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The Lower and Upper Bound Problems for Cubical Polytopes*

William Jockusch

Department of Mathematics, University of Michigan,
Ann Arbor, MI 48109, USA
jockusch@math.lsa.umich.edu

Abstract. We construct a family of cubical polytypes which shows that the upper bound on the number of facets of a cubical polytope (given a fixed number of vertices) is higher than previously suspected. We also formulate a lower bound conjecture for cubical polytopes.

1. Introduction

The *Lower and Upper Bound Problems* (LBP and UBP) for a class of convex polytopes ask: if the number of vertices of a d-polytope P in the class is fixed, what are the smallest and largest possible numbers of i-faces of P? The LBP and UBP for simplicial polytopes have been solved [B1], [B2], [M]. Here we consider the same problems for cubical polytopes; that is, polytopes of which every facet is a combinatorial cube. We construct cubical polytopes for which the number of facets is large compared with the number of vertices. We also construct cubical polytopes for which the number of facets is small compared with the number of vertices; we conjecture that these give a sharp lower bound.

2. Upper Bounds

Kalai, Perles, and Stanley have conjectured [R, problem 3.15] that the maximal number of facets of a cubical polytope with a fixed number of vertices is attained by a cubical zonotope. In particular, this implies that a cubical polytope never

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has more facets than vertices. We show that this is not the case. Indeed, for a cubical d-polytope P, let $r(P) = f_{d-1}(P)/f_0(P)$, where $f_i(P)$ denotes the number of i-faces of P. We show how to construct a cubical 4-polytope P with $r(P)$ arbitrarily close to 5/4. We also show how to construct cubical polytopes P with $r(P)$ as high as one likes, if the dimension is allowed to be arbitrarily large.

We begin by constructing a tiling T of R^3 by combinatorial cubes which locally has five cubes for every four vertices. The vertices are the points of the form $(i \pm \varepsilon, j \pm \varepsilon, k \pm \varepsilon)$, where i, j, and k are integers, all even or all odd, and $\varepsilon < \frac{1}{2}$ is fixed. The tiles are all congruent to one of three combinatorial cubes, whose vertices are given below:

(a) $(\pm\varepsilon, \pm\varepsilon, \pm\varepsilon)$;
(b) $(\pm\varepsilon, \pm\varepsilon, \varepsilon)$; $(\pm\varepsilon, \pm\varepsilon, 2-\varepsilon)$;
(c) $(\pm\varepsilon, \varepsilon, \varepsilon)$; $(\pm\varepsilon, \varepsilon, 2-\varepsilon)$; $(1-\varepsilon, 1-\varepsilon, 1\pm\varepsilon)$; $(-1+\varepsilon, 1-\varepsilon, 1\pm\varepsilon)$.

The tiles are the above cubes, cubes which can be obtained from one of the above by permuting the coordinate axes, and translations of any of these ten cubes by (i, j, k), where i, j, and k are integers, all even or all odd. The best way to understand the tiling is by making a model of a piece of its 1-skeleton; any vertex can be moved to any other vertex by a rigid motion, and the neighbors of $(\varepsilon, \varepsilon, \varepsilon)$ are $(-\varepsilon, \varepsilon, \varepsilon)$, $(\varepsilon, -\varepsilon, \varepsilon)$, $(\varepsilon, \varepsilon, -\varepsilon)$, $(2-\varepsilon, \varepsilon, \varepsilon)$, $(\varepsilon, 2-\varepsilon, \varepsilon)$, $(\varepsilon, \varepsilon, 2-\varepsilon)$, and $(1-\varepsilon, 1-\varepsilon, 1-\varepsilon)$. A model can be made conveniently with $\frac{1}{16}$ in wooden dowels (available at hobby shops), modeling clay, and a clipper to cut the dowels; readers who do not want to make one will find some aids to visualization of the tiling in Fig. 1. Examination of a model will show that there are ten cubes at every vertex: one of type a, three of type b, and six of type c. Since a cube has eight vertices, the tiling locally has five cubes for every four vertices, as claimed.

