THE UNIVERSITY OF MICHIGAN

AFCRL-746 4134-2-F

STUDIES IN NON-LINEAR MODELING - III
ON THE INTERACTION OF ELECTROMAGNETIC FIELDS WITH PLASMAS

by

Kun-Mu Chen

R.E. Kleinman

J. Meixner

O.G. Ruehr

D. Sengupta

31 July 1961

Final Report

The work described in this report was partially supported
by the ADVANCED RESEARCH PROJECTS AGENCY
ARPA Order Nr. 147-60, Project Code Nr. 7600

ARPA Order Nr. 147-60, Project Code Nr. 7600
Contract AF 19(604)-7428

prepared for

Electronics Research Directorate

Air Force Cambridge Research Laboratories

Office of Aerospace Research

Laurence G. Hanscom Field

Bedford, Massachusetts

Requests for additional copies by Agencies of the Department of Defense, their contractors, and other Government agencies should be directed to:

ARMED SERVICES TECHNICAL INFORMATION AGENCY
ARLINGTON HALL STATION
ARLINGTON 12, VIRGINIA

Department of Defense contractors must be established for ASTIA services or have their "need-to-know" certified by the cognizant military agency of their project or contract.

All other persons and organizations should apply to:

U. S. DEPARTMENT OF COMMERCE
OFFICE OF TECHNICAL SERVICES
WASHINGTON 25, D. C.

TABLE OF CONTENTS

ABSTRACT		iv
I.	INTRODUCTION	1
II.	PLASMA SHEATH FORMED BY A STATIONARY PLASMA ON AN INFINITE PLATE	6
III.	PLASMA SHEATH FORMED BY A MOVING PLASMA ON AN INFINITE PLATE	18
IV.	THE INTERACTION OF A HIGH INTENSITY ELECTRO-MAGNETIC FIELD WITH A LOW DENSITY PLASMA	31
V.	THE INTERACTION OF A HGIH INTENSITY ELECTRO-MAGNETIC FIELD WITH A WEAKLY IONIZED GAS	51
VI.	THE ELECTRICAL CONDUCTIVITY OF A PARTIALLY IONIZED GAS	74
VII.	NON-LINEAR ELECTRICAL CONDUCTIVITY OF A FULLY IONIZED GAS	90
APPENDIX A:	NON-LINEAR MODELING OF BESSEL FUNCTIONS	107
APPENDIX B:	LOCAL ALTERATION OF ATMOSPHERIC DENSITY WITH ELECTROMAGNETIC ENERGY	125

ABSTRACT

The results of an investigation of the interaction of electromagnetic fields with plasmas, including the development and use of appropriate non-linear modeling techniques, are presented.

Expressions for density distributions in the sheath and potential induced on a conducting plate in the presence of a plasma (moving or stationary) are found. The potential at one temperature in the sheath is non-linearly modeled to give the potential at another temperature.

Correction terms are found for the standard treatments of the interaction of electromagnetic fields with collisionless plasmas and weakly ionized gases. These corrections are functions of field intensity and give significant results for high intensity fields. The current induced in the plasma with a low intensity field is non-linearly modeled to the high intensity case.

Expressions for the conductivity (both a.c. and d.c.) are found for various plasmas and incident field intensities.

I
INTRODUCTION

This is the final report on contract number AF 19(604)-7428, ARPA Order No. 147-60, concerning an investigation of the interaction of electromagnetic fields with plasmas. An understanding of this phenomenon is essential in treating many of the vital problems associated with the coupling of high intensity electromagnetic energy and plasmas where it is desired to optimize the attenuation of electromagnetic waves by a plasma or to minimize this attenuation without resorting to brute force techniques. This will allow us to optimize systems for telemetering through plumes and re-entry plasmas and also to optimize radar systems for terminal guidance of maneuverable I. C. B. M. 's. In attempting to study these physical problems in the classical tradition by combining knowledge gained from both theoretical and experimental investigations to construct a physical picture, one is confronted with two major obstacles. On the one hand the theoretical problem, the coupling of the Boltzmann equation with the Maxwell equations, is extremely difficult in general and despite much activity in this field, results even in the simplest idealized situations are quite rare. On the other hand useful experimental results are equally, if not more, difficult to obtain. Full scale laboratory experiments of physical importance are, because of the large field strengths, if not impossible, very costly. Furthermore linear modeling, the usual technique for scaling physical phenomena to dimensions feasible for laboratory experiments, is inapplicable since the basic particles of a plasma cannot be modeled correctly.

If reasonable laboratory experiments are to yield significant full scale information for high intensity electromagnetic fields this must necessarily be accomplished by application of suitable non-linear modeling techniques. The concept of non-linear modeling as developed in [1] through [3] provides the basis for theoretical investigations which will enable data obtained in low power laboratory experiments to be transformed to give results in cases of physical significance. In order to develop the non-linear modeling techniques necessary in this problem, work in the Radiation Laboratory under this contract has proceeded by considering a number of special problems involving the interaction of plasmas with electromagnetic fields. These problems were chosen for a variety of reasons. In the first place they are of considerable interest in themselves. Secondly they serve, in a sense, as canonical problems on which the non-linear modeling techniques can be developed and demonstrated. Thirdly the theoretical solutions can be found and used to guide and check the modeling attempts. These solutions themselves represent new results in the field and are presented here for the first time.

In chapter II, the plasma sheath formed by a stationary plasma on the surface of an infinite plate is considered. With the assumption of a low density plasma and a Maxwellian distribution of both ions and electrons up to the plate, exact expressions for the density distributions of electrons and ions in the sheath and the potentials induced in the plate and the sheath are found. The potential in the sheath for one temperature is also non-linearly modeled to yield this data at other temperatures.

Chapter III treats the same problem when the plasma, instead of being stationary, moves toward the plate with constant mainstream velocity. Although the sheath formed is in many respects similar to that formed in the stationary case, the analysis is more complicated, hence approximate rather than exact expressions for the potential and density distributions are found. These expressions include a first order correction to results available in the literature.

In chapter IV, the interaction of a high intensity electromagnetic wave with a collisionless plasma is investigated. The coupled Boltzmann and Maxwell equations are solved without small signal approximations. The zeroth-order analysis yields conventional results; however, the first-order analysis produces new results for the velocity distribution function and quantities derived from it, i.e. conductivity, permittivity, current, and energy density of the plasma. These are evaluated as functions of the intensity of the EM wave. An attempt to use non-linear modeling in this problem is unsuccessful and the difficulty encountered is explained.

The interaction of a high intensity EM wave with a weakly ionized gas is studied in chapter V. A rigorous mathematical treatment is presented with a simple collision model. The zeroth-order analysis gives the plasma parameters as functions of the intensity of the incident wave. The results obtained can be reduced to the well-known results when the intensity of the EM wave is assumed to be very small. A first-order analysis is also presented. Non-linear modeling of the current induced in the plasma by the electric field is found to be possible in this case and the modeling function is presented explicitly.

In chapter VI the electrical conductivity of a low density and partially ionized gas where both electron-neutral particle and Coulomb type collisions play important roles is discussed. Assuming that the ionized gas is perturbed by a weak electric field, the velocity distribution function for the electrons is obtained by solving the Boltzmann equation; the collision between neutral and charged particles is accounted for by the hard sphere model for the particles while the collision between the charged particles is taken care of by the Fokker-Planck equation. Explicit expressions for a.c. and d.c. conductivity are given for various cases. To the extent that the assumptions made for the collision models are valid, the expressions for conductivity are quite general and can be used for any degree of ionization.

The conductivity of a fully ionized gas for arbitrary electric field intensity is investigated in chapter VII. In the d.c. case an instability phenomenon, the runaway effect, limits the intensity of the electric field that may be applied and under this restriction an expression for the conductivity is obtained. In the a.c. case, above a certain critical frequency, the problem of finding an expression for the conductivity becomes one of solving a non-linear first-order differential equation. Approximate expressions are presented in this case.

Not included here is the theoretical treatment of propagation of electromagnetic waves in a plasma column surrounded by an annular isotropic medium in a cylindrical wave guide [16] . With the source of electromagnetic waves taken to be a thin ring source, the formal solution has been obtained for arbitrary angular variation of the current intensity and the case of constant intensity is treated at length as a special case. In this case the excited fields will also be axially symmetric and this problem is solved where the anisotropy of the medium is taken into account by considering both the permittivity and permeability to be tensors.

The most general problem treated in [16] is too complex to be amenable to non-linear modeling techniques at present. However, suitable techniques have been developed to handle some limiting cases where the physical situation is described by a Bessel function. The details of this analysis appear in Appendix A.

A problem of considerable interest involving the interaction of electromagnetic waves with a plasma to alter the density of the atmosphere is discussed in Appendix B. Calculations of the power needed, both on the ground and at various altitudes in the atmosphere, to effect particular temperature (hence density) variations are presented. Appreciation is extended to R. W. Larson for these calculations.

II

PLASMA SHEATH FORMED BY A STATIONARY PLASMA ON AN INFINITE PLATE

In this chapter the plasma sheath formed by a stationary plasma on the surface of an infinite plate is considered. This is a problem frequently met in plasma physics and has been studied by several authors (e.g. [4]), however, the solutions are mostly approximate. It is the purpose of this study to treat the problem more rigorously and an exact solution is attempted. The problem is then examined from the non-linear modeling viewpoint.

Assume a stationary, uniform plasma fills the half space with an infinite plate located in the x - y plane as the boundary. If the potential of the plate is allowed to float, the plate gets charged and the electrical neutrality of the plasma is not preserved in the vicinity of the plate. Due to the fact that the root mean square velocity of the electrons is much higher than that of the positive ions (assumed to be singly charged), the plate must be charged negatively in order to adjust the amount of the electrons and the ions which hit the plate per unit time at equilibrium. This negative potential of the plate results in a low electron density and a high positive ion density in the vicinity of the plate. The deviation from the uniform densities of the electrons and the ions, or the deviation from the electrical neutrality, dies out in the direction away from the plate. The thickness of the layer where the electrical neutrality deviates, or the plasma sheath, and the density distributions of the electrons and the ions in the plasma sheath are found.

Distribution Functions

The velocity distribution function of the electrons in a plasma is

$$f_e = n_e \left(\frac{m_e}{2\pi KT_e} \right)^{3/2} e^{-\frac{m_e c^2}{2KT_e}} \tag{1}$$

the velocity distribution function of the positive ions in the plasma can be written as

$$f_i = n_i \left(\frac{m_i}{2\pi KT_i} \right)^{3/2} e^{-\frac{m_i c^2}{2KT_i}} \tag{2}$$

where n_e and n_i are the number densities and can be expressed as functions of the space. In general, we can write

$$f_e = f_{eo} + f_e' \tag{3}$$

$$f_i = f_{io} + f_i' \tag{4}$$

where

$$f_{eo} = n_o \left(\frac{m_e}{2\pi KT_e} \right)^{3/2} e^{-\frac{m_e c^2}{2KT_e}} \tag{5}$$

$$f_{io} = n_o \left(\frac{m_i}{2\pi KT_i} \right)^{3/2} e^{-\frac{m_i c^2}{2KT_i}} \tag{6}$$

f_e' and f_i' are the perturbed terms and n_o is the unperturbed density of the plasma.

If the velocity distribution functions for the electrons and the ions are assumed to be Maxwellian at any point in the physical space, the Boltzmann equations lead to

$$\frac{\partial f_e}{\partial t} + \vec{c} \cdot \nabla f_e - \frac{e}{m_e} \vec{E} \cdot \nabla_c f_e = 0 \qquad (7)$$

$$\frac{\partial f_i}{\partial t} + \vec{c} \cdot \nabla f_i + \frac{e}{m_i} \vec{E} \cdot \nabla_c f_i = 0 \qquad (8)$$

Since the plasma is stationary, the time derivative terms should drop out. Due to the one-dimensional geometry, the variation is only in the z-direction. Therefore

$$\vec{c} \cdot \nabla f_e = c_z \frac{\partial f_e}{\partial z}$$

$$\vec{E} \cdot \nabla_c f_e = E\hat{z} \cdot \frac{\partial f_e}{\partial c_z} \hat{z} = E \frac{\partial f_e}{\partial c_z}$$

$$= E \frac{\partial f_e}{\partial c^2} \cdot \frac{\partial c^2}{\partial c_z} = - \frac{m_e E}{KT_e} c_z f_e .$$

With these relations, Equation (7) becomes

$$\frac{\partial f_e}{\partial z} + \frac{eE}{KT_e} f_e = 0 \qquad (9)$$

Equation (9) can be solved subject to the boundary condition $f_e \rightarrow f_{eo}$ as $x \rightarrow \infty$, and the relation $\vec{E} = -\nabla\phi = -\frac{\partial\phi}{\partial z}\hat{z}$. The final solution for f_e is

$$f_e = f_{eo}\, e^{\frac{e\phi}{KT_e}} = n_o \left(\frac{m_e}{2\pi KT_e}\right)^{3/2} e^{-\frac{mc^2}{2KT_e}}\, e^{\frac{e\phi}{KT_e}} \tag{10}$$

Where ϕ is the potential in the space and is normalized to zero at infinity. Thus the velocity distribution function of the electrons is determined as a function of space and velocity.

Following similar reasoning, Equation (8) can be reduced to

$$\frac{\partial f_i}{\partial z} - \frac{eE}{KT_i} f_i = 0 \tag{11}$$

and the final solution for f_i is

$$f_i = f_{io}\, e^{-\frac{e\phi}{KT_i}} = n_o \left(\frac{m_i}{2\pi KT_i}\right)^{3/2} e^{-\frac{m_i c^2}{2KT_i}}\, e^{-\frac{e\phi}{KT_i}} . \tag{12}$$

Equation (12) gives the velocity distribution function of the positive ions as a function of space and velocity.

It follows from Equations (10) and (12) that the number densities of the electrons and the positive ions can be expressed as follows:

$$n_e = n_o\, e^{\frac{e\phi}{KT_e}} \tag{13}$$

$$n_i = n_o e^{-\frac{e\phi}{KT_i}} \tag{14}$$

Potential Distribution in the Plasma Sheath

The number densities of the electrons and the positive ions in the plasma sheath are thus determined as functions of the potential in the space. It is now necessary to evaluate the potential distribution in the plasma sheath. The potential in the plasma sheath is the solution of a Poisson's equation as follows:

$$\nabla^2 \phi = -\frac{e}{\epsilon_o}(n_i - n_e)$$

$$= -\frac{n_o e}{\epsilon_o}\left(e^{-\frac{e\phi}{KT_i}} - e^{\frac{e\phi}{KT_e}}\right) . \tag{15}$$

Assume $T_e = T_i = T$ for simplicity, and owing to the one dimensional variation, Equation (15) is reduced to the differential equation

$$\frac{d^2\phi}{dz^2} = \frac{2n_o e}{\epsilon_o} \sinh\left(\frac{e}{KT}\phi\right) . \tag{16}$$

The boundary conditions for ϕ are

$$\phi = \phi_o \quad \text{at } z = 0 \quad \text{(on the plate)}$$

$$\phi = 0, \text{ and } \frac{\partial\phi}{\partial z} = 0 \quad \text{as } z \rightarrow \infty .$$

Equation (16) can be solved as follows:

Rewrite Equation (16) as

$$\frac{d^2\phi}{dz^2} = \alpha \sinh(\beta\phi) \tag{17}$$

with

$$\frac{2n_o e}{\epsilon_o} = \alpha, \quad \frac{e}{KT} = \beta$$

Let

$$\frac{d\phi}{dz} = p(\phi), \quad \frac{d^2\phi}{dz^2} = p\frac{dp}{d\phi}, \quad \text{then}$$

Equation (17) becomes

$$p\frac{dp}{d\phi} = \alpha \sinh(\beta\phi)$$

$$\frac{p^2}{2} = \frac{\alpha}{\beta}\cosh\beta\phi + c_1$$

or

$$\frac{d\phi}{dz} = \pm\sqrt{\frac{2\alpha}{\beta}\cosh\beta\phi + 2c_1} \tag{18}$$

Subject to the boundary condition of

$$\frac{d\phi}{dz} \to 0 \quad \text{as } z \to \infty \quad (\text{or as } \phi \to 0)$$

c_1 is found to be $-\frac{\alpha}{\beta}$. Equation (18) then becomes

$$\frac{d\phi}{dz} = \pm\sqrt{\frac{2\alpha}{\beta}}\sqrt{\cosh\beta\phi - 1} = \pm 2\sqrt{\frac{\alpha}{\beta}}\sinh\frac{\beta\phi}{2}. \tag{19}$$

ϕ is always negative and $\frac{d\phi}{dx}$ is always positive in the plasma sheath. Therefore, the negative sign in Equation (19) is taken. The integration of both sides of Equation (19) leads to

$$\int_{\phi_o}^{\phi} \frac{dt}{\sinh \frac{\beta t}{2}} = -2\sqrt{\frac{\alpha}{\beta}}\, z$$

or

$$\frac{2}{\beta} \log \tanh \frac{\beta\phi}{4} - \log \tanh \frac{\beta\phi_o}{4} = -2\sqrt{\frac{\alpha}{\beta}}\, z \quad .$$

This can be simplified to

$$\tanh \frac{\beta\phi}{4} = \left(\tanh \frac{\beta\phi_o}{4}\right) e^{-\sqrt{\alpha\beta}\, z}$$

or

$$\phi = \frac{4}{\beta} \tanh^{-1} \left[e^{-\sqrt{\alpha\beta}\, z} \tanh \frac{\beta\phi_o}{4} \right] \quad . \qquad (20)$$

The substitutions of $\alpha = \frac{2n_o e}{\epsilon_o}$ and $\beta = \frac{e}{KT}$ in Equation (20) give a final solution for ϕ as follows:

$$\phi = \frac{4KT}{e} \tanh^{-1} \left[e^{-\sqrt{\frac{2n_o e^2}{\epsilon_o KT}}\, z} \tanh \frac{e\phi_o}{4KT} \right] \qquad (21)$$

The solution of ϕ can be checked easily because $\phi \rightarrow 0$ as $z \rightarrow \infty$, and $\phi = \phi_o$ as $z = 0$.

Potential of the Plate

The potential distribution in the plasma sheath is found as a function of some parameters and the potential of the plate, ϕ_o. ϕ_o can be determined from the condition that the same amount of electrons as positive ions will hit a unit area of the plate per unit time at equilibrium. Owing to the fact that the rms velocity of the electrons is much higher than that of the positive ions, more electrons than ions tend to hit the plate per unit time except the plate is charged negatively so that only high energy electrons can reach the plate.

Assume the potential of the plate as ϕ_o, which is negative. A critical velocity of the electron is defined as

$$\frac{1}{2} m_e c_{eo}^2 = |e\phi_o|$$

or

$$c_{eo} = \sqrt{\frac{2|e\phi_o|}{m_e}} \quad . \tag{22}$$

c_{eo} means that only those electrons whose velocities are higher than c_{eo} and point toward the plate can overcome the potential barrier on the plate and reach the plate. Similarly a critical velcoity for the positive ion is defined as

$$c_{io} = \sqrt{\frac{2|e\phi_o|}{m_i}} \quad . \tag{23}$$

Thus, the ion having a velocity lower than c_{io} and points away from the plate may be attracted back to the plate.

The application of the boundary condition at equilibrium yields an equation as follows:

$$-\int_{-\infty}^{-c_{eo}} c_z dc_z \int_{-\infty}^{\infty} dc_x \int_{-\infty}^{\infty} dc_y \left[f_e\right]_{z=\infty} = \left[-\int_{-\infty}^{0} c_z dc_z + \int_{0}^{c_{io}} c_z dc_z\right] \int_{-\infty}^{\infty} dc_x \int_{-\infty}^{\infty} dc_y \left[f_i\right]_{z=\infty}$$

The integrals can be carried out to be

$$n_o \sqrt{\frac{KT}{2\pi m_e}}\; e^{-\frac{|e\phi_o|}{KT}} = n_o \sqrt{\frac{KT}{2\pi m_i}}\; (2 - e^{-\frac{|e\phi_o|}{KT}})$$

or

$$\frac{|e\phi_o|}{KT} = \log\left[\frac{1}{2}\sqrt{\frac{m_i}{m_e}} + \frac{1}{2}\right] . \qquad (24)$$

It is learned that the potential of the plate, ϕ_o, is a function of the temperature and the mass ratio of ion and electron.

Density Distributions of the Electrons and the Ions in the Plasma Sheath

Up to this point the potential distribution in the plasma sheath is completely determined. The density distributions of the electrons and the ions in the plasma sheath are obtained by substituting the potential ϕ of Equation (21) in Equations (13) and (14). The final solutions are

$$n_e = n_o \exp\left[4 \tanh^{-1} \left(e^{-\sqrt{\frac{2n_o e^2}{\epsilon_o KT}}\, z} \tanh \frac{e\phi_o}{4KT} \right) \right] \tag{25}$$

$$n_i = n_o \exp\left[-4 \tanh^{-1} \left(e^{-\sqrt{\frac{2n_o e^2}{\epsilon_o KT}}\, z} \tanh \frac{e\phi_o}{4KT} \right) \right] \tag{26}$$

It is noted that ϕ_o is negative. From Equations (25) and (26) it is observed that both n_e and n_i approach n_o quite rapidly as z increases. The thickness of the plasma sheath, H, is obtained by finding a value of z at which n_e is $0.95\, n_o$ (arbitrary), and after that point n_e and n_i are very close to n_o.

$$H = \sqrt{\frac{\epsilon KT}{2n_o e^2}} \log\left[\frac{\tanh \frac{|e\phi_o|}{4KT}}{\tanh(0.0128)} \right] \tag{27}$$

It is noted that in the numerical calculation the MKS unit system should be used.

As an example, a plasma with $T = 1000^oK$, $\sqrt{\frac{m_i}{m_e}} = \sqrt{1823}$, and $n_o = 10^{12}\ 1/m^3$ is considered. In this case, the numerical results are found to be as follows

$$\frac{|e\phi_o|}{KT} = 3.08, \qquad \phi_o = -0.266 \text{ volt}, \qquad H = 0.605 \text{ cm}.$$

Non-Linear Modeling the Potential Distribution

Equation (16) could also serve as the starting point in a non-linear modeling attempt, where the potential distribution for two different temperatures is sought, with but one experiment. This equation is a representative of a general class of equations treated in some detail in [2] .

Specifically, simplifying the equation somewhat as in (17) we have the two equations

$$\frac{d^2\phi}{dz^2} - \alpha \sinh(\beta_1\phi) = 0 \tag{28}$$

$$\frac{d^2\psi}{dz^2} - \alpha \sinh(\beta_2\psi) = 0 \tag{29}$$

where the only difference is a change in the constant β. We wish to find ϕ as a function of ψ. With [2] we can immediately write, without employing any boundary condition

$$\int \frac{d\phi}{\sqrt{\int \alpha \sinh \beta_1 \phi d\phi}} = \int \frac{d\psi}{\sqrt{\int \alpha \, \mathrm{dinh} \, \beta_2 \psi d\psi}} \tag{30}$$

or

$$\beta_1 \int \frac{d\phi}{\sqrt{\cosh \beta_1 \phi + c_1}} = \beta_2 \int \frac{d\psi}{\sqrt{\cosh \beta_2 \psi + c_2}} \tag{31}$$

where c_1 and c_2 are constants of integration.

In general the integrals in (31) will be elliptic but in the special case when $c_1 = c_2 = -1$, (31) becomes

$$\beta_1 \int \frac{d\phi}{\sinh \frac{\beta_1\phi}{2}} = \beta_2 \int \frac{d\psi}{\sinh \frac{\beta_2\psi}{2}} \tag{32}$$

or

$$\frac{1}{\beta_1} \log \tanh \frac{\beta_1\phi}{4} = \frac{1}{\beta_2} \log \tanh \frac{\beta_2\psi}{4} + c \tag{33}$$

where c is a constant of integration. To evaluate c we apply an initial condition of the form $\psi = \psi_o$ where $\phi = \phi_o$ hence

$$c = \frac{1}{\beta_1} \log \tanh \frac{\beta_1\phi_o}{4} - \frac{1}{\beta_2} \log \tanh \frac{\beta_2\psi_o}{4} \tag{34}$$

Substituting (34) in (33) yields

$$\frac{1}{\beta_1} \log \frac{\tanh \frac{\beta_1\phi}{4}}{\tanh \frac{\beta_1\phi_o}{4}} = \frac{1}{\beta_2} \log \frac{\tanh \frac{\beta_2\psi}{4}}{\tanh \frac{\beta_2\psi_o}{4}} \tag{35}$$

or explicitly

$$\phi = \frac{4}{\beta_1} \tanh^{-1} \left\{ \tanh \frac{\beta_1\phi_o}{4} \left[\frac{\tanh \frac{\beta_2\psi}{4}}{\tanh \frac{\beta_2\psi_o}{4}} \right]^{\frac{\beta_1}{\beta_2}} \right\} \tag{36}$$

Thus in this case it is possible to find a non-linear modeling function and in fact this does correspond to the physically significant case as can be verified by eliminating z from (20) and a similar equation for different values of β.

III

PLASMA SHEATH FORMED BY A MOVING PLASMA ON AN INFINITE PLATE

In this chapter the plasma sheath formed by a moving plasma on an infinite plate is investigated. The significance of this study is to investigate the behavior of a moving plasma on a plane boundary or the behavior of a stationary plasma on the surface of a moving plate. This analysis may be useful in the investigation of the antenna on a space vehicle which is wrapped by a plasma sheath.

The analysis is approximate, however some significant correction terms which are ignored in the conventional studies are evaluated.

Assume a moving plasma, having a mean stream velocity V, moves toward an infinite plate located in the x-y plane. If the potential of the plate is allowed to float, the plate gets charged negatively and a plasma sheath forms on the plate. In this analysis, the distribution functions and the densities of the electrons and the positive ions and the potential distribution in the plasma sheath are obtained.

Distribution Functions

A moving plasma moves toward an infinite plate which is located in the x-y plane. The unperturbed velocity distribution functions for the electrons and the positive ions can be expressed as

$$f_{eo} = n_o \left(\frac{m_i}{2\pi K T_i} \right)^{3/2} e^{-\frac{m_e}{2KT_e} (\vec{c} - \vec{V})^2} \quad (1)$$

$$f_{io} = n_o \left(\frac{m_i}{2\pi K T_i} \right)^{3/2} e^{-\frac{m_i}{2KT_i} (\vec{c} - \vec{V})^2} \quad (2)$$

with $\vec{V} = -V\hat{z}$.

