STUDIES IN RADAR CROSS SECTIONS XLIII - PLASMA SHEATH SURROUNDING A CONDUCTING SPHERICAL SATELLITE AND THE EFFECT ON RADAR CROSS SECTION

by

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PREFACE

This is the forty-third in a series of reports growing out of the study of radar cross sections at The Radiation Laboratory of The University of Michigan. Titles of the reports already published or presently in process of publication are listed on the preceding pages.

When the study was first begun, the primary aim was to show that radar cross sections can be determined theoretically, the results being in good agreement with experiment. It is believed that by and large this aim has been achieved.

In continuing this study, the objective is to determine means for computing the radar cross section of objects in a variety of different environments. This has led to an extension of the investigation to include not only the standard boundary-value problems, but also such topics as the emission and propagation of electromagnetic and acoustic waves, and phenomena connected with ionized media.

Associated with the theoretical work is an experimental program which embraces (a) measurement of antennas and radar scatterers in order to verify data determined theoretically; (b) investigation of antenna behavior and cross section problems not amenable to theoretical solution; (c) problems associated with the design and development of microwave absorbers; and (d) low and high density ionization phenomena.

K. M. Siegel

FOREWORD

This report extends and generalizes the work reported by Professors C. L. Dolph and H. Weil in "Studies in Radar Cross Sections XXXVII - Enhancement of Radar Cross Sections of Warheads and Satellites by the Plasma Sheath", The University of Michigan Radiation Laboratory Report 2778-2-F, RADC-TR-59-239; (December, 1959).

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ABSTRACT

The plasma sheath surrounding a conducting spherical satellite is studied. The density distributions of the positive ions and the electrons in the space are obtained respectively. The satellite is assumed to be charged and the potential of the satellite is determined. The change of the radar cross section of the satellite due to the plasma sheath is evaluated.

I

INTRODUCTION

A conducting sphere is assumed to move with a constant velocity V through a dilute, electrically neutral, ionized atmosphere consisting of oxygen and nitrogen atoms, O^+ and N^+ ions and electrons. The density distributions of the positive ions and the electrons are assumed to be uniform in their undisturbed states. The velocity of the sphere V is assigned to be 8 Km/sec. At the altitude of 500 Km, the root mean square velocity of the positive ions V_i is about one tenth of that of the sphere, and the root mean square velocity of the electrons V_e is at least one order higher than V.

The density distribution of the positive ions is found by using the following model: since the density of the ions is very low at 500 Km altitude, a free molecule model is quite adequate and the interactions between the ions can be ignored. Because $V_i \ll V$, the sphere is traveling at a supersonic velocity compared with that of the ions. The disturbed density distribution surrounding the sphere can be obtained by integrating the zeroth order velocity distribution function and assuming only diffuse reflection on the surface of the sphere. The calculation is facilitated but leads to a same answer if the sphere is assumed to be stationary and the ions are flowing past the sphere with a mean stream velocity V. The final density distribution is a superposition of two distributions, namely: (1) the distribution due to the main stream. This is equal to the density distribution if all the ions are assumed to stick on the surface of the sphere when they hit it. This leads to a distribution similar to a hollow wake behind the sphere; (2) the distribution contributed by the diffuse reflection. The ions which hit the surface of the sphere should be diffusely reflected and satisfy the boundary condition of no absorption or emission from the surface of the sphere. This distribution leads to a pile-up of the ions in the front of the sphere if only the existence of the ions is considered. However, this does not occur because of the existence of the electrons and the conducting surface of the sphere. Suppose the sphere is charged negatively (this will be justified later), all the ions which hit the sphere may be neutralized by the excess of the electrons on the conducting surface of the sphere as the relaxation time of the conductor is extremely short. Therefore, those ions reflected back from the surface of the sphere are already neutralized and there results a pile-up of the neutral particles, instead of the ions in front of the sphere. This argument leads to the conclusion that the final distribution of the ions is equal to the distribution due to the main stream.