We now use the above tiling to construct a cubical 4-polytope P with $r(P)$ arbitrarily close to 5/4. Define $f: R^3 \to R^4$ by $f(x, y, z) = (x^2 + y^2 + z^2, x, y, z)$. Let us call the coordinate axes in R^4 the w, x, y, and z axes, respectively. Note that if r and s are points in R^3, then the distance along a line parallel to the w axis between $f(s)$ and the hyperplane in R^4 tangent to the paraboloid $f(R^3)$ at the point $f(r)$ is the square of the distance rs. Let $V = \{f(v)\colon v \text{ is a vertex of } T\}$. Let U be the convex hull of V, then U is a convex set of infinite extent, with an infinite boundary complex. We claim that f induces a combinatorial isomorphism of T and the boundary complex of U. The best way to see this is to note that, for every cube C of T, there is a sphere $S(C)$ such that every vertex of C lies on $S(C)$ and every other vertex of T lies outside $S(C)$. Then $f(S(C))$ lies in a unique hyperplane I in R^4. (The hyperplane I is parallel to the hyperplane J tangent to $f(R^3)$ at the point $f(O)$, where O is the center of $S(C)$.) Since all vertices of T which are not vertices of C lie outside $S(C)$, all vertices of V which are not images of vertices of C lie on the side of I opposite J; hence the convex hull of $\{f(v)\colon v \text{ is a vertex of } C\}$ is a facet of the boundary complex of U.

We say that a point q' is "visible from" a point q if the interior of the line segment qq' does not intersect U. Let S be the closure of the portion of R^4 invisible from a point $q = (-h, 0, 0, 0)$, where $h > 0$ has been chosen so that q does not lie

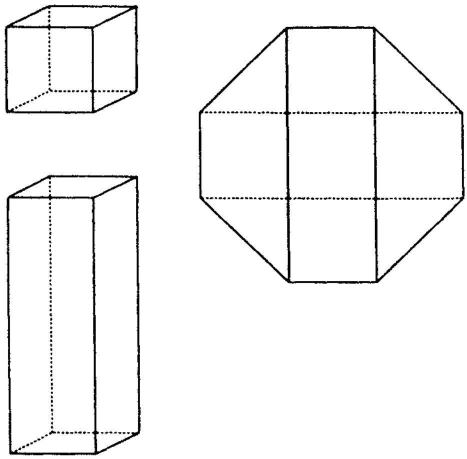

Fig. 1. Combinatorial cubes of type *a* (upper left), *b* (lower left), and *c* (right). The first two are shown viewed from a direction not parallel to any coordinate axis; the perspective for the type *c* cube is parallel to one axis. The type *a* cube is also a geometric cube. The sides of the type *b* cube are two squares and four rectangles. The sides of the type *c* cube are two rectangles and four trapezoids. In the tiling *T*, a square always has a type *a* cube on one side and a type *b* cube on the other. A rectangle always has a type *b* cube on one side and a type *c* cube on the other. A trapezoid always has a type *c* cube on each side. The reader can now piece together the tiling *T* by starting with one cube (any will do, but type *b* may be the most helpful), imagining what lies around it, and extending until the pattern repeats.

in the hyperplane of any facet of U. Then S is an infinite convex region in R^4. Let h' be bigger than the w-coordinate of every vertex of S. Let H denote the hyperplane $w = h'$, and let H^- denote the half-space $w < h'$. Then $S \cap H^-$ is a convex polytope R, and every facet of R except the one lying in H is a combinatorial 3-cube. Let P be the union of R and its reflection about H. Then P is a cubical polytope. The vertices of P are the vertices of U visible from q, their reflections about H, and the vertices of the cubical 3-polytope $H \cap S$. Let Q denote the portion U visible from q. Combinatorially, the boundary complex of P consists of two copies Q_0 and Q_1 of Q, and two copies $\delta Q_0 \times [0, 1]$ and $\delta Q_1 \times [0, 1]$ of $\delta Q \times [0, 1]$; with δQ_i glued to $\delta Q_i \times 0$ and $\delta Q_0 \times 1$ glued to $\delta Q_1 \times 1$. (Geometrically, this last gluing lies in H.) If h is very large, then almost all of the vertices of P are in the interior of one of the Q_i, so $r(P)$ is nearly equal to $r(T) = \frac{5}{4}$.

