SAMPLE SIZE PLANNING AND ESTIMATION OF FORM ERRORS USING A COORDINATE MEASURING MACHINE

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ABSTRACT

In this paper we have developed methods to statistically evaluate form errors using a coordinate measuring machine (CMM). The definitions of form errors in the current standards assume ideal inspection systems. However, there is no such ideal inspection systems actuality. Therefore, we establish practical mathematical definitions of form errors which can be applied for continuous or discrete measurements. They consider the characteristics of manufactured surfaces by assuming that the deviations from the nominal surface follow a Normal distribution. Importantly, these definitions are verified by measuring the real parts. Therefore, these definitions can serve as practical guideline for the inspection of real systems.

In current CMM practice, there are no commonly accepted sample sizes for estimating form errors which have a statistical confidence. Practically, sample size planning is important for the geometric tolerance inspection using a CMM. We determine and validate appropriate sample sizes for form error estimation. Also, we develop form error estimation methods with certain confidence levels based on the obtained sample sizes in various form errors: straightness, flatness, circularity, and cylindricity. The determination of sample sizes use the new approach which is based on the maximum expectation of the straight prediction interval at a certain confidence level. The straight prediction interval is a new development which covers the variations of manufactured surfaces. This approach for estimating form errors, based on the proposed sample sizes, is superior to the current practice because it leads to better measurements

approximation. The proposed sample sizes and estimating method are verified by a simulation study and real part measurements. Furthermore, it considers the characteristics and functionality of manufactured parts.

1. Introduction

High precision manufacturing, and thus increased new and improved high precision processes and machines, is in great demand today. This increasing demand is caused by the need: 1) to eliminate fitting problems and to promote assembly, especially in automatic assembly; 2) to improve interchangeability of components; 3) to improve quality control through higher machine accuracy capability which, in turn, will reduce scrap, rework and conventional inspection; and 4) to achieve further advances in technology. Many advanced (precision) technology products depend entirely on one or more components being manufactured to tolerances or dimensions in the micro- or even nanotechnology range.

One of the tolerances considered in precision manufacturing is the form tolerance. Form tolerance is confirmed by evaluation of form errors. Evaluation and confirmation of form errors are executed by a computer controlled Coordinate Measuring Machine (CMM). It is one of the most widely used tools. Automobile companies alone are estimated to have over 300 CMMs. Generally CMMs are used for discrete measurements. Most CMMs use unique software programs, programs developed by their manufacturer, and, as a result, give different assessment of tolerances. These variations are due to the discrete measurements and mathematical definitions of tolerances built into the programs [Placek(1989) and Weill(1988)]. The tolerance we are dealing with in this paper is form errors.

Formal definitions of form errors are given in current standards (ISO 1101 and ANSI Y14.5). Form errors are the linear distance between two parallel geometrical curves or surfaces. These surfaces contain all the elements of the manufactured object surface. The American National Standards Institute in conjunction with the American Society of Mechanical Engineering offers the following definition of form error: "the error of form is considered as being that deviation from the nominal surface which is not included in surface texture (ANSI/ASME B46.1-1985)". These definitions assume perfect (continuous) measurements. However, continuous measurements are impossible as we can never measure actual maximum (peak) and minimum (valley) points, the points which theoretically contain all the elements of the surface. Therefore, this definition is limited to ideal measurements and does not lead to mathematical definitions for discrete measurements.

To compensate for this limitation in the standards, it is current practice to estimate form errors as the sum of the algebraic maximum and minimum deviations from discrete measurements. These deviations are obtained from estimated surfaces. These surfaces are estimated by the various methods [ElMaraghy, Wu and ElMaraghy (1989), Shunmugam(1987, 1986), Fukuda and Shimokobe (1984), Murthy (1982), Murthy and Abdin(1980), Kakino and Kitazawa (1978), Gota and Lizuka (1977)]. These estimated surfaces vary or change depending on the number of discrete (sample) measurements. Consequently, the estimated form errors also vary.

The evaluation of form errors (e.g. straightness, flatness, circularity, and cylindericity) using a coordinate measuring machine (CMM) relies on discrete measurements. However, definitions of form errors in the current standards (ISO 1101, ANSI Y14.5) assume perfect (continuous) measurements, not discrete measurements. Therefore, there is no commonly accepted method for calculating form errors using discrete measurements; it is current practice to satisfy the definitions of the standards using discretely measured points. However, current practice does not consider the uncertainty of manufactured surfaces. As a result, it is not possible to give statistical confidence to the estimated form errors or to suggest statistically reliable minimum sample points. At the same time, the number of measured points needed to be large enough to provide reliable results.

Theoretically, the minimum number of points to calculate form errors are straight forward. As an example, a minimum of three points are necessary to get a straightness error. Two points are used to estimate a straight line and one point is used to get the information about the uncertainty of the estimated straight line. If all three measured points lie on the perfect straight line, then there is no straightness error because the third point does not give any information. If they are not on a straight line, the third point gives information about the straightness error. However, there are no surfaces or curves whose uncertainty information can be explained by one point. Therefore, the theoretical minimum number of three points are not enough to obtain information about form errors. Additional measurements are needed to get statistically reliable information. By establishing a statistically reliable minimum, the manufacturer does not have to measure an inordinate number of points.

In order to overcome the problem of inconsistency or change, we propose new definitions of form errors which can be used for discrete measurements. Also, these new definitions can be applied to the continuous measurements. In other words, these definitions have the ability to represent the continuous measurements by the discrete measurements.

One of the methods used to approximate continuous events by discrete events is probability distribution. It has often been assumed that there is a Normal (Gaussian) distribution [Greenwood and Williamson (1966)] of the deviations from the general surface shape of all manufactured parts. However, variations [Weckenmann and Heinrichowski (1985), Bourdet, Clement and Weill (1984)] in manufactured surface shape characteristics are significant due to various types of manufacturing processes. We take these effects into account with probability distribution. Then, we classify form errors into two cases depending on the surface shape characteristics; without systematic variation and with systematic variation. Form error without systematic variation occurs when the manufactured surface shape is the same as the specified shape. Here, form error depends on the deviations from the desired surface shape. Form error with systematic variation is the case that when manufactured surface shape is different from the specified shape. Here form error is more affected by the varied shape than by the deviations from the varied shape. In both cases, deviations are assumed to follow normal distribution.

2. Proposed Definition without Systematic Variation

If it is assumed that manufacturing process is noisy, then the deviations of the product surface from the nominal (designed) product surface can be expected to follow a normal distribution. A nominal surface is the intended surface contour which is usually shown and dimensioned on a drawing or descriptive specification. Theoretically, normal distributions have no finite minimum or maximum values. However, such large values are not found among the deviations. Practically, most of the data values lie within $\pm 3\sigma$ (standard deviation) of the nominal surface and these are the range from $+3\sigma$ to -3σ, called the range of natural tolerance limits. If we define a form error as 6σ, then we can say that this range contains all the elements of a manufactured surface from a practical point of view. Therefore, this range satisfies the definition of the standard (ISO 1101). Accordingly we propose the following definition which can be applied to any kind of form error.

> **Form error is 6σ or the range of natural tolerance limits when the deviations from a nominal surface follow a Normal distribution.**

Because we usually do not know the exact value of the standard deviation, we estimate the standard deviation from a sample. When the sample size is large enough to be considered a continuous measurement, we can use the estimated standard deviation to obtain form error. The estimated standard deviation is, however, a random variable. When the sample size is small, the estimated form error could vary depending on sample size and the form error can be overestimated. We will consider that problem in the later section.

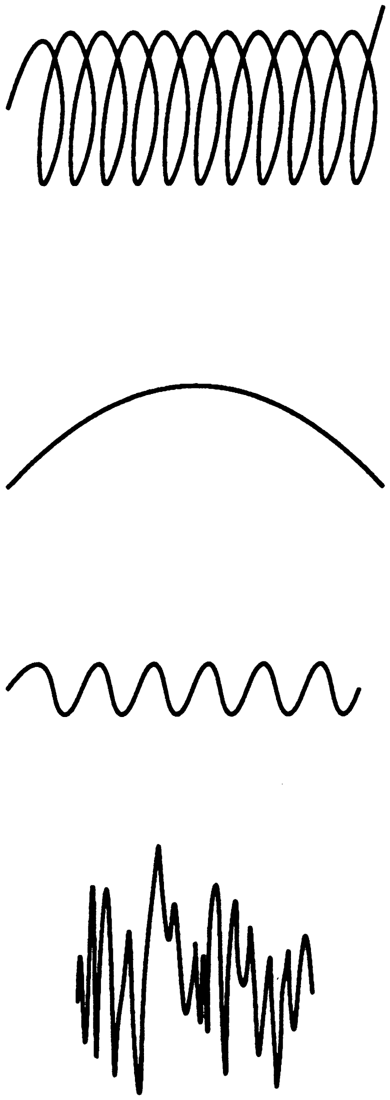
3. Proposed Definition with Systematic Variation

Because the characteristics of manufactured surface vary due to types and noises of manufacturing processes, we need additional definitions of form error to identify those surfaces. Surface characteristics are influenced by clamping setup, residual stresses and tool wear. Even though various surface characteristics have been described by Weckenmann and Heinrichowski [1985], and Bourdet, Clement and Weill [1984], and are shown in Fig. 1, we consider the second order polynomial curve for straightness and the special second order surface for flatness. These are combined with Gaussian case in this section.

When the second order polynomial represents the straight manufactured surface, we can define the straightness error in a new way while still satisfying the definition given in ISO .

When the surface shape is perfectly fit to the second order polynomial, the distance, between the line which passes through two end points and another line which is parallel to the previous line and which is tangential to the second order polynomial, is defined as a straightness error (See Fig. 2).

However, no surface can be perfectly fit to the second order polynomial function because of manufacturing noises. Therefore, the straightness error is estimated by the variation of the shape of second order polynomial model (See Fig. 3).