The density distribution of the electrons can be obtained after the density distribution of the ions is determined. Since $V_e \gg V$, the distribution of the electrons should not be disturbed significantly by the sphere if only the existence of the electrons is supposed. However, in a plasma medium, due to the property of electric neutrality and the low mass of the electrons, the distribution of the electrons must be determined by solving a Poisson's equation and assuming the Boltzmann distribution of the electrons at equilibrium.

It is an essential step for the complete determination of the density distribution of the electrons to determine the potential of the sphere. As $V_e \gg V_i$, it is evident that the sphere must be charged negatively so that the numbers of electrons and ions which hit the surface of the sphere per unit time are adjusted to be equal at equilibrium. Using this condition the potential of the sphere can be obtained.

After the density distribution of the electrons is completely determined, the radar return from the disturbed region is obtained by integrating the Compton scattering from the electrons. The phase factor is taken into account but the secondary scattering and attenuation are ignored.

Numerical results are presented as computed from the theoretical results for some particular values of the parameters.

The analyses in this paper are carried out by emphasizing the physical picture and avoiding the mathematical complexity. Some reasonable approximations are made in order to obtain a more explicit and simpler solution which is amenable to numerical computation.

II

DENSITY DISTRIBUTION OF POSITIVE IONS

A conducting sphere is assumed to travel with a constant velocity V in a uniform plasma medium and we aim to find the disturbed density distribution of the positive ions surrounding the sphere. The coordinate system to be used in the mathematical formulation is shown in Figure 1. As discussed in the Introduction, the density distribution of the ions is found by assuming only the existence of the ions and forgetting the presence of the electrons at this stage. The calculation is facilitated if the sphere is assumed to be stationary and the ions to flow past the sphere with a mean stream velocity V. Of course, this modification leads to the same density distribution surrounding the sphere as one would get by considering the sphere moving with velocity through a stationary plasma.

The density distribution due to the main stream is found first and that due to the diffuse reflection later.

A. The Density Distribution Due to the Main Stream of Ions.

The velocity distribution function of the main stream of the ions is

$$f_i^s = n_o \left(\frac{m_i}{2\pi KT_i}\right)^{3/2} e^{-\frac{m_i}{2KT_i}(\vec{c}-\vec{V})^2}$$

