{mn}^2 H_{mn} &= -\frac{1}{2}[K_{mn}+K_{nm}] \end{aligned}\right\} \quad (A.6)$$

and

$$\left.\begin{aligned} -2\psi_{mn}^2 I_{mn} + (1-\nu)I_{mn}'' - (1+\nu)\psi_{mn}J_{mn}' &= -\frac{1}{2}[N_{mn}-N_{nm}] \\ (1+\nu)\psi_{mn}I_{mn}' + 2J_{mn}'' - (1+\nu)\psi_{mn}^2 J_{mn} &= -\frac{1}{2}[L_{mn}+L_{nm}] \end{aligned}\right\} \quad (A.7)$$

These two pairs of equations characterize the functions of G_{mn}, H_{mn}, I_{mn} and J_{mn}. Note in the case of $\psi_{mn} = 0$, when $m = n$, the second of Eqs. (A.7) becomes

$$J_{mn}'' = -\frac{1}{2}L_{mn} \qquad (A.8)$$

while the first of them does not exist.

Repeating the procedures to $\bar{U}_i^{mn}$, let

$$\left.\begin{aligned} \bar{U}_1^{mn} &= \frac{1}{c}\left[\bar{G}_{mn}(\eta)\sin\phi_{mn}\xi + \bar{I}_{mn}(\eta)\sin\psi_{mn}\xi\right] \\ \bar{U}_2^{mn} &= \frac{1}{c}\left[\bar{H}_{mn}(\eta)\cos\phi_{mn}\xi + \bar{J}_{mn}(\eta)\cos\psi_{mn}\xi\right] \end{aligned}\right\} \quad (A.9)$$

from Eqs. (3.28), one will have

$$\left.\begin{aligned} -2\phi_{mn}^2\bar{G}_{mn} + (1-\nu)\bar{G}_{mn}'' - (1+\nu)\phi_{mn}\bar{H}_{mn}' &= -\frac{1}{2}\left[\bar{M}_{mn}+\bar{M}_{nm}\right] \\ (1+\nu)\phi_{mn}\bar{G}_{mn}' + 2\bar{H}_{mn}'' - (1-\nu)\phi_{mn}^2\bar{H}_{mn} &= -\frac{1}{2}\left[\bar{K}_{mn}+\bar{K}_{nm}\right] \end{aligned}\right\} \quad (A.10)$$

and

$$\left.\begin{aligned} -2\psi_{mn}^2\bar{I}_{mn} + (1\quad)\bar{I}_{mn}' - (1-\nu)\psi_{mn}\bar{J}_{mn}' &= -\frac{1}{2}\left[\bar{N}_{mn}-\bar{N}_{nm}\right] \\ (1+\nu)\psi_{mn}\bar{I}_{mn}' + 2\bar{J}_{mn}' - (1-\nu)\psi_{mn}^2\bar{J}_{mn} &= -\frac{1}{2}\left[\bar{L}_{mn}+\bar{L}_{nm}\right] \end{aligned}\right\} \quad (A.11)$$

where

$$\left.\begin{aligned} \bar{M}_{mn}(\eta) &= -\gamma_m\gamma_n^2\bar{f}_m\bar{f}_n - \frac{1-\nu}{2}\gamma_m\bar{f}_m\bar{f}_n' + \frac{1+\nu}{2}\gamma_n\bar{f}_m'\bar{f}_n' \\ \bar{K}_{mn}(\eta) &= -\bar{f}_m\bar{f}_n'' + \frac{1-\nu}{2}\gamma_n^2\bar{f}_m'\bar{f}_n + \frac{1\ \nu}{2}\gamma_m\gamma_n\bar{f}_m\bar{f}_n' \end{aligned}\right\} \quad (A.12)$$

$$\left.\begin{aligned} \bar{N}_{mn}(\eta) &= \gamma_m\gamma_n^2\bar{f}_m\bar{f}_n - \frac{1-\nu}{2}\gamma_m\bar{f}_m\bar{f}_n'' + \frac{1+\nu}{2}\gamma_n\bar{f}_m'\bar{f}_n' \\ \bar{L}_{mn}(\eta) &= \bar{f}_m'\bar{f}_n'' - \frac{1-\nu}{2}\gamma_n^2\bar{f}_m'\bar{f}_n + \frac{1+\nu}{2}\gamma_m\gamma_n\bar{f}_m\bar{f}_n' \end{aligned}\right\} \quad (A.13)$$