(a) Spiral

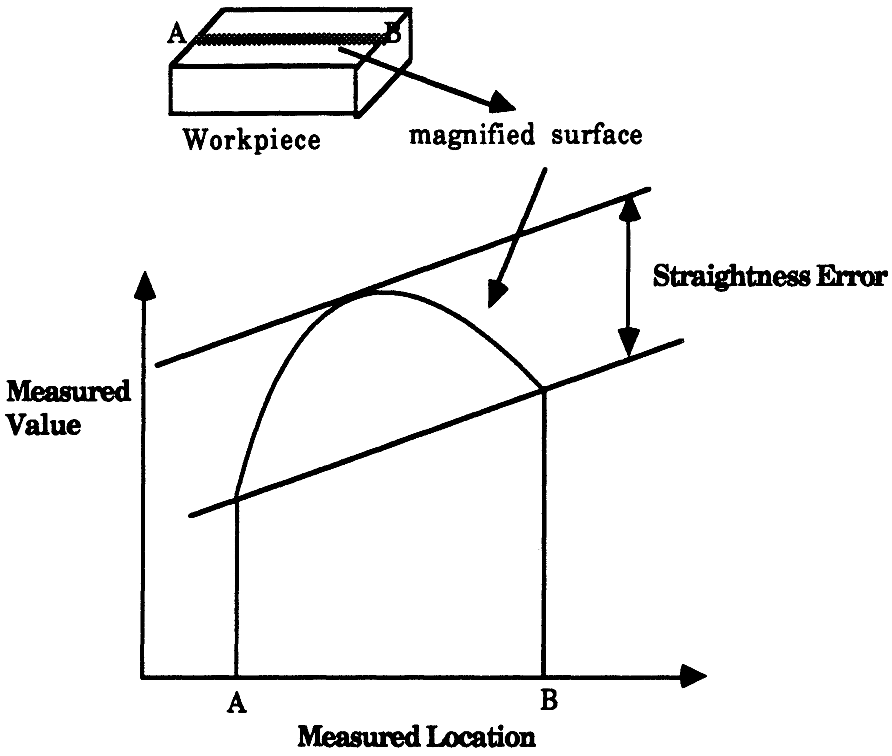
(b) Convex or Concave

(c) Sinusoidal

(d) Gaussian

Note:Each of (a), (b) and (c) can be combined with case (d)

Figure 1 Possible Surface Characteristics



A,B : Two end points of measured workpiece

Figure 2 Straightness Error in Second Order Polynomial Fitting

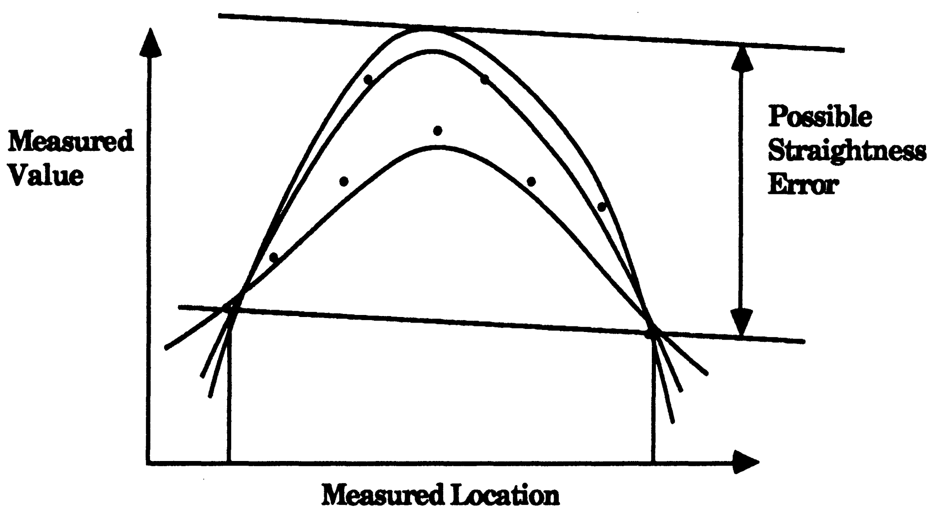


Figure 3 Possible Straightness error in Second Order Polynomial Fitting

The new definition of a straightness error for the second order polynomial can be described in a mathematical expression because the two end points are decided by a specified straightness error given in the specification. A specified straightness error could be for a whole length (total straightness) or a partial length (straightness per unit length). Let the curve shape be a second order polynomial form, then its real function, its estimated function and its two assessment lines are

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \varepsilon \qquad \varepsilon \sim N(0,\sigma^2)$$
$$y = b_0 + b_1 x + b_2 x^2$$
$$y_1 = a_0 + a_1 x$$
$$y_2 = c_0 + a_1 x \tag{1}$$

respectively. x_{max} and x_{min} are two end points of the given straightness range. Then two expected end points will be

$$(x_{min}, b_0 + b_1 x_{min} + b_2 x_{min}^2), (x_{max}, b_0 + b_1 x_{max} + b_2 x_{max}^2) .$$

The slope of two assessment lines, a_1, can be calculated

$$a_1 = \frac{b_2 x_{max}^2 + b_1 x_{max} - b_2 x_{min}^2 - b_1 x_{min}}{x_{max} - x_{min}}$$
$$= b_1 + b_2 (x_{max} + x_{min}) . \tag{2}$$

When $y_1 = a_0 + a_1 x$ is passing through two expected end points, a_0 can be calculated as of confidence based on the confidence interval of the true coefficient γ. $(1 - \alpha)*100$ per cent confidence interval of coefficient γ is

$$C - t(1-\alpha/2; n-4)\, s(C) \leq \gamma \leq C + t(1-\alpha/2; n-4)\, s(C)$$

where n: number of measurements

s(C): estimated standard deviation of γ

$1-\alpha$: confidence level.

We can calculate the possible flatness error using the upper or lower bound value depending on the surface characteristic (lower bound for concave form and upper bound for convex form). The appropriate sample size problem will be discussed in the later section.

4. Various Surface Shapes in Real Parts

We conducted real parts measurement using a CMM to observe the surface shapes and the deviations. This experiment was performed using the Sheffield Cordax RS-30 DCC CMM at the CMM Lab in the University of Michigan. A 165*mm* long bar (Fig. 4), a 200*mm* long bar (Fig. 5), and a 60x30*mm* rectangular bar (Fig. 6) were measured every 1*mm*. Also a 100*mm* long cylinderical bar was measured along the cylinder axis every 1*mm* (Fig. 7). From these measurements we can say that the 165*mm* bar and 100*mm* cylinderical bar surfaces satisfy the condition . The condition is that the deviations from the nominal surface follow the normal distribution. The 200*mm* bar and the 60x30*mm* bar surface satisfy the condition. The condition is that the deviations from the characteristic surface shape follow the normal distribution. The characteristic surface shape depends on the manufacturing process characteristics. Therefore, we can say that our proposed definitions are appropriate for real parts.

