Process Capability Ratio Limits for Tolerance Analysis

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ABSTRACT

In this paper we develop a method for analyzing manufacturing processes and tolerances. This method is based on the study of the process capability ratio (C_p). To compensate for the uncertainty of the estimated process capability ratio, we find the upper and lower limits with a certain confidence level depending on the purpose of the process capability ratio. These limits can be used for general process and tolerance analysis problems.

1. Introduction

Confirmation of geometrical dimensions and tolerances is made more challenging by the smaller size and greater complexity of parts. In addition, conventional measuring instruments are not suited to these size and complexity challenges. Therefore, a coordinate measuring machine (CMM) is widely used to accomplish this task. Generally, the output of a CMM indicates confirmation as well as the effectiveness of conventional process control. However, the conventional process control procedure has a problem which does not take account the uncertainty of the parameter estimation. The conventional process control procedures are control charts and process capability analysis. In this paper, we discuss process capability analysis.

Control charts are the simplest type of statistical process control procedure. A control chart is a device principally used for the study and control of repetitive processes. Dr. Walter A. Shewhart, its originator, suggests that the control chart may serve, first, to define the goal or standard for a process that management might strive to attain; second, it may be used as an instrument for attaining that goal; and, third, it may serve as a means of judging whether the goal has been reached.

Process capability analysis can be defined as the quantification of process variability, as the analysis of this variability relative to product specifications, and as an aid in eliminating or greatly reducing this variability in development and manufacturing. A process capability study usually measures the functional parameters of a product (product characterization), not the process itself (true process capability analysis). However, a true process capability analysis includes observing the manufacturing process and controlling or monitoring the data collection activities. Controlling data collection and knowing the time sequence of the collection allows inferences to be made about the stability of the process over time and allows inferences about the adequacy of the specifications. A true process capability analysis can be contrasted with a product characterization. A product characterization occurs when we have only sample units of a product available and, as a result, there is no direct observation of the process and no time history of production. In a product characterization study we can only estimate product quality and production yield (fraction confirming to specifications); we can say nothing about the dynamic behavior of the process or its state of statistical control (Montgomery [1985]).

When we measure only one dimension of a product such as length, weight, and diameter, it is difficult to describe the dynamic behavior of the process even though we measure 100% of the products. One dimension cannot describe the dynamic behavior of the production process because a product is usually manufactured by various machining processes. However, we can describe the dynamic behavior of the process when we measure two or three dimensions such as geometric tolerances. Because the geometric tolerances are the description of the surface characteristics of the part, they can contain the information of the manufacturing process behavior.

The confirmation of geometrical tolerancing of manufactured parts is usually achieved using a coordinate measuring machine (CMM). The confirmation of geometrical tolerancing requires serial sampling points of a manufactured part. When several parts are measured, these points contain the time sequence of the process. This sequence data can be used for the analysis of the stability of the process and to determine the adequacy of the given specification. In other words, if the process is stable and if the analysis shows that the product is out of tolerance, the given tolerance may be too tight. However, if the process is stable and if the analysis shows that the product is in specification but deviations are lower than the specification, the given tolerance may be too loose. When we inspect the sampled parts over time, the accumulated data can be used for product characterization. Product characterization in geometrical tolerancing can be the analysis of process performance.

Process capability indices are used to perform process capability analysis (Sullivan [1984, 1985], Kane [1986], and Montgomery [1985]). We can evaluate the process capability exactly and compare this value with the recommended minimum value, when the upper and lower specifications and the real process standard deviation are given. However, if we estimate the process standard deviation from a sample size n, then we can evaluate the estimated process capability. Estimated process capability is a random variable because the estimated standard deviation is a random variable. Therefore, we cannot directly compare the estimated process capability with the minimum recommended value of process capability. The minimum recommended values of the process capability ratio are given in Montgomery [1985] and shown in Table 1.

	Two sided Specification	One sided Specification
Existing Process	1.33	1.25
New Process	1.50	1.45
Safety, strength or critical parameter (existing process)	1.50	1.45
Safety, strength or critical parameter (new process)	1.67	1.60

Table 1 Recommended Minimum Values of the Process Capability Ratio

It is necessary to determine the lower and upper limits of the estimated process capability because the estimated process capability is a random variable. This determination leads to two possible conclusions: either the process is capable with a certain confidence or, because of a certain amount of standard deviation increment, it is not capable with a certain confidence. There could be several factors which account for this increment such as tool wear and clamp breakage. Among various process capability indices, we deal with process capability ratio (C_p).

