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NATURAL CONVECTION FROM A PLANE, VERTICAL

SURFACE IN NON-ISOTHERMAL SURROUNDINGS

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ABSTRACT

This paper presents a theoretical investigation of laminar natural convection from a plane, vertical surface in non-isothermal surroundings. Conditions are derived for the existence of similarity solutions. A method is proposed for generalizing the conditions pertaining to existing similarity solutions so as to include the effect of non-isothermal surroundings. Numerical solutions of the ordinary differential equations resulting from the similarity transformation are reported for the special case of an isothermal surface. These results suggest that some variations of surrounding temperature may lead to flow reversal in the boundary layer. Experimental evidence suggests that this may be an unstable condition.

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NOMENCLATURE

A, B, C, D	Constants
C_1, C_2, etc.	Constants
C_p	Specific heat capacity
f	dimensionless similarity stream function $(= \Psi / \varphi_2)$
g	Specific gravitational force
G	Dimensionless temperature $(= g\beta L^3(\theta - \theta_0)/\nu^2)$
k	Thermal conductivity
L	Characteristic length
m, n	Constant parameters
Nu	Nusselt number $(= \dot{Q}''L/k\Delta\theta)$
P	Pressure
Pr	Prandtl number $(= \mu C_p/k)$
$\dot{Q}''$	Heat-transfer rate per area
T	Absolute temperature
t	Time
t*	Dimensionless time $(= \nu t/L^2)$
u	Velocity in the x direction
v	Velocity in the y direction
x	Distance along the plate
X	Dimensionless distance along the plate $(= x/L)$
y	Distance normal to the plate
Y	Dimensionless distance normal to the plate $(= y/L)$

Greek Letters

β	Coefficient of cubical expansion
ϵ	Constant parameter
η	independent similarity variable ($= Y \varphi_1$)
θ	Temperature
μ	Viscosity
ν	Kinematic viscosity
ρ	Density
φ_1, φ_2	Unknown functions
Φ	Dimensionless similarity temperature function

$$[= (G - G_\infty)/(G_w - G_\infty)]$$

ψ	Stream function (u $= \partial \psi / \partial y$, v $= - \partial \psi / \partial x$)
Ψ	Dimensionless stream function ($= \psi / \nu$)
Ω	Body force term in boundary layer equation

Subscripts

w	Conditions on the surface of the heated plate
∞	Conditions outside the boundary layer
o	An arbitrary reference condition

Superscript

'	The prime is used to indicate differentiation with respect to the independent variable

NATURAL CONVECTION FROM A PLANE,
VERTICAL SURFACE IN NON-ISOTHERMAL SURROUNDINGS

Introduction

Following the work of Yang [1], it has been suggested that all the "exact" similarity solutions for the above problem have been explored. However, during a recent experimental investigation (Cheesewright [2]), it became apparent that existing solutions, which are all concerned with isothermal surroundings, did not provide a satisfactory description of the experimental phenomena.

A study of the problem revealed a new class of solutions for non-isothermal surroundings. The derivation of these solutions and the numerical results arising from them are considered in this paper. The form of temperature variation in the surroundings considered is that in which the temperature far away from the heated plate is a function of x, the distance from the leading edge.

The procedure adopted by Yang [1] for the determination of the similarity transformation is followed closely, and after detailed examination of two cases of special interest, a way of generalizing the results of Yang [1] to include cases of non-isothermal surroundings is proposed.

It is believed that the results for the special case of an isothermal surface in non-isothermal surroundings have important applications in the experimental study of transition

to turbulence in natural convection on a plane vertical surface. The results in general make possible the consideration of natural convection from an isolated surface in a cavity of limited extent, and may also facilitate the study of the boundary layer regimes in natural convection in closed cavities. A qualitative study of the effects of a vertical temperature gradient outside the boundary layer in the closed cavity problem has been made by Schwind and Vliet [3].

Analysis

With respect to the coordinate system, the equations of momentum, continuity and energy which govern the flow and heat-transfer in a laminar boundary layer in the presence of a body force are respectively

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \Omega + \nu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{k}{\rho C_p}\frac{\partial^2 \theta}{\partial y^2} \tag{3}$$

with the appropriate boundary conditions.

If consideration is restricted to a semi-infinite plate, having a temperature everywhere greater than that of its

surroundings, we can write $\Omega = -g$

Outside the boundary layer

$$U_\infty \frac{dU_\infty}{dx} + \frac{1}{\rho_\infty} \frac{dP}{dx} + g = 0 \tag{4}$$

(For pure natural convection $U_\infty = 0$)

Eliminating dP/dx between equations (1) and (4), we get

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \rho_\infty U_\infty \frac{dU_\infty}{dx} - g(\rho - \rho_\infty) + \mu \frac{\partial^2 u}{\partial y^2} \tag{5}$$

Following Ostrach [4], property variations are assumed to be important only in so far as they affect the body force term, and the density variation is represented by

$$\rho = \rho_0 \left[1 - \beta(\theta - \theta_0) \right]$$

Where ρ_0 is the density at an arbitrary reference temperature, θ_0 , and β = constant ($1/T_0$ for a perfect gas).

These considerations allow us to rewrite equation (5) as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U_\infty \frac{dU_\infty}{dx} + g\beta(\theta - \theta_\infty) + \nu \frac{\partial^2 u}{\partial y^2} \tag{6}$$

provided $(\theta - \theta_0) \ll T_0$

Equations (2), (3) and (6) govern the flow which we wish to investigate, subject to the following boundary conditions

at $y = 0$ $u = v = 0$ and $\theta = \theta_w(x,t)$

at $y = \infty$ $u = u_\infty(x,t)$ and $\theta = \theta_\infty(x,t)$

It may be noted that equation (6) is identical with the corresponding equation for the case $\theta_\infty =$ constant. The above derivation has been given because it is not felt that this identity is obvious.