As the moving plasma hits the plate, the distribution functions are disturbed in the vicinity of the boundary. In general, the distribution functions can be written as

$$\left. \begin{aligned} f_e &= f_{eo} + f_e' \\ f_i &= f_{io} + f_i' \end{aligned} \right\} \quad (3)$$

f_e' and f_i' represent the perturbed terms. f_e and f_i can be determined from the Boltzmann transport equation. In the case of a moving plasma, f_e and f_i are functions of space and velocity. Moreover, the dependence on the space and the velocity coordinates are found inseparable. Therefore, an approximate approach is devised to determine f_e and f_i.

It is necessary to make some reasonable assumptions here.

(1) As the moving plasma hits the plate, there occurs only the diffuse reflection of the surface. The specular reflection is ignored.

(2) Equal quantities of the electrons and the positive ions hit the unit area of the plate per unit time at equilibrium.

(3) The neutralization taking place on the plate is assumed to be complete. This implies that those positive ions and the electrons reflected from the plate are assumed to be neutralized.

(4) The plate is charged negatively.

With these assumptions the pile up of the reflected particles in the front of the plate can be ignored because they are neutralized particles.

Use the Boltzmann transport equation,

$$\frac{\partial f}{\partial t} + \vec{c} \cdot \nabla f + \frac{e\vec{E}}{m} \cdot \nabla_c f = \left(\frac{\delta f}{\delta t}\right)_{col.} \tag{4}$$

and assume the distribution functions as

$$f_e = n_e(z) \left(\frac{m_e}{2\pi KT_e}\right)^{3/2} e^{-\frac{m_e}{2KT_e}(\vec{c}-\vec{V})^2} \tag{5}$$

$$f_i = n_i(z) \left(\frac{m_i}{2\pi KT_i}\right)^{3/2} e^{-\frac{m_i}{2KT_i}(\vec{c}-\vec{V})^2} \tag{6}$$

Note that Equations (5) and (6) are approximate expressions, because the space and the velocity coordinates are not separable in this case.

The substitution of Equations (5) and (6) in (4) for the steady state give

$$\frac{\partial f_e}{\partial z} + \frac{eE}{KT_e}\left(1 + \frac{V}{c_z}\right) f_e = 0 \tag{7}$$

$$\frac{\partial f_i}{\partial z} - \frac{eE}{KT_i}\left(1 + \frac{V}{c_z}\right) f_i = 0 \tag{8}$$

The integration of Equation (7) in the velocity space yields an equation for n_e if f_e is approximated with f_{eo} in the third term of (7). That is

$$\int \frac{\partial f_e}{\partial z} d^3c + \frac{eE}{KT_e} \int f_e \, d^3c \doteq - \frac{eE}{KT_e} \int \frac{V}{c_z} f_{eo} \, d^3c$$

or

$$\frac{\partial n_e}{\partial z} + \frac{eEn_e}{KT_e} \doteq - \frac{eE}{KT_e} \int \frac{V}{c_z} f_{eo} \, d^3c \tag{9}$$

If the r.h.s. of (9) turns out to be small, the approximation is valid. It is found to be

$$\int \frac{V}{c_z} f_{eo} \, d^3c = - n_o \left[2V' e^{-V'^2} \int_o^{V'} e^{t^2} dt \right] \equiv - \gamma_e n_o \tag{10}$$

where

$$V' = \sqrt{\frac{m_e}{2KT_e}} \, V$$

In the usual case, γ_e turns out to be very small. Therefore, the approximation made in Equation (9) is valid. Now (9) can be solved subject to the boundary condition of $n_e \rightarrow n_o$ as $z \rightarrow \infty$, and the relation $E = - \frac{\partial \phi}{\partial z}$ as follows:

$$n_e = n_o \, e^{\frac{e\phi}{KT_e}} - n_o \gamma_e \frac{eE}{KT_e} \tag{11}$$

Equation (11) gives the electron density as a function of space because ϕ is a function of space. Although (11) is an approximate solution it carries an extra correction term which is usually ignored.

For the case of the positive ions, the approximation made in solving (7) is not valid because the effect of the third term of (8) is enormous. Equation (8) is solved approximately as follows:

$$\int \frac{\partial f_i}{\partial z} d^3c \doteq \frac{eE}{KT_i} \int (1 + \frac{V}{c_z}) f_{io} d^3c$$

or

$$\frac{\partial n_i'}{\partial z} \doteq \frac{eE}{KT_i} n_o + \frac{eE}{KT_i} \int \frac{V}{c_z} f_{io} d^3c \qquad (12)$$

It can be shown that

$$\int \frac{V}{c_z} f_{io} d^3c = - n_o \left[2V'' e^{-V''^2} \int_0^{V''} e^{t^2} dt \right] \equiv - \gamma_i n_o \qquad (13)$$

where

$$V'' = \sqrt{\frac{m_i}{2KT_i}} V$$

γ_i turns out to be near unity in the usual case. Subject to the boundary condition of $n_i \rightarrow n_o$ as $z \rightarrow \infty$ and the relation $E = - \frac{\partial \phi}{\partial z}$, the perturbed term of the positive ion, n_i', can be determined as

$$n_i' = - \frac{e\phi}{KT_i} (1 - \gamma_i) n_o \qquad (14)$$

Therefore,

$$n_i = n_{io} + n_i' = n_o - n_o(1 - \gamma_i) \frac{e\phi}{KT_i} \tag{15}$$

Equation (15) gives the density of the positive ions as a function of space. The factor, $(1 - \gamma_i)$ is very close to zero for V'' greater than 1. Therefore, the second term of (15) serves as a correction term. To get an idea about the magnitude of γ_e and γ_i, these are calculated for the case of $T = 1000^oK$, and $m_i = (16 \times 1825)\, m_e$. The results are: $\gamma_e = 3.2 \times 10^{-3}$ and $\gamma_i \doteq 1$ for $V = 7$ Km/sec, $\gamma_e = 6 \times 10^{-5}$ and $\gamma_i = 0.97$ for $V = 1$ Km/sec. This implies that when V is around or higher than the rms velocity of the positive ions, but much lower than that of the electrons, the assumption of $n_e = n_o e^{\frac{e\phi}{KT}}$ and $n_i = n_o$ is quite accurate. This is equivalent to stating that at this mean stream velocity of the moving plasma the potential of the boundary gives a small effect on the distribution of the positive ions but the distribution of the electrons is entirely governed by the potential of the boundary. For the mean stream velocity of the moving plasma lower than the rms velocity of the positive ions the correction terms should be taken into account. $\int_o^x e^{t^2} dt$ is tabulated to facilitate computations. (See Table I)

Potential Distribution in the Plasma Sheath

The density distributions of the ions and the electrons have been determined as a function of the potential or of the space in the preceding section. The potential distribution in the plasma sheath can be found by solving a Poisson's

TABLE I

$$\int_0^x e^{t^2} dt = A \times 10^P$$

x	A	P	x	A	P	x	A	P
0	0		1.70	6.7035 75983	0	4.8	1.0811 27786	9
.05	5.0041 6992	-2	1.75	7.6850 89817	0	4.9	2.7909 92389	9
.10	1.0033 43383	-1	1.80	8.8543 99688	0	5.0	7.3541 89348	9
.15	1.5113 26386	-1	1.85	1.0254 42272	1	5.1	1.9778 61774	10
.20	2.0269 89793	-1	1.90	1.1939 08605	1	5.2	5.4291 66458	10
.25	2.5530 74677	-1	1.95	1.3976 42572	1	5.3	1.5210 31469	11
.30	3.0924 83086	-1	2.0	1.6452 62808	1	5.4	4.3491 39612	11
.35	3.6483 25877	-1	2.1	2.3190 52389	1	5.5	1.2691 74669	12
.40	4.2239 76023	-1	2.2	3.3452 50713	1	5.6	3.7799 45897	12
.45	4.8231 28889	-1	2.3	4.9398 02230	1	5.7	1.1489 19440	13
.50	5.4498 71235	-1	2.4	7.4676 21621	1	5.8	3.5639 15749	13
.55	6.1087 61080	-1	2.5	1.1556 02507	2	5.9	1.1282 19327	14
.60	6.8049 20923	-1	2.6	1.8302 26303	2	6.0	3.6448 74848	14
.65	7.5441 47374	-1	2.7	2.9659 42341	2	6.1	1.2016 82417	15
.70	8.3330 40927	-1	2.8	4.9165 86436	2	6.2	4.0430 59123	15
.75	9.1791 60408	-1	2.9	8.3346 99927	2	6.3	1.3881 59382	16
.80	1.0091 20769	0	3.0	1.4445 45766	3	6.4	4.8637 72076	16
.85	1.1079 24967	0	3.1	2.5591 06616	3	6.5	1.7390 32700	17
.90	1.2154 98595	0	3.2	4.6331 25068	3	6.6	6.3451 09838	17
.95	1.3332 07308	0	3.3	8.5706 33926	3	6.7	2.3624 56410	18
1.00	1.4626 51863	0	3.4	1.6197 22651	4	6.8	8.9759 06314	18
1.05	1.6057 16168	0	3.5	3.1268 14157	4	6.9	3.4800 01325	19
1.10	1.7646 26158	0	3.6	6.1652 40644	4	7.0	1.3767 78248	20
1.15	1.9420 22025	0	3.7	1.2414 92189	5	7.1	5.5581 34208	20
1.20	2.1410 47271	0	3.8	2.5529 88287	5	7.2	2.2896 64923	21
1.25	2.3654 58828	0	3.9	5.3608 52644	5	7.3	9.6247 57736	21
1.30	2.6197 63680	0	4.0	1.1494 02557	6	7.4	4.1283 84333	22
1.35	2.9093 88950	0	4.1	2.5161 68785	6	7.5	1.8069 26236	23
1.40	3.2408 94369	0	4.2	5.6236 11905	6	7.6	8.0699 03889	23
1.45	3.6222 38609	0	4.3	1.2831 56637	7	7.7	3.6775 76227	24
1.50	4.0631 14269	0	4.4	2.9889 27772	7	7.8	1.7100 87084	25
1.55	4.5753 70719	0	4.5	7.1073 35497	7	7.9	8.1140 17191	25
1.60	5.1735 49665	0	4.6	1.7252 00132	8	8.0	3.9283 73639	26
1.65	5.8755 65931	0	4.7	4.2746 46980	8			

equation as follows:

$$\nabla^2\phi = -\frac{e}{\epsilon_o}(n_i - n_e)$$

$$= \frac{en_o}{\epsilon_o}\left(e^{\frac{e\phi}{KT_e}} - \gamma_e \frac{e\phi}{KT_e} - 1 + (1-\gamma_i)\frac{e\phi}{KT_i}\right) \qquad (16)$$

For simplicity, let $T_e = T_i = T$, $(1 - \gamma_i - \gamma_e) = \gamma$. Equation (16) becomes

$$\frac{\partial^2\phi}{\partial z^2} = \frac{en_o}{\epsilon_o}\left(e^{\frac{e\phi}{KT}} - 1 + \gamma\frac{e\phi}{KT}\right) \qquad (17)$$

The boundary conditions for ϕ are

$$\phi = \phi_o \quad \text{at } z = 0 \text{ (on the plate)}$$

$$\phi = 0 \quad \text{as } z \rightarrow \infty.$$

Equation (17) can be solved as follows:

$$\frac{\partial^2\phi}{\partial z^2} = \alpha(e^{\beta\phi} - 1 + \delta\phi) \qquad (18)$$

with

$$\alpha = \frac{en_o}{\epsilon_o}, \qquad \beta = \frac{e}{KT}, \qquad \delta = \gamma\beta .$$

Let

$$\frac{\partial\phi}{\partial z} = P, \qquad \frac{\partial^2\phi}{\partial z^2} = P\frac{dP}{d\phi},$$

then Equation (18) becomes

$$P \frac{dP}{d\phi} = \alpha (e^{\beta\phi} - 1 + \delta\phi)$$

$$\frac{1}{2} P^2 = \alpha (\frac{1}{\beta} e^{\beta\phi} - \phi + \frac{1}{2} \delta\phi^2) + C_1$$

or
$$\frac{d\phi}{dz} = \pm \sqrt{2\alpha (\frac{1}{\beta} e^{\beta\phi} - \phi + \frac{1}{2} \delta\phi^2) + 2C_1} \tag{19}$$

The arbitrary constant C_1 can be determined subject to the boundary condition of $\frac{\partial\phi}{\partial z} \rightarrow 0$ when $z \rightarrow \infty$ (or as $\phi \rightarrow 0$) as

$$C_1 = - \frac{\alpha}{\beta}$$

Therefore, (19) yields

$$dz = \frac{d\phi}{\sqrt{2\alpha} \sqrt{\frac{1}{\beta}(e^{\beta\phi} - 1) - \phi + \frac{1}{2} \delta\phi^2}} \tag{20}$$

and

$$z = \frac{1}{\sqrt{2\alpha}} \int_0^{\phi} \frac{dt}{\sqrt{\frac{1}{\beta}(e^{\beta t} - 1) - t + \frac{1}{2} \delta t^2}} + C_2$$

The abritrary constant C_2 can be determined by the condition of $\phi = \phi_o$ at $z = 0$. Finally, an implicit solution of ϕ as a function of z can be obtained as

$$z = \frac{1}{\sqrt{2\alpha}} \int_{\phi_o}^{\phi} \frac{dt}{\sqrt{\frac{1}{\beta} (e^{\beta t} - 1) - t + \frac{1}{2} \delta t^2}} \tag{21}$$

In the usual cases the integral can be evaluated approximately by expanding the denominator into a power series. The result is

$$z = \sqrt{\frac{\epsilon_o KT}{n_o e^2(1+\gamma)}} \log \left[\frac{\frac{KT}{e\phi}\left(\sqrt{\frac{1+\gamma}{2} + \frac{e\phi}{6KT} + \frac{1}{24}\left(\frac{e\phi}{KT}\right)^2} + \sqrt{\frac{1+\gamma}{2}}\right) + \frac{1}{6\sqrt{2(1+\gamma)}}}{\frac{KT}{e\phi_o}\left(\sqrt{\frac{1+\gamma}{2} + \frac{e\phi_o}{6KT} + \frac{1}{24}\left(\frac{e\phi_o}{KT}\right)^2} + \sqrt{\frac{1+\gamma}{2}}\right) + \frac{1}{6\sqrt{2(1+\gamma)}}} \right] \tag{22}$$

For the case of $\frac{e\phi}{KT} < 1$, a zeroth order approximate expression of (22) becomes

$$\phi \doteq \phi_o \, e^{-\sqrt{\frac{n_o e^2(1+\gamma)}{\epsilon_o KT}}\, z} \tag{23}$$

Potential of the Plate

The potential distribution in the plasma sheath is found as a function of some parameters and the potential of the plate, ϕ_o. ϕ_o can be determined from the condition that equal quantities of the electrons and the positive ions hit a unit area of the plate per unit time at equilibrium. Owing to the fact that the rms velocity of the electrons is much higher than that of the positive ions, more electrons than ions may hit the plate per unit time except the plate is charged negatively so that only very energetic electrons can reach the plate.

Assume the potential of the plate as ϕ_o, which is negative, and critical velocities for the electrons and the positive ions are defined in the same manner as in Chapter II.

$$C_{eo} = \sqrt{\frac{2\left|e\phi_o\right|}{m_e}} \;, \qquad C_{io} = \sqrt{\frac{2\left|e\phi_o\right|}{m_i}}$$

The condition at equilibrium yields an equation as follows:

$$- \int_{-\infty}^{-C_{eo}} c_z dc_z \int_{-\infty}^{\infty} dc_x \int_{-\infty}^{\infty} dc_y \left[f_e\right]_{z=\infty}$$

$$= \left[- \int_{-\infty}^{0} c_z dc_z + \int_{0}^{C_{io}} c_z dc_z \right] \int_{-\infty}^{\infty} dc_x \int_{-\infty}^{\infty} dc_y \left[f_i\right]_{z=\infty} \; . \qquad (24)$$

Under the conditions

$$\sqrt{\frac{2KT}{m_e}} \gg V, \qquad V^2 \gg \left(\frac{2KT}{m_i}\right) \qquad (25)$$

Equation (24) becomes after the integration as follows:

$$\frac{1}{2\sqrt{\pi}\, V} \sqrt{\frac{2KT}{m_e}}\, e^{-\frac{\left|e\phi_o\right|}{KT}} - \frac{1}{2} \operatorname{erf}\left(\sqrt{\frac{\left|e\phi_o\right|}{KT}}\right)$$

$$= \operatorname{erf}\left(V\sqrt{\frac{m_i}{2KT}}\right) - \frac{1}{2} \operatorname{erf}\left(V\sqrt{\frac{m_i}{2KT}} + \sqrt{\frac{\left|e\phi_o\right|}{KT}}\right) . \qquad (26)$$

The potential of the plate, ϕ_o, can be determined numerically from (26). However, when the conditions in Equation (25) are valid, (26) may be approximated as follows

$$\frac{|e\phi_o|}{KT} \doteq \log\left[\frac{1}{\sqrt{\pi}\, V}\sqrt{\frac{2KT}{m_e}}\right] \tag{27}$$

Density Distributions of the Electrons and the Ions in the Plasma Sheath

Up to this point there is enough information for the determination of the density disbributions of the electrons and the ions in the plasma sheath. The procedures are as follows: (1) For a specific plasma with m_e, m_i, n_o, V and T given, the potential of the plate is determined from Equation (26). (2) The potential distribution in the plasma sheath can be calculated from Equation (22). (3) After the potential in the plasma sheath is found the densities of the electrons and the ions are obtained from Equations (11) and (15).

As an example, a plasma with the following parameters is considered.

$m_i = (16 \times 1825)\, m_e$ (oxygen ion)

$T = 1000^o K$

$V = 7\, Km/sec$

$n_o = 10^6\, 1/cc$

In this case

$$n_e \doteq n_o\, e^{\frac{e\phi}{KT}}, \qquad n_i \doteq n_o.$$

$$\frac{e\phi_o}{KT} \doteq -2.61$$

A distance, H, at which $n_e = 0.95\, n_o$ can be calculated as follows:

If $n_e = 0.95\, n_o$, $\frac{e\phi}{KT} = -0.0513.$

So from Equation (22),

$$H = 0.925 \text{ cm.}$$

IV

THE INTERACTION OF A HIGH INTENSITY ELECTROMAGNETIC FIELD WITH A LOW DENSITY PLASMA

The purpose of this study is to explore the basic properties of a plasma when it interacts with a high intensity EM wave. There are many publications on this subject but they are concerned mainly with the small signal case. The conventional result valid for the small signal case is no longer accurate for the large signal case. In this study it is found that some basic parameters of the plasma vary as the functions of the field intensity.

The approach used in this chapter is to find the velocity distribution function by solving some basic equations exactly. After the velocity distribution functions of a plasma are obtained many basic properties of the plasma are readily found. The conventional approach adopted in many papers 5 through 8 is to assume the velocity distribution function as the sum of an isotropic part and a non-isotropic part varying with the frequency of the incident wave. In the case of a high intensity incident wave this assumption is not valid and the velocity distribution function should be found directly from a Boltzmann equation without any approximation made before solving the equation.

The zeroth-order velocity distribution function obtained exactly from a simplified Boltzmann equation produces some conventional properties of the plasma. The first-order velocity distribution function obtained from a Vlasov's equation can produce some significant results. Some parameters of a plasma are evaluated as functions of the field intensity.

Formulation of the Problem

The interaction of an EM wave with a plasma can be described with two sets of equations, namely the Boltzmann equation and the Maxwell equations. These two sets of equations are mutually coupled and results in the non-linear character of the partial differential equation which is to be solved. Assuming an incident EM wave of high intensity interacts with an infinite plasma, the Boltzmann equation is

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \frac{e}{m} (\vec{E} + \vec{v} \times \vec{B}) \cdot \nabla_v f + \frac{e}{m} (\vec{\xi} + \vec{v} \times \vec{b}) \cdot \nabla_v f = \left(\frac{\partial f}{\partial t}\right)_{coll} \tag{1}$$

where f is the velocity distribution function, $\vec{v}$ is the velocity of the charged particle of the plasma, $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields of the incident EM wave, and $\vec{\xi}$ and $\vec{b}$ are the internal fields induced in the plasma. E and B can be represented as follows:

$$\left.\begin{aligned} \vec{E} &= E \cos \omega t \; \hat{x} \\ \vec{B} &= B \cos \omega t \; \hat{y} = \frac{E}{c_o} \cos \omega t \; \hat{y} \end{aligned}\right\} \tag{2}$$

ξ and b are the solutions of the two wave equations which are derived from the Maxwell equations.

$$\nabla^2\vec{\xi} - \mu_o\epsilon_o \frac{\partial^2\vec{\xi}}{\partial t^2} = \mu_o \frac{\partial}{\partial t}\vec{J} + \frac{1}{\epsilon_o}\nabla\rho = \mu_o \frac{\partial}{\partial t}\sum_A e_A \int \vec{v} f_A d^3v + \frac{1}{\epsilon_o}\nabla \sum_A e_A \int f^A d^3v \tag{3}$$

$$\nabla^2\vec{b} - \mu_o\epsilon_o \frac{\partial^2\vec{b}}{\partial t^2} = -\mu_o \nabla \times \vec{J} = -\mu_o \nabla \times \sum_A e_A \int \vec{v} f^A d^3v \tag{4}$$

To make the problem more specific the following assumptions are made.

(1) The plasma consists of electrons, positive ions (singly charged) and neutral particles.

(2) For the low density plasma the collision term is ignored. This is the case of the ionosphere. The collision term is important in the case of high density plasma which will be investigated in later chapters.

(3) The incident EM wave is assumed to be of high intensity but still low enough that a non-relativistic analysis is valid.

(4) The frequency of the EM wave is higher than the electron plasma frequency of the plasma.

(5) The plasma is of infinite extent and homogenous in its unperturbed state.

(6) In the analysis the spatial variation is neglected. This implies that the velocity function is to be determined as a function of the velocity and the time only.

(7) Although the intensity of the EM wave is probably limited by the

breakdown condition of the plasma, this phenomenon is not considered in the present study.

Based on these assumptions, two groups of basic equations for the electrons and the positive ions of the plasma can be formulated as follows:

(A) For the electrons:

$$\frac{\partial f_e}{\partial t} - \frac{e}{m_e}(\vec{E}+\vec{v}\times\vec{B})\cdot\nabla_v f_e - \frac{e}{m_e}(\vec{\xi}+\vec{v}\times\vec{b})\cdot\nabla_v f_e = 0 \qquad (5)$$

$$\nabla^2\vec{\xi} - \mu_o\epsilon_o\frac{\partial^2\vec{\xi}}{\partial t^2} = \mu_o e\frac{\partial}{\partial t}\int\vec{v}(f_i - f_e)\,d^3v + \frac{e}{\epsilon_o}\nabla\int(f_i - f_e)\,d^3v \qquad (6)$$

$$\nabla^2\vec{b} - \mu_o\epsilon_o\frac{\partial^2\vec{b}}{\partial t^2} = -\mu_o e\nabla\times\int\vec{v}(f_i - f_e)\,d^3v \qquad (7)$$

(B) For the positive ions:

$$\frac{\partial f_i}{\partial t} + \frac{e}{m_i}(\vec{E}+\vec{v}\times\vec{B})\cdot\nabla_v f_i + \frac{e}{m_i}(\vec{\xi}+\vec{v}\times\vec{b})\cdot\nabla_v f_i = 0 \qquad (8)$$

with Equations (6) and (7). These equations are solved in the following sections. After f_e and f_i are evaluated many basic properties of the plasma, e.g., the conductivity, the permittivity, and the energy density etc., can be found easily.

Zeroth-Order Velocity Distribution Function

In the zeroth-order analysis, the incident magnetic field and the internal fields are neglected. The assumption is made on the ground that the effect due to the incident electric field is much greater than that caused by the other fields. This assumption leads to a simplified Boltzmann equation and a zeroth-order velocity distribution function. A refined first-order solution is attempted in a later section.

The simplified Boltzmann equation for the electron is

$$\frac{\partial f_e^{(0)}}{\partial t} - \frac{e}{m_e} E \cos \omega t \frac{\partial f_e^{(0)}}{\partial v_x} = 0 \tag{9}$$

and for the positive ions,

$$\frac{\partial f_i^{(0)}}{\partial t} + \frac{e}{m_i} E \cos \omega t \frac{\partial f_i^{(0)}}{\partial v_x} = 0$$

Equation (9) is solved exactly as follows:

$$\frac{\partial f_e^{(0)}}{\partial t} - \gamma \cos \omega t \frac{\partial f_e^{(0)}}{\partial v_x} = 0 \tag{9a}$$

With $\gamma = \frac{eE}{m_e}$. The value of v_x is between $-\infty$ and ∞ and a steady state solution is sought. One more condition is provided by assuming that $f_e^{(0)}$ is a Maxwell-Boltzmann distribution in the absence of the external field.