2. Limits on C_p

When the upper (U) and lower (L) specifications are given, a conventional measure of process capability is C_p. The process capability ratio (PCR) is defined as

$$C_p = \frac{U - L}{6\sigma} \quad \text{and} \quad \hat{C}_p = \frac{U - L}{6s} .$$

Because we usually do not know the true standard deviation (σ), the estimated standard deviation (s) is used to calculate PCR. Because the estimated standard deviation is a random variable, estimated PCR ($\hat{C}_p$) is also a random variable. Therefore, it is necessary to compare the limits of $\hat{C}_p$ with the recommended minimum values of C_p or vice versa. The limits are either one-sided limits or two-sided limits. One-sided refers to the confirmation of process performance and two-sided refers to both the process performance and tolerance analysis.

The proposed definitions of form errors in previous paper (Chang et al. [1990]) related to the estimation of 6s. Therefore, these values can be used for process capability analysis. When we estimate form error serially, this value contains information about the time sequence of the manufacturing process. It can be used to monitor the dynamic behavior of the process. When we accumulate estimated form errors over time, these values can be used to monitor the process but, also, to analyze tolerance design.

2.1 One Sided Lower Limits on C_p

When $\hat{C}_p$ is given, we can find C_{pL} which is lower limit of C_p with a Type I error of γ by the hypothesis test. Since $\frac{C_p}{\hat{C}_p} = \frac{s}{\sigma}$, we know that $(n-1)(\frac{C_p}{\hat{C}_p})^2$ follows a chi-square distribution with (n-1) degrees of freedom denoted by $\chi^2_{\gamma,n-1}$. Then

H$_0$ (Null Hypothesis): $\sigma_1^2 \leq \sigma_0^2$

H$_1$ (Alternative): $\sigma_1^2 > \sigma_0^2$

where σ_1^2 = unknown variance

σ_0^2 = true variance

$$\Pr\{\text{Type I error}\} = \Pr\{\text{reject } H_0 \mid H_0 \text{ is true}\}$$

$$= \Pr\{ \frac{(n-1)\,\sigma_1^2}{\sigma_0^2} > \chi^2_{\gamma,n-1} \} = \Pr\{ (n-1) (\frac{C_p}{\hat{C}_p})^2 > \chi^2_{\gamma,n-1} \}$$

$$= \Pr\{ \sigma_1^2 > \frac{\sigma_0^2}{n-1} \chi^2_{\gamma,n-1}\} = \Pr\{ C_p^2 > \frac{\hat{C}_p^2}{n-1} \chi^2_{\gamma,n-1}\} = \gamma \ .$$

Therefore, the lower limit of C_p, when $\hat{C}_p$ is given, for determining the process capability with a Type I error of γ, is

$$C_{pL} = \hat{C}_p \sqrt{\frac{(\chi^2_{\gamma,n-1})}{n-1}} \ . \tag{1}$$

We can determine that the process is capable with a Type I error of γ when the recommended value of C_p is at least C_{pL}. We estimate $\hat{C}_p$ from the measured process by taking a random sample of size *n*. However, since the minimum recommended values of C_p are usually given for various process conditions [Montgomery (1985)], we can find the required minimum of $\hat{C}_p$ with a Type I error of γ by representing Eq.(1) in terms of C_{pL}

$$\hat{C}_p = C_{pL} \sqrt{\frac{n-1}{(\chi^2_{\gamma,n-1})}} \ . \tag{2}$$

This result is same as the 100(1-γ)% lower confidence limit on C_p which is derived by Chou, Owen, and Borrego [1990]. Table 2 shows the lower limits of $\hat{C}_p$ with 95% confidence that the process is capable.

n / Cp	4	5	6	7	8	9	10	20	30	40
1.0	2.93	2.37	2.09	1.91	1.80	1.71	1.64	1.37	1.28	1.23
1.1	3.22	2.61	2.29	2.10	1.98	1.88	1.81	1.51	1.41	1.36
1.2	3.51	2.85	2.50	2.30	2.16	2.05	1.97	1.64	1.54	1.48
1.3	3.81	3.09	2.71	2.49	2.33	2.23	2.14	1.78	1.66	1.60
1.4	4.10	3.32	2.92	2.68	2.51	2.40	2.30	1.92	1.79	1.72
1.5	4.39	3.56	3.13	2.87	2.69	2.57	2.47	2.06	1.92	1.85
1.6	4.68	3.80	3.34	3.06	2.87	2.74	2.63	2.19	2.05	1.97
1.7	4.98	4.04	3.54	3.25	3.05	2.91	2.79	2.33	2.18	2.09
1.8	5.27	4.27	3.75	3.44	3.23	3.08	2.96	2.47	2.30	2.22
1.9	5.56	4.51	3.96	3.63	3.41	3.25	3.12	2.60	2.43	2.34
2.0	5.86	4.75	4.17	3.83	3.59	3.42	3.29	2.74	2.56	2.46