Introducing the stream function ψ, [which satisfies equation (2)], equations (6) and (3) become

$$\frac{\partial^2 \psi}{\partial y \partial t} + \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = u_\infty \frac{du_\infty}{dx} + g\beta(\theta - \theta_\infty) + \nu \frac{\partial^3 \psi}{\partial y^3} \tag{7}$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} = \frac{k}{\rho C_p}\frac{\partial^2 \theta}{\partial y^2} \tag{8}$$

It is convenient to make these equations dimensionless by writing $X = x/L$, $Y = y/L$, $\overline{\psi} = \psi/\nu$, $G = \beta g L^3(\theta - \theta_o)/\nu^2$, $G_\infty = \beta g L^3(\theta_\infty - \theta_o)/\nu^2$ and $t^* = \nu t/L^2$ where L is a characteristic length.

Equations (7) and (8) now become

$$\frac{\partial^2 \overline{\psi}}{\partial Y \partial t^*} + \frac{\partial \overline{\psi}}{\partial Y}\frac{\partial^2 \overline{\psi}}{\partial X \partial Y} - \frac{\partial \overline{\psi}}{\partial X}\frac{\partial^2 \overline{\psi}}{\partial Y^2} = \frac{L^2}{\nu^2}u_\infty\frac{du_\infty}{dX} + G - G_\infty + \frac{\partial^3 \overline{\psi}}{\partial Y^3} \tag{9}$$

$$\frac{\partial G}{\partial t^*} + \frac{\partial \overline{\psi}}{\partial Y}\frac{\partial G}{\partial X} - \frac{\partial \overline{\psi}}{\partial X}\frac{\partial G}{\partial Y} = \frac{1}{Pr}\frac{\partial^2 G}{\partial Y^2} \tag{10}$$

where $Pr = \mu C_p / k$

In order to determine the necessary conditions for the existence of similarity solutions, we follow Yang [1] and introduce the new variables

$$\eta = Y \varphi_1(X, t^*)$$

$$f(\eta) = \Psi / \varphi_2(X, t^*)$$

$$\Phi(\eta) = (G - G_\infty)/(G_w - G_\infty)$$

where $G_w = \beta g L^3 (\theta_w - \theta_0)/\nu^2$.

The required conditions are those which enable the introduction of the new variables to transform equations (9) and (10) into ordinary differential equations together with appropriate boundary conditions.

In terms of the new variables the boundary conditions become

$$\Phi = 1 \text{ at } \eta = 0, \quad \Phi = 0 \text{ at } \eta = \infty$$

$$f = df/d\eta = 0 \text{ at } \eta = 0, \quad df/d\eta = u_\infty / \varphi_1 \varphi_2 \text{ at } \eta = \infty$$

The last of these is only acceptable if

$$\frac{u_\infty}{\varphi_1 \varphi_2} = constant = C_1 \tag{11}$$

The conditions for equations (9) and (10) to transform into ordinary differential equations may be seen, by comparison

with the problem treated in Yang [1], to be given by

$$\frac{(G_w - G_\infty)}{\varphi_1^3 \, \varphi_2} = C_2 \tag{12}$$

$$\frac{\partial \varphi_1}{\partial t^*} \, \frac{1}{\varphi_1^3} = C_3 \tag{13}$$

$$\frac{\partial \varphi_2}{\partial t^*} \, \frac{1}{\varphi_1^2 \varphi_2} = C_4 \tag{14}$$

$$\frac{\partial \varphi_1}{\partial x} \, \frac{\varphi_2}{\varphi_1^2} = C_5 \tag{15}$$

$$\frac{\partial \varphi_2}{\partial x} \, \frac{1}{\varphi_1} \cdot = C_6 \tag{16}$$

$$\frac{\partial (G_w - G_\infty)}{\partial t^*} \, \frac{1}{(G_w - G_\infty)\varphi_1^2} = C_7 \tag{17}$$

$$\frac{\partial (G_w - G_\infty)}{\partial x} \, \frac{\varphi_2}{(G_w - G_\infty)\varphi_1} = C_8 \tag{18}$$

$$\frac{\partial G_\infty}{\partial t^*} \, \frac{1}{(G_w - G_\infty)\varphi_1^2} = C_9 \tag{19}$$

$$\frac{\partial G_\infty}{\partial x} \, \frac{\varphi_2}{(G_w - G_\infty)\varphi_1} = C_{10} \tag{20}$$

$$\frac{d\, U_\infty}{d\, x} \, \frac{U_\infty}{\varphi_1^3 \varphi_2} = C_{11} \tag{21}$$

Where the C's are all constants

At this point it is desirable to consider the relationship between the present work and that contained in Yang [1] and other related publications.

The work of Yang [1] and, indeed, all other theoretical treatments of laminar natural convection from a plane surface in an infinite medium, known to the author, are concerned with convection from a surface in isothermal surroundings. The present investigation deals with convection from a surface in surroundings in which the temperature is a function of the distance along the plate and of time. Now it might be supposed that, as for forced convection (e.g. Hansen [5]), only the difference in temperature between the surface and the surroundings is important. However, this is not the case in problems of natural convection, because of the appearance of θ_∞ in the body force term in the momentum equation.

It should be noted that the present work includes all cases treated previously, including those of Yang [1]. This is demonstrated by putting G_∞ = const and U_∞ = 0 in equations (11) to (21), and writing $G^*_w = G_w - G_\infty$, when these equations become identical to equations (16) to (22) in Yang [1].