In the case of $\psi_{mn} = 0$, the pair of equations (A.11) is reduced to

$$\bar{J}''_{mn} = -\frac{1}{2}\bar{I}_{mn} \qquad \text{(A.14)}$$

Because all the conditions along $\xi = 0$ and $\frac{a}{c}$ are satisfied identically by the assumed forms of u_i^{mn} and $\bar{u}_i^{mn}$ (see Eqs. (A.5) and (A.9) respectively), the boundary conditions (3.30) become

$$\left.\begin{aligned} H_{mn} = J_{mn} = 0 \\ G_{mn} = I_{mn} = 0 \end{aligned}\right\} \text{ on } \eta = 0$$

$$\left.\begin{aligned} \bar{H}_{mn} = \bar{J}_{mn} = 0 \\ \bar{G}'_{mn} = \bar{I}'_{mn} = 0 \end{aligned}\right\} \text{ on } \eta = \frac{b}{c} \qquad \text{(A.15)}$$

and the conditions of continuity (3.31) now are, on $\eta = 1$

$$\begin{aligned} G_{mn} = \bar{G}_{mn}, \quad H_{mn} = \bar{H}_{mn}, \quad I'_{mn} = \bar{I}_{mn}, \quad J_{mn} = \bar{J}_{mn} \\ G'_{mn} = \bar{G}'_{mn}, \quad H'_{mn} = \bar{H}'_{mn}, \quad I_{mn} = \bar{I}'_{mn}, \quad J'_{mn} = \bar{J}'_{mn} \end{aligned} \qquad \text{(A.16)}$$

To solve Eqs. (A.6) for $G(\eta)$ and $H(\eta)$, for instance, one may solve for their complementary solutions first. To do this, taking derivative with respect to η to the homogeneous part of the first of Eqs. (A.6), one has

$$-2\phi_{mn} G'_{mn} + (1-\nu) G'''_{mn} - (1+\nu)\phi_{mn} H''_{mn} = 0$$

Solving for G'_{mn} from the homogeneous part of the second of the equations and substituting it and its third derivative,

G_{mn}''' , into the above equation, the result, after simplification, yields

$$H_{mn}'''' - 2\phi_{mn}^2 H_{mn}'' + \phi_{mn}^4 H_{mn} = 0 \tag{A.16}$$

The solution of this equation is easily found as

$$\begin{aligned} H_{mn}^{o}(\eta) &= \left[h_{mn}^{1} + h_{mn}^{2}\phi_{mn}\eta\right]\sin\phi_{mn}\eta \\ &\quad + \left[h_{mn}^{3} + h_{mn}^{4}\phi_{mn}\eta\right]\cosh\phi_{mn}\eta \end{aligned} \tag{A.17}$$

where h_{mn}^{1} , h_{mn}^{2} , h_{mn}^{3} and h_{mn}^{4} are arbitrary constants. The super "O" on H_{mn} indicates it is the complementary solution of H_{mn}. On substitution of the so-found H_{mn}^{o} and hence the obtainable $(H_{mn}^{o})''$ into the homogeneous part of the second of Eqs. (A.6) and after integrating the resulted equation with respect to η once, one will get the complementary solution of G_{mn} as

$$\begin{aligned} G_{mn}^{o}(\eta) &= \left[g_{mn}^{1} + g_{mn}^{2}\phi_{mn}\eta\right]\cosh\phi_{mn}\eta \\ &\quad + \left[g_{mn}^{3} + g_{mn}^{4}\phi_{mn}\eta\right]\sinh\phi_{mn}\eta \end{aligned} \tag{A.18}$$

where

$$\left.\begin{aligned} g_{mn}^{1} &= -\left(h_{mn}^{1} - \tfrac{3-\nu}{1+\nu}h_{mn}^{4}\right), & g_{mn}^{2} &= -h_{mn}^{2}, \\ g_{mn}^{3} &= -\left(h_{mn}^{3} - \tfrac{3-\nu}{1+\nu}h_{mn}^{2}\right), & g_{mn}^{4} &= h_{mn}^{4}. \end{aligned}\right\} \tag{A.18a}$$

The integration constant in the above solution has been omitted. This is based on the consideration of the satisfaction of these solutions to the first of Eqs. (A.6).