Define

$$F(k, t) = \int_{-\infty}^{\infty} f_e^{(0)} e^{ikv_x} dv_x$$

and take the Fourier Transform of (9a). This yields the following equation.

$$\frac{\partial F}{\partial t} + i\gamma k \cos\omega t \; F = 0 \tag{10}$$

Equation (10) can be solved and the solution for F is

$$F(k,t) = F_o(k)\, e^{-i\gamma k \frac{1}{\omega} \sin\omega t} \tag{11}$$

$F_o(k)$ is a constant with respect to t and is to be determined from a boundary condition. Since in the absence of the external field (γ is zero) $f_e^{(0)}$ is MB distribution, $F_o(k)$ can be found as follows:

$$F_o(k) = \int_{-\infty}^{\infty} n_o \left(\frac{m_e}{2\pi KT_e}\right)^{3/2} e^{-\frac{m_e}{2KT_e} v^2} e^{ikv_x} dv_x$$

$$= n_o \left(\frac{m_e}{2\pi KT_e}\right) e^{-\frac{m_e(v_y^2 + v_z^2)}{2KT_e}} e^{-\frac{KT_e}{2m_e} k^2} . \tag{12}$$

$f_e^{(0)}$ is obtained by inverting F(k, t) as follows:

$$f_e^{(0)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k,t)\, e^{-ikv_x} dk$$

$$= \frac{1}{2\pi} n_o \left(\frac{m_e}{2\pi KT_e}\right) e^{-\frac{m_e(v_y^2 + v_z^2)}{2KT_e}} \int_{-\infty}^{\infty} e^{-\frac{KT_e}{2m_e} k^2 - i\gamma \frac{k}{\omega} \sin\omega t - ikv_x} dk$$

$$= n_o \left(\frac{m_e}{2\pi KT_e}\right)^{3/2} e^{-\frac{m_e}{2KT_e}\left[\left(v_x + \frac{\gamma}{\omega}\sin\omega t\right)^2 + v_y^2 + v_z^2\right]} . \tag{13}$$

This is the exact solution of (9a).

Similarly, for the positive ions the zeroth-order velocity distribution can be found as

$$f_i^{(0)} = n_o \left(\frac{m_i}{2\pi K T_i}\right)^{3/2} e^{-\frac{m_i}{2KT_i}\left[\left(v_x - \frac{\gamma'}{\omega}\sin\omega t\right)^2 + v_y^2 + v_z^2\right]} \qquad (14)$$

where γ' stands for $\frac{eE}{m_i}$.

These velocity distribution functions are obtained from a simplified Boltzmann equation without any approximation made in solving the equation. These functions can be used to calculate some approximate properties of a plasma. In effect, these functions produce the conventional results which have been derived by other methods.

Applications of Zeroth-Order Velocity Distribution Function

In this section the zeroth-order values of the current, the conductivity, the permittivity and the energy density of the plasma are obtained by using the zeroth-order velocity distribution functions found in the preceeding section.

Zeroth-Order Current:

By definition, the zeroth-order current is

$$\vec{J}^{(0)} = -e \int \vec{v} f_e^{(0)} d^3v + e \int \vec{v} f_i^{(0)} d^3v$$

After carrying out the integration, it is found that

$$\vec{J}^{(0)} = J_x \hat{x} = \left[\frac{e^2 n_o}{\omega m_e} + \frac{e^2 n_o}{\omega m_i} \right] E \sin \omega t \, \hat{x} \tag{15}$$

This agrees with the known result.

Zeroth-Order Conductivity:

The conductivity is defined as the ratio between the current and the applied electric field. That is

$$\sigma = -j \left[\frac{e^2 n_o}{\omega m_e} + \frac{e^2 n_o}{\omega m_i} \right] \tag{16}$$

Zeroth-Order Permittivity:

If the permittivity is defined as $\epsilon = \epsilon_o (1 + \frac{\sigma}{j\omega\epsilon_o})$, the zeroth-order permittivity of the plasma in the absence of a constant magnetic field is found to be

$$\epsilon = \epsilon_o \left[1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2} \right] \tag{17}$$

Where $\omega_{pe} = \frac{e^2 n_o}{m_e \epsilon_o}$, $\omega_{pi} = \frac{e^2 n_o}{m_i \epsilon_o}$. This result is a well-known one.

Zeroth-Order Energy Density:

With the energy density of the plasma defined as

$$u = \int \frac{1}{2} m_e v^2 f_e^{(0)} d^3v + \int \frac{1}{2} m_i v^2 f_i^{(0)} d^3v \tag{18}$$

The zeroth-order result is

$$u = n_o \left[\frac{3KT_e}{2} + \frac{m_e}{2} \left(\frac{e^2 E^2}{\omega^2 m_e^2} \right) \sin^2 \omega t \right]$$

$$+ n_o \left[\frac{3KT_i}{2} + \frac{m_i}{2} \left(\frac{e^2 E^2}{\omega^2 m_i^2} \right) \sin^2 \omega t \right] . \qquad (19)$$

The time average value of u is

$$\bar{u} = n_o \left[\frac{3KT_e}{2} + \frac{m_e}{4} \left(\frac{e^2 E^2}{\omega^2 m_e^2} \right) \right]$$

$$+ n_o \left[\frac{3KT_i}{2} + \frac{m_i}{4} \left(\frac{e^2 E^2}{\omega^2 m_i^2} \right) \right] . \qquad (20)$$

First-Order Velocity Distribution Function

The zeroth-order velocity distribution functions of the plasma have been found by neglecting the incident magnetic field and the internal fields. These approximations are justified in the small signal case. However, for a high intensity incident EM wave these approximations are not justified, because the velocity of the charged particles of the plasma induced by a strong EM wave could be very high. In order to find some basic properties of the plasma with a higher accuracy and valid for the strong signal case, a first order velocity distribution function is attempted in this section. The basic equations to be considered are the Vlasov's equation and the Maxwell equations.

$$\frac{\partial f^{(1)}}{\partial t} + \frac{e}{m}(\vec{E} + \vec{v} \times \vec{B}) \cdot \nabla_v f^{(1)} + \frac{e}{m}(\vec{\xi} + \vec{v} \times \vec{b}) \cdot \nabla_v f^{(1)} = 0 \tag{21}$$

Equation (21) is the Vlasov's equation without the space variation term.

The internal fields are found as follows:

By neglecting all the space variation terms, the internal fields can be written down from Equations (6) and (7) as

$$\vec{b} = 0 \tag{22}$$

and

$$\frac{\partial^2 \vec{\xi}}{\partial t^2} = -\frac{e}{\epsilon_o} \frac{\partial}{\partial t} \int \vec{v}(f_i - f_e)\, d^3v \quad .$$

With the assumption that $\vec{\xi}$ varies as $e^{i\omega t}$, $\vec{\xi}$ can be found approximately as

$$\vec{\xi} \doteq \frac{-e}{\omega^2 \epsilon_o} \frac{\partial}{\partial t} \int \vec{v} f_e^{(0)}\, d^3v$$

$$= \frac{\omega_{pe}^2}{\omega^2} E \cos \omega t\, \hat{x} \quad . \tag{23}$$

The substition of Equations (22), (23), and (2) in (21) give two basic equations for the electrons and the positive ions as follows:

$$\frac{\partial f_e^{(1)}}{\partial t} - \beta\, \gamma \cos \omega t \frac{\partial f_e^{(1)}}{\partial v_x} + \gamma \cos \omega t \frac{v_z}{c_o} \frac{\partial f_e^{(1)}}{\partial v_x} - \gamma \cos \omega t \frac{v_x}{c_o} \frac{\partial f_e^{(1)}}{\partial v_z} = 0 \tag{24}$$

and

$$\frac{\partial f_i^{(1)}}{\partial t} + \beta\gamma' \cos\omega t \frac{\partial f_i^{(1)}}{\partial v_x} - \gamma' \cos\omega t \frac{v_z}{c_o} \frac{\partial f_i^{(1)}}{\partial v_x} + \gamma' \cos\omega t \frac{v_x}{c_o} \frac{\partial f_i^{(1)}}{\partial v_z} = 0 \qquad (25)$$

where $\beta = (1 + \frac{\omega_{pe}^2}{\omega^2})$, $c_o =$ velocity of light, γ and γ' are defined as before. Equations (24) and (25) can be solved exactly and the method used in solving (24) is briefly outlined as follows:

Write (24) as

$$\frac{1}{\cos\omega t} \frac{\partial f_e^{(1)}}{\partial t} + A \frac{\partial f_e^{(1)}}{\partial v_x} + B \left(v_z \frac{\partial f_e^{(1)}}{\partial v_x} - v_x \frac{\partial f_e^{(1)}}{\partial v_z} \right) = 0 \qquad (26)$$

where $A = -\beta\gamma$, $B = \frac{\gamma}{c_o}$.

Define a new variable τ.

$$\tau = A \frac{\sin\omega t}{\omega}$$

and a new constant

$$C = B/A = -\frac{1}{c_o \beta}$$

Equation (26) then becomes

$$\frac{\partial f_e^{(1)}}{d\tau} + (1 + c v_z) \frac{\partial f_e^{(1)}}{\partial v_x} - c v_x \frac{\partial f_e^{(1)}}{\partial v_z} = 0 \, . \qquad (27)$$

The initial condition for $f_e^{(1)}$ is assumed to be

$$f_e^{(1)}(v, 0) = n_o \left(\frac{m_e}{2\pi K T_e} \right)^{3/2} e^{-\frac{m_e}{2KT_e}(v_x^2 + v_y^2 + v_z^2)} \tag{28}$$

Equation (27) implies, [9],

$$d\tau = \frac{dv_x}{1 + cv_z} = - \frac{dv_z}{cv_x} \quad .$$

Using a well-known technique of integrating first order partial differential equations, it can be shown that (27) has a general solution as

$$f_e^{(1)} = g(u_1, u_2) \tag{29}$$

$$u_1 = \frac{c}{2}(v_x^2 + v_z^2) + v_z \tag{30}$$

$$u_2 = \tau + \frac{1}{c} \sin^{-1} \left\{ \frac{1 + cv_z}{\sqrt{1 + 2cu_1}} \right\} \tag{31}$$

where g is an arbitrary function of u_1 and u_2. To construct an appropriate solution for $f_e^{(1)}$, one can make use of the condition that as $t = 0$ (or $\tau = 0$), $g(u_1, u_2)$ should reduce to (28). By finding an expression for $v_x^2 + v_z^2$ in terms of u_1 and u_2 at $\tau = 0$, $f_e^{(1)}$ can be constructed.

$$v_x^2 + v_z^2 = \frac{2}{c} u_1 - \frac{2}{c^2} (\sqrt{1 + 2cu_1} \ \sin cu_2 - 1), \text{ at } \tau = 0$$

Using (30) and (31), it follows

$$\frac{2}{c} u_1 - \frac{2}{c^2} \left(\sqrt{1+2cu_1} \sin cu_2 - 1 \right) = v_x^2 + v_z^2 + \frac{(1-\cos c\tau)(1+cv_z) - cv_x \sin c\tau}{c^2/2}$$

Hence

$$f_e^{(1)} = n_o \left(\frac{m_e}{2\pi KT_e} \right)^{3/2} \exp \left\{ - \frac{m_e}{2KT_e} \left[v_x^2 + v_y^2 + v_z^2 + \frac{(1-\cos c\tau)(1+cv_z) - cv_x \sin c\tau}{c^2/2} \right] \right\} \tag{32}$$

A further simplification is made as follows:

$$v_x^2 + v_y^2 + v_z^2 + \frac{2}{c^2} (1 - \cos c\tau)(1 + cv_z) - cv_x \sin c\tau$$

$$= v_x^2 + v_y^2 + v_z^2 + 2\beta c_o v_x \sin \left(\frac{\gamma}{\omega c_o} \sin \omega t \right) + 2\beta^2 c_o^2 \left(1 - \frac{v_z}{\beta c_o}\right) \left[1 - \cos \left(\frac{\gamma}{\omega c_o} \sin \omega t \right) \right]$$

$$= \left[v_x + \beta c_o \sin \left(\frac{\gamma}{\omega c_o} \sin \omega t \right) \right]^2 + v_y^2 + \left[v_z - \beta c_o \left(1 - \cos \left(\frac{\gamma}{\omega c_o} \sin \omega t \right) \right) \right]^2$$

Therefore, the final solution for $f_e^{(1)}$ is

$$f_e^{(1)} = n_o \left(\frac{m_e}{2\pi KT_e} \right)^{3/2} \exp \left[- \frac{m_e}{2KT_e} \left\{ \left[v_x + \beta c_o \sin \left(\frac{\gamma}{\omega c_o} \sin \omega t \right) \right]^2 + v_y^2 + \left[v_z - \beta c_o \left(1 - \cos \left(\frac{\gamma}{\omega c_o} \sin \omega t \right) \right) \right]^2 \right\} \right] \tag{33}$$

with

$$\beta = \left(1 + \frac{\omega_{pe}^2}{\omega^2}\right), \qquad \gamma = \frac{eE}{m_e}, \qquad c_o = \text{velocity of light.}$$

By using the same technique, $f_i^{(1)}$ can be found-as

$$f_i^{(1)} = n_o \left(\frac{m_i}{2\pi KT_i}\right)^{3/2} \exp\left[-\frac{m_i}{2KT_i}\left\{\left[v_x - \beta c_o \sin\left(\frac{\gamma'}{\omega c_o}\sin\omega t\right)\right]^2 + v_y^2 + \left[v_z - \beta c_o\left(1 - \cos\left(\frac{\gamma'}{\omega c_o}\sin\omega t\right)\right)\right]^2\right\}\right] \tag{34}$$

with $\gamma' = \frac{eE}{m_i}$.

Equations (33) and (34) are the exact solutions of (24) and (25). These solutions are mathematically very neat and physically very plausible. Some interesting results are found with (33) and (34) and are presented in the next section.

Applications of First-Order Velocity Distribution Function

In this section the first order values of the current, the conductivity, the permittivity and the energy density of the plasma are obtained.

First-Order Current:

With the current defined by:

$$\vec{J}^{(1)} = -e\int \vec{v} f_e^{(1)} d^3v + e\int \vec{v} f_i^{(1)} d^3v$$

the result is found to be

$$\vec{J}^{(1)} = J_x\hat{x} + J_z\hat{y} \tag{35}$$

$$J_x = en_o\left(1+\frac{\omega_{pe}^2}{\omega^2}\right) c_o \sin\left(\frac{eE}{\omega m_e c_o}\sin\omega t\right) + en_o\left(1+\frac{\omega_{pe}^2}{\omega^2}\right) c_o \sin\left(\frac{eE}{\omega m_i c_o}\sin\omega t\right) \quad (36)$$

$$J_z = en_o\left(1+\frac{\omega_{pe}^2}{\omega^2}\right) c_o \cos\left(\frac{eE}{\omega m_e c_o}\sin\omega t\right) - en_o\left(1+\frac{\omega_{pe}^2}{\omega^2}\right) c_o \cos\left(\frac{eE}{\omega m_i c_o}\sin\omega t\right) . \quad (37)$$

The factor $\frac{eE}{\omega m_e c_o}$ is always smaller than unity, because $\frac{eE}{\omega m_e}$ is the velocity of the electron induced by an incident electric field E and this cannot exceed the velocity of light, thus (36) and (37) can be expanded into Fourier series as follows:

$$J_x \doteq en_o\left(1+\frac{\omega_{pe}^2}{\omega^2}\right)\left[\frac{eE}{\omega m_e} - \frac{1}{8c_o^2}\left(\frac{eE}{\omega m_e}\right)^3 + \frac{1}{320}\frac{1}{c_o^4}\left(\frac{eE}{\omega m_e}\right)^5\right.$$

$$\left. + \frac{eE}{\omega m_i} - \frac{1}{8c_o^2}\left(\frac{eE}{\omega m_i}\right)^3 + \frac{1}{320}\frac{1}{c_o^4}\left(\frac{eE}{\omega m_i}\right)^5\right]\sin\omega t$$

$$+ en_o\left(1+\frac{\omega_{pe}^2}{\omega^2}\right)\left[\frac{1}{24}\frac{1}{c_o^2}\left(\frac{eE}{\omega m_e}\right)^3 - \frac{1}{640}\frac{1}{c_o^4}\left(\frac{eE}{\omega m_e}\right)^5\right.$$

$$\left. + \frac{1}{24}\frac{1}{c_o^2}\left(\frac{eE}{\omega m_i}\right)^3 - \frac{1}{640}\frac{1}{c_o^4}\left(\frac{eE}{\omega m_i}\right)^5\right]\sin 3\omega t$$

$$+ en_o\left(1+\frac{\omega_{pe}^2}{\omega^2}\right)\left[\frac{1}{1920}\frac{1}{c_o^4}\left(\frac{eE}{\omega m_e}\right)^5 + \frac{1}{1920}\frac{1}{c_o^4}\left(\frac{eE}{\omega m_i}\right)^5\right]\sin 5\omega t \quad (38)$$

$$J_z \doteq -en_o\left(1+\frac{\omega_{pe}^2}{\omega^2}\right)\left[\frac{1}{4}\frac{1}{c_o}\left(\frac{eE}{\omega m_e}\right)^2 - \frac{1}{64}\frac{1}{c_o^3}\left(\frac{eE}{\omega m_e}\right)^4 - \frac{1}{4}\frac{1}{c_o}\left(\frac{eE}{\omega m_i}\right)^2 + \frac{1}{64}\frac{1}{c_o^3}\left(\frac{eE}{\omega m_i}\right)^4\right]$$

$$+ en_o\left(1+\frac{\omega_{pe}^2}{\omega^2}\right)\left[\frac{1}{4}\frac{1}{c_o}\left(\frac{eE}{\omega m_e}\right)^2 - \frac{1}{48}\frac{1}{c_o^3}\left(\frac{eE}{\omega m_e}\right)^4 - \frac{1}{4}\frac{1}{c_o}\left(\frac{eE}{\omega m_i}\right)^2 + \frac{1}{48}\frac{1}{c_o^3}\left(\frac{eE}{\omega m_i}\right)^4\right]\cos 2\omega t$$

$$+ en_o\left(1+\frac{\omega_{pe}^2}{\omega^2}\right)\left[\frac{1}{192}\frac{1}{c_o^3}\left(\frac{eE}{\omega m_e}\right)^4 - \frac{1}{192}\frac{1}{c_o^3}\left(\frac{eE}{\omega m_i}\right)^4\right]\cos 4\omega t\,. \qquad (39)$$

Some significant points are summarized as follows:

(1) In the absence of a constant magentic field, a strong EM wave produces a drift current in the plasma in the direction of the propagation of the wave. This phenomenon can be visualized from the physical point of view.

(2) The current produced in the direction of the E field of the incident EM wave has odd harmonics and that in the direction of propagation of the incident wave has even harmonics.

(3) The current induced by a strong EM wave is not a linear function of the field intensity of the incident EM wave.

First-Order Conductivity:

If the conductivity of a plasma is defined as the ratio between the fundamental component of the current to the incident electric field, the conductivity is found to be

$$\sigma_{xx} = -j\, en_o\left(1+\frac{\omega_{pe}^2}{\omega^2}\right)\left\{\left[\frac{e}{\omega m_e}+\frac{e}{\omega m_e}\right]-\frac{1}{8c_o^2}\left[\left(\frac{e}{\omega m_e}\right)^3+\left(\frac{e}{\omega m_i}\right)^3\right]E^2\right.$$

$$\left.+\frac{1}{320\, c_o^4}\left[\left(\frac{e}{\omega m_e}\right)^5+\left(\frac{e}{\omega m_i}\right)^5\right]E^4\right\} \tag{40}$$

The conductivity in the z-direction cannot be defined since I_z does not have a fundamental component.

First-Order Permittivity:

According to the definition of the permittivity of the plasma,

$$\epsilon_{xx} = \epsilon_o\left(1+\frac{\sigma_{xx}}{j\omega\epsilon_o}\right)$$

the result is found to be

$$\frac{\epsilon_{xx}}{\epsilon_o} = 1-\left(1+\frac{\omega_{pe}^2}{\omega^2}\right)\left\{\left[\frac{\omega_{pe}^2}{\omega^2}+\frac{\omega_{pi}^2}{\omega^2}\right]-\frac{1}{8c_o^2}\left[\frac{\omega_{pe}^2}{\omega^2}\left(\frac{e}{\omega m_e}\right)^2+\frac{\omega_{pi}^2}{\omega^2}\left(\frac{e}{\omega m_i}\right)^2\right]E^2\right.$$

$$\left.+\frac{1}{320c_o^4}\left[\frac{\omega_{pe}^2}{\omega^2}\left(\frac{e}{\omega m_e}\right)^4+\frac{\omega_{pi}^2}{\omega^2}\left(\frac{e}{\omega m_i}\right)^4\right]E^4\right\} \tag{41}$$

This is a function of the intensity of the incident EM wave.

First-Order Energy Density:

With the energy density of the plasma defined as

$$u = \int \frac{1}{2} m_e v^2 f_e^{(1)} d^3v + \int \frac{1}{2} m_i v^2 f_i^{(1)} d^3v$$

the result is

$$u = n_o \left[\frac{3}{2} KT_e + m_e \left(1 + \frac{\omega_{pe}^2}{\omega^2}\right)^2 c_o^2 \left(1 - \cos\left(\frac{eE}{\omega m_e c_o} \sin\omega t\right)\right) \right]$$

$$+ n_o \left[\frac{3}{2} KT_i + m_i \left(1 + \frac{\omega_{pe}^2}{\omega^2}\right)^2 c_o^2 \left(1 - \cos\left(\frac{eE}{\omega m_i c_o} \sin\omega t\right)\right) \right] \quad (42)$$

The time average value is

$$\bar{u} = n_o \left[\frac{3}{2} KT_e + \frac{1}{4} m_e \left(1 + \frac{\omega_{pe}^2}{\omega^2}\right)^2 \left(\frac{eE}{\omega m_e}\right)^2 - \frac{1}{64} m_e \left(1 + \frac{\omega_{pe}^2}{\omega^2}\right)^2 \frac{1}{c_o^2} \left(\frac{eE}{\omega m_e}\right)^4 \right.$$

$$\left. + \frac{3}{2} KT_i + \frac{1}{4} m_i \left(1 + \frac{\omega_{pe}^2}{\omega^2}\right)^2 \left(\frac{eE}{\omega m_i}\right)^2 - \frac{1}{64} m_i \left(1 + \frac{\omega_{pe}^2}{\omega^2}\right)^2 \frac{1}{c_o^2} \left(\frac{eE}{\omega m_i}\right)^4 \right] . \quad (43)$$

Non-Linear Modeling the Velocity Distribution Function

Equation (9a) can serve as the starting point of a non-linear modeling attempt in the following way.

For different values of γ, say γ_1 and γ_2, we write

$$\frac{\partial f_1}{\partial t} - \gamma_1 \cos\omega t \frac{\partial f_1}{\partial v_x} = 0 \quad (44)$$

and

$$\frac{\partial f_2}{\partial t} - \gamma_2 \cos \omega t \, \frac{\partial f_2}{\partial v_x} = 0 \tag{45}$$

Now we ask if it is possible to find f_2 as a function of f_1 alone.

Assuming that such a relationship does exist we try to determine it as follows. We assume

$$f_2 = f_2(f_1) \qquad \text{hence}$$

$$\frac{\partial f_2}{\partial t} = \frac{df_2}{df_1} \frac{\partial f_1}{\partial t} \qquad \text{and} \qquad \frac{\partial f_2}{\partial v_x} = \frac{df_2}{df_1} \frac{\partial f_1}{\partial v_x}$$

and (45) becomes

$$\frac{df_2}{df_1} \left[\frac{\partial f_1}{\partial t} - \gamma_2 \cos \omega t \, \frac{\partial f_1}{\partial v_x} \right] = 0 \tag{46}$$

or making use of (44)

$$\frac{df_2}{df_1} \cos \omega t \, \frac{\partial f_1}{\partial v_x} (\gamma_1 - \gamma_2) = 0 \tag{47}$$

Since $\frac{\partial f_1}{\partial v_x}$ is not identically zero and γ_1 is chosen $\neq \gamma_2$ it follows that

$$\frac{df_2}{df_1} = 0$$

or f_2 = constant. This result is physically impossible thus we conclude that $f_2 = f_2(f_1)$ is invalid. We might then ask, more modestly, if we can find f_2 as a function of f_1 and t, however for these modeling attempts it seems more appropriate not to work directly with the velocity distribution function but an integral of this function as will become evident in the non-linear modeling discussion of the next example, the high intensity field acting on a weakly ionized gas.

V

THE INTERACTION OF A HIGH INTENSITY ELECTROMAGNETIC FIELD WITH A WEAKLY IONIZED GAS

The previous chapter treated the interaction of a high intensity EM field with a very low density plasma where the collision effect can be neglected. In this chapter the interaction of a high intensity EM field with a weakly ionized gas, or a plasma of low ionization, is considered. The collision effect is important in this case.

A partially ionized gas is assumed to be composed of electrons, the positive ions (singly charged) and neutral particles. The possible collisions are e-n, i-n, e-i, e-e, i-i, and n-n. If the degree of ionization of the gas is low, only the e-n and the i-n collisions are important in the analysis. Fortunately, the mathematical models for these two types of collision are simple, therefore, a rather rigorous analysis is possible. The Boltzmann equation is solved exactly with a simple mathematical model for the collision. Some reasonable approximations are made to derive some useful parameters only after the exact solution of the Boltzmann equation is obtained.

The zeroth-order and the first-order solutions for the velocity distribution function are found. The basic parameters of a plasma are evaluated as the functions of the intensity of the incident EM field.

Formulation of the Problem

The Boltzmann equation is

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \frac{e}{m}(\vec{E}+\vec{v} \times \vec{B}) \cdot \nabla_v f + \frac{e}{m}(\vec{\xi}+\vec{v} \times \vec{b}) \cdot \nabla_v f = \left(\frac{\partial f}{\partial t}\right)_{coll.} \qquad (1)$$

where $\vec{E}$ and $\vec{B}$ are external fields and can be expressed as

$$\begin{cases} \vec{E} = E \cos \omega t \, \hat{x} \\ \vec{B} = B \cos \omega t \, \hat{y} = \dfrac{E}{c_o} \cos \omega t \, \hat{y} \end{cases} \qquad (2)$$

The internal field $\vec{\xi}$ and $\vec{b}$ are found in the previous chapter as

$$\begin{cases} \vec{\xi} \doteq \dfrac{\omega_p^2}{\omega^2} E \cos \omega t \, \hat{x} \\ \vec{b} \doteq 0 \end{cases} \qquad (3)$$

The spatial variation is neglected again in this part of the study.