Two cases are of particular interest, viz., steady natural convection, (a) from an isothermal surface in non-isothermal surroundings; (b) from a non-isothermal surface maintained at a constant temperature differential with its surroundings.

Particular Cases

(i) Steady Natural Convection From An Isothermal Surface
     In Non-Isothermal Surroundings

In this case $u_\infty = 0$, $\theta_w = $ const, $\theta_\infty = \theta_\infty(x)$ and
hence $C_1 = C_3 = C_4 = C_7 = C_9 = C_{11} = 0$ and $C_8 = -C_{10}$.
Equations (12), (15), (16) and (18) determine the conditions
which are imposed on ϕ_1, ϕ_2 and G_∞. A study of the
general solutions to equations (15) and (16) shows that two
cases must now be considered.

Case (a): $C_6/C_5 \neq 1$.

Here $\phi_1 = [C_{14} + (C_5/C_{13})(\varepsilon - 1)x]^{1/(\varepsilon - 1)}$

$\phi_2 = C_{13}[C_{14} + (C_5/C_{13})(\varepsilon - 1)x]^{\varepsilon/(\varepsilon - 1)}$

where $\varepsilon = C_6/C_5$ and C_{13} and C_{14} are constants of integration.
It is convenient to rearrange these equations by writing
$n = (\varepsilon + 3)/(\varepsilon - 1)$ and $C_5 = n - 1$. (This latter step is
possible because of the presence of the arbitrary constants
C_{13} and C_{14} in the equations.)

The equations now become

$$\phi_1 = [C_{14} + 4/C_{13})x]^{(n-1)/4}$$

$$\phi_2 = C_{13}[C_{14} + (4/C_{13})x]^{(n+3)/4}$$

and we also have

$$G_w - G_\infty = C_{13}[C_{14} + (4/C_{13})x]^n$$

Under these conditions, equations (9) and (10) reduce to

$$f''' + (n+3)ff'' - (2n + 2)(f')^2 + \phi = 0 \qquad (22)$$

$$\phi'' + Pr(n+3)f\phi' + 4nPrf'(1- \phi) = 0 \qquad (23)$$

where the prime indicates differentiation with respect to η . These equations, together with the appropriate boundary conditions, may be solved numerically.

Of particular interest are the heat-transfer rate from the surface and the component of velocity parallel to the surface.

$$\dot{Q}'' = -k\left(\frac{\partial\theta}{\partial y}\right)_w = \frac{k\nu^2}{g\beta L^4}\left[C_{14} +(4/C_{13})X\right]^{(5n-1)/4} \phi'_w$$

$$u = \frac{\partial\psi}{\partial y} = \phi_1\phi_2 f' = C_{13}\left[C_{14} + (4/C_{13})X\right]^{(n+1)/2} f'$$

Expressing the heat-transfer rate in dimensionless form in terms of the Nusselt number, we obtain

$$Nu = \frac{\dot{Q}'' x}{k(\theta_w-\theta_\infty)} = \left[\frac{g\beta(\theta_w-\theta_\infty)L^3}{\nu^2 C_{13}}\right]^{1/4} \frac{x}{L} \frac{\phi'_w}{\left[C_{14} +(4/C_{13})X\right]^{1/4}}$$

If C_{14} = 0 (which is shown by a study of the corresponding form of ϕ_1 to be equivalent to the boundary layer having zero thickness at x = 0) this reduces to

$$Nu = (Gr)^{1/4} \phi'_w \blacksquare/(2)^{1/2}$$

where $Gr = g\beta(\theta_w - \theta_\infty)x^3/\nu^2$.

Case (b): $C_6/C_5 = 1$.

Here $\varphi_2 = C_{13}\varphi_1$ and $\varphi_1 = C_{15}e^{C_5X/C_{13}}$ where C_{15} is a constant of integration. It is convenient to write

$m = 4/C_{13}$ and $C_5 = 1$ (the latter is possible because C_{13} and C_{15} are arbitrary) when we get

$$\varphi_1 = C_{15}e^{mX/4}$$

$$\varphi_2 = (4C_{15}/m)e^{mX/4}$$

$$G_w - G_\infty = (4(C_{15})^4/m)e^{mX}$$

and equations (9) and (10) become

$$f''' + ff'' - 2(f')^2 + \phi = 0 \qquad (24)$$

$$\phi'' + Prf\phi' + 4Prf'(1 - \phi) = 0 \qquad (25)$$

The corresponding forms for $\dot{Q}''$ and u are

$$\dot{Q}'' = -k\left(\frac{\partial\theta}{\partial y}\right)_w = \frac{-k\nu^2}{g\beta L^4}\left(4(C_{15})^5/m\right)e^{5mX/4}\phi'_w$$

$$u = \frac{\partial\psi}{\partial y} = \varphi_1\varphi_2 f' = \left(4(C_{15})^2/m\right)e^{mX/2}f'$$

It is thus seen that similarity solutions for steady natural convection from an isothermal surface exist when the temperature of the surroundings varies either as a power of a linear function of x as in the first case or as an

exponential function of x as above. The resulting equations are, in each case, similar but not identical to those for the corresponding cases of variable surface temperature and isothermal surroundings (Yang [1]).

Numerical solutions to equations (22) and (23) have been obtained for a range of values of n, i.e. different distributions of θ_{∞}.

(ii) Steady Natural Convection From a Non-Isothermal Surface Maintained at a Constant Temperature Differential With Its Surroundings.

Here U_{∞} = 0 and $\theta_{w}(x) - \theta_{\infty}(x)$ = constant, so that $C_1 = C_3 = C_4 = C_7 = C_8 = C_9 = C_{11}$ = 0 and equations (12), (15), (16) and (20) provide the conditions which must be satisfied if similarity solutions are to exist.