Table 2 Lower Limits of $\hat{C}_p$ with 95% confidence of process being capable

n / Cp	50	60	70	80	90	100	200	300	400	500
1.0	1.20	1.18	1.16	1.15	1.14	1.13	1.09	1.07	1.06	1.06
1.1	1.32	1.30	1.28	1.27	1.26	1.25	1.20	1.18	1.17	1.16
1.2	1.44	1.42	1.40	1.38	1.37	1.36	1.31	1.29	1.27	1.27
1.3	1.56	1.53	1.51	1.50	1.48	1.47	1.42	1.39	1.38	1.37
1.4	1.68	1.65	1.63	1.61	1.60	1.59	1.53	1.50	1.49	1.48
1.5	1.80	1.77	1.75	1.73	1.71	1.70	1.64	1.61	1.59	1.58
1.6	1.92	1.89	1.86	1.84	1.83	1.81	1.74	1.72	1.70	1.69
1.7	2.04	2.01	1.98	1.96	1.94	1.93	1.85	1.82	1.81	1.79
1.8	2.16	2.12	2.10	2.07	2.06	2.04	1.96	1.93	1.91	1.90
1.9	2.28	2.24	2.21	2.19	2.17	2.15	2.07	2.04	2.02	2.00
2.0	2.40	2.36	2.33	2.30	2.28	2.27	2.18	2.15	2.12	2.11

Table 2 Continued

2.2 One Sided Upper Limits on C_p

When $\hat{C}_p$ is less than the lower limit derived from given C_p by Eq.(2), we claim that the process is not capable with a Type I error of γ. The variance increment can be detected by $\hat{C}_p$ if the cause of the variance increment shows the process is out-of-control. When we have enough information about the causes of the out-of-control state or variance increment, we can detect those causes from $\hat{C}_p$.

When $\hat{C}_p$ is given, we can find C_{pU} which is upper limit of C_p with a Type II error of β by the hypothesis test.

H_0 (Null Hypothesis): $\sigma_1^2 \le k^2 \sigma_0^2$

H_1 (Alternative): $\sigma_1^2 > k^2 \sigma_0^2$

where σ_1^2 = unknown variance

σ_0^2 = true variance

k = variance increment factor

$$\Pr\{\text{Type II error}\} = \Pr\{\text{fail to reject } H_0 \mid H_0 \text{ is false}\}$$

$$= \Pr\{\frac{(n-1)\,\sigma_1^2}{k^2\,\sigma_0^2} < \chi^2_{\beta,n-1}\} = \Pr\{(n-1)\frac{1}{k^2}(\frac{C_p}{\hat{C}_p})^2 < \chi^2_{\beta,n-1}\}$$

$$= \Pr\{\sigma_1^2 < \frac{k^2\,\sigma_0^2}{n-1}\chi^2_{\beta,n-1}\} = \Pr\{C_p^2 < \frac{k^2\,\hat{C}_p^2}{n-1}\chi^2_{\beta,n-1}\} = \beta\,.$$

Therefore, the upper limit of C_p, for determining that the process is not capable with a Type II error of β, is

$$C_{pU} = k\hat{C}_p\sqrt{\frac{(\chi^2_{\beta,n-1})}{n-1}}\,. \qquad (3)$$

We can determine that the process is not capable with a Type II error of β when the recommended value of C_p is at most C_{pL}. We estimate $\hat{C}_p$ from the measured process by taking a random sample of size *n*. We can also find the required maximum of $\hat{C}_p$ with a Type II error of β by representing Eq.(3) in terms of C_{pU}

$$\hat{C}_p = \frac{C_{pU}}{k} \sqrt{\frac{n-1}{(\chi^2_{\beta,n-1})}} \quad . \tag{4}$$

Table 3 gives the upper limits of $\hat{C}_p$ in order for the process to be described as not capable with a 5% Type II error for three different standard deviation increment factors of 1.1, 1.2, and 1.3.