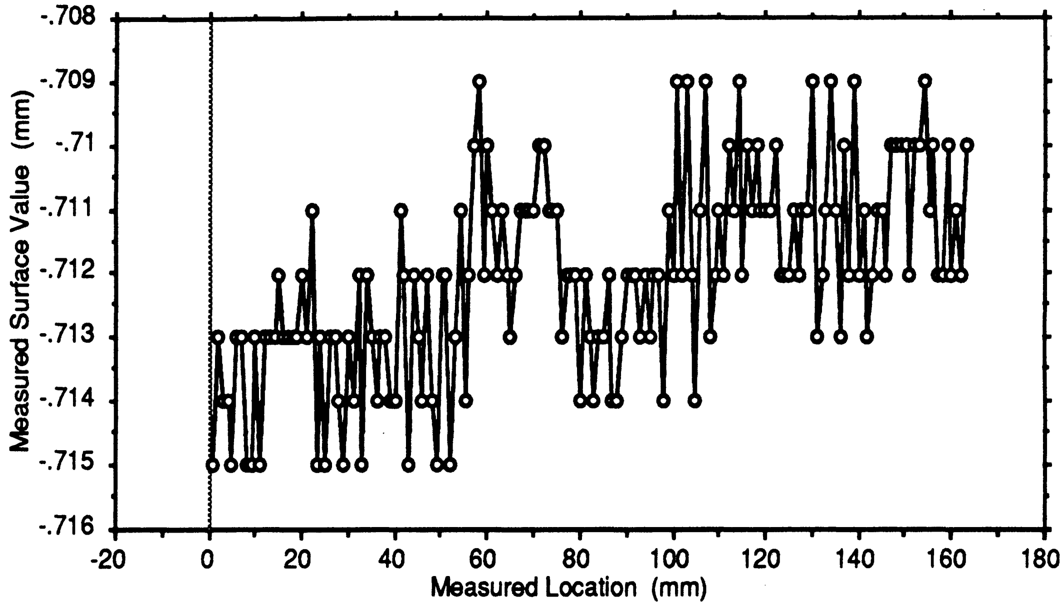


Figure 4 Surface Measurements of 165*mm* long Bar

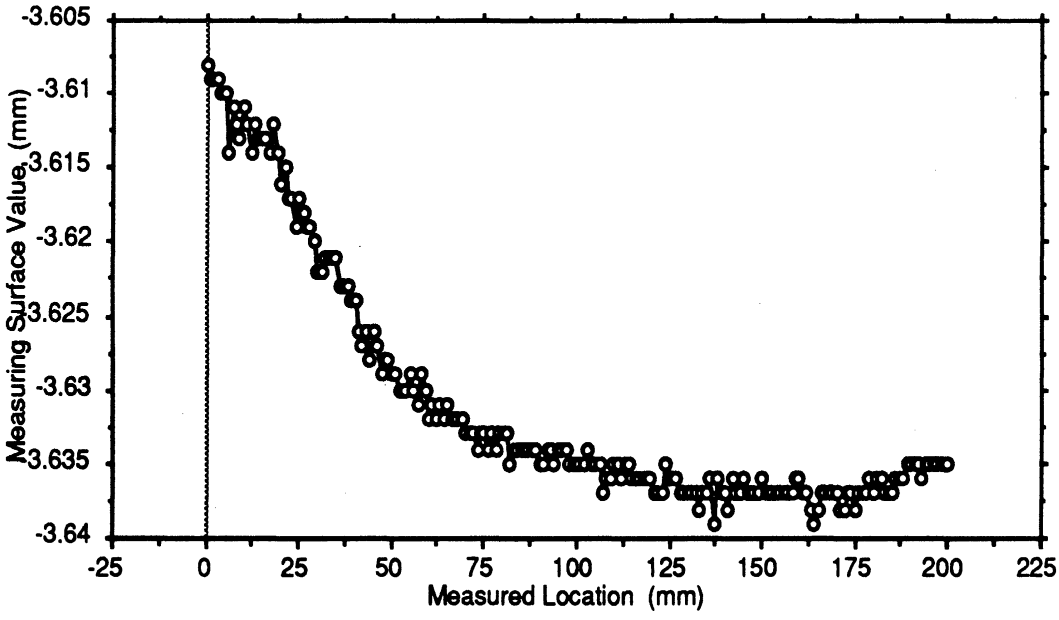


Figure 5 Surface Measurements of 200*mm* long Bar

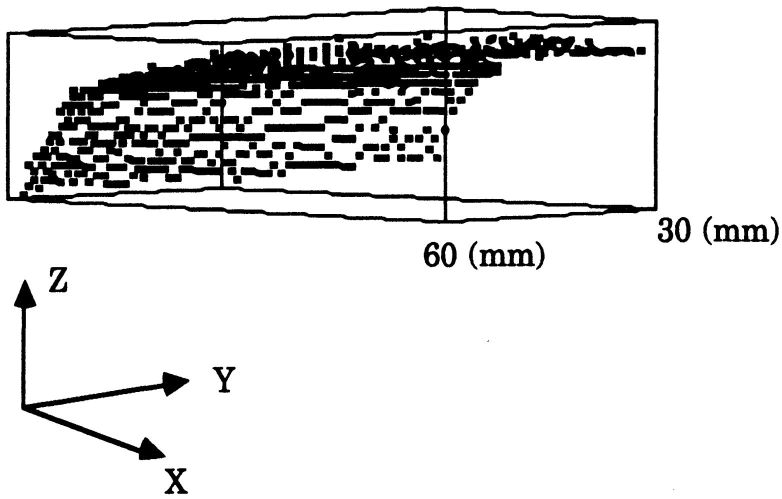


Figure 6 Surface Measurements of 60x30 mm area

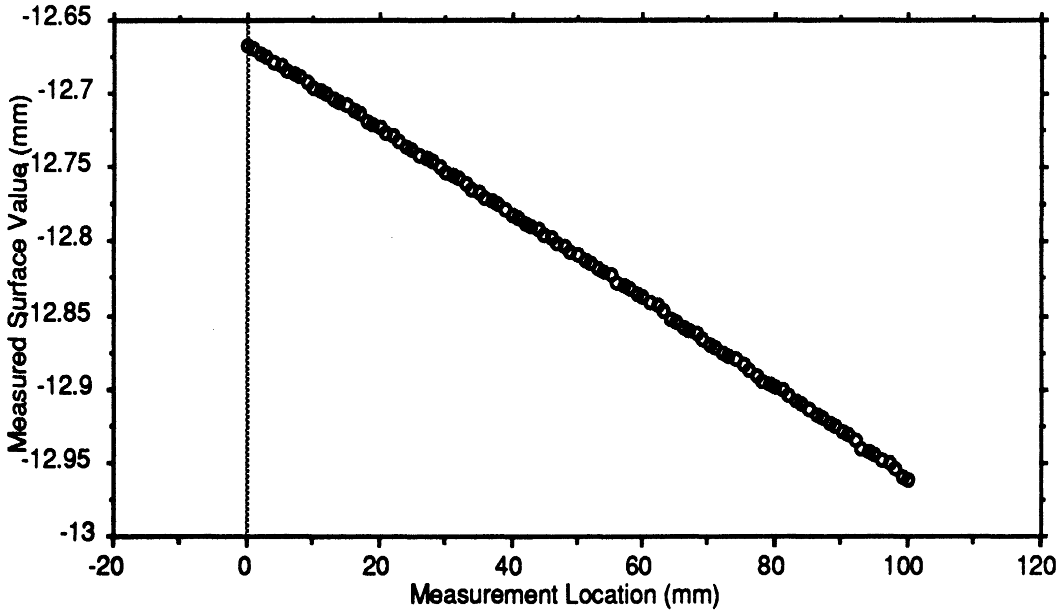


Figure 7 Surface Measurements of 100*mm* long Cylinderical Bar along its axis

5. Proposed Approach

We define form error as 6σ or the function of the surface shape parameter in previous sections. We usually do not know the real values of a standard deviation or a surface shape parameter. They are estimated from a estimated nominal surface. The nominal surface is estimated from sample measurements. It is estimated because we do not know the exact location of real nominal surface (Fig. 8). Therefore, there are two variations in estimating form error: 1) variation in possible location of the nominal surface, and 2) variation within the estimation of standard deviation or surface shape parameter which involves probability distribution. We use the linear regression method to estimate the nominal surface. To use the linear regression method, we make assumptions based on the fact that machining processes are always disturbed by various noises which are independent of the form of the surface. Hence the cumulative effect of these noises is subject to the central limit theorem and is governed by a Normal distribution [Greenwood and Williamson (1966)]. Under these assumptions, we can make basic assertions that involve probability distributions.

Let our manufactured surface be represented by the functional form

$$Z_i = f(X_i, Y_i) + \varepsilon_i$$

where $f(X_i, Y_i)$: function of manufactured surface

$f(X_i, Y_i) = \beta_0 + \beta_1 x_i$ (for simple straightness case)

ε_i : combined noise

1. ε_i is a normal random variable with mean zero and variance σ^2 (unknown), that is,

$$\varepsilon_i \sim N(0, \sigma^2),\ E(\varepsilon_i)=0,\ V(\varepsilon_i)=\sigma^2.$$

2. ε_i and ε_j are uncorrelated, $i \neq j$, so that $Cov(\varepsilon_i, \varepsilon_j) = 0$ and Z_i and Z_j, $i \neq j$, are uncorrelated. Thus

$$E(Z_i) = f(X_i, Y_i), \qquad V(Z_i) = \sigma^2$$

Based on these assumptions, each observation comes from a normal distribution centered vertically at the level implied by the proposed model. The variance of each normal distribution is assumed to be the same.

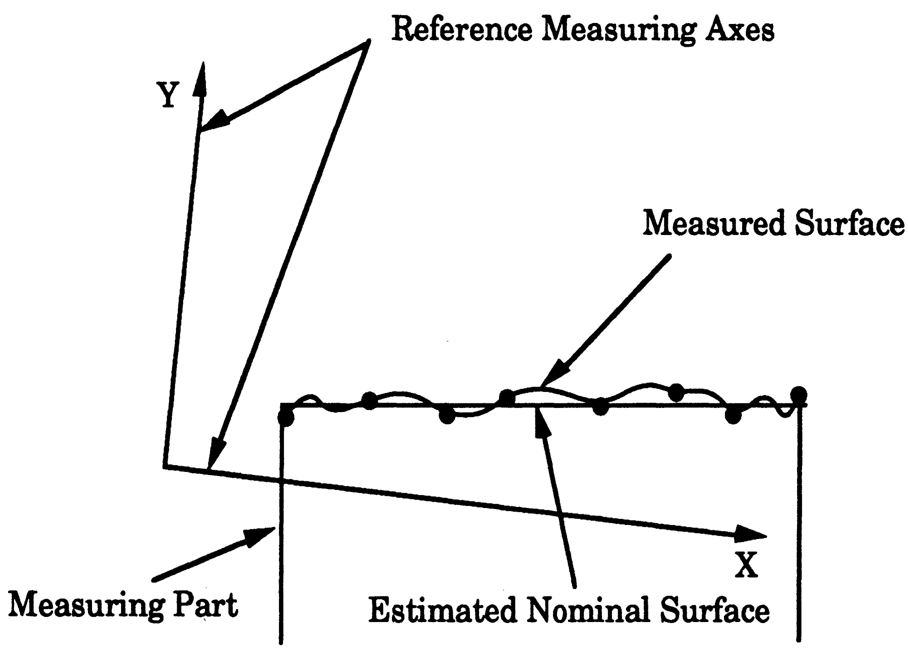


Figure 8 Reference measuring axes in a CMM and estimated nominal surface

We can simply use these estimated standard deviation or surface shape parameter values to obtain form errors without considering statistical confidence. We will not know much about the variations of the real nominal surface and the probability of parameter estimations. Therefore, we use a prediction interval length that considers the variation of real nominal surface and the variation of parameter estimation. The prediction interval length (PI) can be represented by the function of the sample size, a specified point and an estimated standard deviation in general linear regression analysis.

$$PI = 2 * t(n\text{-}p, 1\text{-}\alpha) * \{1 + f(n, P_0)\} * \sqrt{MSE}$$

where $t(n\text{-}p, 1\text{-}\alpha)$: upper $(1\text{-}\alpha)$ percentage point of t-distribution with (n-p) degrees of freedom

$f(n, P_0)$: function of sample size *n* and a specified point P_0

MSE: estimated variance

When a certain PI, with given sample size *n*, confidence level $(1\text{-}\alpha)$ and MSE, is approximately equal to 6σ, we can say that it is an estimated form error. The bands of the PI, however, are curvilinear and our objective is to find linear bands which cover the maximum variations of the nominal surface and its estimation. The maximum PI is chosen at a given sample size (Fig. 9).

However, the interval estimate of the PI is a random variable because the sample standard deviation (or $\sqrt{MSE}$) is a random variable. The expected prediction interval is compared to 6σ. We can say that PI at that sample size is an estimation of form error which has no systematic error when the expected length of the maximum prediction interval, at a certain sample size with a certain confidence level, is approximately equal to 6σ. The upper or lower confidence limits of the surface shape parameter at that sample size can be used to estimate form error. This has a systematic error with the same confidence level because the confidence interval of the surface shape parameter is narrower than the PI with the same sample size.

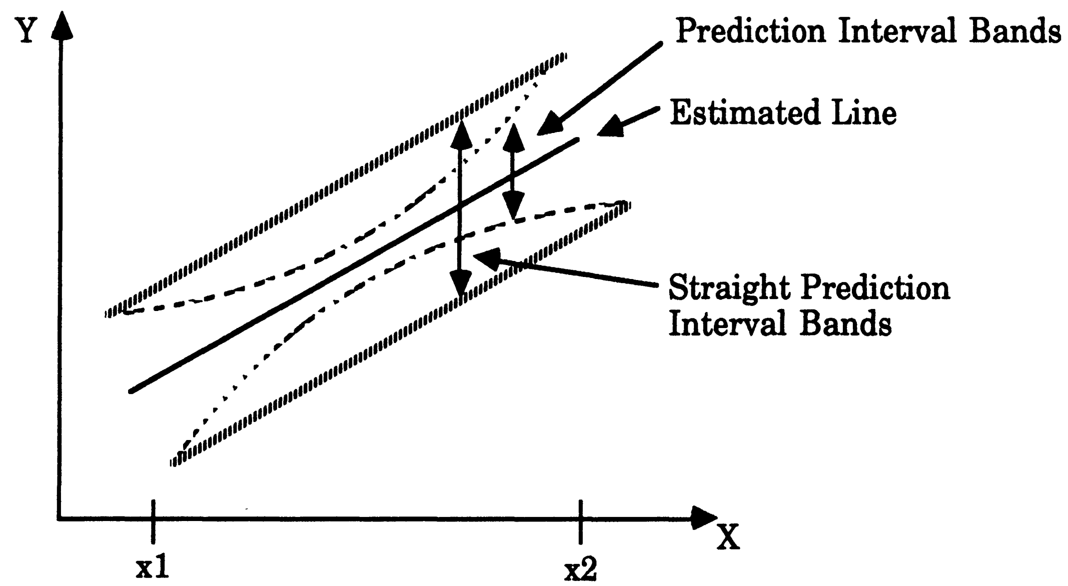


Figure 9 Illustration of Maximum Straight Prediction Interval Bands

The statements above can be represented in mathematical terms as follow;

$$PI(\mathbf{P_0}) = 2 * t_{n-p,1-\alpha/2} \{ 1 + \mathbf{P_0}' (\mathbf{P}'\mathbf{P})^{-1} \mathbf{P_0} \}^{1/2} \sqrt{MSE} \quad (12)$$

where $PI(\mathbf{P_0})$: length of prediction interval at $\mathbf{P_0}$

$\mathbf{P_0}$: specified column vector of $\mathbf{P}$

$\mathbf{P}$: observation design matrix

MSE : estimated variance of least square residuals

p: # of parameters estimated

α: confidence level.

Since the residual error variance (MSE) follows a Chi-square distribution

$$\frac{(n-p)MSE}{\sigma^2} \sim \chi^2_{n-p}$$

the expected length of the prediction interval (E[PI]) can be represented as follows

$$E[PI] = 2\, t_{n-p,1-\alpha/2}\, h(n,\mathbf{P_0}) \frac{\sigma}{\sqrt{n-p}} E[\chi_{n-p}] \qquad (13)$$

where $h(n,\mathbf{P_0}) = \{ 1 + \mathbf{P_0}' (\mathbf{P'P})^{-1} \mathbf{P_0} \}^{1/2}$

$$E[\chi_{n-p}] = \frac{\Gamma[(n-p+1)/2]}{\Gamma[(n-p)/2]} \sqrt{2} \text{ (See Appendix A)}$$

$$\Gamma(n) = \text{Gamma function} = \int_0^\infty x^{n-1} e^{-x}\, dx.$$

If $E[PI] = 6\sigma$, we can determine the appropriate sample size needed to estimate the form error when the confidence level is given. Or, we can determine the appropriate confidence level when the sample size is given which satisfies

$$t_{n-p,1-\alpha/2}\, h(n,\mathbf{P}_0)\, \frac{E[\chi_{n-p}]}{\sqrt{n-p}} = 3. \qquad (14)$$

In this section we are only considering sample size determination when the confidence level is given for straightness, flatness, circularity, and cylindericity errors. The proposed new approach for determining the appropriate sample size and for estimating form errors not only satisfies the proposed definitions but also accounts for the possible variations in the estimating procedure.