Equations (15) and (16) together yield a general solution

$$\varphi_2 = C_{16} \, \varphi_1^{\,\epsilon}$$

where ϵ = C_6/C_5 and C_{16} is a constant of integration. Substituting back into equations (15) and (16) we obtain, for the case $\epsilon \neq 1$ (the case ϵ = 1 need not be considered as it does not yield a solution for this particular problem)

$$\varphi_1 = [C_{17} + (C_5/C_{16})(\epsilon - 1)X]^{1/(\epsilon - 1)} \tag{26}$$

$$= C_{16}[C_{17} + (C_5/C_{16})(\epsilon - 1)X]^{\epsilon/(\epsilon - 1)} \tag{27}$$

where C_{17} is a constant of integration.

Equation (12) may be rewritten as

$$\phi_1^3 \phi_2 = (G_w - G_\infty)/C_2 = C_{12} \tag{28}$$

Substituting equations (26) and (27) into equation (28)

gives $\varepsilon = -3$ and $C_2 C_{16} = (G_w - G_\infty)$.

Examination of this result shows that except for the

constant of integration C_{17}, the functions ϕ_1 and ϕ_2

are identical with the corresponding functions occurring

in the analysis of steady natural convection from an isothermal

surface in isothermal surroundings.

We still have to satisfy equation (20) which may be

rewritten as

$$\frac{d G_\infty}{d X} = \frac{C_{10} (G_w - G_\infty)}{C_{16} [C_{17} - (4 C_5/C_{16}) X]}$$

Thus

$$G_\infty = -\frac{C_{10} (G_w - G_\infty) \ln[C_{17}(G_w - G_\infty) - 4 C_2 C_5 X]}{4 C_5} + C_{18}$$

where C_{18} is a constant of integration.

In order that this may be applied to physically

meaningful problems, G_∞ must remain finite. This imposes

the following mathematical restrictions

    (i) C_{17} cannot be zero

    (ii) $C_{17}(G_w - G_\infty)$ cannot be equal to $4C_2C_5X$ for any X

A study of the corresponding forms of ϕ_1 shows that the

restrictions have the common interpretation that the existence

of a constant temperature differential between a surface
and its surroundings, is incompatible with the existence of
a point of zero boundary layer thickness, except for the
special case of an isothermal surface in isothermal
surroundings (C_{10} = 0).

When $C_{17} \neq 0$, the boundary layer has a non-zero
thickness at X = 0 and the magnitudes of the temperature
and velocity profiles are specified by the value of C_{17}.

For this case, equations (9) and (10) reduce to the
ordinary differential equations

$$f''' + 3ff'' - 2(f')^2 + \phi = 0 \tag{29}$$

$$\phi'' + 3Prf\phi' - C_{10}f' = 0 \tag{30}$$

Because the appearance of C_{10} in equations (30) places
a severe restriction on the generality of any solution and
because of the limitations imposed on C_{17}, no numerical
solutions of these equations have been attempted.

(iii) Other Cases of Convection to Non-Isothermal
        Surroundings

A careful study of equations (11) to (21) shows that
the conditions imposed by these equations are closely
related to those which must be satisfied in problems with
isothermal surroundings. Equations (11) to (18) and (21)
impose the same conditions on ($G_w - G_\infty$) as are usually
imposed on G_w. Equations (19) and (20) then impose
conditions on G_∞ as well.

As an example, let us consider the case of steady natural convection from a non-isothermal surface to non-isothermal surroundings, excluding the case where $G_w - G_\infty$ is constant.

Similarity solutions are possible for $(G_w - G_\infty)$ $= (A + BX)^n$ and for $G_w - G_\infty = Ae^{mX}$ provided G_∞ also satisfies equation (20) which requires that $G_\infty = C(A + BX)^n$ (for $n \neq 0$) in one case and $G_\infty = De^{mX}$ in the other case (A, B, C and D are constants). The similarity in the form of $(G_w - G_\infty)(X)$ and $G_\infty (X)$ results from the similarity of equations (17) and (19), and (18) and (20). This gives a means of generalizing existing results to include cases of non-isothermal·surroundings. Restrictions previously derived concerning allowable variations in surface temperature may be used to describe allowable variations in temperature difference, surface to surroundings, provided that the same conditions are imposed on the variations of surrounding temperature alone. This generalization may be applied to all know similarity solutions for natural convection (see for example Yang [1], Pau-Chang Lu [6], Sparrow and Gregg [7], Eichhorn [8], etc.) and also to similarity solutions for combined forced and free convection (see Sparrow, Eichhorn and Gregg [9] and Brindley [10]).

It should be noted that the resulting ordinary differential equations in all these cases will not be the same as those for the corresponding cases of isothermal surroundings,

but will, in each case, contain an additional term in the 'energy' equation. Numerical solutions have not been obtained for any of the new equations, because practical applications have not yet arisen which would justify such work.

Results and Discussion

Numerical solutions of equations (22) and (23) have been obtained on the University of London Atlas computer, for $-0.30 \leqslant n \leqslant 0.6$ and $Pr = 0.708$ (air). Both Runge-Kutta integration and iterated integration techniques were used. See Cheesewright [2] for details and a comparison of the methods.

Figures 1, 2, and 3 show the influence of n on the local heat-transfer rate, the temperature profiles and the velocity profiles respectively. Because of the difficulty of showing fine detail on the graphs, the full solutions are tabulated for the cases of $n = -0.15$ and $n = -0.3$, in tables 1 and 2.

It should be noted that $n < 0$ corresponds to θ_∞ increasing with increasing X while $n > 0$ corresponds to θ_∞ decreasing with increasing X. The former of these is the more likely to occur in practical situations; in fact, the condition $n = 0$ (θ_∞ = const) is very rarely achieved, although in many cases departures from it are small enough to be ignored. Cases of $n > 0$ may not occur in practical situations because this would in general represent an unstable situation.