n	4			5			6			7			8		
k / **Cp**	**1.1**	**1.2**	**1.3**	**1.1**	**1.2**	**1.3**	**1.1**	**1.2**	**1.3**	**1.1**	**1.2**	**1.3**	**1.1**	**1.2**	**1.3**
1.0	2.66	2.44	2.25	2.16	1.98	1.83	1.90	1.74	1.60	1.74	1.59	1.47	1.63	1.50	1.38
1.1	2.93	2.68	2.48	2.37	2.18	2.01	2.09	1.91	1.76	1.91	1.75	1.62	1.80	1.65	1.52
1.2	3.19	2.93	2.70	2.59	2.37	2.19	2.27	2.09	1.92	2.09	1.91	1.77	1.96	1.80	1.66
1.3	3.46	3.17	2.93	2.81	2.57	2.37	2.46	2.26	2.09	2.26	2.07	1.91	2.12	1.95	1.80
1.4	3.73	3.42	3.15	3.02	2.77	2.56	2.65	2.43	2.25	2.43	2.23	2.06	2.29	2.10	1.93
1.5	3.99	3.66	3.38	3.24	2.97	2.74	2.84	2.61	2.41	2.61	2.39	2.21	2.45	2.25	2.07
1.6	4.26	3.90	3.60	3.45	3.16	2.92	3.03	2.78	2.57	2.78	2.55	2.35	2.61	2.39	2.21
1.7	4.52	4.15	3.83	3.67	3.36	3.10	3.22	2.95	2.73	2.96	2.71	2.50	2.78	2.54	2.35
1.8	4.79	4.39	4.05	3.88	3.56	3.29	3.41	3.13	2.89	3.13	2.87	2.65	2.94	2.69	2.49
1.9	5.06	4.64	4.28	4.10	3.76	3.47	3.60	3.30	3.05	3.30	3.03	2.80	3.10	2.84	2.63
2.0	5.32	4.88	4.50	4.32	3.96	3.65	3.79	3.48	3.21	3.48	3.19	2.94	3.27	2.99	2.76

Table 3 The upper limits of $\hat{C}_p$ for which the process is not capable with 5% of Type II error

n	9			10			20			30			40		
k Cp	1.1	1.2	1.3	1.1	1.2	1.3	1.1	1.2	1.3	1.1	1.2	1.3	1.1	1.2	1.3
1.0	1.56	1.43	1.32	1.49	1.37	1.26	1.25	1.14	1.05	1.16	1.07	0.98	1.12	1.03	0.95
1.1	1.71	1.57	1.45	1.64	1.51	1.39	1.37	1.26	1.16	1.28	1.17	1.08	1.23	1.13	1.04
1.2	1.87	1.71	1.58	1.79	1.64	1.52	1.49	1.37	1.26	1.40	1.28	1.18	1.34	1.23	1.14
1.3	2.02	1.85	1.71	1.94	1.78	1.64	1.62	1.48	1.37	1.51	1.39	1.28	1.46	1.33	1.23
1.4	2.18	2.00	1.84	2.09	1.92	1.77	1.74	1.60	1.48	1.63	1.49	1.38	1.57	1.44	1.33
1.5	2.33	2.14	1.98	2.24	2.05	1.90	1.87	1.71	1.58	1.74	1.60	1.48	1.68	1.54	1.42
1.6	2.49	2.28	2.11	2.39	2.19	2.02	1.99	1.83	1.69	1.86	1.71	1.57	1.79	1.64	1.52
1.7	2.65	2.43	2.24	2.54	2.33	2.15	2.12	1.94	1.79	1.98	1.81	1.67	1.90	1.75	1.61
1.8	2.80	2.57	2.37	2.69	2.47	2.28	2.24	2.06	1.90	2.09	1.92	1.77	2.02	1.85	1.71
1.9	2.96	2.71	2.50	2.84	2.60	2.40	2.37	2.17	2.00	2.21	2.03	1.87	2.13	1.95	1.80
2.0	3.11	2.85	2.63	2.99	2.74	2.53	2.49	2.28	2.11	2.33	2.13	1.97	2.24	2.05	1.90

Table 3 Continued

n	50			60			70			80			90		
k / **Cp**	**1.1**	**1.2**	**1.3**	**1.1**	**1.2**	**1.3**	**1.1**	**1.2**	**1.3**	**1.1**	**1.2**	**1.3**	**1.1**	**1.2**	**1.3**
1.0	1.09	1.00	0.92	1.07	0.98	0.91	1.06	0.97	0.90	1.05	0.96	0.89	1.04	0.95	0.88
1.1	1.20	1.10	1.02	1.18	1.08	1.00	1.16	1.07	0.99	1.15	1.06	0.97	1.14	1.05	0.97
1.2	1.31	1.20	1.11	1.29	1.18	1.09	1.27	1.16	1.07	1.26	1.15	1.06	1.25	1.14	1.05
1.3	1.42	1.30	1.20	1.40	1.28	1.18	1.38	1.26	1.16	1.36	1.25	1.15	1.35	1.24	1.14
1.4	1.53	1.40	1.29	1.50	1.38	1.27	1.48	1.36	1.25	1.47	1.34	1.24	1.45	1.33	1.23
1.5	1.64	1.50	1.39	1.61	1.48	1.36	1.59	1.46	1.34	1.57	1.44	1.33	1.56	1.43	1.32
1.6	1.75	1.60	1.48	1.72	1.57	1.45	1.69	1.55	1.43	1.68	1.54	1.42	1.66	1.52	1.41
1.7	1.86	1.70	1.57	1.82	1.67	1.54	1.80	1.65	1.52	1.78	1.63	1.51	1.76	1.62	1.49
1.8	1.97	1.80	1.66	1.93	1.77	1.63	1.91	1.75	1.61	1.89	1.73	1.60	1.87	1.71	1.58
1.9	2.08	1.90	1.76	2.04	1.87	1.73	2.01	1.84	1.70	1.99	1.82	1.68	1.97	1.81	1.67
2.0	2.18	2.00	1.85	2.15	1.97	1.82	2.12	1.94	1.79	2.09	1.92	1.77	2.08	1.90	1.76