6. Sample Size Determination

In this section, we explain the procedure for determining sample size using the prediction interval approach for various functional forms. Observation matrix **P** is constructed with the assumption that each measurement is equi-distance in every dimension.

6.1 Simple Straight Line Function

The general simple straight line regression function from sample size n is represented by

$$Y = b_0 + b_1X.$$

In the simple straight line case, the maximum value of $\mathbf{P_0}'(\mathbf{P'P})^{-1}\mathbf{P_0}$ can be obtained at one of two end points. As an example, we have $(\frac{1}{3} + 0.5)$ when n=3 as follows:

$$\mathbf{P} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad (\mathbf{P'P})^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix} \qquad \mathbf{P_0} = (1,\ -1) \text{ or } (1,\ 1)$$

$$\mathbf{P_0}'(\mathbf{P'P})^{-1}\mathbf{P_0} = (\frac{1}{3} + 0.5).$$

In the same way we can get the maximum value of $\mathbf{P_0}'(\mathbf{P'P})^{-1}\mathbf{P_0}$ for different sample sizes. Then, we can find the appropriate sample size which satisfies the following condition:

$$\{\, 2\, t_{n-2,1-\alpha/2}\, (\mathbf{P_0}'(\mathbf{P'P})^{-1}\mathbf{P_0})^{1/2} \frac{E[\chi_{n-p}]}{\sqrt{n-2}} \,\} \approx 6 \qquad (15)$$

where $1-\alpha$: confidence coefficient of prediction interval.

The appropriate sample sizes with 95% (α=0.05) and 99% (α=0.01) confidence for the simple straight line function are 7 and 24, and are shown in Tables 1 and 2 respectively. Numerical values for $E[\chi_{n-p}]$ are in Appendix B.

Sample Size	$t_{n-2,1-0.05/2}$	$E[\chi_{n-2}]/\sqrt{n-2}$	$1+X_0'(X'X)^{-1}X_0$	$E[PI]/\sigma$
3	12.706	.7979	$1+\frac{1}{3}+0.50$	27.454
4	4.303	.8862	$1+\frac{1}{4}+0.45$	9.944
5	3.182	.9213	$1+\frac{1}{5}+0.40$	7.416
6	2.776	.9400	$1+\frac{1}{6}+0.36$	6.448
7	2.571	.9513	$1+\frac{1}{7}+0.32$	5.916
8	2.447	.9594	$1+\frac{1}{8}+0.29$	5.585

Table 1 Sample Size for Simple Straight Line Function with α=0.05

Sample Size	$t_{n-2,1-0.01/2}$	$E[\chi_{n-2}]/\sqrt{n-2}$	$1+X_0'(X'X)^{-1}X_0$	$E[PI]/\sigma$
22	2.845	.9876	$1+\frac{1}{22}+0.12$	6.066
23	2.831	.9882	$1+\frac{1}{23}+0.12$	6.030
24	2.819	.9887	$1+\frac{1}{24}+0.12$	6.008
25	2.807	.9892	$1+\frac{1}{25}+0.11$	5.955

Table 3.2 Sample Size for Simple Straight Line Function with α=0.01

6.2 Second Order Polynomial Curve Function

The general second order polynomial regression function from sample size *n* is represented by

$$Y = b_0 + b_1X + b_2X^2.$$

As an example, the maximum value of $\mathbf{P_0}'(\mathbf{P'P})^{-1}\mathbf{P_0}$ will be $(\frac{1}{4} + 0.7)$ when n=4 as follows:

$$\mathbf{P} = \begin{bmatrix} b & X & X^2 \\ 1 & 1 & 1 \\ 1 & 1/3 & 1/9 \\ 1 & -1/3 & 1/9 \\ 1 & -1 & 1 \end{bmatrix} \qquad (\mathbf{P'P})^{-1} = \begin{bmatrix} 41/64 & 0 & -45/64 \\ 0 & 9/20 & 0 \\ -45/64 & 0 & 81/64 \end{bmatrix}$$

$$\mathbf{P_0}' = (1,\ 1,\ 1) \text{ or } (1,\ -1,\ 1)$$

$$\mathbf{P_0}'(\mathbf{P'P})^{-1}\mathbf{P_0} = (\frac{1}{4} + 0.7).$$

In the same way we can get the maximum value of $\mathbf{P_0}'(\mathbf{P'P})^{-1}\mathbf{P_0}$ for different sample sizes. The appropriate sample sizes with 95% and 99% confidence for the second order polynomial curve function are 9 and 36.

6.3 Simple Plane Function

The general simple plane regression function from sample size n is represented by

$$Z = b_0 + b_1X + b_2Y.$$

As an example, the maximum value of $\mathbf{P_0}'(\mathbf{P'P})^{-1}\mathbf{P_0}$ will be $(\frac{1}{4} + 0.5)$ when n=4 as follows:

$$\mathbf{P} = \begin{bmatrix} b & X & Y \\ 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \qquad (\mathbf{P'P})^{-1} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad \mathbf{P}_0' = (1,\ 1,\ 1) \text{ or } (1,\ -1,\ -1)$$

$$\mathbf{P}_0'(\mathbf{P'P})^{-1}\mathbf{P}_0 = (\frac{1}{4} + 0.5).$$

In the same way, we can get the maximum value of $\mathbf{P}_0'(\mathbf{P'P})^{-1}\mathbf{P}_0$ for different sample sizes. The result is similar to the simple straight line function except for the number of parameters to be estimated. The appropriate sample sizes with 95% and 99% confidence for the simple plane function are 8 and 25.

6.4 Second Order Surface Function

In this case we consider only the specific form of a surface

$$Z = b_0 + b_1X + b_2X^2 + b_3Y.$$

As an example, the maximum value of $\mathbf{P}_0'(\mathbf{P'P})^{-1}\mathbf{P}_0$ will be $(\frac{1}{5} + 0.8)$ when n=5 as follows:

$$\mathbf{P} = \begin{bmatrix} b & X & Y & X^2 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad (\mathbf{P'P})^{-1} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ -1 & 0 & 0 & 5/4 \end{bmatrix}$$

$$\mathbf{P_0}' = (1,\ 1,\ 1,\ 1) \text{ or } (1,\ -1,\ -1,\ 1)$$

$$\mathbf{P_0}'(\mathbf{P}'\mathbf{P})^{-1}\mathbf{P_0} = (\frac{1}{5} + 0.8).$$

In the same way, we can get the maximum value of $\mathbf{P_0}'(\mathbf{P}'\mathbf{P})^{-1}\mathbf{P_0}$ for different sample sizes. The result is similar to the simple straight line function except for the number of parameters. The appropriate sample sizes with 95% and 99% confidence for the simple plane function are 9 and 36.

6.5 Circular Function

The linearized deviation (Fig. 10) is used [Shunmugam (1986)] to estimate the circle from *n* observations which are represented by polar coordinates (r_i, θ_i):

$$e_i = r_i - (R_0 + x_0 Cos\theta_i + y_0 Sin\theta_i) \tag{16}$$

where R_0 = radius of the estimated circle

x_0, y_0 = coordinates of origin of the estimated circle.

Then, the desired regression function can be written as follows:

$$r_i = R_0 + x_0 Cos\theta_i + y_0 Sin\theta_i \tag{17}$$

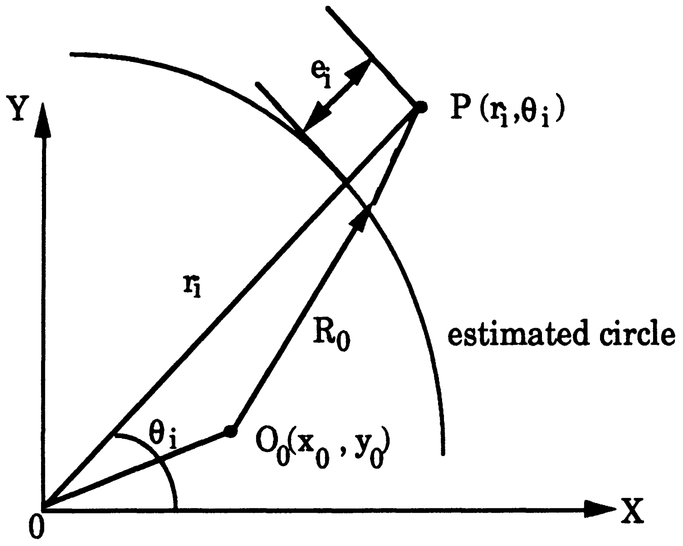


Figure 10 Linearized Deviation from Circle

and if $Y = r_i$, $b_0 = R_0$, $b_1 = x_0$, $b_2 = y_0$, $X_1 = \text{Cos}\theta_i$ and $X_2 = \text{Sin}\theta_i$, then

$$Y = b_0 + b_1X_1 + b_2X_2. \tag{18}$$

Because Cosθ cannot be represented by the linear combination of Sinθ and there are three parameters to be estimated, Eq.(18) is exactly same as the simple plane function. Therefore, the appropriate sample sizes with 95% and 99% confidence for circular function are 8 and 25.

6.6 Cylindrical Function

The linearized deviation (Fig. 11) is used Shunmugam (1986)] to estimate the cylinder from *n* observations which are represented by cylindrical coordinates (r_i, θ_i, z_i):

$$e_i = r_i - [R_0 + (x_0 + l_0 z_i)\text{Cos}\theta_i + (y_0 + m_0 z_i)\text{Sin}\theta_i] \quad (19)$$

where R_0 = radius of estimated cylinder

x_0 = x coordinate of origin of estimated cylinder

y_0 = y coordinate of origin of estimated cylinder

l_0, m_0 = slopes of estimated cylinder axis.

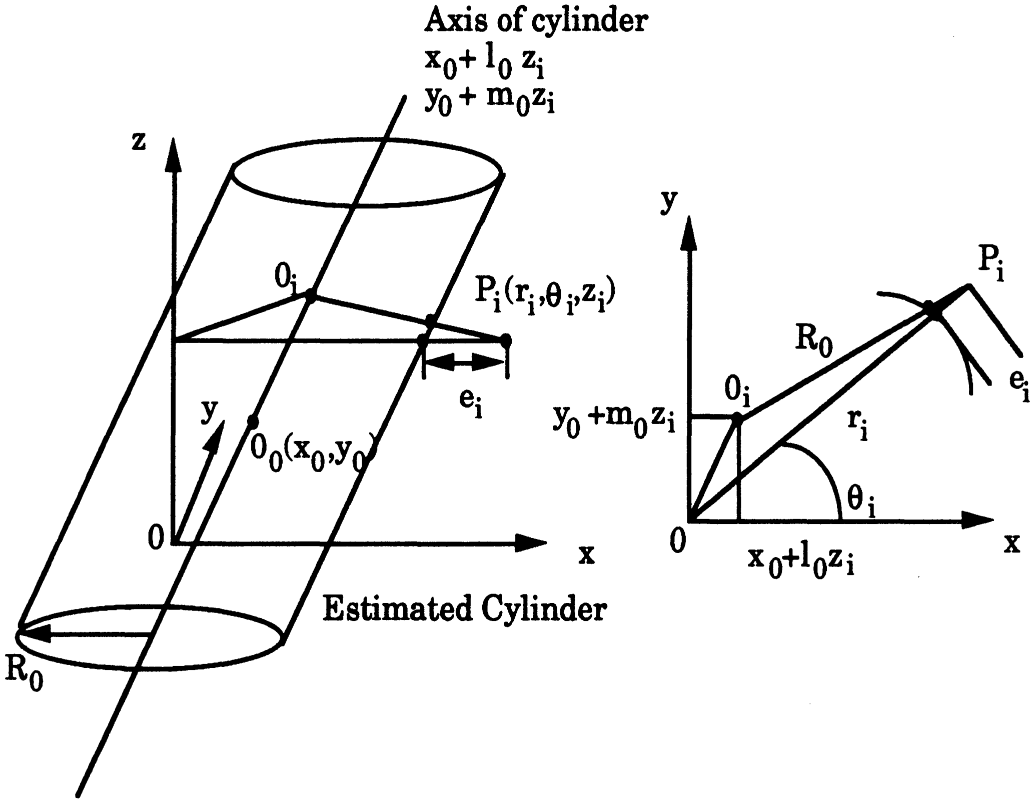


Figure 11 Linearized Deviation from Cylinder

Then, the desired regression function can be written as follows:

$$r_i = R_0 + x_0 \mathrm{Cos}\theta_i + y_0 \mathrm{Sin}\theta_i + l_0 z_i \mathrm{Cos}\theta_i + m_0 z_i \mathrm{Sin}\theta_i \qquad (20)$$

and if $Y = r_i$, $b_0 = R_0$, $b_1 = x_0$, $b_2 = y_0$, $b_3 = l_0$, $b_4 = m_0$, $X_1 = \mathrm{Cos}\theta_i$, $X_2 = \mathrm{Sin}\theta_i$, $X_3 = z_i\mathrm{Cos}\theta_i$ and $X_4 = z_i\mathrm{Sin}\theta_i$, then

$$Y = b_0 + b_1X_1 + b_2X_2 + b_3X_3 + b_4X_4. \qquad (21)$$

As an example, the maximum value of $\mathbf{P_0}'(\mathbf{P'P})^{-1}\mathbf{P_0}$ will be $(\frac{1}{6} + 0.82)$ when n=6 as follows:

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 \\ 1 & \frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{3}{10} & -\frac{3\sqrt{3}}{10} \\ 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{10} & -\frac{\sqrt{3}}{10} \\ 1 & -1 & 0 & -\frac{2}{10} & 0 \\ 1 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & -\frac{3}{10} & -\frac{3\sqrt{3}}{10} \\ 1 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{5}{10} & -\frac{5\sqrt{3}}{10} \end{bmatrix}$$

$$(\mathbf{P'P})^{-1} = \begin{bmatrix} \frac{3}{2} & \frac{2}{3} & -\frac{2\sqrt{3}}{9} & -\frac{5}{3} & \frac{5\sqrt{3}}{3} \\ \frac{2}{3} & \frac{71}{105} & -\frac{32\sqrt{3}}{315} & \frac{19}{21} & \frac{17\sqrt{3}}{21} \\ -\frac{2\sqrt{3}}{9} & -\frac{32\sqrt{3}}{315} & \frac{149}{315} & -\frac{13\sqrt{3}}{63} & -\frac{19}{21} \\ \frac{5}{3} & \frac{19}{21} & -\frac{13\sqrt{3}}{63} & -\frac{55}{21} & \frac{40\sqrt{3}}{21} \\ \frac{5\sqrt{3}}{3} & \frac{17\sqrt{3}}{21} & -\frac{19}{21} & \frac{40\sqrt{3}}{21} & \frac{45}{7} \end{bmatrix}$$

$$\mathbf{P_0}' = (1,\ 1,\ 0,\ -1,\ 0) \text{ or } (1,\ \frac{1}{2},\ -\frac{\sqrt{3}}{2},\ \frac{5}{10},\ -\frac{5\sqrt{3}}{10})$$

$$\mathbf{P_0}'(\mathbf{P'P})^{-1}\mathbf{P_0} = (\frac{1}{6} + 0.82).$$

In the same way we can get the maximum value of $\mathbf{P_0}'(\mathbf{P'P})^{-1}\mathbf{P_0}$ for different sample sizes. The appropriate sample sizes with 95% and 99% confidence for the cylindrical function are 13 and 55.

7. Testing the Aptness of the Proposed Sample Sizes

Appropriate sample sizes were obtained based on the expected length of the prediction interval. These sample sizes require a minimum number at a certain confidence level. Estimated form errors are the length of the maximum prediction interval at the same confidence level. To verify the aptness of this proposal, a simulation study was conducted and real parts measurements were made.

7.1 Simulation

In the previous section, we obtained the appropriate sample sizes for various functions with two different confidence levels. To test the aptness of these sample sizes, we conducted a simulation study for the simple straight line function. We expect that similar results will be obtained in other functions.

We generated 1000 normal random number sets, each of which consists of 1000 numbers, with a mean of zero and a standard deviation of

0.001, to test the aptness of the sample sizes. We assume that each data set represents all the surface elements of a measured part. By observing each data set, we can say that straightness error of each set is 0.006 (6σ).

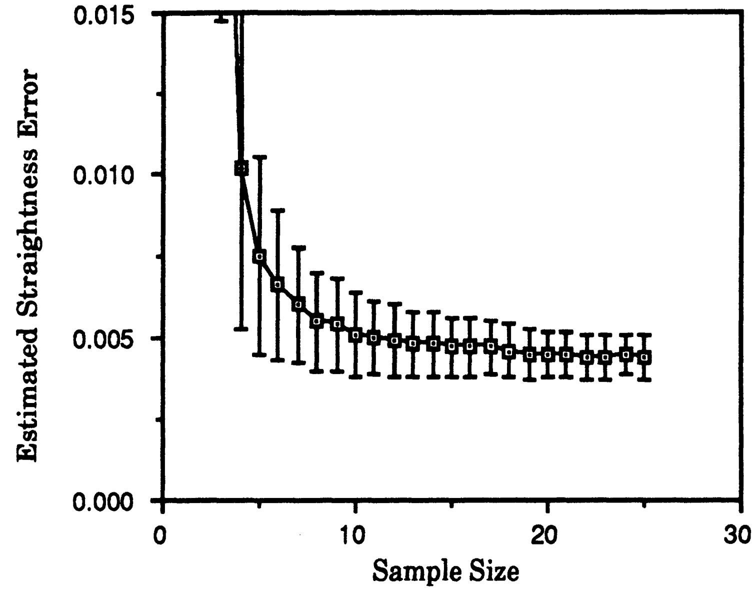
We collected sample sizes from 3 through 25 numbers from each set with equi-intervals. In other words, we assigned the number 1 to 1000 for each number in each data set assuming that each number represented a serial measurement location. As an example, with sample size 7, we collected values which have assigned numbers of 1, 167, 333, 500, 667, 833 and 1000. Next, we estimated a straight line using each sample based on the least squares estimation. From each straight line estimation, we collected square roots of the mean square error (MSE) values and calculated estimated straightness error, and calculates mean and standard deviation of those values (Table 3 and Fig. 12). The constants multiplied by $\sqrt{\text{MSE}}$ to estimate straightness error are different depending on the sample size and the confidence level. Exact values of these constants are obtained using Eq.(13). Even though the constant values at sample size 7 and 24 are not exactly 6 (Table 4), the estimated straightness errors are not significantly different from that of multiplied by 6 because the $\sqrt{\text{MSE}}$ value is very small in practical form error measurements. The same mean value of 0.006 was the estimated straightness error of the sample sizes 7 and 24 for 95% and 99% confidence level, respectively.

Based on these results, we can say that our sample sizes were appropriate for each confidence level. Also, the mean value of the estimated straightness error was calculated by simply multiplying 6 by $\sqrt{\text{MSE}}$ for the practical purpose (Table 5). We can observe that our proposed definition is appropriate. This means that if we do not have to worry about the confidence level, then simply estimated straightness error can be a reasonable estimation. We can determine the confidence level at each sample size using Eq.(14). However, we did not calculate these confidence levels because our objective is to verify the sample size at a certain confidence level (95% and 99%).

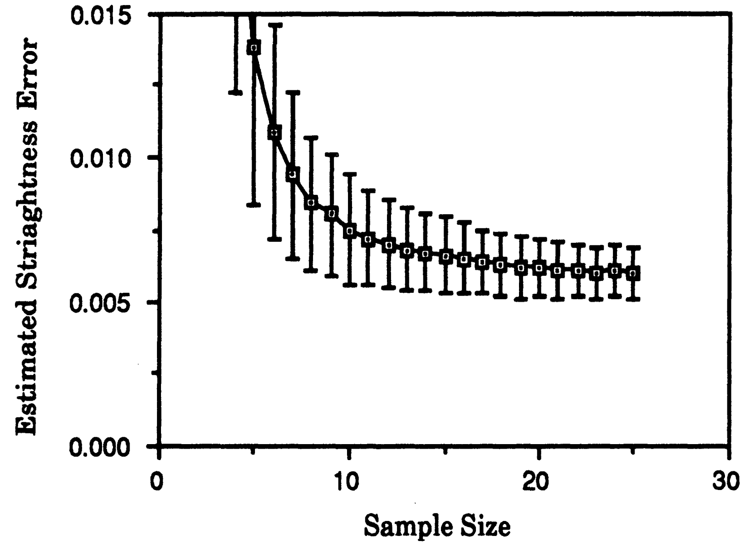
Sample Size	Mean of 95% Confidence SE	Std.of 95% Confiednce SE	Mean of 99% Confidence SE	Std. of 95% Confiednce SE
3	0.0337	0.0190	0.1686	0.0953
4	0.0102	0.0049	0.0236	0.0113
5	0.0075	0.0030	0.0138	0.0055
6	0.0066	0.0023	0.0109	0.0037
7	0.0060	0.0018	0.0094	0.0029
8	0.0055	0.0015	0.0084	0.0023
9	0.0054	0.0014	0.0080	0.0021
10	0.0051	0.0013	0.0075	0.0019
11	0.0050	0.0011	0.0072	0.0016
12	0.0049	0.0011	0.0070	0.0015
13	0.0048	0.0010	0.0068	0.0014
14	0.0048	0.0010	0.0067	0.0013
15	0.0047	0.0009	0.0066	0.0013
16	0.0047	0.0009	0.0065	0.0012
17	0.0047	0.0008	0.0064	0.0011
18	0.0046	0.0008	0.0063	0.0011
19	0.0045	0.0008	0.0062	0.0011
20	0.0045	0.0007	0.0062	0.0010
21	0.0045	0.0007	0.0061	0.0010
22	0.0044	0.0007	0.0061	0.0009
23	0.0044	0.0007	0.0060	0.0009
24	0.0044	0.0006	0.0060	0.0009
25	0.0044	0.0007	0.0060	0.0009

Note: SE: Straightness Error Std.: Standard Deviation Real SE = 0.006

Table 3 Mean and Standard Deviation of estimated straightness error with 95% and 99% confidence for different sample sizes



(a) 95% Confidence



(b) 99% Confidence

Note: Real Straightness Error = 0.006

Figure 12 Mean and Standard Deviation of Estimated Straightness Error with **(a)** 95% and **(b)** 99% confidence

Sample Size	Constant for 95% Confidence	Constant for 99% Confidence
3	34.408	172.384
4	11.221	25.881
5	8.049	14.777
6	6.860	11.378
7	6.219	9.753
8	5.821	8.819
9	5.559	8.224
10	5.359	7.796
11	5.200	7.471
12	5.068	7.208
13	4.974	7.019
14	4.895	6.862
15	4.824	6.726
16	4.773	6.610
17	4.706	6.507
18	4.655	6.414
19	4.608	6.329
20	4.584	6.279
21	4.543	6.209
22	4.504	6.142
23	4.487	6.102
24	4.471	6.077
25	4.439	6.020

Table 4 Constants multiplied by $\sqrt{\text{MSE}}$ for estimating Straightness Error

Sample Size	Straightness Error ($6*\sqrt{MSE}$)	Standard Deviation
3	.0059	.0033
4	.0055	.0026
5	.0056	.0022
6	.0058	.0020
7	.0058	.0018
8	.0057	.0016
9	.0058	.0016
10	.0058	.0014
11	.0058	.0013
12	.0058	.0013
13	.0058	.0012
14	.0058	.0012
15	.0059	.0012
16	.0059	.0011
17	.0059	.0010
18	.0059	.0010
19	.0059	.0010
20	.0059	.0009
21	.0059	.0009
22	.0059	.0009
23	.0059	.0009
24	.0060	.0009
25	.0060	.0009

Note: Real Straightness Error = 0.006

Table 5 Mean value and Standard Deviation of Simply Estimated Straightness Error without considering Confidence Level

7.2 Type I and II errors of the proposed approach

The purpose for estimating form error is to evaluate the conformance of a product to its tolerance. However, we cannot guarantee 100% assurance unless we measure all the surface points of the product. Therefore, we determined the minimum sample size with a specified confidence level. Now we want to see the tolerance conformity in terms of Type I and Type II errors.