The collision term is approximated as follows:

For the electron case, among all types of collision the e-n collision is predominant when the degree of ionization is low. To take into account this collision the simplest mathematical model is to assume

$$\left(\frac{\partial f}{\partial t}\right)_{coll.} = -\nu(f - f_o) \qquad (4)$$

ν is the collision frequency of the electrons and the neutral particles. ν is known to be nearly constant for the low velocity electrons and nearly proportional to the velocity for the high velocity electrons. A more accurate behavior of ν was found in experiments but is too complicated for the theoretical analysis. In order to improve the accuracy of the collision model expressed in (4), f_o must be modified. f_o is the equilibrium state of f and can be assumed to be a Maxwellian distribution in the small signal case. However, if the incident EM field is strong, f_o may be quite different from Maxwellian. It has been found that the isotropic part of f which is constant with respect to time varies as a function of the intensity of the incident EM field [5]. After the substitution of equation (4) in (1), the comparison of both sides of (1) suggests that f_o in (4) should be the isotropic part of f because the left hand side of (1) is dependent on time when no constant external field is present. By modifying f_o and specifying ν carefully, the collision model expressed in (4) is acceptable for the present analysis.

The zeroth-order analysis is made by neglecting the incident magnetic field and the internal fields. These neglected fields are taken into account in the first-order analysis.

Zeroth-Order Velocity Distribution Function

For the zeroth-order analysis, the effects due to the incident magnetic field and the internal fields are neglected. The simplified Boltzmann equations to be solved are as follows:

For the electrons:

$$\frac{\partial f_e^{(1)}}{\partial t} - \gamma \cos \omega t \, \frac{\partial f_e^{(0)}}{\partial v_x} = -\nu (f_e^{(0)} - f_{eo}) \tag{5}$$

where $\gamma = \frac{eE}{m_e}$, ν is the collision frequency of the electrons and the neutral particles, and f_{eo} is the equilibrium state of $f_e^{(0)}$. f_{eo} will be specified carefully later.

For the positive ions:

$$\frac{\partial f_i^{(0)}}{\partial t} + \gamma' \cos\omega t \frac{\partial f_i^{(0)}}{\partial v_x} = -\nu'(f_i^{(0)} - f_{io}) \tag{6}$$

where $\gamma' = \frac{eE}{m_i}$, ν' is the collision frequency of the positive ions and the neutral particles, and f_{io} is the equilibrium state of $f_i^{(0)}$. f_{io} can be assigned to be Maxwellian due to the large mass of the positive ions.

The analysis is carried out for the electron case only.

Equation (5) implies the following relations:

$$dt = \frac{dv_x}{-\gamma\cos\omega t} = \frac{df_e^{(0)}}{-\nu(f_e^{(0)} - f_{eo})} \tag{7}$$

From (7) two equations are obtained as follows:

$$v_x + \frac{\gamma}{\omega}\sin\omega t = C = \text{constant} \tag{8}$$

$$\frac{df_e^{(0)}}{dt} + \nu f_e^{(0)} = \nu f_{eo} \tag{9}$$

Equation (9) leads to

$$e^{\int \nu dt} f_e^{(0)} - \left[e^{\int \nu dt} f_e^{(0)} \right]_{t=-\infty} = \int_{-\infty}^{t} \nu f_{eo} e^{\int \nu dt} dt.$$

If the collision frequency ν is assumed to be independent of time, t, $f_e^{(0)}$ can be written as

$$e^{\nu t} f_e^{(0)} = \int_{-\infty}^{t} \nu f_{eo} e^{\nu t} dt \quad . \tag{10}$$

In evaluating the integral, f_{eo} is considered to be a function of t. Originally f_{eo} is a function of v_x, v_y, and v_z. However, with (8) f_{eo} in this integral should be rewritten as

$$f_{eo}(v_x, v_y, v_z) \rightarrow f_{eo}(C - \frac{\Upsilon}{\omega} \sin \omega t, v_y, v_z)$$

or (10) is rewritten as

$$e^{\nu t} f_e^{(0)} = \int_{-\infty}^{t} \nu f_{eo}(C - \frac{\Upsilon}{\omega} \sin \omega s, v_y, v_z) \, e^{\nu s} ds \quad . \tag{10a}$$

In order to put the integral on the right of (10a) more explicitly, a new variable u = s - t is introduced. This changes (10a) to

$$e^{\nu t} f_e^{(0)} = \int_{-\infty}^{0} \nu f_{eo}(C - \frac{\Upsilon}{\omega} \sin \omega(u+t), v_y, v_z) \, e^{\nu(u+t)} du \tag{11}$$

The constant, C, is eliminated by substituting (8) in (11). Thus

$$f_e^{(0)} = \int_{-\infty}^{0} \nu f_{eo} \left[v_x + \frac{\Upsilon}{\omega} \sin \omega t - \frac{\Upsilon}{\omega} \sin \omega(u+t), v_y, v_z \right] e^{\nu u} du$$

Changing the variable u into -u, the final solution for $f_e^{(0)}$ is obtained as

$$f_e^{(0)} = \int_0^{\infty} \nu f_{eo} \left[v_x + \frac{\gamma}{\omega} (\sin \omega u \cos \omega t + (1 - \cos \omega u) \sin \omega t),\ v_y,\ v_z \right] e^{-\nu u}\, du \tag{12}$$

Equation (12) is the exact solution of (5) subject to the assumption that ν is independent of time. The argument of f_{eo} appearing in (12) means that the v_x term in the original form of f_{eo} is to be replaced by

$$v_x + \frac{\gamma}{\omega} \left[\sin \omega u \cos \omega t + (1 - \cos \omega u) \sin \omega t \right] .$$

Equation (12) is a definite integral and can be evaluated immediately once f_{eo} and ν are specified.

In order to have an accurate form of f_{eo} which is also appropriate for the formulation (5), the following cases are considered.

As the first choice, the result obtained by Margenau [5] will be used. That is

$$f_{eo} = A \exp \left[- \int_0^{v^2} \frac{m_e / 2dc^2}{KT + M\gamma^2 \lambda^2 / 6(c^2 + \omega^2 \lambda^2)} \right] \tag{13}$$

where M is the mass of neutral particles, λ is the mean free path of the electron and the neutral particles. A is a constant defined in such a way that

$$\int f_{eo}\, d^3 v = n_o$$

If (13) is used, $f_e^{(0)}$ can be expressed explicitly as

$$f_e^{(0)} = \int_0^\infty \nu e^{-\nu u} du\ A \exp\left[- \int_0^{(v_x + H(u,t))^2 + v_y^2 + v_z^2} \frac{m_e/2dc^2}{KT + M\gamma^2\lambda^2/6(c^2+\omega^2\lambda^2)} \right] \tag{14}$$

where $H(u,t) \equiv \frac{\gamma}{\omega}\left[\sin \omega u \cos \omega t - (1 - \cos \omega u) \sin \omega t\right]$. In the evaluation of (14), ν can be assumed to be a constant or v/λ depending on the circumstance.

As some special cases the following are considered:

(1) For the case of $KT > \frac{M\gamma^2}{\omega^2}$, $\omega^2 > \nu^2$, or when the frequency of the EM field is higher than the collision frequency and the kinetic energy added by the incident EM field is smaller than the thermal energy of the electrons, an approximate expression for $f_e^{(0)}$ can be found as

$$f_e^{(0)} = \int_0^\infty \nu e^{-\nu u} du\ A \exp\left\{ - \frac{m_e}{2KT_e} \left[(v_x + H(u,t))^2 + v_y^2 + v_z^2 \right] \right\} \tag{15}$$

with

$$T_e = T + \frac{M\gamma^2}{6K\left[\omega^2 + \frac{3}{\lambda^2 m_e}\left(KT + \frac{M\gamma^2}{6\omega^2}\right)\right]} \tag{16}$$

(2) For the case of $\frac{M\gamma^2}{\omega^2} \gg KT$, or the very strong EM field case, $f_e^{(0)}$ can be expressed as

$$f_e^{(0)} = \int_0^\infty \nu e^{-\nu u} du\ A \exp\left\{ - \frac{3m_e}{2M\gamma^2\lambda^2} \left[(v_x + H(u,t))^2 + v_y^2 + v_z^2 \right]^2 + 2\omega^2\lambda^2 \left[(v_x + H(u,t))^2 + v_y^2 + v_z^2 \right] \right\} \tag{17}$$

corresponding to the limiting case of f_{eo},

$$f_{eo} = A \exp\left[-\frac{3m_e(v^4 + 2\omega^2\lambda^2 v^2)}{2M\gamma^2\lambda^2}\right]$$

(3) For the case of $\nu^2 > \omega^2$, or as the frequency of the EM field is lower than the collision frequency, the result obtained by Chapman and Cowling [10] seems more appropriate. In this case

$$f_{eo} = A \exp\left[-\int_0^v \frac{3m_e c^3 dc}{3KTc^2 + M\gamma^2\lambda^2}\right] \tag{18}$$

and

$$f_e^{(0)} = \int_0^\infty \nu e^{-\nu u} du \; A \exp\left[-\int_0^{\left[(v_x + H(u,t))^2 + v_y^2 + v_z^2\right]^{1/2}} \frac{3m_e c^3 dc}{3KTc^2 + M\gamma^2\lambda^2}\right] \tag{19}$$

(4) For the case of $\nu^2 > \omega^2$ and $M\gamma^2 \frac{\lambda^2}{c^2} \gg KT$, the limiting case of (18) or the result obtained by Druyvesteyn is used. That is

$$f_{eo} = A \exp\left[-\frac{3m_e v^4}{4M\gamma^2\lambda^2}\right] \tag{20}$$

and

$$f_e^{(0)} = \int_0^\infty \nu e^{-\nu u} du \; A \exp\left\{-\frac{3m_e}{4M\gamma^2\lambda^2}\left[(v_x + H(u,t))^2 + v_y^2 + v_z^2\right]\right\} \tag{21}$$

As a matter of interest and with the purpose of checking the theory, a well known result of f_e for the small signal case is reproduced from (12).

When the incident EM field is small, it is accurate to assign f_{eo} as

$$f_{eo} = n_o \left(\frac{m_e}{2\pi KT} \right)^{3/2} e^{-\frac{m_e v^2}{2KT}}$$

and it is also reasonable to put

$$f_{eo}\left[v_x + \frac{\gamma}{\omega}(\sin\omega u \cos\omega t + (1-\cos\omega u)\sin\omega t),\ v_x,\ v_z \right]$$

$$\doteq n_o \left(\frac{m_e}{2\pi KT} \right)^{3/2} e^{-\frac{m_e v^2}{2KT}} \left[1 - \frac{m_e}{KT}\frac{\gamma}{\omega} v_x (\sin\omega u \cos\omega t + (1-\cos\omega u)\sin\omega t) \right]$$

The substitution of this expression in (12) gives

$$f_e^{(0)} = f_{eo}\left[1 - \frac{m_e}{KT} v_x \frac{\gamma\nu}{\omega^2+\nu^2}\cos\omega t - \frac{m_e}{KT} v_x \frac{\gamma\omega}{\omega^2+\nu^2}\sin\omega t \right]$$

$$= f_{eo} + \gamma v_x \frac{\lambda}{v^2+\omega^2\lambda^2}\left(\frac{\partial f_{eo}}{\partial v} \right)\cos\omega t + \gamma v_x \frac{\omega}{v^2+\omega^2\lambda^2}\left(\frac{\partial f_{eo}}{\partial v} \right)\sin\omega t \qquad (22)$$

This result was derived by Margenau [5] .

Application of Zeroth-Order Velocity Distribution Function

In this section the zeroth-order values of the current, the conductivity and the permittivity are obtained by using the zeroth-order velocity distribution function found in the preceeding section.

Zeroth-Order Current:

With the zeroth-order current defined as

$$\vec{J}^{(0)} \doteq -e \int \vec{v} f_e^{(0)} \, d^3 v$$

it can be found that

$$\vec{J}^{(0)} = I_x \hat{x}$$

and

$$J_x = -e \iiint_{-\infty}^{\infty} v_x dv_x dv_y dv_z \; A \exp \left[- \int_0^{(v_x + H(u,t))^2 + v_y^2 + v_z^2} \frac{m_e/2dc^2}{KT + M\gamma^2\lambda^2/6(c^2 + \omega^2\lambda^2)} \right] \cdot$$

$$\cdot \int_0^{\infty} \nu e^{-\nu u} \, du \qquad (23)$$

To evaluate this complex integral approximately, a new variable $v_x' = v_x + H(u, t)$ is introduced and ν is held constant at this step. Equation (23) then becomes

$$J_x \doteq 4\pi e \int_0^{\infty} du \int_0^{\infty} dv \, v^2 \, A \, \exp \left[- \int_0^{v^2} \frac{m_e/2dc^2}{KT + M\gamma^2\lambda^2/6(c^2 + \omega^2\lambda^2)} \right] H(u,t) \, \nu e^{-\nu u}$$

Think of ν as v/λ, and find a value of v which corresponds to a point near the peak of the integrand of the above integral. This value of v is a root of the following equation.

$$\frac{\partial}{\partial v}\left\{ v^3 \exp\left[- \int_0^{v^2} \frac{m_e/2dc^2}{KT + M\gamma^2\lambda^2/6(c^2+\omega^2\lambda^2)} \right] \right\} = 0$$

The root of this equation is found to be

$$\bar{v} = \frac{1}{2\sqrt{m_e}} \left[\sqrt{8m_e M\gamma^2\lambda^2 + (2m_e\omega^2\lambda^2 + 6\,KT)^2} - 2m_e\omega^2\lambda^2 + 6\,KT \right]^{1/2} \tag{24}$$

J_x is approximated as

$$J_x \doteq 4\pi e \int_0^{\infty} v^2 dv\, A \exp\left[- \int_0^{v^2} \frac{m_e/2dc^2}{KT + M\gamma^2\lambda^2/6(c^2+\omega^2\lambda^2)} \right] \int_0^{\infty} H(u,t)\, \frac{\bar{v}}{\lambda}\, e^{-\frac{\bar{v}}{\lambda}u} du$$

$$\doteq en_o\gamma \frac{\bar{v}/\lambda}{\omega^2 + (\bar{v}/\lambda)^2} \cos\omega t + en_o\gamma \frac{\omega}{\omega^2 + (\bar{v}/\lambda)^2} \sin\omega t \tag{25}$$

It is noted that (25) is the approximate value of the current. If the harmonic components of the current are needed, (23) must be evaluated more accurately.

Zeroth-Order Conductivity:

The conductivity is defined as the ratio between the current and the applied electric field. It is found to be

$$\sigma = \sigma_r - i\sigma_i \tag{26}$$

$$\sigma_r = \frac{n_o e^2}{m_e} \frac{\left[\sqrt{\frac{M\gamma^2}{2m_e\lambda^2} + \left(\frac{\omega^2}{2} + \frac{3KT}{2m_e\lambda^2}\right)^2} - \frac{\omega^2}{2} + \frac{3KT}{2m_e\lambda^2}\right]^{1/2}}{\sqrt{\frac{M\gamma^2}{2m_e\lambda^2} + \left(\frac{\omega^2}{2} + \frac{3KT}{2m_e\lambda^2}\right)^2} + \frac{\omega^2}{2} + \frac{3KT}{2m_e\lambda^2}} \tag{27}$$

$$\sigma_i = \frac{n_o e^2}{m_e} \frac{\omega}{\sqrt{\frac{M\gamma^2}{2m_e\lambda^2} + \left(\frac{\omega^2}{2} + \frac{3KT}{2m_e\lambda^2}\right)^2} + \frac{\omega^2}{2} + \frac{3KT}{2m_e\lambda^2}} \tag{28}$$

Zeroth-Order Permittivity:

If the equivalent permittivity is defined as $\epsilon = \epsilon_o(1 - \frac{\sigma_i}{\omega\epsilon_o})$, the zeroth-order permittivity of the plasma in the absence of a constant magnetic field is found to be

$$\epsilon = \epsilon_o \left[1 - \frac{\omega_{pe}^2}{\sqrt{\frac{M\gamma^2}{2m_e\lambda^2} + \left(\frac{\omega^2}{2} + \frac{3KT}{2m_e\lambda^2}\right)^2} + \frac{\omega^2}{2} + \frac{3KT}{2m_e\lambda^2}} \right] \tag{29}$$

where $\omega_{pe}^2 = \frac{n_o e^2}{m_e \epsilon_o}$.

It is interesting to show that a well known result for the conductivity in the small signal case can be derived from (23). When $KT \gg \frac{M\gamma^2}{\omega^2}$, (23) can be written as

$$J_x \doteq -e \iiint_{-\infty}^{\infty} v_x dv_x dv_y dv_z \, A \, e^{-\frac{m_e v^2}{2KT}} \left[1 - \frac{m_e}{KT} v_x H(u,t)\right] \int_0^{\infty} \nu e^{-\nu u} \, du$$

$$\doteq \frac{4\pi e}{3} \frac{m_e}{KT} \int_0^{\infty} v^4 dv \, A \, e^{-\frac{mv^2}{2KT}} \int_0^{\infty} H(u,t) \, \nu e^{-\nu u} \, du$$

Letting $u = \sqrt{\frac{m_e}{2KT}} \, v$, J_x becomes

$$J_x = \frac{8}{3\sqrt{\pi}} \gamma n_o e \int_0^{\infty} e^{-u^2} \frac{\nu u^4}{\omega^2 + \nu^2} \, du \cos \omega t$$

$$+ \frac{8}{3\sqrt{\pi}} \gamma n_o e \int_0^{\infty} e^{-u^2} \frac{\omega u^2}{\omega^2 + \nu^2} \, du \sin \omega t \qquad (30)$$

and the conductivity can be written as

$$\sigma_r = \frac{n_o e^2}{m_e} \frac{8}{3\sqrt{\pi}} \int_0^{\infty} e^{-u^2} \frac{\nu u^2}{\omega^2 + \nu^2} \, du$$

$$\sigma_i = \frac{n_o e^2}{m_e} \frac{8}{3\sqrt{\pi}} \int_0^{\infty} e^{-u^2} \frac{\omega u^2}{\omega^2 + \nu^2} \, du$$

These are the results obtained by Margenau [8] .

First-Order Velocity Distribution Function

The Boltzmann equations to be solved for the first-order analysis are as follows:

(A) For the electrons,

$$\frac{\partial f_e^{(1)}}{\partial t} - \beta\gamma \cos\omega t \frac{\partial f_e^{(1)}}{\partial v_x} + \gamma \cos\omega t \frac{v_z}{c_o} \frac{\partial f_e^{(1)}}{\partial v_x} - \gamma \cos\omega t \frac{v_x}{c_o} \frac{\partial f_e^{(1)}}{\partial v_x}$$

$$= -\nu(f_e^{(1)} - f_{eo}) \tag{31}$$

where $\beta = (1 + \frac{\omega_{pe}^2}{\omega^2})$, c_o = velocity of light, γ and ν are defined as before.

(B) For the positive ions,

$$\frac{\partial f_i^{(1)}}{\partial t} + \beta\gamma' \cos\omega t \frac{\partial f_i^{(1)}}{\partial v_x} - \gamma' \cos\omega t \frac{v_z}{c_o} \frac{\partial f_i^{(1)}}{\partial v_x} + \gamma' \cos\omega t \frac{v_x}{c_o} \frac{\partial f_i^{(1)}}{\partial v_z}$$

$$= -\nu'(f_i^{(1)} - f_{io}) \tag{32}$$

where γ' and ν' are defined as before.

Equation (31) implies

$$dt = \frac{dx}{-(\beta - \frac{v_z}{c_o})\gamma \cos\omega t} = \frac{dz}{-\frac{v_x}{c_o}\gamma \cos\omega t} = \frac{df_e^{(1)}}{-\nu(f_e^{(1)} - f_{eo})} \tag{33}$$

Three equations are obtained from (33) as follows:

$$v_x + (\beta - \frac{v_z}{c_o}) \frac{\gamma}{\omega} \sin\omega t = c_1 \tag{34}$$

$$v_z + \frac{v_x}{c_o}\frac{\gamma}{\omega}\sin\omega t = c_2 \tag{35}$$

$$\frac{df_e^{(1)}}{dt} + \nu f_e^{(1)} = \nu f_{eo} \tag{36}$$

or

$$v_x = \frac{c_1 - \left(\beta - \frac{c_2}{c_o}\right)\frac{\gamma}{\omega}\sin\omega t}{1 + \frac{1}{c_o^2}\frac{\gamma^2}{\omega^2}\sin^2\omega t} \tag{37}$$

$$v_z = \frac{c_2 - \left(\frac{c_1}{c_o} - \frac{\beta}{c_o}\frac{\gamma}{\omega}\sin\omega t\right)\frac{\gamma}{\omega}\sin\omega t}{1 + \frac{1}{c_o^2}\frac{\gamma^2}{\omega^2}\sin^2\omega t} \tag{38}$$

Following the same procedure as in the zeroth-order analysis, an expression for $f_e^{(1)}$ can be found to be

$$f_e^{(1)} = \int_0^{\infty} \nu e^{-\nu u}\, du\, f_{eo}(v_x', v_y', v_z') \tag{39}$$

$f_{eo}(v_x', v_y', v_z')$ means that v_x, v_y, v_z terms in the original form of f_{eo} are to be replaced by v_x', v_y', v_z'. v_x', v_y', v_z' are expressed in terms of v_x, v_y, v_z, t, u, and some other parameters as follows:

$$v'_x = \frac{1}{1 + \frac{1}{c_o^2}\frac{\gamma^2}{\omega^2}\sin^2\omega(t-u)} \left\{ v_x \left[1 - \frac{1}{c_o^2}\frac{\gamma^2}{\omega^2}\frac{1}{2}(\sin\omega u \sin 2\omega t - \cos\omega u(1-\cos 2\omega t))\right] - \frac{v_z}{c_o}\frac{\gamma}{\omega}\left[\sin\omega u \cos\omega t + (1-\cos\omega u)\sin\omega t\right] + \beta\frac{\gamma}{\omega}\left[\sin\omega u\cos\omega t + (1-\cos\omega u)\sin\omega t\right]\right\} \quad (40)$$

$$v'_y = v_y \quad (41)$$

$$v'_z = \frac{1}{1 + \frac{1}{c_o^2}\frac{\gamma^2}{\omega^2}\sin^2\omega(t-u)} \left\{ v_z \left[1 + \frac{1}{c_o^2}\frac{\gamma^2}{\omega^2}\sin^2\omega t\right] - \frac{\beta}{c_o}\frac{\gamma^2}{\omega^2}\frac{1}{2}\sin\omega u \sin 2\omega t - \frac{\beta}{c_o}\frac{\gamma^2}{\omega^2}\frac{1}{2}(1-\cos\omega u)(1-\cos 2\omega t)\right\} \quad (42)$$

After the substitution of (40) – (42) in (39), $f_e^{(1)}$ can be theoretically evaluated as a function of velocity, time, and the intensity of the EM field. The actual evaluation of (39) may be quite impossible without some approximations. The applications of the first-order velocity distribution function are straightforward but are omitted here to avoid the lengthy mathematical formulas.

Non-Linear Modeling the Induced Current

We may use this example in a non-linear modeling attempt in the following way. In one system we measure the effect of applying a weak EM field to a plasma, while in another system we would like information when a strong field is applied.

In this example, our purpose is to predict the result of the strong field experiment from the data obtained in the weak field experiment. This problem can be solved if we can model the basic differential equations which govern the two systems. That is to say we aim to obtain a quantity in system II as a function of the quantity in system I theoretically. The quantities of interest are the induced currents in two plasma systems to which two different electric fields are applied. We assume that the two systems have identical plasma of weakly ionized gas type. If an electric field of $E \cos \omega t$ is applied to the first system, there will be an induced current i_1. The question is what will be the current i_2 in the second system if a strong field, $\mathcal{l} E \cos \omega t$, is applied to it. $\mathcal{l}$ is a constant much bigger than unity. If we can succeed in solving for i_2 as a function of i_1 theoretically, we can predict i_2 from i_1 which can be obtained from a much easier experiment.

For the two systems of plasma assumed above, we can formulate two Boltzmann equations to describe them as follows:

$$\frac{\partial f_1}{\partial t} - \frac{eE}{m}\cos\omega t \frac{\partial f_1}{\partial v_x} = -\nu_1\left(f_1 - \frac{n_1(t)}{n_o} f_{10}\right) \tag{43}$$

$$\frac{\partial f_2}{\partial t} - \mathcal{L}\frac{eE}{m}\cos\omega t \frac{\partial f_2}{\partial v_x} = -\nu_2\left(f_2 - \frac{n_2(t)}{n_o} f_{20}\right) \tag{44}$$

f_1 and f_2 represent the velocity distribution functions of the electrons in system I and II. To system I, an electric field, $E\cos\omega t$, is applied in the x-direction and to system II, a strong electric field of $\mathcal{L} E\cos\omega t$ is applied in the same direction. $\mathcal{L}$ is a constant much larger than unity. ν_1 and ν_2 are the collision frequency of the electrons with the neutral particles in the two systems. ν_1 and ν_2 are the functions of the gas temperature and the intensity of the applied field. The collision model adopted in this analysis is the elastic collision type with conservation of particle during the collision. The conservation of particle is a necessary condition for the formulation of (43) and (44) in which the spatial variation terms are dropped. $n_1(t)$ and $n_2(t)$ are assumed to be the fluctuating densities and n_o is the unperturbed density for both systems. It can be proved later that $n_1(t)$ and $n_2(t)$ are independent of t from (43) and (44).

f_1 and f_2 are functions of velocity and time and the attempt to solve f_2 as a function of f_1 is difficult. However, there is a trick to sidestep this difficulty by eliminating one of the independent variables before solving (43) and (44) in the velocity space.

Thus, from (43)

$$\frac{\partial}{\partial t}\int f_1\, d^3v - \frac{eE}{m}\cos\omega t\int \frac{\partial f_1}{\partial v_x}\, d^3v = -\nu_1\int \left(f_1 - \frac{n_1(t)}{n_o} f_{10}\right) d^3v$$

or, since

$$\int \frac{\partial f_1}{\partial v_x}\, d^3v = 0, \qquad \int f_{10}\, d^3v = n_o, \quad \text{and} \int f_1\, d^3v = n_1$$

$$\frac{\partial}{\partial t} n_1 = 0 \tag{45}$$

The same operation on equation (44) gives

$$\frac{\partial}{\partial t} n_2 = 0 \tag{46}$$

If the unperturbed densities in the two systems are the same and equal to n_o, then we can put $n_1 = n_2 = n_o$.