Table 3 Continued

n	100			200			300			400			500		
k Cp	1.1	1.2	1.3	1.1	1.2	1.3	1.1	1.2	1.3	1.1	1.2	1.3	1.1	1.2	1.3
1.0	1.03	0.94	0.87	0.99	0.91	0.84	0.98	0.89	0.83	0.97	0.89	0.82	0.96	0.88	0.81
1.1	1.13	1.04	0.96	1.09	1.00	0.92	1.07	0.98	0.91	1.06	0.97	0.90	1.06	0.97	0.89
1.2	1.24	1.13	1.05	1.19	1.09	1.01	1.17	1.07	0.99	1.16	1.06	0.98	1.15	1.06	0.97
1.3	1.34	1.23	1.13	1.29	1.18	1.09	1.27	1.16	1.07	1.26	1.15	1.06	1.25	1.14	1.06
1.4	1.44	1.32	1.22	1.39	1.27	1.17	1.37	1.25	1.16	1.35	1.24	1.14	1.34	1.23	1.14
1.5	1.55	1.42	1.31	1.49	1.36	1.26	1.46	1.34	1.24	1.45	1.33	1.23	1.44	1.32	1.22
1.6	1.65	1.51	1.40	1.59	1.45	1.34	1.56	1.43	1.32	1.54	1.42	1.31	1.53	1.41	1.30
1.7	1.75	1.61	1.48	1.69	1.54	1.43	1.66	1.52	1.40	1.64	1.50	1.39	1.63	1.49	1.38
1.8	1.85	1.70	1.57	1.78	1.64	1.51	1.76	1.61	1.49	1.74	1.59	1.47	1.73	1.58	1.46
1.9	1.96	1.79	1.66	1.88	1.73	1.59	1.85	1.70	1.57	1.83	1.68	1.55	1.82	1.67	1.54
2.0	2.06	1.89	1.74	1.98	1.82	1.68	1.95	1.79	1.65	1.93	1.77	1.63	1.92	1.76	1.62

Table 3 Continued

[Example]

Suppose the required $C_p \geq 1.2$ and that the upper and lower specifications are given. We take a random sample of size *n* and calculate $\hat{C}_p$. If the required Type I error is 0.05 and the Type II error is 0.05, then the claim of process capability will depend on the values calculated from Eqs. (3) and (4).

Let n=20, then from Eq.(3)

$C_{pL} = 1.64$

and from Eq.(4)

$C_{pU} = 1.49$ for k=1.1

$C_{pU} = 1.37$ for k=1.2

$C_{pU} = 1.27$ for k=1.3

1) If $\hat{C}_p \geq 1.64$, we can claim that the process is capable with 95% confidence.
2) If $1.49 \leq \hat{C}_p < 1.64$, we can claim that the process is not capable. It may be due to an increase in the standard deviation of up to 10% with 5% of Type II error.
3) If $1.37 \leq \hat{C}_p < 1.49$, we can claim that the process is not capable. It may be due to an increment in the standard deviation of more than 10% and up to 20% with a 5% Type II error.
4) If $1.27 \leq \hat{C}_p < 1.37$, we can claim that the process is not capable. It may be due to an increase in the standard deviation of more than 20% and up to 30% with a 5% Type II error.

2.3 Two Sided Limits on C_p

In previous section, we considered only the confirmity of process capability not the adequacy of tolerance. In this section we focus on tolerance. Geometrical tolerances are confirmed by sample points measurements. These measurements should satisfy a geometrical tolerance. In addition, each point has its own tolerance which each point should satisfy. The geometrical tolerance and the point tolerance may not be the same because of surface functionality. Therefore, even though all the sample points satisfy the geometrical tolerance requirement, any point might not satisfy its own tolerance or vice versa. When these things are happen, we cannot use the results of previous section. Instead, a two-sided test is necessary to analyze the tolerances which are too tight or too loose. Therefore, we modify one-sided test for a Type I error to two-sided test as follow.