We estimate a form error using the length of the prediction interval at a specific sample size. However, it is a random variable because of $\sqrt{MSE}$. This interval has its own confidence interval with a given specified confidence level (1-γ). When we assume that the manufacturing process is in-control, we are interested in determining the out-of-control state or the increment of variance (Type I error). A Type I error is (γ*100)% assuming that this specification is the upper confidence limit of the estimated form error with (1-γ) confidence. Therefore, when the specification is given, the Type I error can be obtained as follows:

$$\begin{aligned} &H_0 \text{ (Null Hypothesis):} && \sigma^2 \leq \sigma_0^2 \\ &H_a \text{ (Alternative):} && \sigma^2 > \sigma_0^2 \end{aligned} \qquad (22)$$

where σ^2: estimated variance

σ_0^2: desired (or tolerable) variance

$$PI(\mathbf{P_0}) = 2 * t_{n-p,1-\alpha/2} \{ 1 + \mathbf{P_0}' (\mathbf{P'P})^{-1} \mathbf{P_0} \}^{1/2} \sqrt{MSE}$$

and we already know $\dfrac{(n-p)MSE}{\sigma_0^2} \sim \chi^2_{n-p}$

we want to find the probability to reject the Null Hypothesis

$$Pr\left(\frac{(n\text{-}p)\,MSE}{\sigma_0^2} > \chi^2_{n\text{-}p,\gamma} \right) = \gamma$$

$$Pr\left(\sqrt{MSE} > \sigma_0 \sqrt{\frac{(\chi^2_{n\text{-}p,\gamma})}{n\text{-}p}} \right) = \gamma$$

$$Pr\left(PI > \frac{PI}{\sqrt{MSE}} \sigma_0 \sqrt{\frac{(\chi^2_{n\text{-}p,\gamma})}{n\text{-}p}} \right) = \gamma \tag{23}$$

Therefore, we can find Type I error γ by solving the equation when the specification is represented by the multiplication of the standard deviation ($C*\sigma_0$):

$$C = 2 * t_{n\text{-}p,1\text{-}\alpha/2} \{ 1 + \mathbf{P_0}' (\mathbf{P'P})^{-1} \mathbf{P_0} \}^{1/2} \sqrt{\frac{(\chi^2_{n\text{-}p,\gamma})}{n\text{-}p}} . \tag{24}$$

When we assume that the manufacturing process is out-of-control, we are interested in determining the in-control state. This decision is a Type II error. A Type II error can be obtained when the state of out-of-control is given. We assume that the state of out-of-control is the increment of variance by the multiplication of the standard deviation. It also can be expressed as follows:

$$\begin{aligned} &H_0 \text{ (Null Hypothesis):} && (K\sigma)^2 \geq (C\sigma_0)^2 \\ &H_a \text{ (Alternative):} && (K\sigma)^2 < (C\sigma_0)^2 \end{aligned} \tag{25}$$

where $(K\sigma)^2$: estimated variance

$C\sigma_0$: specification

K: out-of-control state

we want to find the probability to reject the Null Hypothesis

$$\Pr\left(\frac{(n-p)\,MSE}{(C\sigma_0)^2} < \chi^2_{n-p,1-\beta} \right) = 1-\beta$$

$$\Pr\left(\frac{\sqrt{MSE}}{K} < C\sigma_0 \sqrt{\frac{(\chi^2_{n-p,1-\beta})}{n-p}} \right) = 1-\beta$$

$$\Pr\left(PI \leq \frac{PI}{\sqrt{MSE}} \frac{C}{K} \sigma_0 \sqrt{\frac{(\chi^2_{n-p,1-\beta})}{n-p}} \right) = 1-\beta \qquad (26)$$

Then, we can obtain the Type II error β by solving the equation

$$\frac{C}{K} = 2 * t_{n-p,1-\alpha/2} \{ 1 + \mathbf{P_0}' (\mathbf{P'P})^{-1} \mathbf{P_0} \}^{1/2} \sqrt{\frac{(\chi^2_{n-p,1-\beta})}{n-p}} . \qquad (27)$$

In order to test the procedure for obtaining Type I and II errors, we generated 5000 normal random number sets with a mean of zero, and a standard deviation of 0.001. Each of them had 1000 numbers. We collected samples of size 7 and 24 with equi-intervals from each set for 95% and 99% confidence, respectively. Then, estimated a straightness error for each sample size.

For both sample sizes, we obtained a mean value of 0.006 which is the same as the estimated straightness error. However, their variation (Fig. 14) is different because of the difference of the sample sizes. When we assumed that the specification is 9σ or 0.009, the Type I error for sample size 7 was a little bit greater than 0.05 and for sample size 24 was much less than 0.005. It was obtained by following procedures.

$$C = 2 * t_{n-p,1-\alpha/2} \{ 1 + \mathbf{P_0}' (\mathbf{P'P})^{-1} \mathbf{P_0} \}^{1/2} \sqrt{\frac{(\chi^2_{n-p,\gamma})}{n-p}} \quad \text{from Eq.(3.13)}$$

where $C = 9, p = 2$

For n=7, α= 0.1

$$2 * t_{n-p,1-\alpha/2} \{ 1 + \mathbf{P_0}' (\mathbf{P'P})^{-1} \mathbf{P_0} \}^{1/2} = 6.22$$

For n= 24, α=0.02

$$2 * t_{n-p,1-\alpha/2} \{ 1 + \mathbf{P_0}' (\mathbf{P'P})^{-1} \mathbf{P_0} \}^{1/2} = 6.05$$

$\chi^2_{5,0.05} = 11.07$	$\chi^2_{22,0.005} = 42.80$
$\chi^2_{5,\gamma} = 10.47$	$\chi^2_{22,\gamma} = 48.69$
$\chi^2_{5,0.1} = 9.24$	$\chi^2_{5,0.01} = 40.29$

Then, γ is a little bit greater than 0.05 for n=7, α= 0.1

γ is much less than 0.005 for n= 24, α=0.02.

To test for the Type II error, we did the same procedure except for the standard deviation of 0.002. The mean value of the estimated straightness error was 0.012 as expected. Their distributions are shown in Fig. 15. We can obtain the Type II error for sample size 7 as a little bit greater than 0.1 and for sample size 24 as a little bit less than 0.05 because we increased the standard deviation two times. It can be also obtained by following procedure.

$$\frac{C}{K} = 2 * t_{n-p,1-\alpha/2} \{ 1 + \mathbf{P_0}' (\mathbf{P'P})^{-1} \mathbf{P_0} \}^{1/2} \sqrt{\frac{(\chi^2_{n-p,1-\beta})}{n-p}} \quad \text{from Eq.(3.16)}$$

where C=9, K=2, p=2

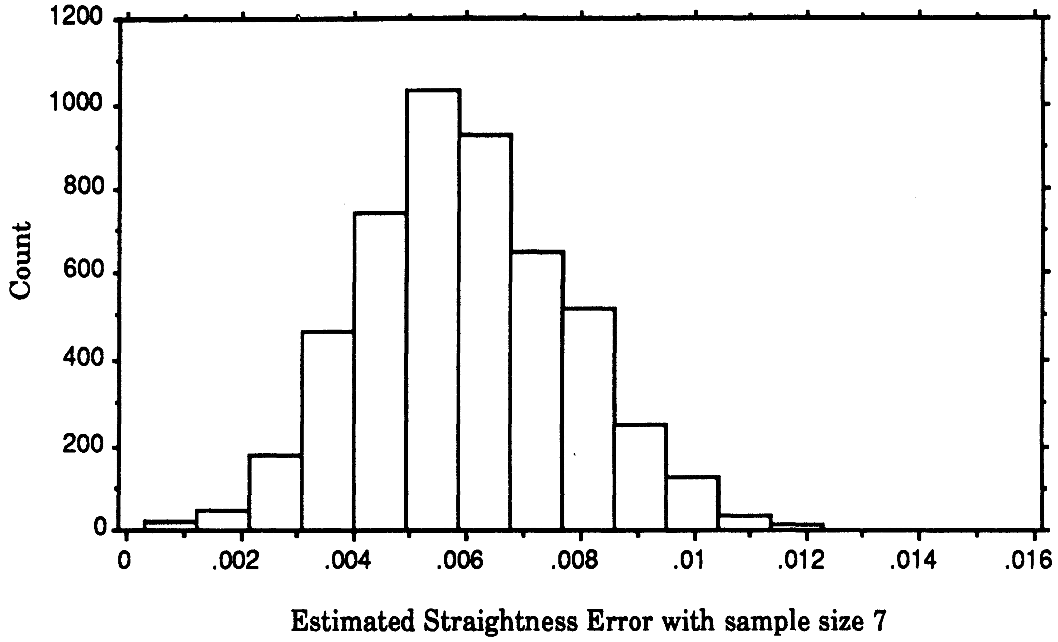
$\chi^2_{5,1-0.5} = 4.35$ $\chi^2_{22,1-0.05} = 12.34$

$\chi^2_{5,1-\beta} = 2.62$ $\chi^2_{22,1-\beta} = 12.17$

$\chi^2_{5,1-0.1} = 1.61$ $\chi^2_{5,1-0.025} = 10.98$

Then, β is a little bit greater than 0.1 for n=7, α= 0.1

β is a little bit less than 0.05 for n= 24, α=0.02.