Figure 1 shows that for $n < 0$, the local heat-transfer rate is increased as compared with $n = 0$ while for $n > 0$ it is reduced. This is in keeping with the physical picture of the phenomenon. For negative n one would expect that at a particular section X, the temperatures at all points in the boundary layer would be less than would have existed if n had been zero and $\theta_w - \theta_\infty$ had been everywhere equal to its local value. This picture is confirmed by the temperature and velocity profiles in figures 2 and 3.

Table 1 shows that for $n = -0.15$, the temperature in a part of the boundary layer is less than that outside the boundary layer. Table 2 shows that for $n = -0.3$ the effect is more pronounced and is sufficient to cause flow reversal in the outer part of the boundary layer as indicated by the negative values of f'.

It must be emphasized that although these effects are small in numerical magnitude, they are genuine and are not due to numerical inaccuracy. This was confirmed by calculating one set of results to eight-figure accuracy.

The physical picture of the phenomena in that the rate of heat-transfer from the plate to fluid in the outer part of the boundary layer is not sufficient to keep its temperature in step with the temperature outside the boundary layer, as it moves upwards.

The possibility that the above described temperature minimum, and flow reversal, may constitute an unstable

condition is suggested by experiments in which negative n existed (or was suspected). The evidence is all indirect and no experiments have yet been carried out to check this idea directly. In all cases the variation of n was imposed by conditions not controlled during the experiment so that the condition $\theta_w - \theta_\infty = Ax^n$ was not satisfied. Nonetheless, it is felt that an estimation of the effects can be obtained by the representation of $(\theta_w - \theta_\infty)(x)$ as Ax^n in a piecewise manner.

In experiments reported by Cheesewright [2], n was always negative and may have been as low as -0.1 in some cases. The 'laminar' boundary layer in these experiments was almost always unsteady. It is believed that the only two days during a period of several months on which the boundary layer was steady were characterized by $n \simeq 0$, but definite evidence on this point does not exist. The introduction of artificial disturbances into the laboratory, outside the boundary layer, did not appear to affect the steadiness in either the steady or the unsteady situation. Further details on these points are given in [2].

These results are in keeping with those reported by a number of authors who have studied boundary layer flow regimes in closed cavity natural convection. (Elder [11], Carlson [12], Gaster and Murgatroyd[13], Watson [14] and Hammitt [15]). In all cases it has been reported that it is very difficult, or impossible, to obtain steady laminar

flow. In all cases substantial variations of θ_∞ (considering the boundary layers on the cell walls) with respect to X occurred, and while it is realized that the closed cavity imposes more severe conditions with regard to stability than a free flow, it is felt that the phenomena are generally the same.

The possibility that unstable flows may occur for n less than some particular value, could explain the unusual results of Tritton [16] who reported a change in the stability of the laminar boundary layer due to a change in laboratory conditions which he was not able to specify. His laboratory was generally similar to that used by Cheesewright [2] so that similar values of n may have occurred.

Cheesewright [2] has also reported increased local heat-transfer rates under conditions for which n was known to be negative. The increase was always greater than that predicted by the laminar steady state solution. The difference is believed to be due to the unsteadiness discussed above.

Conclusions

1. Similarity solutions exist for problems of laminar natural convection from a plane vertical surface in non-isothermal surroundings. The conditions for these solutions may be obtained by the generalization of conditions for solutions in isothermal surroundings.

2. For the special case of an isothermal surface in non-isothermal surroundings ($\theta_w - \theta_\infty$) $= Ax^n$) with $n < 0$, a temperature minimum and a region of reversed flow occurring within the boundary layer. Experimental evidence suggests that this is an unstable situation.

3. For the above special case, the effect of negative n is to increase the local heat-transfer rate while positive n decreases it.

Acknowledgements

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Fig. 1 — The Variation of the Local Heat-
Transfer Rate With The Temperature Gradient
Outside the Boundary Layer.

$$(\; \theta_\infty = Bx^n - \theta_w \;, \qquad \theta_w = const. \quad Pr = 0.708)$$