The operation, $\int v_x d^3v$, on equation (43) yields

$$\frac{\partial}{\partial t}\int f_1 v_x d^3v - \frac{eE}{m}\cos\omega t\int \frac{\partial f_1}{\partial v_x}\, v_x d^3v = -\nu_1\int \left(f_1 - \frac{n_1(t)}{n_o} f_{10}\right) v_x d^3v$$

or

$$\frac{\partial}{\partial t}(n_1 u_1) + \frac{eE}{m}\cos\omega t\, n_1 = -\nu_1 n_1 u_1 \tag{47}$$

where

$$n_1 u_1 = \int f_1 v_x d^3v$$

and f_{10} is the equilibrium state of f_1 and is assumed to be isotropic in the velocity space.

The same operation on (44) gives

$$\frac{\partial}{\partial t}(n_2 u_2) + \ell \frac{eE}{m} \cos \omega t \, n_2 = -\nu_2 n_2 u_2 \tag{48}$$

The combination of equations (43)-(48) gives two differential equations as follows:

$$\frac{d}{dt} u_1 + \nu_1 u_1 = - \frac{eE}{m} \cos \omega t \tag{49}$$

$$\frac{d}{dt} u_2 + \nu_2 u_2 = - \ell \frac{eE}{m} \cos \omega t \tag{50}$$

Now, we have transformed (43) and (44) to (49) and (50). The quantities u_2 and u_2 are to be found instead of f_1 and f_2. The problem is simplified because u_1 and u_2 are functions of t only.

The next step is to try to solve u_2 as a function of u_1. An equation relating u_1 and u_2 is immediately found as follows:

$$\frac{d}{dt} u_2 + \nu_2 u_2 = \ell \left(\frac{d}{dt} u_1 + \nu_1 u_1 \right) \tag{51}$$

An expression of u_2 as a function of u_1 can be produced from (51) in the following way:

$$u_2 = e^{-\nu_2 t} \int \ell \left(\frac{d}{dt} u_1 + \nu_1 u_1 \right) e^{\nu_2 t} \, dt$$

$$= \ell e^{-\nu_2 t} \left[\int \frac{d}{dt} \left(u_1 e^{\nu_2 t} \right) dt + (\nu_1 - \nu_2) \int u_1 e^{\nu_2 t} \, dt \right] .$$

That is

$$u_2 = \ell u_1 + \ell(\nu_1 - \nu_2)\, e^{-\nu_2 t} \int u_1 e^{\nu_2 t}\, dt \tag{52}$$

Equation (52) shows that the mean velocity of the electrons does vary non-linearly with the applied electric field.

There are the following ways to express u_2 as a function of u_1 more explicitly.

(1) If we can determine u_1 from an experiment, u_2 will be known immediately after u_1 is substituted in the integral appearing in (52).

(2) Solve (52) by successive approximations. That is, consider the second term on the right of (52) as a correction term and substitute u_1 with $\frac{1}{\ell} u_2$ in the first approximation.

(3) If we can determine theoretically an approximate solution of u_1 (this is possible in many cases) then u_2 can be obtained after the approximate solution of u_1 is substituted in the integral in (52). In this case u_2 will be an approximate solution. It is perhaps interesting to show the result for this particular example. From (50), a solution (exact in this particular example) of u_1 is found to be

$$u_1 = -\frac{eE}{m} \frac{1}{\nu^2 + \omega^2} (\nu_1 \cos \omega t + \omega \sin \omega t). \tag{53}$$

The substitution of (53) in (52) gives

$$u_2 = \mathcal{L} u_1 + \mathcal{L} \frac{eE}{m} \frac{(\nu_2 - \nu_1)}{(\nu_1^2 + \omega^2)(\nu_2^2 + \omega^2)} \left[(\nu_1 \nu_2 - \omega^2) \cos \omega t + (\nu_1 + \nu_2) \omega \sin \omega t \right] \tag{54}$$

If the electric currents are defined as

$$i_1 = -en_1 u_1 \quad , \qquad i_2 = -en_2 u_2$$

and

$$n_1 = n_2 = n_o \; ,$$

then the relation between two currents can be expressed as

$$i_2 = \mathcal{L} i_1 - \mathcal{L} \frac{n_o e^2 E}{m} \frac{(\nu_2 - \nu_1)}{(\nu_1^2 + \omega^2)(\nu_2^2 + \omega^2)} \left[(\nu_1 \nu_2 - \omega^2) \cos \omega t + (\nu_1 + \nu_2) \omega \sin \omega t \right] \tag{55}$$

Thus i_2 is determined as a function of i_1 and t. This time dependence will not appear in the actual experiment. The quantity we can measure in an experiment is the magnitude of the current. For this quantity the relation is

$$|i_2| = \left[\left(\mathcal{L} \mathcal{R}e\, i_1 - \mathcal{L} \frac{n_o e^2 E}{m} \frac{(\nu_2 - \nu_1)(\nu_1 \nu_2 - \omega^2)}{(\nu_1^2 + \omega^2)(\nu_2^2 + \omega^2)} \right)^2 + \left(\mathcal{L} \mathcal{I}m\, i_1 - \mathcal{L} \frac{n_o e^2 E}{m} \frac{(\nu_2 - \nu_1)(\nu_1 + \nu_2)\omega}{(\nu_1^2 + \omega^2)(\nu_2^2 + \omega^2)} \right)^2 \right]^{1/2} \tag{56}$$

This relation can be checked experimentally.

Note that

$$\nu_1 = \frac{1}{\lambda}\sqrt{\frac{3KT_{1e}}{m}}\,, \qquad T_{1e} = T + \frac{M\gamma^2}{6K\left[\omega^2 + \frac{3}{\lambda^2 m}\left(KT + \frac{M\gamma^2}{6\omega^2}\right)\right]}$$

$$\nu_2 = \frac{1}{\lambda}\sqrt{\frac{3KT_{2e}}{m}}\,, \qquad T_{2e} = T + \frac{\ell^2 M\gamma^2}{6K\left[\omega^2 + \frac{3}{\lambda^2 m}\left(KT + \frac{\ell^2 M\gamma^2}{6\omega^2}\right)\right]}$$

where λ = mean free path of the electrons and the neutral particles,

T = temperature of gas,

$\gamma = \frac{eE}{m}$,

M = mass of neutral particles.

VI

THE ELECTRICAL CONDUCTIVITY OF A PARTIALLY IONIZED GAS

In this chapter an expression for the electrical conductivity of a partially ionized gas is derived where both electron-neutral particle and Coulomb type collisions between particles play important roles. As mentioned previously, considerable work has been done along this line. Margenau [5], [6] and his group published a series of papers dealing mostly with low intensity fields and weakly ionized gas. Spitzer [11] and his co-workers on the other hand dealt with the small signal static conductivity of a fully ionized gas. In a weakly ionized gas Coulomb collisions between charged particles are neglected. Boltzmann's equation is solved in these cases by considering only collisions between the electrons and neutral particles. In the fully ionized case, however, it is the Coulomb collision which determines the electron velocity distribution function.

This chapter deals with an intermediate case of the interaction between a low-intensity electromagnetic field and a partially ionized gas where neither the electron-neutral particle nor the Coulomb collisions can be neglected. The term conductivity, as used here, is defined in the usual manner as the ratio between the current produced in the ionized gas to the amplitude of the incident electric field. The effects of inelastic collisions and a steady magnetic field are neglected in the present analysis.

The contribution to the collision term in Boltzmann's equation due to collisions between neutral particles and electrons is accounted for by the standard analysis [10], [12]. The contribution due to Coulomb collisions, however, is rather complicated to analyze. Here, we shall use the Coulomb collision model derived by Dreicer [13] from the Fokker-Planck equation. After assuming that the interacting electromagnetic field is of low intensity, an expression for the electron velocity distribution function is derived by solving Boltzmann's equation with the above two models for the collision terms. Expressions for the electrical conductivity in various cases are then derived. To the extent that the assumptions for the collision models are valid, the expressions derived here for the conductivity are quite general in nature. The expressions reduce to the well-known relations for both the limiting cases of fully and weakly ionized gases.

Basic Formulation of the Problem

In this section we shall formulate the problem in general terms. The basic parameter which has to be determined before one can obtain any information about the ionized gas is the electron velocity distribution function. Let us define the electron velocity distribution function $F(v, t)$ such that $F d^3v$ gives the number of electrons whose velocities lie in the element of volume d^3v situated around the point v in the velocity space. It is assumed that the macroscopic properties of the gas do not vary from point to point. Then the distribution function F satisfies

Boltzmann's equation:

$$\frac{\partial F}{\partial t} + \vec{a}(t) \cdot \nabla_v F = \left(\frac{\partial F}{\partial t}\right)_{coll.} \tag{1}$$

where

$\vec{a}(t)$ is the force per unit mass on the electrons,

∇_v is the gradient operator in velocity space,

and $\left(\frac{\partial F}{\partial t}\right)_{coll.}$ is the rate of change of the distribution function due to various types of collisions. Here we assume,

$$\left(\frac{\partial F}{\partial t}\right)_{coll.} = \left(\frac{\partial F}{\partial t}\right)_{cn} + \left(\frac{\partial F}{\partial t}\right)_{cc} , \tag{2}$$

where the subscript cn means collision between electron and neutral particles, and cc means Coulomb collisions between the charged particles. Explicit expressions for the collision terms depend on the type of model one assumed for the gas.

In general there should be considered two other equations similar to (1) in order to account for the velocity distributions of the heavy neutral particles and positive ions. However, for simplicity of analysis it is assumed here that the heavy particles are stationary relative to the motions of the electrons. This is a reasonable assumption compatible with the physical cases where the mass of a heavy particle is about 1830 times heavier than that of an electron.

In the present problem the acceleration of the electrons is assumed to be of the form

$$\vec{a}(t) = -\frac{eE_o \cos \omega t}{m} \tag{3}$$

where

E_o is the amplitude of the incident electric field (polarized along the z-axis),

ω is the angular frequency of the incident field,

$-e$ is the charge of an electron,

m is the mass of an electron.

The effect of the alternating magnetic field is neglected. After the distribution function F is determined from (1), the current produced in the gas may be obtained from the relation,

$$J = -n_o e \bar{v}_z = -e \int F d^3 v \, v \cos \theta \tag{4}$$

where,

J is the z-component of the current, assumed positive along the positive direction of the z-axis,

$\bar{v}_z$ is the mean velocity of the electrons along the positive z-direction,

θ is the angle between the z-axis and the velocity vector $\vec{v}$,

n_o is the electron density. The integration in (4) extends over the entire velocity space.

Discussion of the Collision Terms

As mentioned in the introduction we shall consider only two types of elastic collisions—electron-neutral particles and electron-heavy positive ion. If the gas is fully ionozed and of high density, then collisions between charged particles of the same kind (for example electron-electron, ion-ion) should also be considered. The electron-neutral particle interaction is a short-range phenomenon and explicit expressions for this are taken from the standard analysis [10], [12]. Since Coulomb force is a long-range phenomenon, the cumulative effects of small deflections suffered by the electrons at large distances become very important [11] and they cannot be accounted for in the same way as is done in the first case. Random two-body encounters associated with distances smaller than the Debye length λ are assumed to be the sole mechanism for Coulomb interaction considered here. This type of interaction is in general accounted for by the Fokker-Planck equation. The detailed derivation of the collision term from the Fokker-Planck equation is given by Dreicer [13]. Here, we take our collision term after applying the relevant approximations to the general Coulomb collision term give by Dreicer.

Evaluation of the Electron Distribution Function

In the analysis we shall make use of the Lorentz approximation which states that the collisions between various particles produce a spherically symmetric velocity distribution—small deviations from spherical symmetry are then explained accurately enough by the second coefficient $F^1(v)$ in the spherical harmonic expansion of F:

$$F(\cos\theta, v) = \sum_{n=o}^{\infty} F^n(v)\, P_n(\cos\theta) \doteq F^0(v) + \cos\theta\, F^1(v) \tag{5}$$

where,

the polar axis is chosen to be the z-axis,

P_n's are the Legendre polynomials.

For the small signal case it is assumed that $F^1 << F^0$. This condition physically means that the average velocity of the electron gas is small compared to the root-mean square electron speed. This is reasonable if the perturbing field is small.

By using the orthogonality property of spherical harmonics the following relations are obtained after the integrations over the angles are carried out.

$$\text{electron concentration } n_o = \int_o^{\infty} F^0 4\pi v^2 dv, \tag{6}$$

$$\text{the electric current } J_z = -\frac{4\pi e}{3} \int_o^{\infty} F^1 v^3 dv. \tag{7}$$

After expanding the term $\vec{a}(t) \cdot \nabla_v F$ in equation (1) into its spherical coordinate components in the velocity space and substituting (5) into (1) and equating the terms independent of $\cos\theta$ and those dependent on $\cos\theta$ the following relations are obtained.

$$\frac{\partial F^0}{\partial t} - \frac{\gamma \cos \omega t}{3v^2} \frac{\partial}{\partial v} (v^2 F^1) = \left(\frac{\partial F^0}{\partial t}\right)_{cn} + \left(\frac{\partial F^0}{\partial t}\right)_{cc} \qquad (8)$$

$$\frac{\partial F^1}{\partial t} - \gamma \cos \omega t \left(\frac{\partial F^0}{\partial v}\right) = \left(\frac{\partial F^1}{\partial t}\right)_{cn} + \left(\frac{\partial F^1}{\partial t}\right)_{cc} , \qquad (9)$$

where

$$\gamma = \frac{eE_o}{m} .$$

The contributions due to electron-neutral particle collisions are [10] , [12] ,

$$\left(\frac{\partial F^0}{\partial t}\right)_{cn} = \frac{1}{v^2} \frac{m}{M} \frac{\partial}{\partial v} (v^3 F^0 \nu_e(v)) + \frac{KT}{Mv^2} \frac{\partial}{\partial v} (v^2 \nu_e(v) \frac{\partial F^0}{\partial v}) \qquad (10)$$

$$\left(\frac{\partial F^1}{\partial t}\right)_{cc} = - F^1 \nu_e(v) \qquad (11)$$

where

$$\nu_e(v) = 2\pi N v \int_0^{\pi} (1 - \cos \beta) \sin \beta \sigma_e(\beta, v) \, d\beta, \qquad (12)$$

N = the number density of neutral particles,

$\sigma_e(\beta, v)$ = differential cross section for elastic scattering through the angle β,

M = the mass of the neutral particles,

T = temperature of the electron gas

K = Boltzmann constant.

The factor $\nu_e(v)$ may be identified with the collision frequency of electron-neutral particle collisions. In general $\nu_e(v)$ is a function of v; but for a hydrogen plasma $\nu_e(v)$ is independent of v for v above several electron volts.

The Coulomb collision terms, subject to the Lorentz approximation and the condition $\frac{m}{M} \ll 1$, are given by [13],

$$\left(\frac{\partial F^1}{\partial t}\right)_{cc} = -\frac{n_o \Gamma_{ei}}{v^3} \tag{13}$$

$$\left(\frac{\partial F^0}{\partial t}\right)_{cc} = 0 \tag{14}$$

where

$$\Gamma_{ei} = 4\pi \left(\frac{e\,e_i}{4\pi\epsilon_o m}\right)^2 \ln(\lambda/p_o), \tag{15}$$

p_o = the average impact parameter (distance of closest approach between the two colliding particles),

e_i = charge on a heavy ion = Ze, Z being the degree of ionization,

λ = the Debye shielding distance,

$4\pi\epsilon_o \doteq [1/9\pi]\ 10^{-9}$ Coulomb-volt^{-1}-meter^{-1}.

After substituting the relations (10), (11), (13) and (14) into (8) and (9), the following two equations are obtained.

$$\left(\frac{\partial F^0}{\partial t}\right) - \frac{\gamma\cos\omega t}{3v^2}\frac{\partial}{\partial v}(v^2F^1) = \frac{1}{v^2}\frac{m}{M}\frac{\partial}{\partial v}(v^3F^0\nu_e(v))$$

$$+ \frac{KT}{Mv^2}\frac{\partial}{\partial v}\left(v^2\nu_e(v)\frac{\partial F^0}{\partial v}\right) \tag{15}$$

$$\frac{\partial F^1}{\partial t} - \gamma \cos \omega t \left(\frac{\partial F^0}{\partial v} \right) = - \left[\nu_e(v) + \frac{n_o \Gamma_{ei}}{v^3} \right] F^1 \qquad (16)$$

It is interesting to note that, subject to the present approximations made, the Coulomb interaction effectively increases the collision frequency from $\nu_e(v)$ to $\left[\nu_e(v) + n_o \Gamma_{ei}/v^3\right]$ in the equation determining the perturbed distribution F^1. Equations (15) and (16) together determine the distribution function F.

In order to solve the two simultaneous equations (15) and (16) we assume that the isotropic part of the distribution function is independent of time. After applying the technique of solving differential equations by Laplace transforms, it can be shown that the solution F^1 of (16) in terms of F^0 is given by the following

$$F^1 = F^1 (\text{at } t=0)\, e^{-\left[\nu_e(v) + \frac{n_o \Gamma_{ei}}{v^3}\right] t}$$
$$+ \gamma \left(\frac{\partial F^0}{\partial v} \right) \left[\frac{\omega}{\omega^2 + \left[\nu_e(v) + \frac{n_o \Gamma_{ei}}{v^3} \right]^2} \sin \omega t \right.$$
$$+ \frac{\nu_e(v) + n_o \Gamma_{ei}/v^3}{\omega^2 + \left[\nu_e(v) + \frac{n_o \Gamma_{ei}}{v^3} \right]^2} \cos \omega t$$
$$\left. - \frac{\nu_e(v) + n_o \Gamma_{ei}/v^3}{\omega^2 + \left[\nu_e(v) + \frac{n_o \Gamma_{ei}}{v^3} \right]^2}\, e^{-\left[\nu_e(v) + \frac{n_o \Gamma_{ei}}{v^3}\right] t} \right] \qquad (17)$$

It can be seen from (17) that the combined collision effect (electron-neutral particle and Coulomb) relaxes the perturbed distribution function F^1 to the following steady state value

$$F^1 = \gamma\left(\frac{\partial F^0}{\partial v}\right)\left[\frac{\omega}{\omega^2 + \left[\nu_e(v) + \frac{n_o \Gamma_{ei}}{v^3}\right]^2} \sin\omega t + \frac{\nu_e(v) + n_o \Gamma_{ei}/v^3}{\omega^2 + \left[\nu_e(v) + \frac{n_o \Gamma_{ei}}{v^3}\right]^2} \cos\omega t\right] \tag{18}$$

After substituting (18) into (15) and taking the time average, the following is obtained,

$$-\frac{\gamma^2}{6v^2}\frac{\partial}{\partial v}\left[v^2 \frac{\partial F^0}{\partial v} \frac{\nu_e(v) + n_o \Gamma_{ei}/v^3}{\omega^2 + \left[\nu_e(v) + n_o \Gamma_{ei}/v^3\right]^2}\right]$$

$$= \frac{1}{v^2}\frac{m}{M}\frac{\partial}{\partial v}\left(v^3 F^0 \nu_e(v)\right) + \frac{KT}{Mv^2}\frac{\partial}{\partial v}\left(v^2 \nu_e(v)\frac{\partial F^0}{\partial v}\right) \tag{19}$$

From (19) F^0 can be written in the following form,

$$F^0 = A\exp - \int_0^v \frac{m\,v\,dv}{\frac{M\gamma^2}{6\nu_e(v)} \frac{\nu_e(v) + n_o \Gamma_{ei}/v^3}{\omega^2 + \left[\nu_e(v) + \frac{n_o \Gamma_{ei}}{v^3}\right]^2} + KT} \tag{20}$$

where A is a constant. If Coulomb collisions are neglected the distribution function (20) reduces to the one given by Margenau [5], [6] for the constant mean free path case. If the thermal energy of the electrons is large and the electric field E_o is sufficiently small, then the term KT in the denominator of the integrand in (20) is predominant and F^0 reduces to the Maxwellian distribution

$$F^0 = n_o \left(\frac{m}{2\pi KT}\right)^{3/2} e^{-mv^2/KT} \tag{21}$$

where the constant A has been determined by using the relation (6). Equation (20) indicates that, strictly speaking, F^0 cannot be assumed to be Maxwellian and independent of the field intensity E^0. As we shall see later this fact makes the electrical conductivity of the ionized gas a non-linear function of the field intensity. We are now in a position to calculate the conductivity of the gas. This is done in the next section.

Expressions for Electrical Conductivity

After using (7) and (18) the following relation is obtained for the electric current,

$$J = -\frac{4\pi e^2 E_o}{3m} \int_0^\infty \left[\frac{\omega v^3}{\omega^2 v^6 + \left[v^3 \nu_e(v) + n_o \Gamma_{ei}\right]^2} \sin \omega t + \frac{\left[\nu_e(v) v^3 + n_o \Gamma_{ei}\right]}{\omega^2 v^6 + \left[\nu_e(v) v^3 + n_o \Gamma_{ei}\right]^2} \cos \omega t \right] v^6 \left(\frac{\partial F^0}{\partial v}\right) dv \tag{22}$$

Defining complex conductivity as $\sigma = \sigma_r - i\sigma_i$, it can be shown that,

$$\sigma_i = -\frac{4\pi e^2}{3m} \int_0^\infty \frac{\omega v^3}{\omega^2 v^6 + \left[v^3 \nu_e(v) + n_o \Gamma_{ei}\right]^2} v^6 \left(\frac{\partial F^0}{\partial v}\right) dv \qquad (23)$$

$$\sigma_r = -\frac{4\pi e^2}{3m} \int_0^\infty \frac{\nu_e(v) v^3 + n_o \Gamma_{ei}}{\omega^2 v^6 + \left[\nu_e(v) v^3 + n_o \Gamma_{ei}\right]^2} v^6 \left(\frac{\partial F^0}{\partial v}\right) dv \qquad (24)$$

For the d.c. case $\omega = 0$, $\sigma_i = 0$, $\sigma_r = \sigma_{D.C}$. If Coulomb collisions are neglected, and F^0 is assumed to be given by (21), then (23) and (24) can be written in the following forms after introducing the dimensionless parameter $x^2 = \frac{mv^2}{2KT}$,

$$\sigma_i = \frac{4\pi e^2 n_o}{3m} \frac{2}{(\pi)^{3/2}} \int_0^\infty \frac{\omega}{\omega^2 + \nu_e^2(x)} e^{-x^2} x^4 dx \qquad (25)$$

$$\sigma_r = \frac{4\pi e^2 n_o}{3m} \frac{2}{(\pi)^{3/2}} \int_0^\infty \frac{\nu_e(x)}{\omega^2 + \nu_e^2(x)} e^{-x^2} x^4 dx \qquad (26)$$

In general $\nu_e(x)$ is a complicated function of x and the integrals in (25) and (26) are not always amenable to integration in closed forms. For constant ν_e, σ_i and σ_r as given above reduce to the familiar forms.

Conductivity of a Fully Ionized Gas

In this section we shall derive expressions for the a.c. conductivity of a fully ionized gas. In order to simplify the analysis it is assumed that F^0 is given by (21). Since the gas is fully ionized $\nu_e(v) = 0$ and hence (23) and (24) may be written as follows:

$$\sigma_r = \frac{4\pi e^2 n_o}{3m} \frac{2}{(\pi)^{3/2}} \left(\frac{2KT}{m}\right)^{3/2} \int_0^\infty \frac{n_o \Gamma_{ei}}{\omega^2 \left(\frac{2KT}{m}\right)^3 x^6 + n_o^2 \Gamma_{ei}^2} x^7 e^{-x^2} dx \qquad (27)$$

$$\sigma_i = \frac{4\pi e^2 n_o}{3m} \frac{2\omega}{(\pi)^{3/2}} \left(\frac{2KT}{m}\right)^{3/2} \int_0^\infty \frac{x^{10} e^{-x^2}}{\omega^2 \left(\frac{2KT}{m}\right)^3 x^6 + n_o^2 \Gamma_{ei}^2} dx. \qquad (28)$$

For the d.c. case, $\omega = 0$ Consequently, $\sigma_i = 0$ and the d.c. conductivity is given by:

$$\sigma_{D.C} = \frac{4\pi e^2}{3m \Gamma_{ei}} \frac{2}{(\pi)^{3/2}} \left(\frac{2KT}{m}\right)^{3/2} \int_0^\infty x^7 e^{-x^2} dx. \qquad (29)$$

But $\int_0^\infty x^7 e^{-x^2} dx = 3$ and therefore,

$$\sigma_{D.C.} = \frac{4\pi e^2}{m \Gamma_{ei}} \frac{2}{(\pi)^{3/2}} \left(\frac{2KT}{m}\right)^{3/2} \qquad (30)$$

Expression (30) can be transformed into the familiar form for resistivity $(1/\sigma_{D.C.})$ of a fully ionized gas a given by Spitzer [14] .