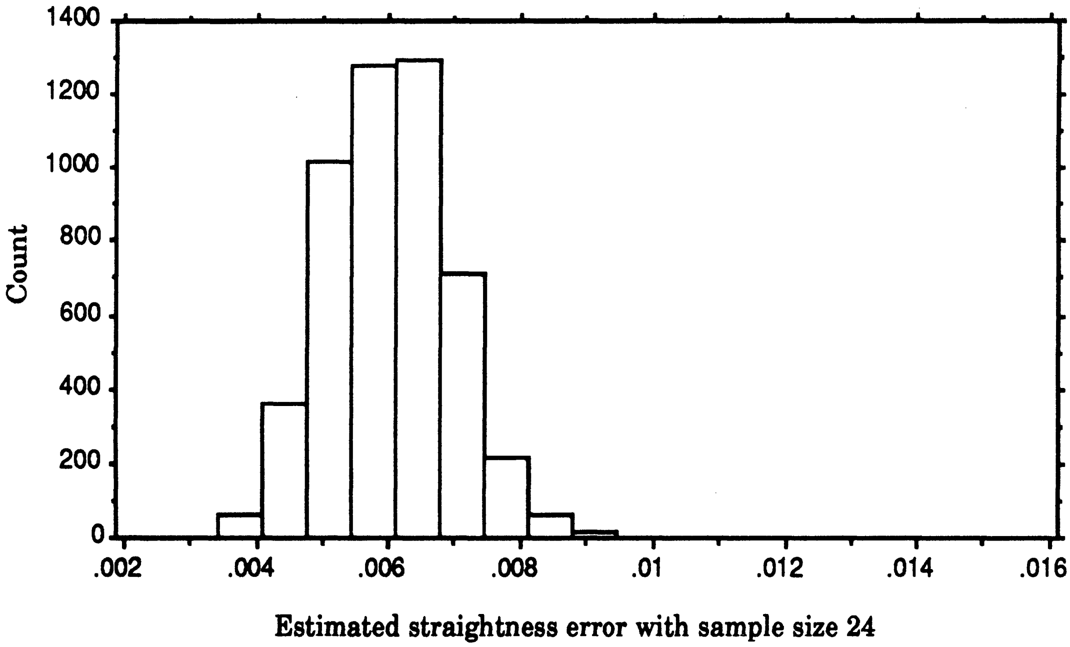
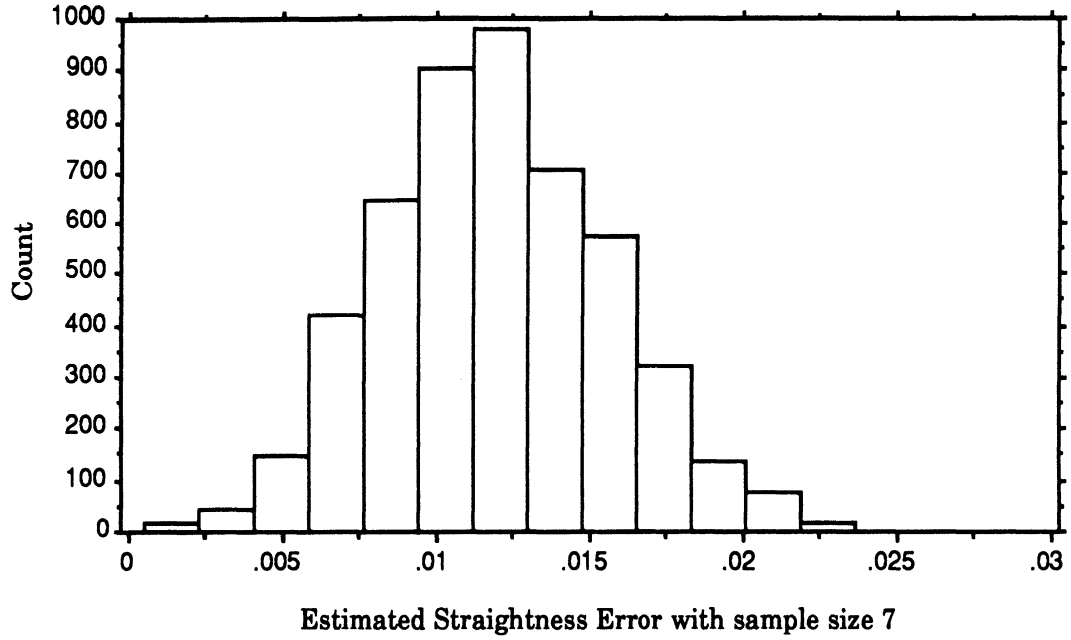


Figure 14 Distribution of Estimated Straightness Error



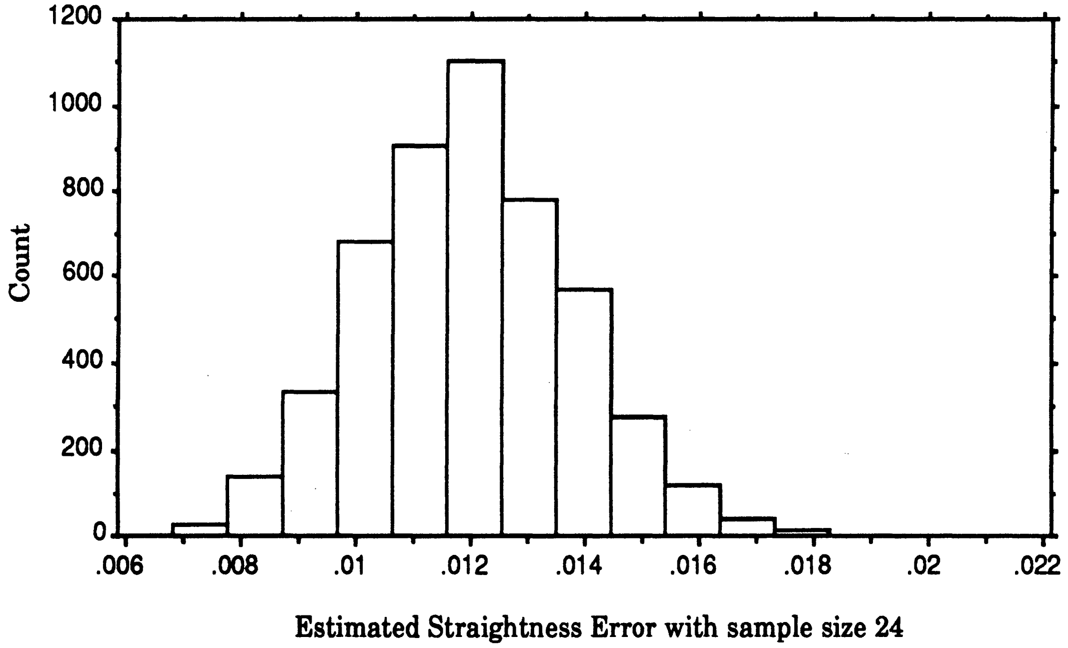


Figure 15 Distribution of Estimated Straightness Error when standard deviation is doubled

7.3 Experiments with Real Data

In addition to the simulation, we conducted real part measurements using the CMM to assess the straightness error. This experiment was performed using Sheffield Cordax RS-30 DCC CMM at the CMM Lab in the University of Michigan. A 165mm long rectangular bar was measured in increment of 1mm (Fig. 4). Based on these measurements we concluded that its straightness error is approximately 0.006mm. We collected sample sizes 3 to 25 from these data with almost equi-interval. For each sample size, we estimate 95% and 99% confidence straightness errors (Table 6). A 200mm long bar was measured in the same way (Fig. 5) and the estimated real straightness error was 0.017mm. Sample sizes 4 to 15 and 36, with almost equi-interval, were collected. These Estimated straightness errors, using Eq.(9), are shown in Table 7 and Figure 17.

The estimated straightness errors have a tendency to decrease when the sample size is increased even though there are some fluctuations (Figs. 16 and 17). We expected those fluctuations because our form error estimation approach uses $\sqrt{\text{MSE}}$, . Even though we did not get the exact value of a straightness error at the desired sample size, we got reasonably estimated values around the desired sample size.

Sample Size	95% Confidence (*mm*)	99% Confidence (*mm*)
3	0.014	0.070
4	0.007	0.016
5	0.008	0.015
6	0.009	0.015
7	0.005	0.007
8	0.006	0.008
9	0.005	0.007
10	0.005	0.007
11	0.008	0.011
12	0.005	0.008
13	0.006	0.008
14	0.004	0.005
15	0.008	0.011
16	0.005	0.007
17	0.007	0.010
18	0.006	0.009
19	0.005	0.007
20	0.005	0.007
21	0.007	0.010
22	0.005	0.007
23	0.005	0.006
24	0.007	0.009
25	0.005	0.006

Table 6 Estimated Straightness Error for 165*mm* long bar

Sample Size	95% Confidence (*mm*)	99% Confidence (*mm*)
4	0.050	0.191
5	0.026	0.041
6	0.018	0.021
7	0.019	0.022
8	0.019	0.021
9	0.019	0.021
10	0.019	0.021
11	0.017	0.018
12	0.020	0.022
13	0.017	0.018
14	0.018	0.019
15	0.017	0.019
36	0.015	0.016

Table 7 Estimated Straightness Error for 200*mm* long bar

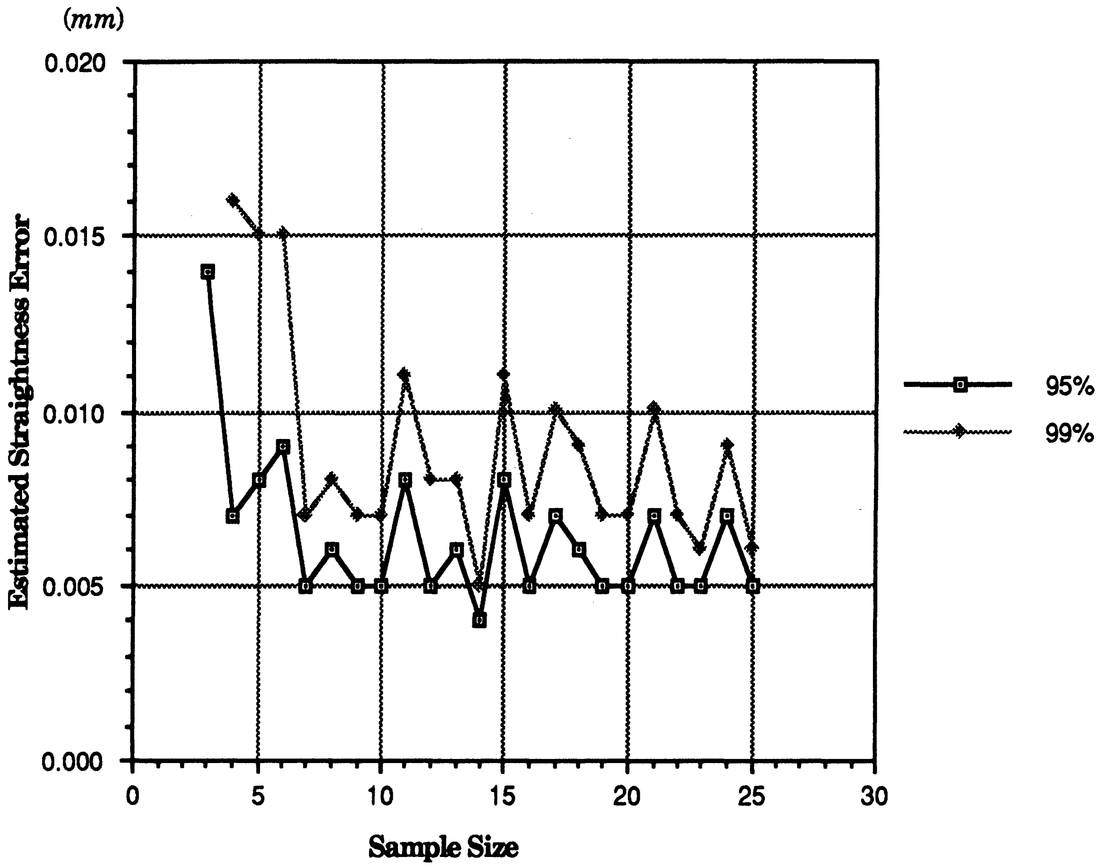


Figure 16 Variation of Estimated Straightness Error according to the sample size for 165*mm* long bar