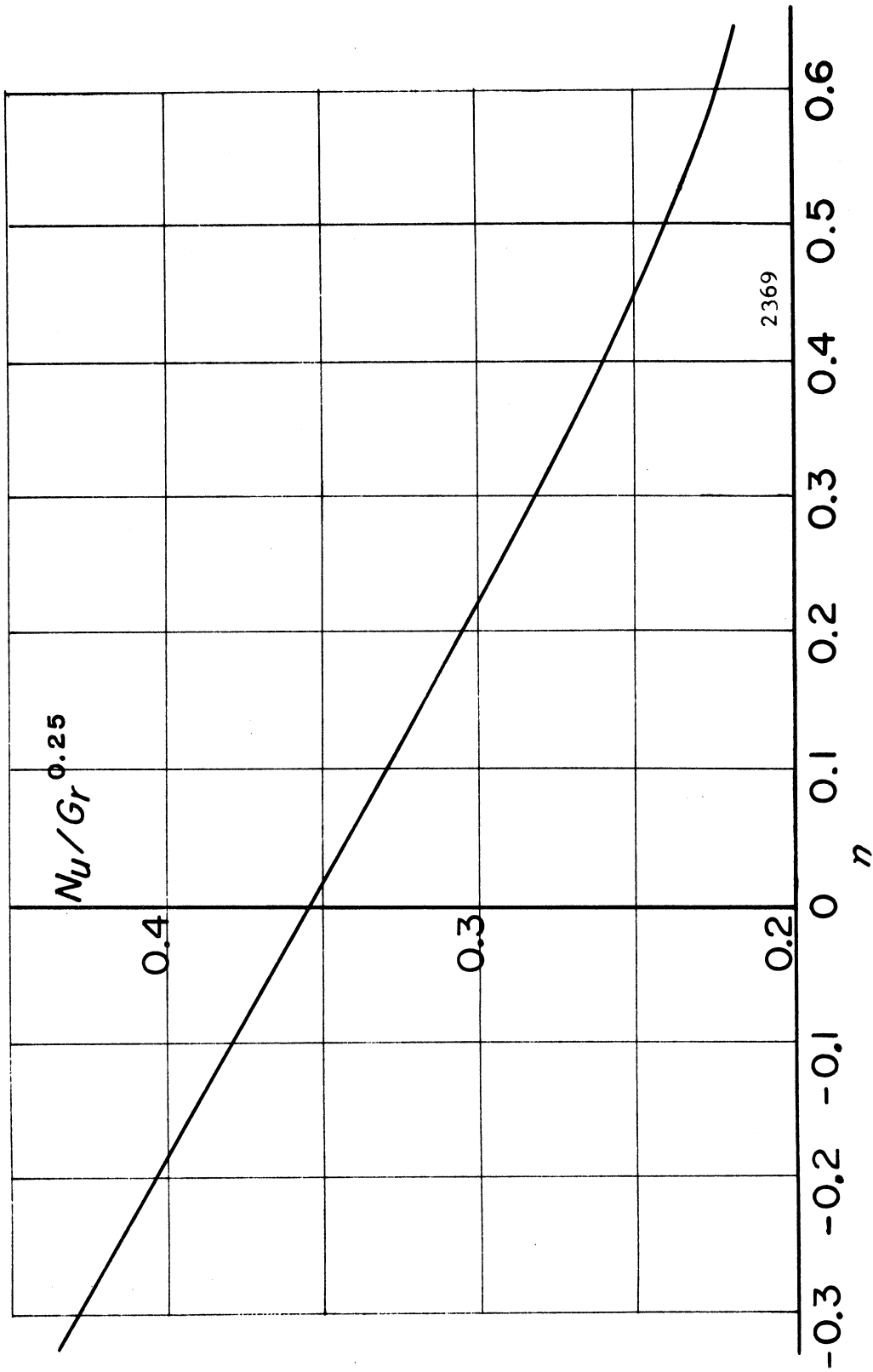Fig. 2 - The Variation of the Dimensionless Temperature Profiles With The Temperature Gradient Outside the Boundary Layer.

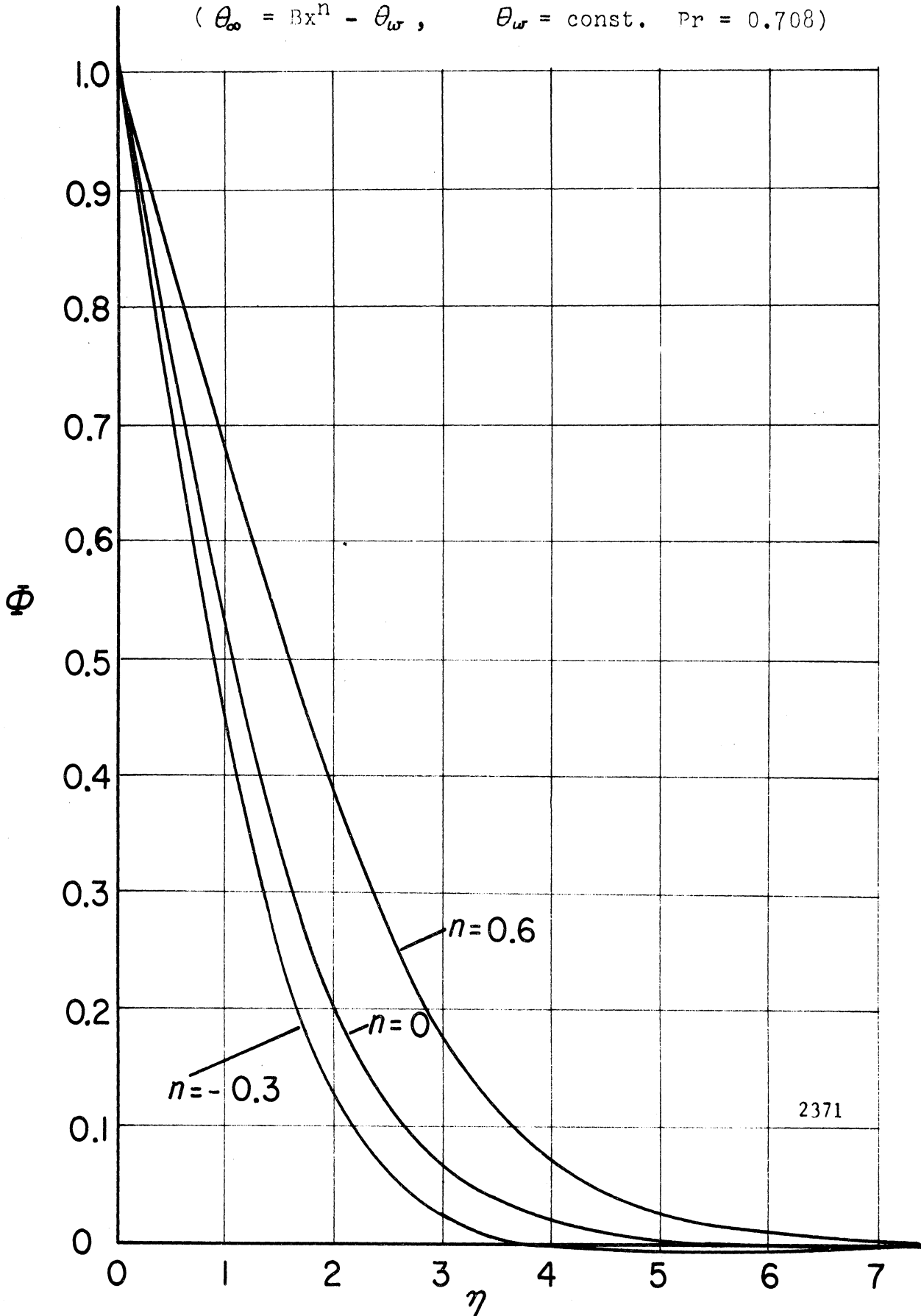($\theta_\infty = Bx^n - \theta_w$, θ_w = const.   Pr = 0.708)