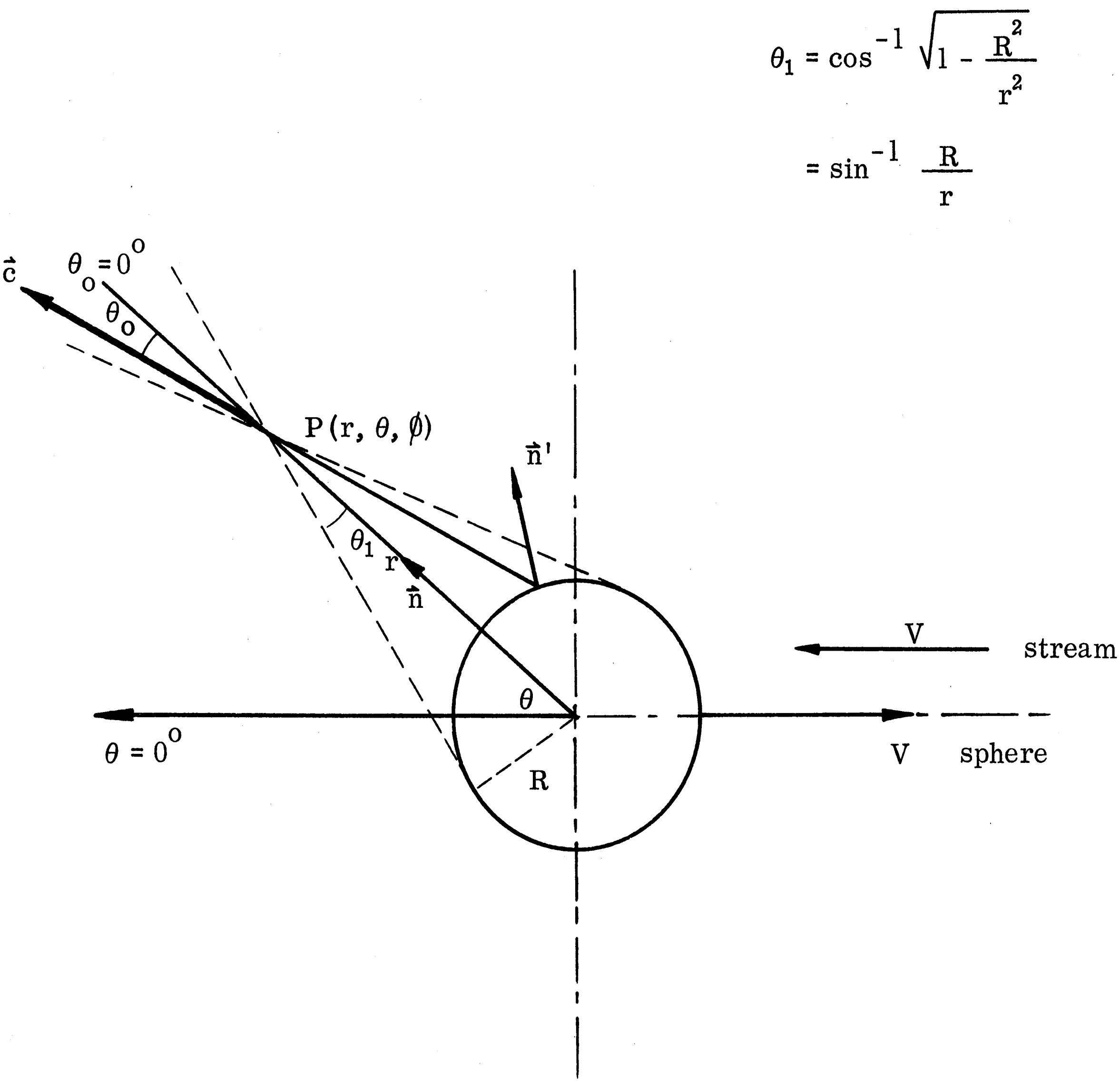

FIG. 1: COORDINATE SYSTEM FOR THE DETERMINATION OF DENSITY DISTRIBUTIONS.

For convenience the following normalizations are made:

$$V' = \frac{V}{\sqrt{\frac{2KT_i}{m_i}}} \quad , \quad c' = \frac{c}{\sqrt{\frac{2KT_i}{m_i}}}$$

After dropping the primes from the normalized velocities, a new velocity distribution function for the main stream of the ions is

$$f_i^s = \frac{n_o}{\pi^{3/2}} \, e^{-(\vec{c}-\vec{V})^2} \tag{1}$$

The ion density due to the main stream at point **P** (r, θ, ϕ) can be written as

$$n_i^s = n_o - \frac{n_o}{\pi^{3/2}} \int_0^{\infty} c^2 \, dc \int_0^{2\pi} d\phi_o \int_0^{\theta_1} \sin\theta_o \, d\theta_o \, e^{-(\vec{c}-\vec{V})^2} \tag{2}$$

Equation (2) implies that the density at point **P** is less than the unperturbed value by the amount which could possibly be intercepted by the presence of the sphere. The integral on the right hand side of equation (2) is just an integration in the velocity space to sum up all the ions whose velocity vectors point away from the sphere and lie within the solid angle subtended by the sphere at point **P**. In this integration, the vector $\vec{r}$ is taken as the polar axis of a new spherical coordinate system (See Fig. 1).