We shall now calculate σ_r as given by (27). For this we need to evaluate the integral

$$I = \int_0^\infty \frac{x^7 e^{-x^2}}{x^6 + a^3} dx \qquad (31)$$

where

$$a^3 = \frac{n_o^2 \Gamma_{ei}^2}{\omega^2 \left(\frac{2KT}{m}\right)^3} \tag{32}$$

After a change of variable $t = x^2$, the integral (31) can be written as follows:

$$I = \frac{1}{2} \int_0^{\infty} \frac{t^3 e^{-t}}{t^3 + a^3} \, dt. \tag{33}$$

After breaking the term $1/t^3 + a^3$ into partial fractions and making use of the known result $\int_o^{\infty} \frac{t^n e^{-t}}{t_1 + t} dt = K_n(t_1)$, where K_n is the modified Bessel function of the second kind and order n, the following is obtained,

$$I = \frac{1}{6a^2} K_3(a) + \frac{1}{12a^2} \frac{\pi}{2} \left[H_3^{(1)}(ae^{-\pi i/6}) + H_3^{(2)}(ae^{\pi i/6}) \right]$$

$$+ \frac{i}{4\sqrt{3}a^2} \frac{\pi}{2} \left[H_3^{(1)}(ae^{-\pi i/6}) - H_3^{(2)}(ae^{\pi i/6}) \right] \tag{34}$$

where $H_3^{(1)}$ and $H_3^{(2)}$ are the usual notations for Hankel functions. Using the result [15] that $H_3^{(1)}(ae^{-\pi i/6})$ is the complex conjugate to $H_3^{(2)}(ae^{\pi i/6})$, (34) may be written as follows:

$$I = \frac{1}{6a^2} K_3(a) + \frac{\pi}{12a^2} \rho \cos \phi + \frac{\pi}{4\sqrt{3}a^2} \rho \sin \phi , \tag{35}$$

where

$$H_3^{(1)}(ae^{-\pi i/6}) = \rho \cos \phi - i\rho \sin \phi.$$

Thus the following is obtained for σ_r:

$$\sigma_r = \frac{4\pi e^2 n_o}{3m} \frac{2}{(\pi)^{3/2}} \left(\frac{2KT}{m}\right)^{3/2} \frac{a}{n_o \Gamma_{ei}} \left[\frac{K_3(a)}{6} + \frac{\pi\rho}{12}\cos\phi + \frac{\pi\rho}{4\sqrt{3}}\sin\phi\right] \quad (36)$$

The integral in (28) can be similarly evaluated and the following expression is obtained for σ_i:

$$\sigma_i = \frac{4\pi e^2 n_o}{m} \frac{2\omega}{(\pi)^{3/2}} \left(\frac{2KT}{m}\right)^{3/2} \frac{a}{n_o^2 \Gamma_{ei}^2}$$

$$\cdot \left[\frac{K_{9/2}(a)}{6} - \frac{K_{9/2}(-ae^{\pi i/3}) + K_{9/2}(-ae^{-\pi i/3})}{12} - \frac{i}{4\sqrt{3}}\left[K_{9/2}(-ae^{\pi i/3}) - K_{9/2}(-ae^{-\pi i/3})\right]\right] \quad (37)$$

If F^0 is assumed to be non-Maxwellian, in general, one has to take recourse to numerical methods in order to evaluate σ_r and σ_i.

Discussion

In the above the electrical conductivity of a partially ionized gas has been discussed in detail. The general expressions derived for conductivity may be applied to specific cases keeping in mind the assumptions made in the analysis. For non-Maxwellian distribution for the unperturbed distribution function and collision frequency depending on velocity, the conductivity has to be calculated by numerical methods. Depending on the type of gas and the degree of ionization, one can, however, make some reasonable physical assumptions which will greatly simplify the calculation. It is evident from the above analysis that even in the small signal case and for weakly ionized gas, strictly speaking the electrical conductivity turns out to be a non-linear function of the field intensity. From this, at least qualitatively, one would expect that in the case of high intensity field the electrical conductivity, if it can be defined in the usual sense, will heavily depend on the field amplitude. This conclusion should be true for weakly as well as strongly ionized gas. The case of the interaction between high intensity EM field and strongly ionized gas will be investigated in the next chapter.

VII

NON-LINEAR ELECTRICAL CONDUCTIVITY OF A FULLY IONIZED GAS

The electrical conductivity of a fully ionized gas was investigated by several authors [11] , [13] for a small d.c. electric field case, in which case the analysis is linearized and small perturbation approximation is justified. However, when the impressed electric field is not small the analysis can be much more complicated and the conductivity may have non-linear behavior. The purpose of this chapter is to study the general behavior of the electrical conductivity of a fully ionized gas based on the Fokker-Planck equation. A method which may be useful in the study of electrical conductivity is developed.

An electric field of arbitrary intensity is assumed to be applied uniformly throughout a fully ionized gas which is of infinite extent and spatially homogeneous. To find the electrical conductivity the mean velocity of electrons induced in the plasma is first obtained and from that the induced electrical current and the electrical conductivity of the plasma are determined. In the d.c. case, an instability phenomenon which is called the runaway effect can be observed. This instability automatically restricts the intensity of the impressed electric field to be lower than a critical value. Under this restriction, the d.c. electrical conductivity is obtained as a function of the intensity of the electric field and other parameters. In the a.c. case, the runaway effect loses its significance if the impressed frequency is higher than a critical value. For high intensity microwaves, the electrical conductivity can be very non-linear. In this chapter a simple way of obtaining an approximate electrical conductivity is also presented.

The Basic Equations

An electric field, $E \cos \omega t$, is assumed to be applied along the x-direction to a fully ionized gas which is of infinite extent and spatially homogeneous. The basic equation which describes the system is a Fokker-Planck equation as follows:

$$\frac{\partial f_e}{\partial t} - \frac{eE}{m_e} \cos \omega t \frac{\partial f_e}{\partial v_x} = -\sum_i \frac{\partial}{\partial v_i} (f_e \langle \Delta v_i \rangle) + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial v_i \partial v_j} (f_e \langle \Delta v_i \Delta v_j \rangle) \tag{1}$$

f_e is the velocity distribution function of the electrons of the plasma. The spatial variation term is neglected so that f_e is to be determined as a function of velocity, v, and time, t, only. The right hand side of (1) represents the collision effects of electrons with positive ions and electrons due to the friction and the dispersion in the velocity space. These two terms can be represented [17] by

$$\sum_i \frac{\partial}{\partial v_i} (f_e \langle \Delta v_i \rangle) = - \sum_i \frac{\partial}{\partial v_i} \left(f_e \frac{\partial H_e}{\partial v_i}\right) \tag{2}$$

$$\frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial v_i \partial v_j} (f_e \langle \Delta v_i \Delta v_j \rangle) = \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial v_i \partial v_j} \left(f_e \frac{\partial^2 G_e}{\partial v_i \partial v_j}\right) \tag{3}$$

and

$$H_e = H_{ep} + H_{ee} = \Gamma_e \left[\int \frac{f_p(\vec{v}', t)}{|\vec{v} - \vec{v}'|} d^3 v' + 2 \int \frac{f_e(\vec{v}', t)}{|\vec{v} - \vec{v}'|} d^3 v' \right] \tag{4}$$

$$G_e = G_{ep} + G_{ee} = \Gamma_e \left[\int |\vec{v} - \vec{v}'| \; f_p(\vec{v}', t)\, d^3v' + \int |\vec{v} - \vec{v}'| \; f_e(\vec{v}', t)\, d^3v' \right] \quad (5)$$

$$\Gamma_e = \frac{4\pi e^4}{m_e^2} \ln\left(\frac{h}{p_o}\right) .$$

h = Debye shielding length

p_o = average impact parameter for a 90° Coulomb deflection.

The positive ions of the plasma are assumed to have a Maxwellian distribution and undisturbed by the impressed electric field. The velocity distribution function of the positive ions is

$$f_p = n_o \frac{\alpha_p^3}{(\pi)^{3/2}} e^{-\alpha_p^2 v^2} \quad (6)$$

with $\alpha_p = \sqrt{\frac{m_p}{2KT}}$, n_o = unperturbed density of the plasma.

With the information expressed in (1) to (6), we are, in principle, able to solve for f_e. However, it is hopelessly complicated, especially if (1) is considered without making any approximation. Fortunately, for the purpose of studying the electrical conductivity we can find it much easier to solve the problem by integrating (1) in the velocity space and solving some moment equations which are thus obtained.

First of all, the operation $\int d^3v$ on (1) gives

$$\frac{\partial n_e}{\partial t} = \int \left(\frac{\partial f_e}{\partial t}\right)_{coll.} d^3v = 0 \tag{7}$$

This is valid as long as the conservation of particles during the collision is asserted. Equation (7) implies that n_e is independent of time. This is actually a necessary condition if the spatial variation of f_e is assumed to be zero. We can, therefore, let

$$n_e = n_o \tag{8}$$

Secondly, the operation $\int v_x d^3v$ on (1) gives

$$\frac{\partial}{\partial t} n_e u + \frac{eE}{m_e} \cos\omega t \; n_e = \int \left(\frac{\partial f_e}{\partial t}\right)_{coll.} v_x d^3v \tag{9}$$

where

$$n_e u = \int f_e v_x d^3v$$

In carrying out the right hand side of (9) the following facts are used.

(1) The collisions between like particles do not alter the total momentum of the parent gas.

(2) The dispersion in velocity space leaves the momentum of a gas element unchanged.

Based on these reasons and using (2) to (5) the right hand side of (9) is simplified to the following form.

$$\int \left(\frac{\partial f_e}{\partial t}\right)_{coll} v_x d^3v = - \int_v \frac{\partial}{\partial v_x}\left(f_e \frac{\partial H_{ep}}{\partial v_i}\right) v_x d^3v$$

$$= \Gamma_e \int_v f_e d^3v \left[\frac{\partial}{\partial v_i} \int_{v'} \frac{f_p(\vec{v}', t)}{|\vec{v} - \vec{v}'|} d^3v'\right]$$

$$= \Gamma_e \int_v f_e d^3v \frac{\partial}{\partial v_x} \int_{v'} \frac{f_p(\vec{v}', t)}{|\vec{v} - \vec{v}'|} d^3v' \qquad (10)$$

In deriving (10), f_p is assumed to be Maxwellian and f_e is assumed to be an even function of v_y and v_z. The substitution of (10) in (9), with (8), yields

$$\frac{\partial u}{\partial t} + \frac{eE}{m_e} \cos \omega t = \frac{\Gamma_e}{n_o} \int_v f_e d^3v \frac{\partial}{\partial v_x} \int_{v'} \frac{f_p(\vec{v}', t)}{|\vec{v} - \vec{v}'|} d^3v' \qquad (11)$$

Equation (11) is to be solved. From it the mean velocity of electrons, u, can be determined.

Simplification of Equation (11)

The crucial part of solving (11) is to evaluate its right hand side. It will be evaluated in a form convenient for further study in this section.

The integral

$$M(\vec{v}, t) = \int_{v'} \frac{f_p(\vec{v}', t)}{|\vec{v} - \vec{v}'|} d^3v'$$

can be evaluated exactly if f_p is assumed to be Maxwellian as expressed in (6)

That is

$$M(\vec{v}) = n_o \frac{\alpha_p^3}{(\pi)^{3/2}} \int_{v'} \frac{e^{-\alpha_p^2 v'^2}}{|\vec{v} - \vec{v}'|} d^3v'$$

This integral is analogous to the potential integral in electrostatics. Analogously the integral gives the potential at the point v in the velocity space produced by a spherically symmetrical charge density expressed as

$$4\pi n_o (\alpha_p^3/\pi^{3/2}) e^{-\alpha_p^2 v^2} .$$

This potential is well known in electrostatics and the exact answer is

$$M(\vec{v}) = 4\pi n_o \frac{\alpha_p^3}{(\pi)^{3/2}} \left[\frac{1}{v} \int_o^v e^{-\alpha_p^2 v'^2} v'^2 dv' + \int_v^\infty e^{-\alpha_p^2 v'^2} v' dv' \right]$$

$$= n_o \frac{1}{v} \operatorname{erf}(\alpha_p v) \tag{12}$$

The right hand side of (11) then becomes

$$\text{r.h.s.} = \Gamma_e \int_v d^3v f_e \frac{\partial}{\partial v_x} \left[\frac{1}{v} \operatorname{erf}(\alpha_p v) \right]$$

$$= - \Gamma_e \int_v d^3v\, v_x f_e \left[\frac{1}{v^3} \operatorname{erf}(\alpha_p v) - \frac{2}{\sqrt{\pi}} \frac{\alpha_p}{v^2} e^{-\alpha_p^2 v^2} \right] \tag{13}$$

Therefore, (11) finally becomes

$$\frac{\partial u}{\partial t} + \frac{eE}{m_e} \cos\omega t = - \Gamma_e \int_v d^3v \, v_x f_e \left[\frac{1}{v^3} \operatorname{erf}(\alpha_p v) - \frac{2}{\sqrt{\pi}} \frac{\alpha_p}{v^2} e^{-\alpha_p^2 v^2} \right] \tag{14}$$

Approximate Electrical Conductivity

For the purpose of showing the usefulness of (14), the approximate electrical conductivity will be found in this section.

We can interpret the r.h.s. of (14) as

$$- \Gamma_e \int_v d^3v \, v_x f_e \left[\frac{1}{v^3} \operatorname{erf}(\alpha_p v) - \frac{2}{\sqrt{\pi}} \frac{\alpha_p}{v^2} e^{-\alpha_p^2 v^2} \right] = - \frac{e^2}{m_e} n_o \int du R(v) \tag{15}$$

where

$du = \frac{1}{n_o} f_e v_x d^3v$ = density of the mean velocity of electrons in the velocity space,

$u = \frac{1}{n_o} \int_v f_e v_x d^3v$ = total mean velocity of electrons,

$$R(v) = \frac{\Gamma_e m_e}{e^2} \left[\frac{1}{v^3} \operatorname{erf}(\alpha_p v) - \frac{2}{\sqrt{\pi}} \frac{\alpha_p}{v^2} e^{-\alpha_p^2 v^2} \right] \tag{16}$$

= electrical resistivity in the velocity space.

For practical purposes, in many cases, we can write (15) approximately as

$$-\frac{e^2}{m_e}\int_V d^3v\, v_x f_e R(v) \doteq -\frac{e^2}{m_e} R(\bar{v}) \int_V d^3v\, v_x f_e = -\frac{e^2}{m_e} R(\bar{v})\, n_o u \qquad (17)$$

where $\bar{v}$ is an appropriate value of v which is to be substituted in R(v) so that (17) is valid. For example, $\bar{v}$ can be assigned as the rms velocity of electrons.

If the r.h.s. of (14) can be approximated in the way expressed in (17), (14) becomes

$$\frac{\partial u}{\partial t} + \frac{eE}{m_e}\cos\omega t = -\left(\frac{e^2}{m_e} n_o R(\bar{v})\right) u \qquad (18)$$

Equation (18) gives a steady state solution immediately as

$$u = -\frac{eE}{m_e} \frac{1}{\left[\frac{e^2}{m_e} n_o R(\bar{v})\right]^2 + \omega^2} \left[\frac{e^2}{m_e} n_o R(\bar{v})\cos\omega t + \omega\sin\omega t\right] \qquad (19)$$

From this the induced current, $J = -en_o u$, and the complex electrical conductivity, $\sigma = J/E = \sigma_r - \sigma_i$ are easily determined.

The approximate electrical conductivity is then

$$\sigma_r = \frac{n_o e^2}{m_e} \frac{\frac{e^2}{m_e} n_o R(\bar{v})}{\left[\frac{e^2}{m_e} n_o R(\bar{v})\right]^2 + \omega^2} \qquad (20)$$

$$\sigma_i = \frac{n_o e^2}{m_e} \frac{\omega}{\left[\frac{e^2}{m_e} n_o R(\bar{v})\right]^2 + \omega^2} \qquad (21)$$

and

$$R(\bar{v}) = \frac{\Gamma_e m_e}{e^2}\left[\frac{1}{\bar{v}^3}\,\mathrm{erf}(\alpha_p \bar{v}) - \frac{2}{\sqrt{\pi}}\,\frac{\alpha_p}{\bar{v}^2}\, e^{-\alpha_p^2 \bar{v}^2}\right]$$

$$\bar{v} = \sqrt{\frac{2KT}{m_e}}\,, \qquad \alpha_p = \sqrt{\frac{m_p}{2KT}}$$

In the next two sections, the electrical conductivity for the d.c. and a.c. cases are analysed more exactly from (14). The non-linear behavior will then appear.

D.C. Electrical Conductivity of a Fully Ionized Gas

For the d.c. case, or $\omega = 0$, (14) can be reduced to

$$E = -\frac{\Gamma_e m_e}{e}\int d^3v\, v_x f_e \left[\frac{1}{v^3}\,\mathrm{erf}(\alpha_p v) - \frac{2}{\sqrt{\pi}}\,\frac{\alpha_p^2}{v^2}\, e^{-\alpha_p^2 v^2}\right] \tag{22}$$

If the electrons acquire a mean velocity, u, after a d.c. electric field, E, is applied to the plasma, f_e can be assumed to have a form as

$$f_e = n_o\,\frac{\alpha_3^3}{(\pi)^{3/2}}\, e^{-\alpha_e^2\left[(v_x - u)^2 + v_y^2 + v_z^2\right]} \tag{23}$$

with $\alpha_e = \sqrt{\dfrac{m_e}{2KT}}$

Upon the substitution of (23) in (22), the mean velocity, u, can be determined. In evaluating the integral in (22), it is learned that

$$\alpha_p v \sim \alpha_p / \alpha_e = \sqrt{\frac{m_p}{m_e}} \gg 1$$

Hence, (22) can be approximated as

$$E = -\frac{\Gamma_e m_e}{e} n_o \frac{\alpha_e^3}{(\pi)^{3/2}} \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z \, v_x \frac{e^{-\alpha_e^2\left[(v_x - u)^2 + v_y^2 + v_z^2\right]}}{\left[v_x^2 + v_y^2 + v_z^2\right]^{3/2}}$$

with a very high degree of accuracy. The integral can be evaluated exactly (details are included at the end of this chapter) and (22) finally becomes

$$E = -\frac{\Gamma_e m_e}{e} n_o \left[\frac{1}{u^2} \operatorname{erf}(\alpha_e u) - \frac{2}{\sqrt{\pi}} \frac{\alpha_e}{u} e^{-\alpha_e^2 u^2}\right] \tag{24}$$

Equation (24) shows that the mean velocity, u, is non-linearly dependent on E and other parameters. u can be solved as a function of E from (24) at least numerically. After u is determined as a function of E, the current and the electrical conductivity are obtained immediately

The behavior of the induced current, J, as a function of the electric field, E, is shown graphically in Fig. 1. We observe a very important phenomenon at this point. That is when E is increased higher than a critical value, E_c, the induced current increases monotonically and shows instability. This effect is called the runaway effect. If can be found numerically that E_c corresponds approximately to $\alpha_e u = 1$ and its value is

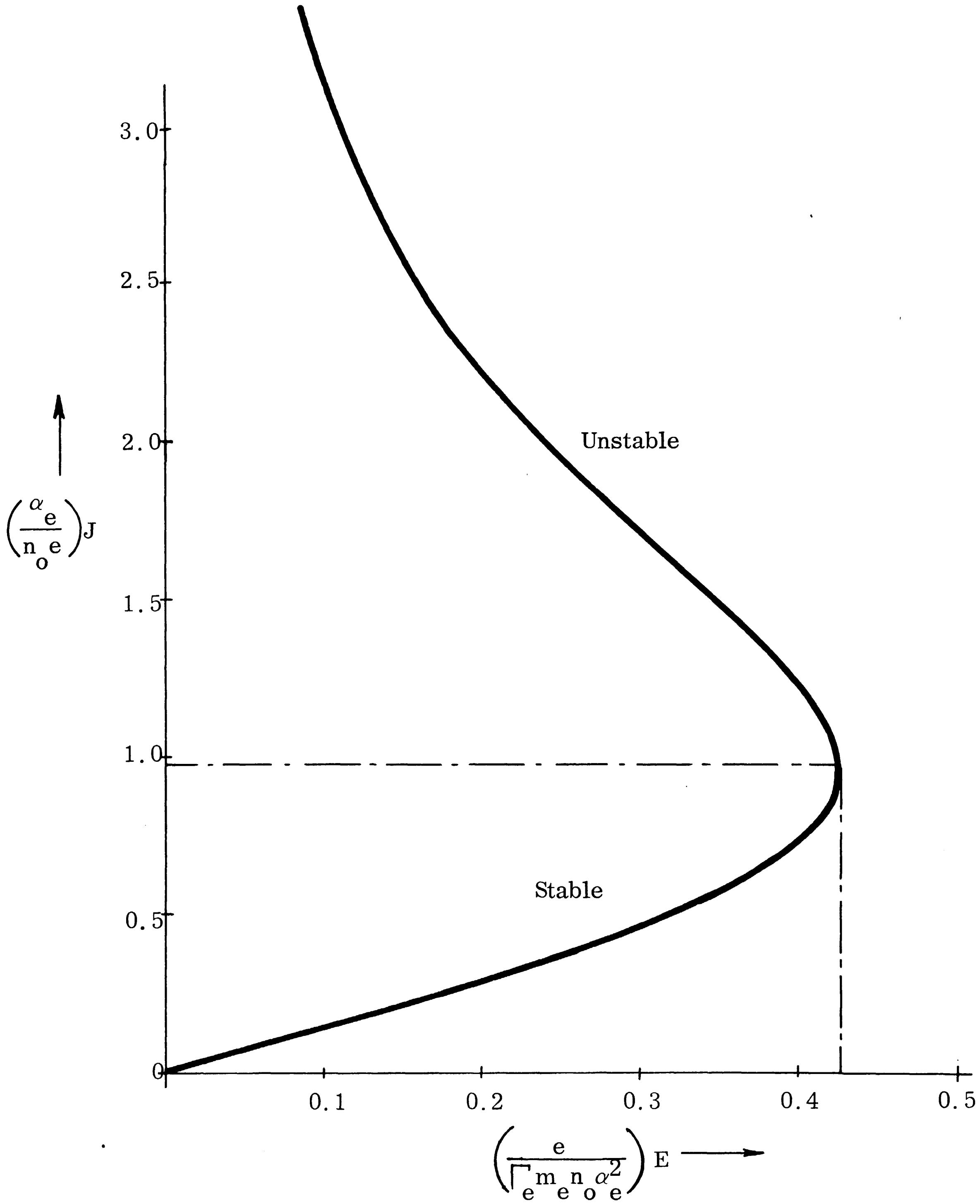

FIGURE 1: THE INDUCED CURRENT IN A FULLY IONIZED GAS AS A FUNCTION OF D.C. ELECTRIC FIELD

$$E_c = 0.43\, n_o \left(\frac{\Gamma_e m_e}{e}\right)\left(\frac{m_e}{2KT}\right) = 0.86\,\pi\, \frac{e^3}{K} \ln\left(\frac{h}{p_o}\right) \frac{n_o}{T} \tag{25}$$

Because of this phenomenon the analysis of the d. c. electrical conductivity of a fully ionized gas should be restricted to the case of

$$E < E_c, \qquad \text{or} \qquad u < \sqrt{\frac{2KT}{m_e}} \tag{26}$$

if sensible results are expected.

Under the condition expressed in (26), (24) can be written as

$$E \doteq -\frac{\Gamma_e m_e}{e}\, n_o \frac{2}{\sqrt{\pi}} \alpha_e^3 \left[\frac{2}{3} u - \frac{2}{5} \alpha_e^2 u^3 + \frac{1}{7} \alpha_e^4 u^5\right]$$

If the d. c. electrical conductivity is defined as

$$\sigma_{D.C.} = \frac{J}{E} = -en_o \frac{u}{E}$$

$\sigma_{D.C.}$ can be expressed as follows:

$$\sigma_{D.C} = \frac{3\sqrt{\pi}}{4} \frac{e^2}{\Gamma_e m_e} \left(\frac{2KT}{m_e}\right)^{3/2} \left[1 - \frac{27\pi}{80}\left(\frac{e}{\Gamma_e m_e n_o}\right)^2 \left(\frac{2KT}{m_e}\right)^2 E^2 + \frac{243\pi^2}{3584}\left(\frac{e}{\Gamma_e m_e n_o}\right)^4 \left(\frac{2KT}{m_e}\right)^4 E^4\right]^{-1} \tag{27}$$

This expression shows that $\sigma_{D.C.}$ is dependent on E, n_o and other parameters. When E is very small $\sigma_{D.C.}$ reduces to

$$\sigma_{D.C.} = \frac{3\sqrt{\pi}}{4} \frac{e^2}{\Gamma_e m_e} \left(\frac{2KT}{m_e}\right)^{3/2}$$

which is about 1/2 that obtained by Spitzer [11] .

A.C. Electrical Conductivity of a Fully Ionized Gas

If f_e has a form as expressed in (23), (14) can be written as

$$\frac{\partial u}{\partial t} + \frac{eE}{m_e} \cos \omega t = - \Gamma_e n_o \left[\frac{1}{u^2} \operatorname{erf}(\alpha_e u) - \frac{2}{\sqrt{\pi}} \frac{\alpha_e}{u} e^{-\alpha_e^2 u^2} \right] \qquad (28)$$

To determine u we have to solve the above non-linear differential equation. Analogous to the d.c. case, first of all we have to find a critical condition which serves as a limitation to avoid instability. The easiest way to assure a stable solution for u is to make the linear term always larger than the non-linear term in (23). That is

$$\frac{\partial u}{\partial t} > \Gamma_e n_o \left[\frac{1}{u^2} \operatorname{erf}(\alpha_e u) - \frac{2}{\sqrt{\pi}} \frac{\alpha_e}{u} e^{-\alpha_e^2 u^2} \right]$$

Since $\frac{\partial u}{\partial t}$ is roughly equal to ωu and the right hand side of the above inequality has a maximum at $u \doteq \frac{1}{\alpha_e}$, the following relation can be obtained. That is

$$\omega u > \Gamma_e n_o \left[\frac{1}{u^2} \operatorname{erf}(\alpha_e u) - \frac{2}{\sqrt{\pi}} \frac{\alpha_e}{u} e^{-\alpha_e^2 u^2} \right] .$$

and a critical value of ω is obtained if u is replaced with $\frac{1}{\alpha_e}$ as follows:

$$\omega_c = 0.43\, n_o \Gamma_e \alpha_e^3 = 0.43\, n_o \left(\frac{m_e}{2KT}\right)^{3/2} \left(\frac{4\pi e^4}{m_e^2}\right) \ln\left(\frac{h}{p_o}\right) \tag{29}$$

Therefore, we can state that in the analysis of the a.c. electrical conductivity of a fully ionized gas the electric field E can be of arbitrary value if $\omega > \omega_c$.

In the actual case, the determination of u from (28) is quite involved and only special cases are discussed here.