Fig. 3 — The Variation of the Dimensionless
Velocity Profiles With The Temperature
Gradient Outside the Boundary Layer.

($\theta_\infty = Bx^n - \theta_w$, θ_w = const.   Pr = 0.708)

TABLE 1

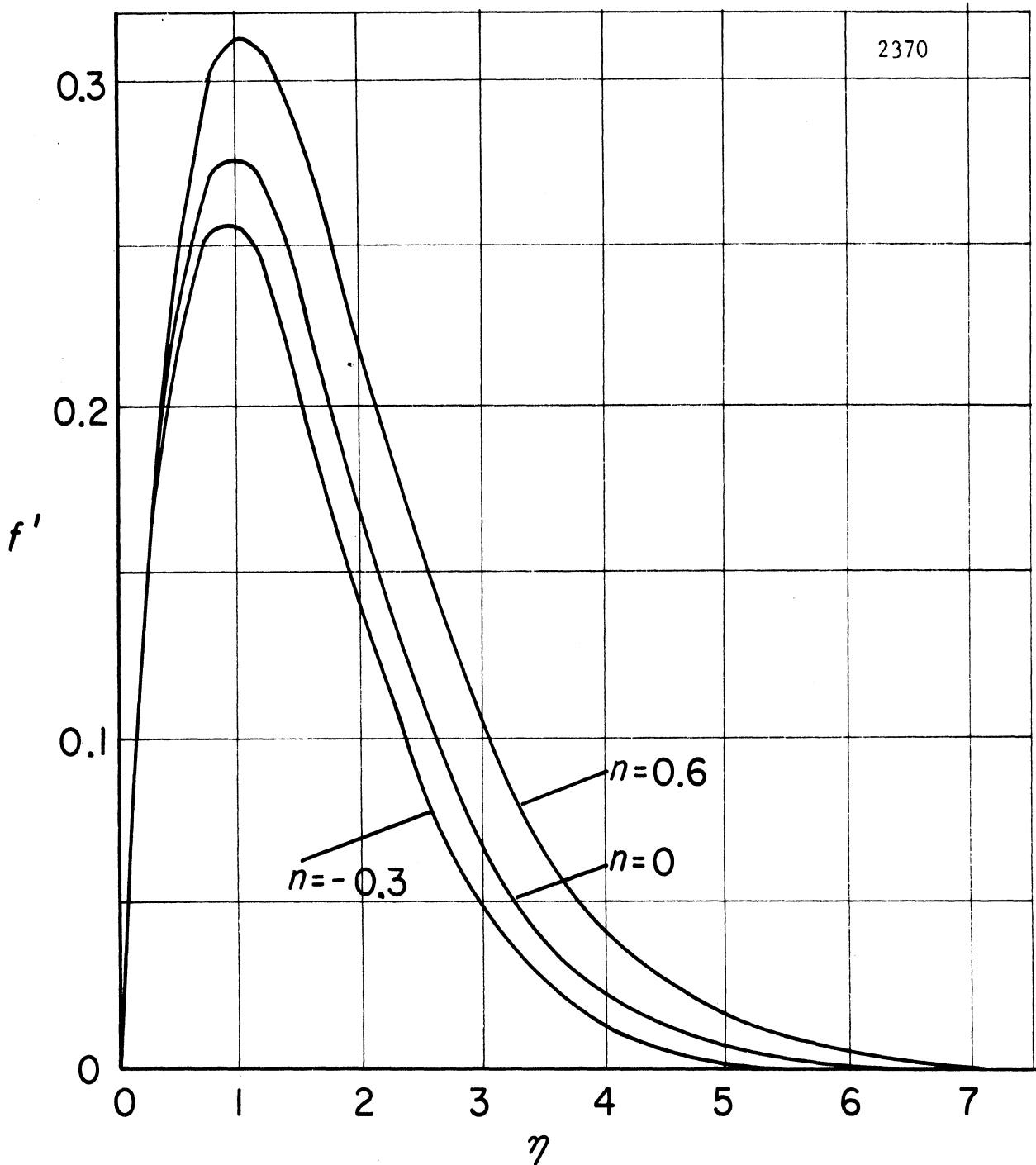Theoretical Solution   n = -0.15   Pr = 0.708