Proceeding to find out the particular solutions of Eqs. (A.6), one computes the functions M_{mn} and K_{mn} first, from Eqs. (A.3a) and (A.4a) with the functions f_m given by Eq. (3.22b). The results are in the following forms

$$
\begin{aligned}
\frac{1}{2}\left[M_{mn}+M_{nm}\right] = \frac{1}{2}\Big[& (a^1_{mn}+a^1_{nm})\cos(\alpha_m-\alpha_n)\eta + (a^2_{mn}+a^2_{nm})\cos(\alpha_m+\alpha_n)\eta \\
& + (a^3_{mn}+a^3_{nm})\cosh(\beta_m-\beta_n) + (a^4_{mn}+a^4_{nm})\cosh(\beta_m+\beta_n)\eta \\
& + (a^5_{mn}+a^6_{nm})\sinh\beta_m\eta\sin\alpha_n\eta + (a^6_{mn}+a^5_{nm})\sin\alpha_m\eta\sinh\beta_n\eta \\
& + (a^7_{mn}+a^8_{nm})\cosh\beta_m\eta\cos\alpha_n\eta + (a^8_{mn}+a^7_{nm})\cos\alpha_m\eta\cosh\beta_n\eta \Big]
\end{aligned}
\qquad (A.19)
$$

where

$$
\left.
\begin{aligned}
\begin{Bmatrix} a^1_{mn} \\ a^2_{mn} \end{Bmatrix} &= \frac{1}{4}\left[\mp 2\gamma_m\gamma_n^2 \mp (1-\nu)\gamma_m\alpha_n^2 + (1+\nu)\gamma_n\alpha_m\alpha_n\right] \\
\begin{Bmatrix} a^3_{mn} \\ a^4_{mn} \end{Bmatrix} &= \frac{1}{4}\left[\pm 2\gamma_m\gamma_n^2 \mp (1-\nu)\gamma_m\beta_n^2 + (1+\nu)\gamma_n\beta_m\beta_n\right] \\
a^5_{mn} &= \frac{1}{2}B_m\left[-2\gamma_m\gamma_n^2 - (1-\nu)\gamma_m\alpha_n^2\right] \\
a^6_{mn} &= \frac{1}{2}B_n\left[-2\gamma_m\gamma_n^2 + (1-\nu)\gamma_m\beta_n^2\right] \\
a^7_{mn} &= \frac{1}{2}B_m\left[1+\nu\right]\gamma_n\beta_m\alpha_n \\
a^8_{mn} &= \frac{1}{2}B_n\left[1+\nu\right]\gamma_n\alpha_m\beta_n
\end{aligned}
\right\} \qquad (A.19a)
$$

and

$$
\begin{aligned}
\tfrac{1}{2}[K_{mn}+K_{nm}] = \tfrac{1}{2}\Big[&(b^1_{mn}+b^1_{nm})\sin(\alpha_m+\alpha_n)\eta+(b^2_{mn}-b^2_{nm})\sin(\alpha_m-\alpha_n)\eta \\
&+(b^3_{mn}+b^3_{nm})\sinh(\beta_m+\beta_n)\eta+(b^4_{mn}-b^4_{nm})\sinh(\beta_m-\beta_n)\eta \\
&+(b^5_{mn}+b^8_{nm})\cosh\beta_m\eta\sin\alpha_n\eta+(b^6_{mn}+b^7_{nm})\cos\alpha_m\eta\sinh\beta_n\eta \\
&+(b^7_{mn}+b^6_{nm})\sinh\beta_m\eta\cos\alpha_n\eta+(b^8_{mn}+b^5_{nm})\sin\alpha_m\eta\cosh\beta_n\eta\Big]
\end{aligned}
\tag{A.20}
$$