When E is small and $\alpha_e^2 u^2 \ll 1$, (26) can be written as

$$\frac{\partial u}{\partial t} + \left(\frac{4}{3\sqrt{\pi}} \Gamma_e n_o \alpha_e^3\right) u = -\frac{eE}{m_e} \cos \omega t \tag{30}$$

The steady state solution for u is

$$u = -\frac{eE}{m_e} \frac{1}{\left(\frac{4}{3\sqrt{\pi}} \Gamma_e n_o \alpha_e^3\right)^2 + \omega^2} \left[\frac{4}{3\sqrt{\pi}} \Gamma_e n_o \alpha_e^3 \cos \omega t + \omega \sin \omega t\right] \tag{31}$$

With the definitions $J = -n_o eu$, and $\sigma = \frac{J}{E} = \sigma_r - i\sigma_i$, the a.c. electrical conductivity can be found as

$$\sigma_r = \frac{n_o e^2}{m_e} \frac{\frac{4}{3\sqrt{\pi}} \Gamma_e n_o \alpha_e^3}{\left(\frac{4}{3\sqrt{\pi}} \Gamma_e n_o \alpha_e^3\right)^2 + \omega^2} \tag{32}$$

$$\sigma_i = \frac{n_o e^2}{m_e} \frac{\omega}{\left(\frac{4}{3\sqrt{\pi}} \Gamma_e n_o \alpha_e^3\right)^2 + \omega^2} \tag{33}$$

When E is large and $\alpha_e^2 u^2 \gg 1$ but $\omega > \omega_c$, (28) can be written as

$$\frac{\partial u}{\partial t} + \frac{\Gamma_e n_o}{u^2} = - \frac{eE}{m_e} \cos \omega t \; . \tag{34}$$

u can be obtained if this non-linear differential equation can be solved.

Evaluation of a Definite Integral

A definite integral appeared in deriving (24), the evaluation of which was omitted to avoid obscuring detail, but is presented here for completeness. The following integral is to be evaluated as a function of the parameters α and c:

$$I = \int_{-\infty}^{\infty} x\,dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \; \frac{e^{-\alpha^2\left[(x-c)^2 + y^2 + z^2\right]}}{\left[x^2 + y^2 + z^2\right]^{3/2}} \tag{35}$$

Let: $z' = \alpha x$

$x' = \alpha z \qquad\qquad s = \alpha c \qquad (36)$

$y' = \alpha y$

$$I = \frac{1}{\alpha} \int_{-\infty}^{\infty} z'dz' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dx' \; \frac{e^{-\left[(z'-s)^2 + y'^2 + x'^2\right]}}{\left[(z')^2 + (y')^2 + (x')^2\right]^{3/2}}$$

Note that αI is a function of s only. Define $I_1(s) = \alpha I$. Introduce spherical coordinates:

$$x' = r\sin\theta\cos\phi$$
$$y' = r\sin\theta\sin\phi \qquad (37)$$
$$z' = r\cos\theta$$

One obtains:

$$I_1(s) = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\cos\theta\, d\theta \int_0^{\infty} dr\ e^{-\left[(r - s\cos\theta)^2 + s^2\sin^2\theta\right]} \qquad (38)$$

Note that the integrand is independent of ϕ. Let $t = r - s\cos\theta$. It follows that:

$$I_1(s) = 2\pi \int_0^{\pi} e^{-s^2\sin^2\theta} \sin\theta\cos\theta\, d\theta \left[\frac{\sqrt{\pi}}{2} + \int_0^{s\cos\theta} e^{-t^2}\, dt\right] \qquad (39)$$

Integrate by parts:

$$u = \frac{\sqrt{\pi}}{2} + \int_0^{s\cos\theta} e^{-t^2}\, dt$$

$$du = -s\sin\theta\ e^{-s^2\cos^2\theta}\, d\theta$$

$$dv = e^{-s^2\sin^2\theta} \sin\theta\cos\theta\, d\theta$$

$$v = \frac{-e^{-s^2\sin^2\theta}}{2s^2}$$

$$I_1(s) = 2\pi \left[\frac{-1}{2s^2}\left\{\left(\frac{\sqrt{\pi}}{2} + \int_0^{-s} e^{-t^2}\, dt\right) - \left(\frac{\sqrt{\pi}}{2} + \int_0^{s} e^{-t^2}\, dt\right)\right\} - \frac{1}{2s}\int_0^{\pi} \sin\theta\ e^{-s^2}\, d\theta\right] \qquad (40)$$

$$I_1(s) = \frac{\pi}{s^2} \left[\sqrt{\pi} \text{ erf } s - 2s\, e^{-s^2} \right] \tag{41}$$

In terms of the original parameters we write I:

$$I(\alpha, c) = \frac{\pi}{\alpha^3 c^2} \left[\sqrt{\pi} \text{ erf}(\alpha c) - 2\alpha c\, e^{-\alpha^2 c^2} \right] \tag{42}$$

APPENDIX A
NON-LINEAR MODELING OF BESSEL FUNCTIONS

This appendix treats the non-linear modeling of two systems described by Bessel functions. The domain of regularity of the modeling function or transformation between the systems is investigated and an explicit series representation of the modeling function is obtained. The corresponding results [2] for trigonometric functions are included (and expanded), not only for comparison, but also because they suggest the structure of the coefficients in the modeling function series.

Statement of the Problem

Suppose a (physical) system is described by the following differential equation with initial conditions:

$$\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} + K_1^2 y = 0 \tag{1}$$

at $t = 0$:

$$y(0) = 1 \quad , \quad \frac{dy}{dt}(0) = 0 \, . \tag{2}$$

Another system is described in terms of the independent variable s:

$$\frac{d^2x}{ds^2} + \frac{1}{s}\frac{dx}{ds} + K_2^2 x = 0 \tag{3}$$

at $s = 0$: $x(0) = 1 \quad , \quad \frac{dx}{ds}(0) = 0 \qquad (4)$

Let the independent variables of the two systems be related by a change of scale:

$$s = at \tag{5}$$

The initial conditions on x remain unchanged, (4), and the differential equation for x as a function of t becomes:

$$\frac{d^2x}{dt^2} + \frac{1}{t}\frac{dx}{dt} + a^2K_2^2 x = 0 \tag{6}$$

A non-linear modeling function is a transformation, $y[x]$, giving the solution y(t) in terms of x(t). We assume that $y[x]$ is not explicitly a function of t. Thus

$$\frac{dy}{dt} = \frac{d}{dx} y[x] \cdot \frac{dx}{dt} \tag{7}$$

Essentially, finding $y[x]$ means eliminating t from the two systems. Suppose the solution of (1) and (2) is:

$$y = g(K_1 t) \tag{8}$$

It is not difficult to show that x(t) is given by:

$$x = g(K_2 s) = g(K_2 at) \tag{9}$$

We can obtain $y[x]$ formally or symbolically as:

$$y[x] = g\left[\frac{K_1}{K_2 a} g^{-1}(x)\right] = g\left[Kg^{-1}(x)\right] \tag{10}$$

where

$$K = \frac{K_1}{K_2 a} \tag{11}$$

The characteristics of the transformation y [x] thus depend on the parameter K. This can be seen also by making the change $\tau = aK_2t$ in (1) and (6). Restrictions of K which make y [x] single-valued are called similitude conditions [1]. We will attempt to investigate these conditions by expressing y [x] in a power series whose coefficients depend on K.

Mappings

Before proceeding to the development of the modeling function, y [x], we digress to a discussion of the systems (1)-(4) and the corresponding situation for g = cos t in which we suppose that the solutions are known. Strictly speaking, if non-linear modeling is to be fruitful, we cannot solve the original equations analytically, at least not in a practical representation. We hope to find the transformation y [x] between two "unsolvable" systems and by performing one, x(t), experimentally, then to deduce a solution, y(t), of the other through y [x].

However, one must distinguish between determination of modeling transformations on the one hand and the theory or study of such determinations on the other. The latter, of course, properly includes some questions of technique as well as of existence and limitations. By considering modeling of "known" equations we hope to develop general principles and methods more readily. In later sections we determine modeling functions for systems corresponding to Bessel and trigonometric functions without solving the original differential equations. In this section, by assuming known properties of the solutions, i.e. y(t) and x(t), we obtain information which allows us to interpret and extend these later results.

Consider then, the solutions of the systems (1)-(4). (We assume the simplification of parameters introduced previously.)

$$y(\tau) = J_o(K\tau) \tag{12}$$

$$x(\tau) = J_o(\tau) \tag{13}$$

The modeling function is obtained symbolically as

$$y[x] = J_o\left[K J_o^{-1}(x)\right] \tag{14}$$

It is clear that $y[x]$ is not necessarily uniquely defined for all values of x and K since J_o^{-1} is a multi-valued function. Even so, it is conceivable that $y[x]$ is single-valued for some choices of K. Consider systems defined by:

$$y_1(\tau) = \cos(K\tau) \tag{15}$$

$$x_1(\tau) = \cos(\tau) \tag{16}$$

$$y_1[x] = \cos\left[K\cos^{-1}(x)\right] \tag{17}$$

When K is an integer, n, (and for no other choice) $y_1[x]$ is single-valued [1] [2].

$$K = n \Rightarrow \quad y_1[x] = T_n(x), \qquad \text{Tschebyscheff polynomial} \tag{18}$$

We see that not only is $\dot{y}_1[x]$ single-valued, it is actually an entire function of x considered as a complex variable. For example, the case n = 2 is:

$$y_1[x] = T_2(x) = 2x^2 - 1 \tag{19}$$

To obtain all values of x and y, however, we must consider that τ in equations (3) and (4) also takes on complex values. For suppose τ is real. Then $|x|$ does not exceed unity and we obtain only a portion of the function defined by (19)

$$y_1[x] = T_2(x) = 2x^2 - 1 \qquad |x| \leqslant 1 \tag{20}$$

Suppose τ varies over a strip:

$$\tau = \sigma + i\eta \tag{21}$$

$$0 \leqslant \sigma < \pi \quad , \quad 0 \leqslant \eta < \infty \tag{22}$$

$$0 < \sigma \leqslant \pi \quad , \quad -\infty < \eta < 0 \tag{23}$$

Now we obtain all complex values of x just once, i.e. the function $\cos\tau$ maps the strip onto an entire x plane. Hence $\cos^{-1}x$ can be defined so that it is one-valued onto the strip. For this restriction we would expect $y_1[x]$ to be unique for all K (and not just integer cases). What is the behavior of this function? To investigate singularities we compute the derivative of $y_1[x]$. Here, of course, we are again freely using knowledge of the solutions of the original systems.

$$y_1' = \frac{dy_1[x]}{dx} = \frac{\frac{dy_1(\tau)}{d\tau}}{\frac{dx_1(\tau)}{d\tau}} = \frac{-K\sin K\tau}{-\sin\tau} \tag{24}$$

Singularities will occur only when $\sin \tau$ is zero but not necessarily even there. If $\sin K\tau$ is also zero the singularity is removable. This happens when K is the ratio of two zeros of the sine function, in this case, when K is an integer.

The question arises: suppose we consider a region analogous to the strip for the Bessel function, i.e. a region of the complex τ plane which maps under $x = J_o(\tau)$ onto the x plane one-to-one. We have here again a proper (i.e. single-valued) definition for $J_o^{-1}(x)$. Thus the function $y[x]$ can be properly defined for all complex x (and for any K) as long as x results from taking τ within the fundamental region. This gives rise, in general, to many different possibilities (one for each fundamental region) for $y[x]$, each restricting the range of τ. For example the fundamental region containing $\tau = 0$ and small positive values of τ extends along the real axis up to $\tau = \tau_1$ such that $J_1(\tau_1) = 0$.

If we represent $y[x]$ for this τ-region, say in a power series, we would expect a singularity (divergence) when $x = J_o(\tau_1)$ unless K were a ratio of two zeroes of the function $J_1(\tau)$, since

$$\frac{dy[x]}{dx} = \frac{-KJ_1(K\tau)}{J_1(\tau)} \tag{25}$$

We expect these numbers $K_i = \frac{\lambda_i}{\tau_1}$, where $J_1(\lambda_i) = 0$, to be significant in the power series representation for $y[x]$ just as in the trigonometric example.

Taking these K values is no guarantee that various branches of $y[x]$ will have the same representation, however. The periodicity of the trigonometric function is apparently what allows this for $y_1[x]$.

Derivation of Differential Equations for the Modeling Function

No loss of generality (see (11)) is suffered by lumping all parameters into one constant K in terms of which the similitude conditions are described. Consider, then, y(t) and x(t) defined by:

$$\ddot{y} + \frac{\dot{y}}{t} + K^2 y = 0 \tag{26}$$

$$\ddot{x} + \frac{\dot{x}}{t} + x = 0 \tag{27}$$

at t = 0: $$y(0) = x(0) = 1 \tag{28}$$

$$\dot{y}(0) = \dot{x}(0) = 0$$

It is desired to find the modeling function $y[x]$ in terms of K (i.e. by eliminating t) without solving the differential equations. From (26) and (27) we determine a differential equation for $y[x]$. From (28) we can obtain sufficient boundary conditions to determine $y[x]$ uniquely. Let primes denote derivatives of y with respect to x. Since $y(t) = y\left[x(t)\right]$ we have:

$$\dot{y} = y'\dot{x} \tag{29}$$

$$\ddot{y} = y''\dot{x}^2 + y'\ddot{x} \tag{30}$$

From (1), (2), (4), and (5):

$$\dot{x}^2 y'' - xy' + K^2 y = 0 \qquad (31)$$

Since the modeling functions are non-linear in general ($y'' \not\equiv 0$) we see that $\dot{x}^2$ is a function, say f, of x only:

$$\dot{x}^2 = \frac{xy' - K^2 y}{y''} = f(x) \qquad (32)$$

$$f(x)y'' - xy' + K^2 y = 0 \qquad (33)$$

From (27) and (32) we determine an equation for f(x):

$$f' = \frac{\dot{f}}{\dot{x}} = 2\ddot{x} = -2x - \frac{2\dot{x}}{t} \qquad (34)$$

$$f' + 2x = -\frac{2\dot{x}}{t} \qquad (35)$$

Differentiate again with respect to x:

$$f'' + 2 = \frac{2}{t^2} - \frac{2\ddot{x}}{\dot{x}t} \qquad (36)$$

$$f(f'' + 2) = \frac{2(\dot{x})^2}{t^2} - \frac{\dot{x}}{t} f' \qquad (37)$$

Substitute from (35):

$$f(f'' + 2) = \frac{1}{2}(f' + 2x)^2 + \frac{f'}{2}(f' + 2x) \qquad (38)$$

Finally:

$$f(f''+2) = (f'+2x)(f'+x) \qquad (39)$$

From (39) we can obtain f as a function of x. Then (33) determines the modeling function $y[x]$. From the conditions (28) on y and x as functions of t we find at $x = 1$ (i.e. at $t = 0$):

$$y[1] = 1 \qquad (40)$$

$$f(1) = 0 \qquad (41)$$

$$f'(1) = 2\ddot{x} \qquad (42)$$

from which (see below) $f'(1) = -1$ (43)

In (42) we have encountered a removable singularity in the term $\frac{\dot{x}}{t}$. A similar situation occurs in determining y' at $x = 1$:

$$y' = \frac{\dot{y}}{\dot{x}} = \frac{\ddot{y}}{\ddot{x}} \qquad (44)$$

From (1) and (2) we find that, as $t \rightarrow 0$:

$$\ddot{x} + \frac{\dot{x}}{t} + x \rightarrow \ddot{x} + \ddot{x} + x \rightarrow 0 \qquad (45)$$

$$\therefore \quad \ddot{x} \rightarrow -\frac{x}{2} \rightarrow -\frac{1}{2} \qquad (46)$$

Similarly

$$\ddot{y} + \frac{\dot{y}}{t} + K^2 y \rightarrow 2\ddot{y} + K^2 y \rightarrow 0 \qquad (47)$$

$$\therefore \quad \ddot{y} \rightarrow -\frac{K^2 y}{2} \rightarrow -\frac{K^2}{2} \qquad (48)$$

Hence from (44), (46), and (48):

$$y'(1) = K^2 \qquad (49)$$

Summarizing from (33), (39), (40), (41), (43), and (49) we have the following system for the modeling function $y[x]$:

$$fy'' - xy' + K^2 y = 0 \tag{50}$$

$$f(f'' + 2) = (f' + 2x)(f' + x) \tag{51}$$

$$y(1) = 1 \qquad y'(1) = K^2 \tag{52}$$

$$f(1) = 0 \qquad f'(1) = -1 \tag{53}$$

Series Solutions for the Modeling Functions

A brief description of the situation corresponding to $g = \cos t$ will be helpful. These results have been obtained elsewhere, [1], [2] but the emphasis here is on the power series representation.

Let:

$$\ddot{y} + K^2 y = 0 \quad , \quad y(0) = 1, \quad \dot{y}(0) = 0 \tag{54}$$

$$\ddot{x} + x = 0 \quad , \quad x(0) = 1, \quad \dot{x}(0) = 0 \tag{55}$$

$y = y[x]$ again implies (31),

$$fy'' - xy' + K^2 y = 0 \,. \tag{56}$$

Since $f = \dot{x}^2$ we obtain, using (55), a differential equation for f as a function of x:

$$f' = \frac{\dot{f}}{\dot{x}} = \frac{2\dot{x}\ddot{x}}{\dot{x}} = 2\ddot{x} = -2x \tag{57}$$

$$f = c - x^2 \tag{58}$$

when t = 0, x = 1, and f = 0. Hence f(1) = 0.

$$f = 1 - x^2 \tag{59}$$

Boundary conditions on $y[x]$ near x = 1 from (54) are:

$$y[1] = 1, \qquad y'[1] = \frac{\dot{y}}{\dot{x}} = \frac{\ddot{y}}{\ddot{x}} = K^2 \tag{60}$$

A power series solution is now written down for the equation for $y[x]$ using conditions (60).

$$(1 - x^2)y'' - xy' + K^2 y = 0 \tag{61}$$

$$y[x] = 1 + \sum_{m=1}^{\infty} s_m(K^2)(x - 1)^m \tag{62}$$

$$s_1 = K^2 \tag{63}$$

$$s_2 = \frac{K^2(K^2 - 1)}{3 \cdot 2!} \tag{64}$$

$$s_3 = \frac{K^2(K^2 - 1)(K^2 - 4)}{3 \cdot 5 \cdot 3!} \tag{65}$$

$\vdots$

$$s_m = \prod_{i=0}^{m-1} \frac{(K^2 - i^2)}{m!\,[3 \cdot 5 \cdot \ldots (2m - 1)]} \tag{66}$$

The zeroes of s_m are just the integers with absolute value less than m. Thus they are cumulative and trivially approach these values as m increases. Consider the zeroes, η_i, of the derivative of g,

$$g(t) = \cos t \tag{67}$$

$$\dot{g}(t) = -\sin t \tag{68}$$

$$\sin \eta_i = 0 \tag{69}$$

$$\eta_i = \pi i \qquad i = 0, \pm 1, \pm 2, \ldots \tag{70}$$

$$\eta_1 = \pi \tag{71}$$

$$\frac{\eta_i}{\eta_1} = i \tag{72}$$

We find that the coefficients of the power series expansion of the modeling function are polynomials whose zeroes approach the ratio of the zeroes of the derivative of the function being modeled to the first such zero. If we take $K = i$ the series (62) becomes finite. It is the series for $T_i(x)$ (Tschebyscheff polynomial) expanded near $x = 1$. A series solution for $y[x]$ corresponding to the Bessel function is now obtained near $x = 1$ by first obtaining a series for $f(x)$ from (51) and (53), then substituting this into (50), and using (52).

Let:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-1)^n \tag{73}$$

Inserting this series into the equation $ff'' - (f')^2 = 3xf' - 2f + 2x^2$, we obtain, near $x = 1$:

$$\sum_{n=o}^{\infty} a_n(x-1)^n \sum_{m=o}^{\infty} a_m m(m-1)(x-1)^{m-2}$$

$$- \sum_{n=o}^{\infty} n\, a_n(x-1)^{n-1} \sum_{m=o}^{\infty} m\, a_m(x-1)^{m-1}$$

$$= 3 \sum_{n=o}^{\infty} n\, a_n(x-1)^n + 3 \sum_{n=o}^{\infty} n\, a_n(x-1)^{n-1}$$

$$- 2 \sum_{n=o}^{\infty} a_n(x-1)^n + 2(x-1)^2 + 4(x-1) + 2 \tag{74}$$

After computing the Cauchy products and changing indices we have:

$$\sum_{n=o}^{\infty} \sum_{j=o}^{n} a_j a_{n-j} \left[j(j-1) - j(n-j)\right] (x-1)^{n-2}$$

$$= \sum_{n=2}^{\infty} (3n-8)\, a_{n-2}(x-1)^{n-2} + 3 \sum_{n=1}^{\infty} (n-1)\, a_{n-1}(x-1)^{n-2}$$

$$+ 2(x-1)^2 + 4(x-1) + 2 \tag{75}$$

Equating like powers of $(x-1)$ yields:

$$0 \cdot a_0^2 = 0 \tag{76}$$

$$0 \cdot a_1 a_0 = 3 \cdot 0 \cdot a_0 \tag{77}$$

$$-a_1^2 = -2a_0 + 3a_1 + 2 \tag{78}$$

$$-2a_1a_2 + 0a_2a_1 + 6a_3a_0 = a_1 + 6a_2 + 4 \qquad (79)$$

$$-3a_1a_3 - 2a_2^2 + 3a_3a_1 + 12a_4a_0 = 4a_2 + 9a_3 + 2 \qquad (80)$$

Finally for $n \geqslant 5$:

$$\sum_{j=0}^{n} ja_ja_{n-j}(2j-n-1) = (3n-8)a_{n-2} + 3(n-1)a_{n-1} \qquad (81)$$

The conditions on $f(x)$ obtained in (53) imply

$$a_0 = 0 \qquad (82)$$

$$a_1 = -1 \qquad (83)$$

Clearly (82) and (83) satisfy (76), (77), and (78). Equations (79) and (80) determine a_2 and a_3 uniquely:

$$a_2 = -3/4 \qquad (84)$$

$$a_3 = -1/72 \qquad (85)$$

Noticing that (81) is a linear equation in the highest subscript of a for $n \geqslant 5$, we see that the coefficients are uniquely determined in a recursive way. Since we need them later we list the values of a_4 and a_5:

$$a_4 = \frac{1}{288} = \frac{2}{(4!)^2} \qquad (86)$$

$$a_5 = \frac{-13 \cdot 4}{3 \cdot (5!)^2} = \frac{-52}{3 \cdot (5!)^2} \qquad (87)$$

For this application (i.e. starting with $a_0 = 0$ and $a_1 = -1$) (81) can be simplified:

$$\sum_{j=2}^{n-1} a_j a_{n-(j-1)}\, j(2j-n-2) = (3n-5)\, a_{n-1} + n^2 a_n \qquad n \geqslant 4 \qquad (88)$$

Assuming a series solution for f(x) near x = 1 is known, we can solve $fy'' - xy' + K^2 y = 0$ near x = 1 as follows. Let

$$y(x) = \sum_{m=o}^{\infty} b_m (x-1)^m \qquad (89)$$

From the equation for y (x) :

$$\sum_{n=o}^{\infty} a_n (x-1)^n \quad \sum_{m=o}^{\infty} m(m-1)\, b_m (x-1)^{m-2}$$

$$- \sum_{m=o}^{\infty} m\, b_m (x-1)^m - \sum_{m=o}^{\infty} m\, b_m (x-1)^{m-1} + K^2 \sum_{m=o}^{\infty} b_m (x-1)^m = 0 \qquad (90)$$

$$\sum_{m=o}^{\infty} \sum_{i=o}^{m} i(i-1)\, b_i a_{m-i} (x-1)^{m-2} - \sum_{m=2}^{\infty} (m-2)\, b_{m-2} (x-1)^{m-2}$$

$$- \sum_{m=1}^{\infty} (m-1)\, b_{m-1} (x-1)^{m-2} + K^2 \sum_{m=2}^{\infty} b_{m-2} (x-1)^{m-2} = 0 \qquad (91)$$

Equating like powers of (x - 1) we obtain:

$$0 \cdot b_0 a_0 = 0 \qquad (92)$$

$$0 \cdot b_0 a_1 + 0 \cdot b_1 a_0 - 0 \cdot b_1 = 0 \qquad (93)$$

Equations (92) and (93) impose no restrictions.

For $m \geqslant 2$,

$$\sum_{i=2}^{m} i(i-1)\, b_i a_{m-i} = (m-1)\, b_{m-1} + (m - K^2 - 2)\, b_{m-2} \tag{94}$$

We can determine the coefficients b_m uniquely in terms of the a_m. A few terms are calculated for reference: (We assume a_0 is zero for all applica tions)

$$b_1 = K^2 b_0 \tag{95}$$

(This agrees with (52).)

$$b_2 = \frac{(K^2 - 1)\, b_1}{2(1 - a_1)} \tag{96}$$

$$b_3 = \frac{(K^2 - 2 + 2a_2)\, b_2}{3(1 - 2a_1)} \tag{97}$$

$$b_4 = \frac{(K^2 - 3 + 6a_2)\, b_3 + 2a_3 b_2}{4(1 - 3a_1)} \tag{98}$$

Taking the values of a_n previously found, putting them into (96) - (98), and continuing the process we find an expansion for the modeling function, $y\ x$, near $x = 1$.

$$y[x] = \sum_{m=o}^{\infty} b_m (x-1)^m = 1 + \sum_{m=o}^{\infty} \frac{R_m(K^2)(x-1)^m}{(m!)^2} \tag{99}$$

$$
\begin{aligned}
R_1 &= K^2 \\
R_2 &= K^2(K^2 - 1) \\
R_3 &= K^2(K^2 - 1)(K^2 - 7/2) \\
R_4 &= K^2(K^2 - 1)(K^4 - 11K^2 + 26) \qquad (100) \\
R_5 &= K^2(K^2 - 1)\left(K^6 - 24K^4 + \frac{503}{3}K^2 - \frac{997}{3}\right) \\
R_6 &= K^2(K^2 - 1)\left(K^8 - 44K^6 + \frac{1287}{2}K^4 - \frac{7263}{2}K^2 + \frac{13,001}{2}\right) \\
&\vdots
\end{aligned}
$$

The zeroes of these polynomials possess some very interesting properties. Up to m = 6 they are all real. Their squares are listed in Table II.