	f	f'	f"	Φ	Φ'
0.00	0.00000	0.00000	0.65949	1.00000	-0.55423
0.50	0.06312	0.21703	0.23471	0.72515	-0.53705
1.00	0.18846	0.26296	-0.02227	0.47457	-0.45359
1.50	0.31151	0.22080	-0.12471	0.27956	-0.32318
2.00	0.40539	0.15430	-0.13975	0.15026	-0.19855
2.50	0.46734	0.09612	-0.09939	0.07500	-0.10901
3.00	0.50448	0.05537	-0.06454	0.03525	-0.05515
3.50	0.52530	0.03012	-0.03810	0.01570	-0.02623
4.00	0.53640	0.01568	-0.02109	0.00662	-0.01184
4.50	0.54208	0.00786	-0.01113	0.00261	-0.00508
5.00	0.54488	0.00381	-0.00564	0.00093	-0.00205
5.50	0.54622	0.00178	-0.00277	0.00027	-0.00076
6.00	0.54683	0.00081	-0.00131	0.00004	-0.00024
6.50	0.54711	0.00035	-0.00060	-0.00002	-0.00005
7.00	0.54723	0.00014	-0.00026	-0.00003	0.00001
7.50	0.54727	0.00006	-0.00011	-0.00002	0.00002
8.00	0.54729	0.00002	-0.00004	-0.00002	0.00002
8.50	0.54730	0.00001	-0.00002	-0.00001	0.00001
9.00	0.54730	0.00000	-0.00000	-0.00001	0.00001
9.50	0.54730	-0.00000	-0.00000	0.00000	0.00000
10.00	0.54730	-0.00000	0.00000	0.00000	0.00000
10.50	0.54730	-0.00000	0.00000	0.00000	0.00000
11.00	0.54729	-0.00000	0.00000	0.00000	0.00000
11.50	0.54729	-0.00000	0.00000	0.00000	0.00000
12.00	0.54729	-0.00000	0.00000	0.00000	0.00000
12.50	0.54729	0.00000	0.00000	0.00000	0.00000
13.00	0.54729	0.00000	0.00000	0.00000	0.00000
13.50	0.54729	0.00000	0.00000	0.00000	0.00000
14.00	0.54729	0.00000	0.00000	0.00000	0.00000
14.50	0.54730	0.00000	0.00000	0.00000	0.00000
15.00	0.54730	0.00000	0.00000	0.00000	0.00000
15.50	0.54730	0.00000	0.00000	0.00000	0.00000
16.00	0.54730	0.00000	0.00000	0.00000	0.00000
16.50	0.54730	0.00000	-0.00000	0.00000	0.00000
17.00	0.54730	0.00000	-0.00000	0.00000	0.00000
17.50	0.54730	0.00000	-0.00000	0.00000	0.00000
18.00	0.54730	0.00000	-0.00000	0.00000	0.00000
18.50	0.54730	0.00000	-0.00000	0.00000	0.00000
19.00	0.54730	0.00000	-0.00000	0.00000	0.00000
19.50	0.54730	-0.00000	-0.00000	0.00000	0.00000
20.00	0.54730	-0.00000	-0.00000	0.00000	0.00000

TABLE 2

Theoretical Solution   n = -0.3   Pr = 0.708

η	f	f'	f"	ϕ	ϕ'
0.00	0.00000	0.00000	0.64099	1.00000	-0.60414
0.50	0.06090	0.20845	0.21979	0.70089	-0.58180
1.00	0.18034	0.24846	-0.03083	0.43271	-0.47702
1.50	0.29536	0.20374	-0.12652	0.23255	-0.32119
2.00	0.38072	0.13771	-0.12746	0.10863	-0.18113
2.50	0.43491	0.08191	-0.09355	0.04342	-0.08782
3.00	0.46572	0.04426	-0.05822	0.01371	-0.03684
3.50	0.48176	0.02204	-0.03249	0.00211	-0.01282
4.00	0.48946	0.01011	-0.01664	-0.00145	-0.00298
4.50	0.49286	0.00422	-0.00787	-0.00193	0.00037
5.00	0.49421	0.00153	-0.00341	-0.00150	0.00111
5.50	0.49465	0.00041	-0.00131	-0.00096	0.00098
6.00	0.49474	0.00002	-0.00041	-0.00055	0.00066
6.50	0.49471	-0.00009	-0.00007	-0.00029	0.00039
7.00	0.49467	-0.00009	0.00004	-0.00014	0.00021
7.50	0.49463	-0.00006	0.00005	-0.00006	0.00011
8.00	0.49460	-0.00004	0.00004	-0.00003	0.00005
8.50	0.49459	-0.00002	0.00003	-0.00001	0.00002
9.00	0.49458	-0.00001	0.00002	0.00000	0.00001
9.50	0.49457	-0.00000	0.00001	0.00000	0.00000
10.00	0.49457	-0.00000	0.00000	0.00000	0.00000
10.50	0.49457	-0.00000	0.00000	0.00000	-0.00000
11.00	0.49457	0.00000	0.00000	0.00000	0.00000
11.50	0.49457	0.00000	0.00000	0.00000	0.00000
12.00	0.49457	0.00000	-0.00000	0.00000	0.00000
12.50	0.49457	0.00000	-0.00000	0.00000	0.00000
13.00	0.49457	0.00000	-0.00000	0.00000	0.00000
13.50	0.49457	-0.00000	-0.00000	0.00000	0.00000
14.00	0.49457	-0.00000	-0.00000	0.00000	0.00000
14.50	0.49457	-0.00000	-0.00000	0.00000	-0.00000
15.00	0.49457	-0.00000	-0.00000	0.00000	-0.00000
15.50	0.49457	-0.00000	-0.00000	0.00000	-0.00000
16.00	0.49457	-0.00000	-0.00000	0.00000	-0.00000
16.50	0.49457	-0.00000	-0.00000	0.00000	-0.00000
17.00	0.49457	-0.00000	0.00000	0.00000	-0.00000
17.50	0.49457	-0.00000	0.00000	0.00000	-0.00000
18.00	0.49457	-0.00000	0.00000	0.00000	-0.00000
18.50	0.49457	-0.00000	0.00000	0.00000	-0.00000
19.00	0.49457	0.00000	0.00000	0.00000	-0.00000
19.50	0.49457	0.00000	0.00000	0.00000	-0.00000
20.00	0.49457	0.00000	0.00000	0.00000	0.00000