where

$$
\left.
\begin{aligned}
\begin{bmatrix} b^1_{mn} \\ b^2_{mn} \end{bmatrix} &= \frac{1}{4}\left[\mp 2\alpha_m\alpha_n^2 \mp (1-\nu)\gamma_n^2\alpha_m+(1+\nu)\gamma_m\gamma_n\alpha\right] \\
\begin{bmatrix} b^3_{mn} \\ b^4_{mn} \end{bmatrix} &= \frac{1}{4}\left[\pm 2\beta_m\beta_n^2 \mp (1-\nu)\gamma_n^2\beta_m+(1+\nu)\gamma_m\gamma_n\beta_n\right] \\
b^5_{mn} &= \frac{1}{2}\left[2\beta_m\alpha_n^2+(1-\nu)\beta_m\gamma_n^2\right] \\
b^6_{mn} &= \frac{1}{2}\left[-2\alpha_m\beta_n^2+(1-\nu)\alpha_m\gamma_n^2\right] \\
b^7_{mn} &= \frac{1}{2}B_n(1+\nu)\gamma_m\gamma_n\alpha_n \\
b^8_{mn} &= \frac{1}{2}B_n(1+\nu)\gamma_m\gamma_n\beta_n
\end{aligned}
\right\}
\tag{A.20a}
$$

Then let the particular solutions of G_{mn} and H_{mn} respectively be

$$G^p_{mn} = C^1_{mn}\cos(\alpha_m-\alpha_n)\eta + C^2_{mn}\cos(\alpha_m+\alpha_n)\eta$$
$$+ C^3_{mn}\cosh(\beta_m-\beta_n)\eta + C^4_{mn}\cosh(\beta_m+\beta_n)\eta$$
$$+ C^5_{mn}\sinh\beta_m\eta\sin\alpha_n\eta + C^6_{mn}\sin\alpha_m\eta\sinh\beta_n\eta \qquad (A.21)$$
$$+ C^7_{mn}\cosh\beta_m\eta\cos\alpha_n\eta + C^8_{mn}\cos\alpha_m\eta\cosh\beta_n\eta$$

and

$$H^p_{mn} = d^1_{mn}\sin(\alpha_m-\alpha_n)\eta + d^2_{mn}\sin(\alpha_m+\alpha_n)\eta$$
$$+ d^3_{mn}\sinh(\beta_m-\beta_n)\eta + d^4_{mn}\sinh(\beta_m+\beta_n)\eta$$
$$+ d^5_{mn}\cosh\beta_m\eta\sin\alpha_n\eta + d^6_{mn}\sin\alpha_m\eta\cosh\beta_n\eta \qquad (A.22)$$
$$+ d^7_{mn}\sinh\beta_m\eta\cos\alpha_n\eta + d^8_{mn}\cos\alpha_m\eta\sinh\beta_n\eta$$

where C_{mn}'s and d_{mn}'s are constant coefficients. They can be determined by substituting the two assumed functions of G^p_{mn} and H^p_{mn} into the two equations (A.6), with $\frac{1}{2}[M_{mn}+M_{nm}]$ and $\frac{1}{2}[K_{mn}+K_{nm}]$ given by Eqs. (A.19) and (A.20) and equating the constant coefficients of each corresponding terms on each side of the equations and then solving for them from the resulted algebraic equations.

For the interior part, following a similar approach, first one will get an equation parallel to Eq. (A.16) for the homogeneous solution of $\bar{H}_{mn}$ from Eqs. (A.10). That is

$$\bar{H}_{mn}'''' - 2\phi_{mn}^2 \bar{H}_{mn}'' + \phi_{mn}^4 \bar{H}_{mn} = 0 \tag{A.23}$$

of which the solution is

$$\begin{aligned} \bar{H}_{mn}^{o} &= (p_{mn}^{1} + p_{mn}^{2}\phi_{mn}\eta)\, e^{\phi_{mn}\eta} \\ &\quad + (p_{mn}^{3} + p_{mn}^{4}\phi_{mn}\eta)\, e^{-\phi_{mn}\eta} \end{aligned} \tag{A.24}$$

where p_{mn}^{1}, p_{mn}^{2}, p_{mn}^{3} and p_{mn}^{4} are arbitrary constants. The homogeneous solution of its conjugate function $\bar{G}_{mn}$, obtained from the homogeneous part of equations (A.10) is

$$\begin{aligned} \bar{G}_{mn}^{o} &= (q_{mn}^{1} + q_{mn}^{2}\phi_{mn}\eta)\, e^{\phi_{mn}\eta} \\ &\quad + (q_{mn}^{3} + q_{mn}^{4}\phi_{mn}\eta)\, e^{-\phi_{mn}\eta} \end{aligned} \tag{A.25}$$

in which

$$\left.\begin{aligned} q_{mn}^{1} &= -\left(p_{mn}^{1} - \frac{3-\nu}{1+\nu}\, p_{mn}^{4}\right), & q_{mn}^{2} &= -p_{mn}^{2} \\ q_{mn}^{1} &= -\left(p_{mn}^{3} - \frac{3-\nu}{1+\nu}\, p_{mn}^{2}\right), & q_{mn}^{4} &= q_{mn}^{4} \end{aligned}\right\} \tag{A.25a}$$