H_0 (Null Hypothesis): $\sigma_1^2 = \sigma_0^2$

H_1 (Alternative): $\sigma_1^2 \neq \sigma_0^2$

where σ_1^2 = unknown variance

σ_0^2 = true variance

$$\Pr\{\text{Type I error}\} = \Pr\{\text{reject } H_0 \mid H_0 \text{ is true}\}$$
$$= 1 - \Pr\{\chi^2_{\gamma/2,n-1} \leq \frac{(n-1)\,\sigma_1^2}{\sigma_0^2} \leq \chi^2_{1-\gamma/2,n-1}\}$$
$$= 1 - \Pr\{\chi^2_{\gamma/2,n-1} \leq (n-1)\,(\frac{C_p}{\hat{C}_p})^2 \leq \chi^2_{1-\gamma/2,n-1}\}$$
$$= 1 - \Pr\{\frac{\sigma_0^2}{(n-1)}\chi^2_{\gamma/2,n-1} \leq \sigma_1^2 \leq \frac{\sigma_0^2}{(n-1)}\chi^2_{1-\gamma/2,n-1}\}$$
$$= 1 - \Pr\{\frac{\hat{C}_p^2}{n-1}\chi^2_{\gamma/2,n-1} \leq C_p^2 \leq \frac{\hat{C}_p^2}{n-1}\chi^2_{1-\gamma/2,n-1}\}$$
$$= \gamma\ .$$

Therefore, two sided limits of C_p, when $\hat{C}_p$ is given, for determining the tolerance tightness with Type I error of γ, is

$$C_{pTL} = \hat{C}_p\sqrt{\frac{(\chi^2_{\gamma/2,n-1})}{n-1}} \quad \text{or} \quad \hat{C}_p = C_{pTL}\sqrt{\frac{n-1}{(\chi^2_{\gamma/2,n-1})}}$$

$$C_{pTU} = \hat{C}_p\sqrt{\frac{(\chi^2_{1-\gamma/2,n-1})}{n-1}} \quad \text{or} \quad \hat{C}_p = C_{pTU}\sqrt{\frac{n-1}{(\chi^2_{1-\gamma/2,n-1})}}\,. \qquad (5)$$

We can decide that the point tolerance is too tight or too loose when we estimate $\hat{C}_p$ for the measured point of surface by taking a random sample of size n. The tolerance is too tight when the recommended value of C_p is greater than C_{pTL}. The tolerance is too loose when the recommended value of C_p is less than C_{pTU}. In both cases the Type I error is γ. However, when the minimum recommended values of C_p are given, we can find the required limits of $\hat{C}_p$ with a Type I error of γ. Table 4 shows the lower and upper limits of $\hat{C}_p$ with 95% confidence. The tolerance is too loose when $\hat{C}_p$ is greater than the upper limit. The tolerance is too tight when $\hat{C}_p$ is less than the lower limit.

n	3		4		5		6		7	
Cp	TU	TL	TU	TL	TU	TL	TU	TL	TU	TL
1.0	6.32	0.52	3.69	0.57	2.89	0.60	2.45	0.64	2.20	0.64
1.1	6.96	0.57	4.06	0.62	3.18	0.66	2.70	0.70	2.42	0.71
1.2	7.59	0.62	4.43	0.68	3.46	0.72	2.95	0.76	2.64	0.77
1.3	8.22	0.68	4.80	0.74	3.75	0.78	3.19	0.83	2.86	0.84
1.4	8.85	0.73	5.17	0.79	4.04	0.84	3.44	0.89	3.08	0.90
1.5	9.49	0.78	5.54	0.85	4.33	0.90	3.68	0.95	3.30	0.97
1.6	10.12	0.83	5.91	0.91	4.62	0.96	3.93	1.02	3.52	1.03
1.7	10.75	0.88	6.28	0.96	4.91	1.02	4.17	1.08	3.74	1.10
1.8	11.38	0.94	6.65	1.02	5.20	1.08	4.42	1.14	3.96	1.16
1.9	12.02	0.99	7.02	1.08	5.48	1.14	4.66	1.21	4.18	1.22
2.0	12.65	1.04	7.39	1.13	5.77	1.20	4.91	1.27	4.40	1.29

Table 4 Upper and Lower Limits of $\hat{C}_p$ with Type I error of 5%

n	8		9		10		15		20	
Cp	TU	TL	TU	TL	TU	TL	TU	TL	TU	TL
1.0	2.04	0.66	1.92	0.68	1.83	0.69	1.58	0.73	1.58	0.76
1.1	2.24	0.73	2.11	0.74	2.01	0.76	1.73	0.81	1.74	0.84
1.2	2.44	0.79	2.30	0.81	2.19	0.83	1.89	0.88	1.89	0.91
1.3	2.65	0.86	2.49	0.88	2.37	0.89	2.05	0.95	2.05	0.99
1.4	2.85	0.93	2.68	0.95	2.56	0.96	2.21	1.02	2.21	1.06
1.5	3.05	0.99	2.87	1.01	2.74	1.03	2.37	1.10	2.37	1.14
1.6	3.26	1.06	3.07	1.08	2.92	1.10	2.52	1.17	2.52	1.22
1.7	3.46	1.12	3.26	1.15	3.10	1.17	2.68	1.24	2.68	1.29
1.8	3.66	1.19	3.45	1.22	3.29	1.24	2.84	1.32	2.84	1.37
1.9	3.87	1.26	3.64	1.28	3.47	1.31	3.00	1.39	3.00	1.44
2.0	4.07	1.32	3.83	1.35	3.65	1.38	3.15	1.46	3.16	1.52