We can construct a cubical $3n + 1$-polytope P with $r(P)$ arbitrarily close to $5^n/4^n$ by taking the product of n copies of T, then using the method of the previous paragraph to create a polytope from the tiling of R^{3n} thus obtained. Hence there is no dimension-independent upper bound on r for cubical polytopes.

We have no reason to believe that any of these constructions are even close to best possible. For example, we do not know if there is an upper bound on r for cubical 4-polytopes.

3. Lower Bounds

There is a construction for cubical polytopes which seems similar to the "stacking" construction used to attain the lower bounds on f_i for simplicial polytopes [B_1], [B_2]. Given a cubical polytope P, we say that a polytope Q is a *capped* polytope over P if there is a combinatorial cube C such that $Q = P \cup C$ and $P \cap C$ is a facet of P. We say that a cubical polytope is *stacked* if it can be obtained from a combinatorial cube by repeated capping.

The capping operation cuts out the facet F from the boundary complex of P and replaces it by the complement of F in the boundary of a d-cube. Hence capping raises f_i by

$$2^{d-i}\binom{d}{i} - 2^{d-i-1}\binom{d-1}{i}$$

if $i < d - 1$, or by $2(d - 1)$ if $i = d - 1$. We conjecture that, for any cubical d-polytope P,

$$f_i(P) \geq \left[2^{d-i}\binom{d}{i} - 2^{d-i-1}\binom{d-1}{i}\right]\left[\frac{f_0(P)}{2^{d-1}} - 2\right] + 2^{d-i}\binom{d}{i}$$

if $0 \leq i < d - 1$ and that this bound is sharp when v is a multiple of 2^{d-1}.

4. Remarks

1. In constructing a cubical polytope with a large number of facets per vertex, we need to make the sum of the solid angles at the vertices of a facet as small as possible.

2. (Due to the author and Bob MacPherson.) There are tilings of R^3 with more combinatorial cubes per vertex than the tiling T above. For example, we can move each of the eight vertices closest to the origin slightly away from the first coordinate axis (say their new coordinates are $(\pm\varepsilon, \pm\varepsilon', \pm\varepsilon')$, where $\varepsilon < \varepsilon' < \frac{1}{2}$), delete the cubes containing the origin, $(\frac{1}{2}, 0, 0)$, and $(-\frac{1}{2}, 0, 0)$, and replace these by five combinatorial cubes. One of the new cubes will have vertices $(\pm(1 - \varepsilon), \pm\varepsilon, \pm\varepsilon)$; the other four are rotations about the first axis of the cube with vertices $(\pm(1 - \varepsilon), \pm\varepsilon, \varepsilon)$ and $(\pm\varepsilon, \pm\varepsilon', \varepsilon')$. Repeated use of this and similar operations gives tilings with arbitrarily close to three cubes per two vertices. Whether these tilings can be used to obtain cubical 4-polytopes with r near 3/2 is unclear; the above construction fails because the tiling has cubes whose vertices do not lie on a sphere.

3. By a lemma of Perles published in [B2], the above lower bound conjecture for edges is as strong as the general statement.

4. It is not the case that a cubical polytope for which the above lower bound is obtained must be combinatorially equivalent to a stacked polytope; for example, take any cubical 3-polytope which is not stacked.

5. Blind and Blind have shown [BB] that a polytope with no triangular 2-faces has at least 2^d vertices whenever $d > 2$. It might be asked: what are the possible numbers of vertices of a cubical d-polytope? It seems likely that there will be some gaps for numbers slightly larger than 2^d but that all numbers of vertices will be possible past a certain point. However, R. Blind points out (private communication) that one is unaware of any cubical d-polytope, $d > 3$, with an odd number of vertices.

6. Bob MacPherson and Richard Stanley noticed independently that there are some nice immersed manifolds in cubical polytopes. For example, to get an immersed circle, start at the midpoint of any subfacet, go across any facet to the opposite subfacet, and continue until it cycles. Similarly, we can get an immersed 2-manifold by starting at the midpoint of any sub-subfacet, going across any adjacent face to the midpoints of the three "parallel" sub-subfacets, etc. These manifolds can be partially ordered by inclusion. Can they be used to prove any combinatorial properties of cubical polytopes?

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Note added in proof. Blind and Blind have shown that *dn* even-dimensional cubical polytope of dimension at least four always has an even number of vertices.