The particular solutions of $\bar{G}_{mn}$ and $\bar{H}_{mn}$ can be obtained by the same way as that done for G^p_{mn} and H^p_{mn}. Compute $\bar{M}_{mn}$ and $\bar{K}_{mn}$ first; assume two functions for $\bar{G}^p_{mn}$ and $\bar{H}^p_{mn}$ which are the same in forms as that of $\bar{M}_{mn}$ (or $\bar{K}_{mn}$); then determine their coefficients through the differential equations. The results are:

$$\begin{aligned}\bar{G}^p_{mn} = &\left(K^1_{mn} + K^2_{mn}\phi_{mn}\eta + K^3_{mn}\phi^2_{mn}\eta^2\right)e^{\phi_{mn}\eta} \\ &+\left(K^4_{mn} + K^5_{mn}\phi_{mn}\eta + K^6_{mn}\phi^2\eta^2\right)e^{-\phi_{mn}\eta} \\ &+\left(K^7_{mn} + K^8_{mn}\psi_{mn}\eta + K^9_{mn}\psi^2_{mn}\eta^2\right)e^{\psi_{mn}\eta} \\ &+\left(K^{10}_{mn} + K^{11}_{mn}\psi_{mn}\eta + K^{12}_{mn}\psi^2_{mn}\eta^2\right)e^{-\psi_{mn}\eta}\end{aligned} \tag{A.26}$$

for $m \neq n$; while $m = n$,

$$\begin{aligned}\bar{G}^p_{mn} = &\left(K^1_{mn} + K^2_{mn}\phi_{mn}\eta + K^3_{mn}\phi^2_{mn}\eta^2\right)e^{\phi_{mn}\eta} \\ &+\left(K^4_{mn} + K^5_{mn}\phi_{mn}\eta + K^3_{mn}\phi^2_{mn}\eta^2\right)e^{-\phi_{mn}\eta} \\ &+K^7_{mn} + K^8_{mn}\phi_{mn}\eta + K^9_{mn}\phi^2_{mn}\eta^2.\end{aligned} \tag{A.26a}$$

$\bar{H}^p_{mn}$ are also in the above forms but with different coefficients.

The eight arbitrary constants involved in the complementary solutions of G_{mn}, H_{mn}, $\bar{G}_{mn}$ and $\bar{H}_{mn}$ are determined by the eight conditions; four from the boundary conditions

(A.15) and the other four from the conditions of continuity (A.16).

Following the same procedures as that did for G_{mn}, H_{mn} and $\bar{G}_{mn}$, $\bar{H}_{mn}$, one can find the solutions for I_{mn}, J_{mn} and $\bar{I}_{mn}$, $\bar{J}_{mn}$. Except when m = n, I_{mn} and $\bar{I}_{mn}$ vanish and

$$\left.\begin{aligned} J_{mn} &= -\frac{1}{2}\iint L_{mn}(\eta)\,d\eta\,d\eta + \delta^1_{mn}\eta + \delta^2_{mn} \\ \bar{J}_{mn} &= -\frac{1}{2}\iint \bar{L}_{mn}(\eta)\,d\eta\,d\eta + \delta^3_{mn}\eta + \delta^4_{mn} \end{aligned}\right\} \quad (A.27)$$

which can be easily integrated. The final solutions for I_{mn} and J_{mn} are in the same forms as that of G_{mn} and H_{mn} respectively but change ϕ_{mn} to ψ_{mn}. They have been given in Section 3. (See Eqs. 3.32b) and (3.32c)). Those for $\bar{I}_{mn}$ and $\bar{J}_{mn}$ are in the same form as that of $\bar{G}_{mn}$. (See Eqs. (3.33b) and (3.33c)).

The complete solutions of G_{mn}, H_{mn}, I_{mn}, J_{mn} and $\bar{G}_{mn}$, $\bar{H}_{mn}$, $\bar{I}_{mn}$, $\bar{J}_{mn}$ of which the coefficients are computed after the constants given by Table I, are tabulated in Tables II to IV.

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