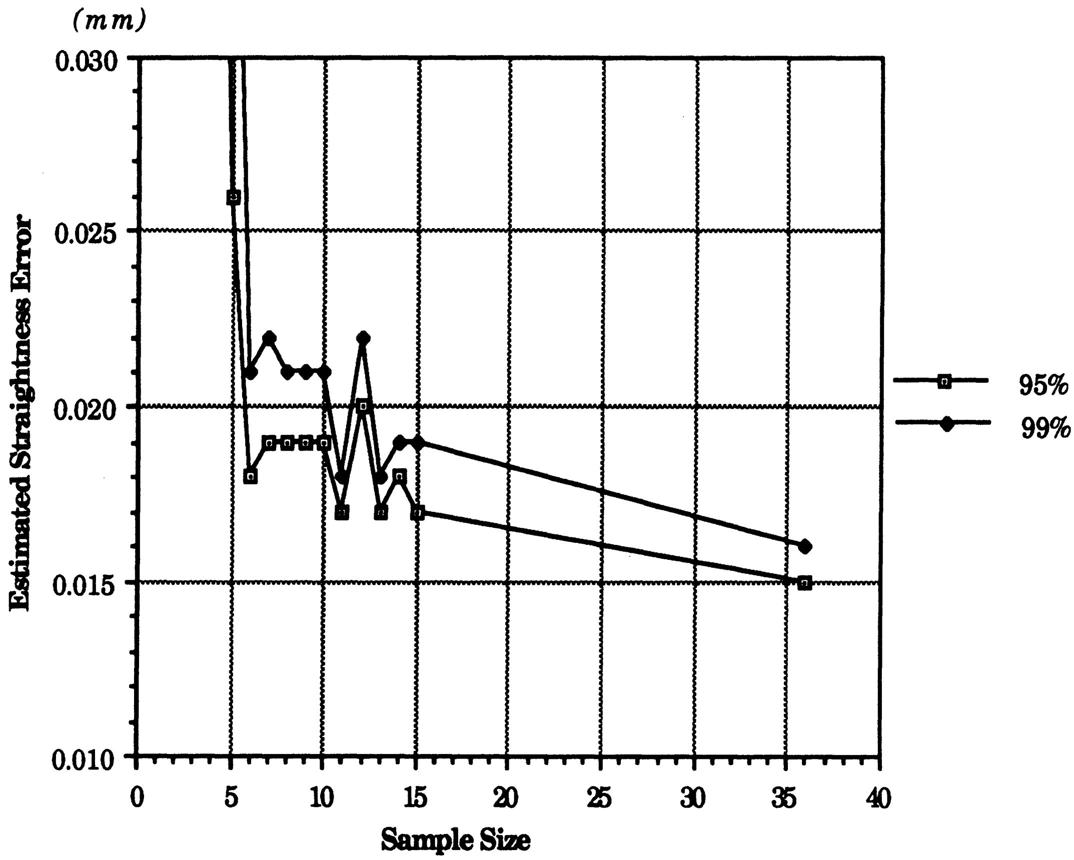


Figure 17 Variation of Estimated Straightness Error according to the sample size for 200*mm* long bar

8. Comparison to other Techniques using Simulation

We proposed an alternate approach for estimating form errors in previous sections. We can compare the results of this proposal with those of current approaches in order to test its aptness. The Least Squares (LS) and the Minimum Deviation (MD) approaches are used for comparison purpose. We only considered straightness for simplicity because other form error cases will give the similar results, .

We used normal random numbers to test the various approaches because it has often been assumed that there is a Gaussian distribution [Thomas (1982)] to all manufactured parts that are being measured. However, whatever approach is applied to estimate the straightness error, the estimated value is a random variable because it is estimated from the combination of random numbers. We compared the expected values as well as several example cases.

We generated 5 sets of 1000 normal random numbers with a mean of zero and a standard deviation of 0.001 to represent 5 different surfaces. From these numbers, we said that the real straightness error is approximately 0.006 for each surface. We collected samples of size 7 and 24 from each set. We collected 1st, 167th, 333rd, 500th, 667th, 833rd and 1000th numbers at equi-distances of sample size 7. The same procedure was applied to sample size 24.

By applying three different approaches -- prediction interval, least squares and minimum deviation -- we estimated straightness errors for each simulated surface. Actually we did not calculate the results of the minimum deviation approach because they are always less than or equal to those of the least squares approach. The estimated straightness errors of each approach are given in Table 8 for sample sizes 7 and 24.

Case	Real	PI Method		LS Method		MD Method	
Sample Size		7	24	7	24	7	24
Surface 1	0.006	0.005	0.006	0.002	0.004	≤ 0.002	≤ 0.004
Surface 2	0.006	0.004	0.008	0.002	0.005	≤ 0.002	≤ 0.005
Surface 3	0.006	0.007	0.005	0.003	0.003	≤ 0.003	≤ 0.003
Surface 4	0.006	0.005	0.006	0.002	0.005	≤ 0.002	≤ 0.005
Surface 5	0.006	0.004	0.006	0.002	0.004	≤ 0.002	≤ 0.004

where PI: Prediction Interval
LS: Least Squares
MD: Minimum Deviation

Table 8 Estimated Straightness Errors using Different Approaches

We compared the expected values of each approaches to show the generality even though the prediction interval approach gives results close to the real value in the cases above. The expected value of the prediction interval approach was given in previous section as follows:

$$E[PI] = 2\, t_{n-p,1-\alpha/2}\, h(n,\mathbf{P_0}) \frac{\sigma}{\sqrt{n-p}} w(n-p) \tag{28}$$

where $h(n,\mathbf{P_0}) = \{ 1 + \mathbf{P_0}' (\mathbf{P'P})^{-1} \mathbf{P_0} \}^{1/2}$

$$w(n\text{-}p) = E[\chi_{n-p}] = \frac{\Gamma[(n-p+1)/2]}{\Gamma[(n-p)/2]}\sqrt{2}$$

$$\Gamma(n) = \text{Gamma function}$$

$$= \int_0^{\infty} x^{n-1} e^{-x}\, dx$$

For the least squares approach we used the expected values of the range, $X_{(n)} - X_{(1)}$, in the order statistics [Sarhan and Greenberg (1962), Harter (1969)]. We did not use the residual because the difference between maximum and minimum residuals is always less than or equal to the difference between maximum and minimum values in certain sample size from order statistics. If we have a sample of *n* observations, $X_1, X_2, \ldots, X_n$, and rearrange them in ascending order of magnitude as

$$X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)},$$

we call $X_{(r)}$ is the *rth* order statistic. X_i are assumed to be statistically independent and identicaly distributred. The expected value of the *k*th largest observation, in a sample of size *n* from a standard normal population (μ=0, σ^2=1), is given by

$$E(X_{(k)}) = \frac{n!}{(n-k)!\,(k-1)!} \int_{-\infty}^{\infty} X\,[\,\frac{1}{2} - \Phi(X)]^{k-1}\,[\,\frac{1}{2} + \Phi(X)]^{n-k}\,\phi(X)\,dX \qquad (29)$$

where $$\phi(X) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{X^2}{2})$$

$$\Phi(X) = \int_0^X \phi(X)dX$$

Utilizing Eqs.(28) and (29), we obtained the expected values of the estimated straightness error for the different sample sizes as given in Table 9.

Case	Real	PI Method		LS Method		MD Method	
Sample Size		7	24	7	24	7	24
Straightness	0.006	0.006	0.006	0.003	0.004	≤ 0.003	≤ 0.004

* We assume that the population is a normal distribution with $\mu=0$ and $\sigma=0.001$.

Table 9 Expected Values of Estimated Straightness Error in different Approaches

9. Conclusion

This paper has presented two definitions for various form errors which can be operationalized and represented in mathematical terms even with discrete measurements. This chapter has also presented a procedure for determining the appropriate sample size and a formulation for evaluating form errors using the CMM.

These new definitions have the following characteristics.

1) They consider the characteristics of manufactured surfaces.

2) They can be represented in mathematical terms.

3) They can be used to determine appropriate sample sizes with a certain confidence level.
4) They were carefully tested by measuring the real surfaces.

The approach for sample size determination have the following charaterisitcs.

1) It determines the sample size with a new criterion which is applied to the expectation of prediction interval with various confidence levels (95% and 99%).
2) It can be used to determine the confidence level when the sample size is given.
3) It uses the least squares criterion to estimate the desired feature in functional form.
4) It can be used to calculate Type I and II errors when the specification is given because it is statistically well defined.

The results of testing and verifying of this new sample size determination approach are as follows.

1) It was carefully tested for determining the sample size for straightness, flatness, circularity and cylindericity.
2) The formulation was carefully tested for determining the straightness and flatness errors from simulated data and real measurement data.
3) Finally and most importantly, the results were tested and shown to be successful and satisfactory.

The approach proposed in this paper can provide a useful basis for further research for estimating form errors using the CMM. The formulations developed for straightness and flatness errors can be extended to a higher order of dimensional geometric tolerances. Consequently, the formulation can be established to estimate true geometric errors using the CMM.

Appendix

A. Expectation of χ_k

The probability density function of χ_k^2 distribution is

$$f(x) = \frac{1}{\Gamma(\frac{k}{2})} (\frac{1}{2})^{\frac{k}{2}} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}$$

The expectation of χ_k can be calculated by substituting $\sqrt{x}$ into x in applying the definition of expectation,

$$E[\sqrt{x}] = \int_0^\infty \sqrt{x} \frac{1}{\Gamma(\frac{k}{2})} (\frac{1}{2})^{\frac{k}{2}} x^{\frac{k}{2}-1} e^{-\frac{x}{2}} dx$$

$$= \int_0^\infty \frac{1}{\Gamma(\frac{k}{2})} (\frac{1}{2})^{\frac{k}{2}} x^{\frac{k}{2}-\frac{1}{2}} e^{-\frac{x}{2}} dx$$

$$= \int_0^\infty \frac{1}{\Gamma(\frac{k}{2})} (\frac{1}{2})^{\frac{k}{2}} x^{\frac{k+1}{2}-1} e^{-\frac{x}{2}} dx$$

$$= \int_0^{\infty} \frac{1}{\Gamma(\frac{k+1}{2})} (\frac{1}{2})^{\frac{k+1}{2}} x^{\frac{k+1}{2} - 1} e^{-\frac{x}{2}} \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})} (\frac{1}{2})^{-\frac{1}{2}} dx$$

$$= \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})} (\frac{1}{2})^{-\frac{1}{2}} \int_0^{\infty} \frac{1}{\Gamma(\frac{k+1}{2})} (\frac{1}{2})^{\frac{k+1}{2}} x^{\frac{k+1}{2} - 1} e^{-\frac{x}{2}} dx$$

$$= \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})} (\frac{1}{2})^{-\frac{1}{2}}$$

B. Table for Numerical Values of $E[\chi_{n-p}]/\sqrt{n-p}$

n-p	$E[\chi_{n-p}]/\sqrt{n-p}$
1	0.7979
2	0.8862
3	0.9213
4	0.9400
5	0.9513
6	0.9594
7	0.9650
8	0.9693
9	0.9727
10	0.9754
11	0.9776
12	0.9794
13	0.9810
14	0.9823
15	0.9835
16	0.9845
17	0.9854
18	0.9862
19	0.9869
20	0.9876
21	0.9882
22	0.9887
23	0.9892
24	0.9896
25	0.9901
26	0.9904
27	0.9908
28	0.9911
29	0.9914
30	0.9917

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