TABLE II

m	Squares of Zeroes of R_m Ordered in Magnitude					
1	0					
2	0	1				
3	0	1	3.5000			
4	0	1	3.4385	7.5615		
5	0	1	3.4103	7.3735	13.2162	
6	0	1	3.3944	7.2799	12.8431	20.4826
	⋮	⋮	⋮	⋮	⋮	⋮
$\left(\frac{\lambda_i}{\lambda_1}\right)^2$	0	1	3.3523	7.0493	12.091	18.4772

The last line is a listing of the quantities $\left(\frac{\lambda_i}{\lambda_1}\right)^2$, where the λ_i are defined as the non-negative zeroes of the Bessel function J_1. The natural and obvious conjecture is that the zeroes of the polynomials R_m approach $\frac{\lambda_i}{\lambda_1}$. This conjecture has not, as yet, been proved. However, similar results are obtained for other cases and a result of this nature is believed valid for a rather large class of modeling functions of the type:

$$y = g\left[Kg^{-1}(x)\right] \tag{101}$$

APPENDIX B

LOCAL ALTERATION OF ATMOSPHERIC DENSITY WITH ELECTROMAGNETIC ENERGY

Summary

It is assumed that by some means a spherical domain in a homogeneous atmosphere can be uniformly heated so that the density reduces by a factor $1/\gamma$ at the center of the sphere. First the steady state with this density ratio is considered and the power needed to maintain the steady state is evaluated. Second, there is assumed uniform atmosphere for $t < 0$, the power supply starting at $t = 0$. Then the time is evaluated after which a density ratio of $1/\gamma$ at the center of the sphere is reached. The density and temperature distribution are given and a simple formula is derived for the power supply needed to achieve a density ratio $1/\gamma$ in a spherical domain of radius a and in a time interval t_o. For $\gamma = 10$ and $a = 100$ cm, the total energy to be delivered until this density ratio is obtained is of the order of 3.10^6 Joules.

Problem I

In an infinitely extended gas a spherical volume of radius a is constantly heated with a heat production P per unit volume and unit time. Determine the steady state and the density distribution.

Make the center of the sphere the origin of a coordinate system. Since we are concerned with the steady state the continuity equation is trivially satisfied.

The equation of motion gives a constant pressure throughout the whole space. The energy law gives

$$-\nabla\cdot(\sigma\nabla T) = P, \qquad (1)$$

where σ is the heat conductivity. Finally we have the equation of state which is

$$\rho T = \rho_{\infty} T_{\infty} \quad . \qquad (2)$$

The suffix ∞ refers to infinite distance, ρ is the density.

The heat conductivity is practically independent of the density, however varying with temperature as $T^{1/2}$. We neglect this variation, which will not change the qualitative picture. We have then in polar coordinates

$$\sigma\left[\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr}\right] = -P \qquad (r < a)$$
$$= 0 \qquad (r > a) \; . \qquad (3)$$

We have the boundary conditions

$$T \text{ and } \frac{dT}{dr} \text{ continuous at } r = a, \; T = T_{\infty} \text{ at } r = \infty. \qquad (4)$$

The unique solution is

$$T = T_{\infty} + \frac{P}{3\sigma}\frac{a^3}{r} \qquad (r > a) \qquad (5)$$

$$T = T_{\infty} + \frac{P}{6\sigma}(3a^2 - r^2) \qquad (r < a). \qquad (6)$$

The density distribution is then given by

$$\rho = \rho_\infty \left[1 + \frac{P}{3\sigma T_\infty} \frac{a^3}{r} \right]^{-1} \qquad (r > a) \tag{7}$$

$$\rho = \rho_\infty \left[1 + \frac{P}{6\sigma T_\infty} (3a^2 - r^2) \right]^{-1} \qquad (r < a) \tag{8}$$

The problem contains only one essential parameter which is

$$\alpha = \frac{Pa^2}{6\sigma T_\infty} \tag{9}$$

Put r = as, then

$$\frac{\rho}{\rho_\infty} = \left[1 + \frac{2\alpha}{s} \right]^{-1} \qquad (s > 1) \tag{10}$$

$$\frac{\rho}{\rho_\infty} = \left[1 + \alpha(3 - s^2) \right]^{-1} \qquad (s < 1)\ . \tag{11}$$

Figure 2 gives a rough sketch of ρ/ρ_∞ for $\alpha = 1$ and $\alpha = 3$.

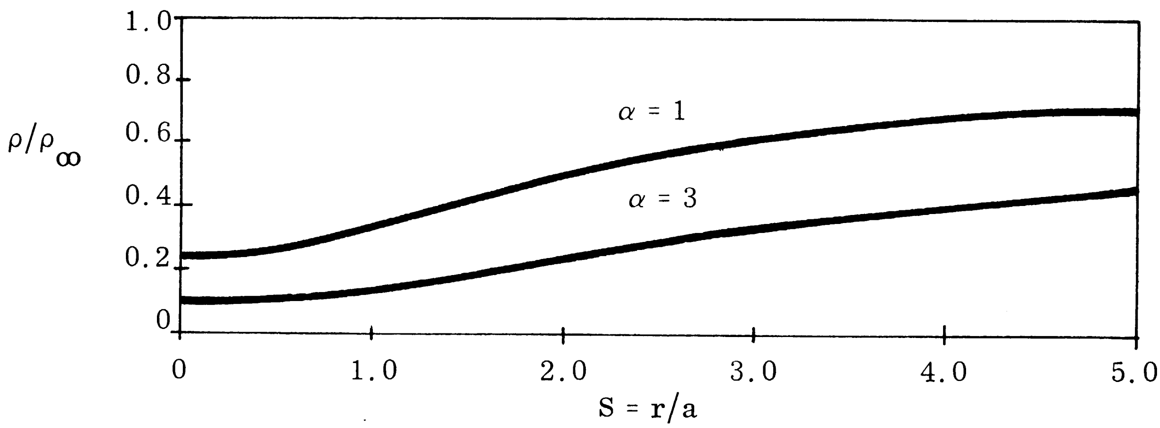

FIGURE 2

As is seen, one has $\rho(0)/\rho_{\infty} = 0.1$ if $\alpha = 3$. Then $\rho/\rho_{\infty} = 0.5$ for $s = 6$. The value $\alpha = 3$ is equivalent to

$$P a^2 = 18 \sigma T_{\infty}. \tag{12}$$

The total production of heat per unit time is

$$\frac{4\pi}{3} a^3 P = 24\pi \, a \sigma T_{\infty}. \tag{13}$$

The heat needed to make $\rho(0)/\rho_{\infty} = 0.1$ is therefore proportional to a, that is, to the radius of the sphere inside which an appreciable drop in density is desired. It is remarkable that a, σ, T_{∞} are the only parameters entering the necessary power.

A rough estimate is given with $\sigma \sim 2.10^4$ erg/cm sec $^{\circ}$K, $T_{\infty} = 300^{\circ}$K, and $a = 100$ cm.

$$\frac{4\pi}{3} a^3 P \approx 5.10^{10} \text{erg s}^{-1} = 5000 \text{ J s}^{-1} \tag{14}$$

Of course, one has also to compensate for the loss by black body radiation, which is certainly not negligible for $T(0) \approx 3000^{\circ}$K. Hence the above result gives a lower limit of the power needed to maintain the steady state.

Problem II

In an infinitely extended gas a spherical volume of radius a is constantly heated with a heat production P per unit volume and time. Determine the temperature, the density and the velocity field in the initial stage after the power supply has started, if the gas was at equilibrium before.

Assumptions:

1. viscosity can be neglected

2. heat conduction can be neglected

3. the pressure is practically constant

While the first assumption is certainly a good approximation, since there are no boundaries and hence no boundary layers, the second assumption is certainly not generally correct. It is, however, a good approximation in the initial stage and the duration of the initial stage can be well defined. The temperature conductivity $k = \frac{\sigma}{\rho c_p}$ is approximately $k = 2\,cm^2/sec$; this says that heat spreads over a distance of 1 cm in about 1 second, or over a distance of 100 cm in about 10,000 seconds. In our problem the initial stage is determined by the condition that heat conduction takes place only over a distance which is small as compared to a. This is equivalent to

$$t \ll a^2/k. \qquad (15)$$

As long as t satisfies this condition the neglect of heat conductivity should not alter the results essentially.

The third assumption is usually made in the theory of flames and its justification is given there. It can certainly be applied in our case as well. As a consequence we can disregard the equation of motion which would only give the small accelerations in which we are not interested.

First we have the equation of state which is written

$$\rho T = \rho_o T_o \tag{16}$$

ρ_o and T_o are the initial values of the density and of the temperature respectively. Obviously we have $\rho_o = \rho_\infty$, and $T_o = T_\infty$, where ρ_∞ and T_∞ refer to infinite distance.

The continuity equation is

$$\frac{d\rho}{dt} + \rho \nabla \cdot \vec{v} = 0 \tag{17}$$

with d/dt being the substantial derivative.

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla \tag{18}$$

The temperature satisfies the equation

$$\rho c_p \frac{dT}{dt} = P(r < a), \quad = 0 \quad (r > a). \tag{19}$$

and c_p is the specific heat at constant pressure.

From (16), (17), and (18) we obtain

$$0 = \rho \frac{dT}{dt} + T \frac{d\rho}{dt} = \frac{P}{c_p} - T\rho \nabla \cdot \vec{v} \ (r < a), = -T\rho \nabla \cdot \vec{v} \ (r > a) \tag{20}$$

or

$$\nabla \cdot \vec{v} = \frac{P}{T_o \rho_o c_p} \ (r < a), \quad \nabla \cdot \vec{v} = 0 \quad (r > a)$$

Because of spherical symmetry we may assume that v has only a radial component which depends only on r. Then (20) can be easily integrated to give

$$v = \frac{r}{\tau} \; (r < a), \qquad v = \frac{a^3}{\tau r^2} \; (r > a) \, , \tag{21}$$

τ , which has the dimension of a time, is defined by

$$\tau = \frac{3}{P} c_p T_o \rho_o \quad . \tag{22}$$

In (21) we have assumed that v is continuous at r = a. This can be concluded from T and hence ρ being continuous as well as the mass flow $\rho \vec{v}$.

With the velocity field as given in (21) we integrate the continuity equation (17), which we write now in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \quad \text{or} \quad \frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho v) = 0. \tag{23}$$

We have to distinguish three domains of the variable r. The first domain is $0 \leqslant r \leqslant a$. Upon insertion of (21) into (23) one finds

$$\rho = \rho_o e^{-3t/\tau} \qquad (0 \leqslant r \leqslant a). \tag{24}$$

The second domain extends from a to $\xi(t)$ where $\xi(t)$ is the distance from the center which has been reached by the gas originally at r = a in the process of expansion. Beyond $r = \xi(t)$, the gas is at temperature T_o, since heat conduction is neglected and therefore the density is also unchanged. We have now to determine $\xi(t)$.

In the interval $a < r < \xi$, we have from (21) and (23)

$$\frac{\partial \rho}{\partial t} + \frac{a^3}{\tau r^2} \frac{\partial \rho}{\partial r} = 0 \qquad (25)$$

Hence ρ is a function of $t - \frac{r^3 \tau}{3a^3}$ which assumes the value (24) at $r = a$. Thus we obtain

$$\rho(r) = \rho_o \, e^{-3t/\tau + r^3/a^3 - 1} \qquad (a \leqslant r \leqslant \xi(t)). \qquad (26)$$

Now we are able to determine $\xi(t)$. It is given by

$$\rho(\xi) = \rho_o \qquad \text{or}$$

$$\xi^3 = a^3 \left[1 + \frac{3t}{\tau} \right] \qquad (27)$$

Figure 3 gives the density as a function of r/a for $\tau/t = 3/4$, $\rho(0, \frac{3}{4}\tau) \approx 0.1\rho_o$; $\xi \approx 1.5\,a$, and for $t/\tau = 1/3$, $\rho(0, \frac{1}{3}\tau) \approx 0.36\rho_o$, $\xi \approx 1.25\,a$.

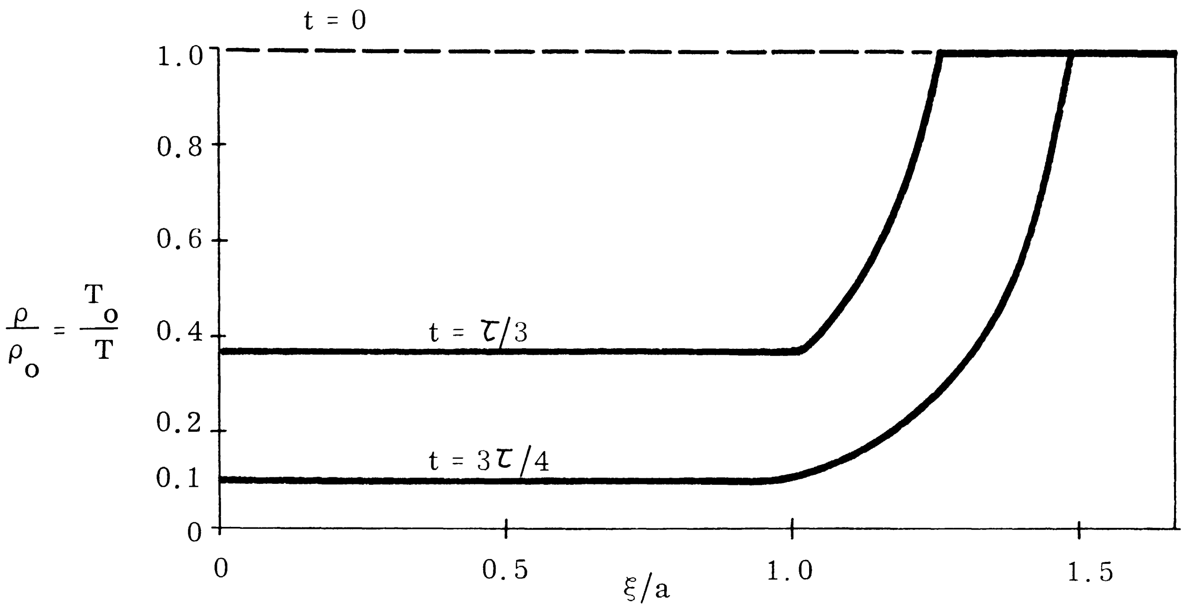

FIGURE 3

The whole picture is independent of the special values of P and a. Their values enter only in the scales. Of course the picture will change if the condition (15) is violated.

It is convenient to express P in terms of the P in (9) which is the power per unit volume needed to maintain the steady state with a density ratio $(1+3\alpha)^{-1}$ at the center. (see (11)). This P will be denoted by P_{st}. We have then

$$P = \beta P_{st} \tag{28}$$

with some numerical constant β. It is clear from the outset that a density ratio $(1+3\alpha)^{-1}$ will not be ultimately reached if $\beta < 1$. It is also clear that $\beta = 1$ will bring us ultimately down to this density ratio; but the neglect of heat conduction will not permit us to apply our results down to this density ratio. If, however, $\beta >> 1$, let us say $\beta = 10$, then we reach this density ratio $(1+3\alpha)^{-1}$ during a time interval $0 < t < t_o$ which complies with the inequality (15). The ultimate density for $t \rightarrow \infty$ will, of course, be lower than $(1+3\alpha)^{-1}$.

To prove this, we insert (28) with the value (9) of P_{st} into (22) to obtain

$$\tau = \frac{1}{2\alpha\beta} \frac{a^2}{k} . \tag{29}$$

Then we have from the definition of t_o and from (24)

$$e^{3t_o/\tau} = 1 + 3\alpha = 1 + \frac{3a^2}{2\beta k \tau} . \tag{30}$$

or

$$\frac{kt_o}{a^2} = \frac{1}{3}\frac{k\tau}{a^2}\ln\left(1+\frac{3a^2}{2\beta k\tau}\right) \tag{31}$$

The right hand number is always less than $(2\beta)^{-1}$ so for $\beta = 10$, (15) is very well satisfied.

So far we have only proved that for $\beta \gg 1$ the neglect of heat conductivity is permitted up to $t = t_o$. Now we give an expression for the time t_o needed to achieve a density ratio $(1+3\alpha)^{-1} = \gamma^{-1}$. From (29) and (30) we obtain

$$t_o = \frac{a^2}{2\beta(\gamma-1)k}\ln\gamma . \tag{32}$$

Furthermore we evaluate the total power delivered up to t_o. It is, from (22) and (24)

$$\frac{4\pi}{3}a^3 Pt_o = \frac{4\pi}{3}a^3 \cdot 3\rho_o c_p T_o \frac{t_o}{\tau} = \frac{4\pi}{3}a^3 \cdot c_p\rho_o T_o \ln\gamma. \tag{33}$$

or, with $T_o = 300^oK$, $\rho_o \approx 10^{-3}\,gm\,cm^{-3}$, $c_p \approx 0.24\,cal/gm\,grad = 10^7\,erg/gm^oK$

$$\frac{4\pi}{3}a^3 Pt_o = \frac{4\pi}{3}a^3\ln\gamma \cdot 3.10^6\,erg\,cm^{-3} . \tag{34}$$

We obtain in particular for $a = 100\,cm$ and $\gamma = 10$,

$$\frac{4\pi}{3}a^3 Pt_o = 3.10^6\,Joule \tag{35}$$

It is, of course, evident that β should not enter the equation (34). This quantity β served only to indicate the power needed in this problem to achieve a certain density ratio in relation to the power needed to maintain the steady state at this density ratio. As is seen, by comparison with (14), in this numerical example βt_o is of the order of 600.

Using the equation of state of air and assuming a specific heat $= \frac{7}{2}$ R/M (M = average molecular weight of air), we can simplify (33) into

$$\frac{4\pi}{3} a^3 P t_o = \frac{4\pi}{3} a^3 \cdot 7/2\, p \ln \gamma \;, \tag{36}$$

which reveals clearly the dependence on pressure.

Our assumption of a constant pressure over the whole space is, of course not exactly correct. One important consequence of this assumption is the steady velocity field which we have obtained in (21) while we would expect that a disturbance travels at a finite speed and does not affect the initial state at some distance until after some time. However, if the process of heating is relatively slow so that t_o is large compared to the time during which a disturbance travels over a distance a, we can expect our results to be valid though (21) would no longer be correct at least outside the zone reached by the disturbance. But it is certainly a good approximation in the domain of interest, that is inside the sphere of radius $\xi(t)$ which is less than 1.5 a in the numerical examples. It is possible to give a justification by integrating the equation of motion

$$\rho \frac{d\vec{v}}{dt} = - \text{grad } p \tag{37}$$

for the pressure by using (21). We restrict to $r < a$ and find

$$p(r,t) = p(0,t) - \rho_o \, e^{-3t/\tau} \, \frac{r^2}{2\tau^2} \tag{38}$$

p(0, t) does, according to our assumptions, not depend on t. Inserting now the equation of state

$$p_o = R/M \, \rho_o T_o . \tag{39}$$

we obtain

$$\frac{p(r,t)}{p_o} = 1 - \frac{M}{RT_o} \, \frac{r^2}{2\tau^2} \, e^{-3t/\tau} \quad (0 < r < a) \tag{40}$$

Now $1.4RT_o/M$ is the square of the sound velocity v_s. The correction term in (40) is therefore $0.7(r/\tau v_s)^2 \, e^{-3t/\tau}$ which is in fact small if the time sound needs to travel over a distance a is small compared to τ.

The preceding analysis is applied to find power required to heat a 1 meter sphere in the ionosphere between 60 and 100 Km so that the density at the center is 1/4 the ambient value. The results are shown in Fig. 4. The power required on the ground was computed assuming the heating is accomplished by high frequency power absorption and that the power given by (9) is increased by two factors. The first factor accounts for the small fraction of the power which will be absorbed in traveling a distance a (taken as 1 meter). For frequencies greater than 3 Mcs (greater than expected plasma and collision frequency), this factor is given approximately by:

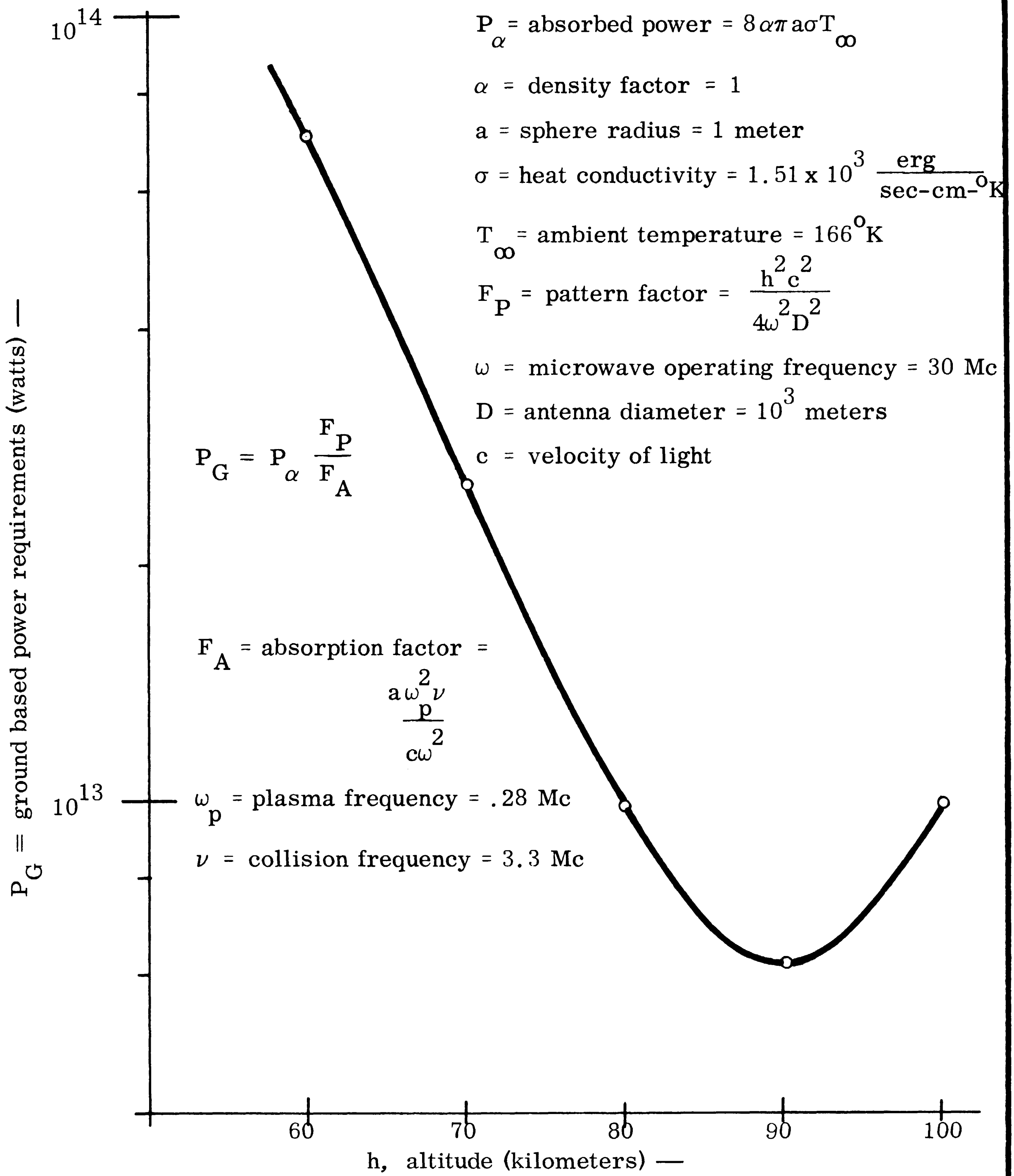

FIGURE 4: POWER REQUIRED TO EFFECT A 75% REDUCTION IN DENSITY AT THE CENTER OF A 1 METER ATMOSPHERIC SPHERE AS A FUNCTION OF ALTITUDE

$$F_A \simeq 2ka, \qquad k \simeq \frac{\omega_p^2 \nu}{2c\omega^2}$$

where k is the absorption coefficient, ω_p the plasma frequency, ν the collision frequency and ω the microwave frequency. The second factor accounts for the fact that at these high altitudes, even very large antennas with relatively narrow beams will illuminate a much larger area of the ionosphere than the desired π square meters, and is given by

$$F_P \simeq \frac{h^2 \lambda^2}{4D^2}$$

where D is the antenna diameter, h the altitude, and λ the wavelength. Using (9) with $\alpha = 1$, values of ionospheric temperature and heat conductivity given in the 1959 ARDC model ionosphere [18] , and the above factors, the ground based power requirements can be quickly estimated, and are given in Fig. 4. It is found that this power is relatively independent of frequency in the frequency range indicated. This power requirement is seen to be large even though the value given by (9) is not.

REFERENCES

1. Ritt, R.K., IRE Trans. AP-4, 3, (July 1958).

2. Belyea, J.E., et al, "Studies in Radar Cross Sections XXXVIII - Non-Linear Modeling of Maxwell's Equations", University of Michigan Radiation Laboratory Report No. 2871-4-T (December 1959).

3. Belyea, J.E., et al, "Studies in Radar Cross Sections XLV - Studies in Non-Linear Modeling II", University of Michigan Radiation Laboratory Report No. 2871-6-F (December 1960).

4. Tonks, L. and Langmuir, I., Phys. Rev. 34, 876 (1929).

5. Margenau, H., Phys. Rev. 69, 508, (1946).

6. Margenau, H., Phys. Rev. 73, 309, (1948).

7. Hartman, L.M., Phys. Rev. 73, 316, (1948).

8. Margenau, H., Phys. Rev. 109, 6, (1958).

9. Webster, A.G. Partial Differential Equations of Mathematical Physics, (New York: Dover Publishing Co.) p. 59.

10. Chapman, S., and Cowling, T.G., The Mathematical Theory of Non-Uniform Gases, (Cambridge: Cambridge University Press, 1939) p. 348

11. Cohen, R.S., et al, Phys. Rev. 80, (2), 230-238, (1950).

12. Holstein, T., Phys. Rev. 70, 367, (1946).

13. Dreicer, H., Phys. Rev. 17, (2), 343-354, (1960).

14. Spitzer, L. Jr., Physics of Fully Ionized Gases, (New York: Interscience Publishers, Inc., 1956) p. 83.

15. Jahnke and Emde, Tables of Functions, (New York: Dover Publishing Co.) p. 134.

16. Samaddar, S.N., "Wave Propagation in an Anisotropic Column with Ring Source Excitation", (To be published).

17. Rosenbluth, N.M., et al, Phys. Rev. 107, 1, (1957).

18. Minzner, R.A., et al, "The ARDC Model Ionosphere", Air Force Survey in Geophysics No. 115, AFCRC-TR-59-267, (August, 1959).