Table 4 Continued

[Example]

Suppose the required $C_p \geq 1.5$ and that the difference between the upper and lower specifications are given by 9σ. We take a random sample of size n and calculate $\hat{C}_p$. If the required Type I error is 0.05 and the Type II error is 0.05, then the claim of process capability and specifications will depend on the values obtained from Tables 2, 3, and 4.

Let n=20, then from Table 2

$C_{pL} = 2.06$

and from Table 3

$C_{pU} = 1.87$ for k=1.1

$C_{pU} = 1.71$ for k=1.2

$C_{pU} = 1.58$ for k=1.3

and from Table 4

$C_{pTU} = 2.37$

$C_{pTL} = 1.14$

1) If $\hat{C}_p \geq 2.37$, we can claim that the specification is too loose with 95% confidence even though the process is capable.
2) If $\hat{C}_p \geq 2.06$, we can claim that the process is capable with 95% confidence.
3) If $1.87 \leq \hat{C}_p < 2.06$, we can claim that the process is not capable. It may be due to an increase in the standard deviation of up to 10% with 5% of Type II error.
4) If $1.71 \leq \hat{C}_p < 1.87$, we can claim that the process is not capable. It may be due to an increase in the standard deviation of more than 10% and up to 20% with 5% of Type II error.
5) If $1.58 \leq \hat{C}_p < 1.71$, we can claim that the process is not capable. It may be due to an increase in the standard deviation of more than 20% and up to 30% with 5% of Type II error.
6) If $\hat{C}_p < 1.14$, we can claim that the specification is too tight with 95% confidence when we cannot find any causes of the process being out-of-control.

3. Examples in Geometrical Tolerance Problem

Assuming that we measure straightness errors of machined surfaces, we analyze the process capability using the estimated straightness errors. The specification is $0.009mm$ and the process standard deviation is $0.001mm/mm$ and the minimum C_p is greater than or equal to 1.3. We conducted four case studies: when the process is capable, when the process is not capable because the process standard standard deviation is increased by the factor of 1.2, when the specification is changed to $0.012mm$ (too loose), and when the specification is changed to $0.006mm$ (too tight). We generate random normal numbers to conduct these studies.

Case 1] The process is capable.

We assume that we measure four parts which are collected serially or randomly over time. On each part, five points, which are generated by normal random number generator with mean zero and standard deviation 0.001, are measured by equi-distance (Table 5).

X-Coordinate	Y-Coordinate			
	Part 1	Part 2	Part 3	Part 4
0	0.0004	0.0005	0.0003	-0.001
1	-0.0011	-0.0005	-0.0011	0.0019
2	0.0014	0.0014	-0.001	0.0002
3	0.0006	-0.0006	-0.0006	-0.0008
4	0.0006	-0.0003	-0.0007	-0.0005

Table 5 Coordinate values of four measured parts

Straightness errors are estimated by converting the data in Table 5. Table 6 shows the estimated straightness errors and the converted data. We assume that there is no sign of the process being out-of-control. No part is out pf specification. Then, we calculate the process capability ratio using these converted data of 20 points. The process is capable with 95%

confidence because the estimated process capability ratio is greater than the minimum required value 1.78 from Table 2 with n=20, C_p=1.3.

$$\hat{C}_p = \frac{0.009}{6*0.0008} = 1.875$$

Case 2] The process is not capable.

We assume that the data collection procedure is same as case 1 except that the data is generated with mean zero and standard deviation 0.0012 (Table 7). Estimated straightness errors and converted data are shown in Table 8. One part is out-of specification. Then, we calculate process capability ratio using the converted data, and as a result, the process is not capable and the process standard deviation is increased up to 10% with a 5% Type II error. The actual process standard deviation was increased by 20%. This decision is based on Table 3 with n=20 and k=1.1.

$$1.62 \leq \hat{C}_p = \frac{0.009}{6*0.0009} = 1.667 \leq 1.78$$

	Part 1	Part 2	Part 3	Part 4
E.S.E.	0.0059	0.0056	0.0035	0.0079
	0.0004	0.0001	0.0006	-0.0013
	-0.0013	-0.0008	-0.0006	0.0018
	0.001	0.0013	-0.0004	0.0002
	1.0000E-5	-0.0005	0.0002	-0.0006
	-0.0002	-0.0001	0.0002	-0.0001

Note: E.S.E.= Estimated Straightness Error
Real Straightness Error = 0.006

Table 6 Estimated Straightness Errors and Converted data

X-Coordinate	Y-Coordinate			
	Part 1	Part 2	Part 3	Part 4
0	0.0005	-0.0012	0.0003	0.0006
1	-0.0014	0.0023	-0.0013	-0.0006
2	0.0017	0.0003	-0.0012	0.0017
3	0.0007	-0.0009	-0.0007	-0.0007
4	0.0007	-0.0006	-0.0009	-0.0004

Table 7 Coordinate values of four measured parts

	Part 1	Part 2	Part 3	Part 4
E.S.E.	0.0073	0.0095	0.0040	0.0067
	0.0006	-0.0016	0.0007	0.0001
	-0.0016	0.0021	-0.0007	-0.0009
	0.0013	0.0003	-0.0004	0.0016
	1.0000E-5	-0.0007	0.0002	-0.0006
	-0.0002	-0.0002	0.0002	-0.0001

Note: E.S.E.= Estimated Straightness Error
Real Straightness Error = 0.006

Table 8 Estimated Straightness Errors and Converted data

Case 3] Specification is changed to 0.012.

We use the same data as case 1 except that the specification is changed to 0.012*mm*. We can say that the specification is too loose with 95% confidence because the estimated process capability ratio value 2.5 is too high relative to the minimum recommended value 2.05 in Table 4 with n=20 and C_p=1.3.

$$\hat{C}_p = \frac{0.012}{6*0.0008} = 2.5$$

Case 4] Specification is changed to 0.006.

We use the same data as case 1 except that the specification is changed to 0.006*mm*. Three parts are out-of-specification. Then, we first determine if the process is out-of-control. If it is not and there is no assignable cause of out-of-control, we can say that the specification is too tight even though we cannot detect it from the estimated process capability ratio comparison from Table 4 with n=20 and C_p=1.3. In this case, we should measure another set of data to make sure the determination.

$$\hat{C}_p = \frac{0.006}{6*0.0008} = 1.25 \geq 0.99$$

4. Conclusion

This chapter has given a procedure for determining the upper and lower limits for the estimated process capability ratio, $\hat{C}_p$, when the recommended minimum C_p is given. When $\hat{C}_p$ is used to determine whether the process is capable or not, one sided limits (C_{pL}, C_{pU}) are used. When $\hat{C}_p$ is used to analyze the given tolerance, two sided limits (C_{pTL}, C_{pTU}) are used. Depending on the purpose of the process capability ratio, comparison between $\hat{C}_p$ and the upper and lower limits affects the determination. These determinations are summarized below.

1) The process is capable

   when $\hat{C}_p > C_{pL}$.

2) The process standard deviation is shifted by more than Δs_1 and less than Δs_2.

   when $C_{pU}(\Delta s_1) < \hat{C}_p < C_{pU}(\Delta s_2)$.

3) The given specification is not appropriate

   when $\hat{C}_p > C_{pTU}$ or $\hat{C}_p < C_{pTL}$.

   If $\hat{C}_p > C_{pTU}$

   then, the given specification (tolerance) is too loose.

   If $\hat{C}_p < C_{pTL}$

   then, the given specification is too tight.

The limits of the process capability ratio can be used for general process capability analysis and, also, for geometrical tolerance analysis problems.

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References

Chang, S. H., G. Herrin and S. M. Wu [1990], "Sample Size Planning and Estimation of Form Errors using a Coordinate Measuring Machine", *Department of Industrial & Operations Engineering Technical Report 90-35*, University of Michigan, Ann Arbor.

Chou, Y. M., D. B. Owen and S. Borrego [1990], "Lower Confidence Limits on Process Capability Indices", *Journal of Quality Technology*, Vol. 22, No. 3, pp. 223-229.

Duncan, J. D. [1986], *Quality Control and Industrial Statistics*, 5th Ed. Irwin, Homewood, Ill.

Kane, V. E. [1986], "Process Capability Indices", *Journal of Quality Technology*, Vol. 18, No. 1, pp. 41-52.

Montgomery, D. C. [1985], *Introduction to Statistical Quality Control*, John Wiley & Sons, New York, NY.

Sullivan, L. P. [1984], "Reducing Variability: A New Approach to Quality", *Quality Progress* , 17, pp. 15-21.

Sullivan, L. P. [1985], "Letters", *Quality Progress* , 18, pp. 7-8.