With the understanding that the density deviates significantly from the unperturbed value only in the region behind the sphere or where θ is small when V (normalized velocity) is much bigger than 1, a reasonable approximation is made to facilitate the integration. That is

$$(\vec{c}-\vec{V})^2 = c^2 + V^2 - 2\vec{c}\cdot\vec{V} \doteq c^2 + V^2 - 2cV \cos\theta \cos\theta_o .$$

This approximation is poor in the region immediately near the surface of the sphere and will be improved later. The above approximation leads to

$$\frac{n_o}{\pi^{3/2}} \int_0^\infty c^2 dc \int_0^{2\pi} d\phi_o \int_0^{\theta_1} \sin\theta_o d\theta_o \, e^{-(\vec{c}-\vec{V})^2} = \frac{n_o}{2}\Big[\mathrm{erfc}(-V\cos\theta)\, e^{-V^2\sin^2\theta}$$

$$- \cos\theta_1 \,\mathrm{erfc}(-V\cos\theta\cos\theta_1)\, e^{-V^2(1-\cos^2\theta\cos^2\theta_1)}\Big] .$$

With

$$\cos\theta_1 = \sqrt{1 - \frac{R^2}{r^2}} .$$

Note that erfc stands for the complimentary error function.

n_i^s at point $P(r, \theta, \phi)$ is obtained as

$$n_i^s = n_o - \frac{n_o}{2}\left[\text{erfc}\,(-V\cos\theta)\; e^{-V^2\sin^2\theta}\right.$$
$$\left. - \sqrt{1-\frac{R^2}{r^2}}\;\text{erfc}\,\left(-V\sqrt{1-\frac{R^2}{r^2}}\cos\theta\right)\; e^{-V^2\left(\sin^2\theta + \frac{R^2}{r^2}\cos^2\theta\right)}\right] \quad (3)$$

In order to improve the accuracy of equation (3) in the immediate neighborhood of the sphere, the density distribution of the ions on the surface of the sphere is studied.

At $r = R$, n_i^s can be obtained easily if rectangular coordinates are used and the vector $\vec{r}$ is made the z-axis.

$$n_i^s = \frac{n_o}{\pi^{3/2}} \int_{-\infty}^{0} dc_x \int_{-\infty}^{\infty} dc_y \int_{-\infty}^{\infty} dc_z \; e^{-(\vec{c}-\vec{V})^2} = \frac{n_o}{2}\,\text{erfc}\,(V\cos\theta). \quad (4)$$

From the comparison between equations (3) and (4) and the asymptotic behavior of equation (2), it is found that the following form improves the accuracy of equation (3) quite appreciably:

$$n_i^s = n_o - \frac{n_o}{2}\left[\text{erfc}\,(-V\cos\theta)\; e^{-V^2\sin^2\theta\,(1-\frac{R}{r})}\right.$$
$$\left. - \sqrt{1-\frac{R^2}{r^2}}\;\text{erfc}\,\left(-V\sqrt{1-\frac{R^2}{r^2}}\cos\theta\right)\; e^{-V^2\left(\sin^2\theta + \frac{R^2}{r^2}\cos^2\theta\right)}\right] \quad (5)$$

Equation (5) leads to an exact solution when $\theta = 0^o$, or $\theta = \pi$, and $r = R$.

When V is about 10, equation (5) can be simplified further because

$$\operatorname{erfc}(-V\cos\theta) \doteq \begin{cases} 2 & \text{for } 0 \leqslant \theta < 90^o \\ 0 & \text{for } 90^o < \theta \leqslant 180^o \\ 1 & \text{for } \theta = 90^o \end{cases}$$

and $\operatorname{erfc}\left(-V\sqrt{1-\frac{R^2}{r^2}}\cos\theta\right)$ can be approximated similarly except at $R \sim r$. The careful study of the whole formula of equation (5) suggests a neat approximate expression for n_i^s as follows:

$$\begin{cases} n_i^s = n_o \left[1 - e^{-V^2\sin^2\theta}\left(e^{V^2\sin^2\theta\,\frac{R}{r}} - \sqrt{1-\frac{R^2}{r^2}}\; e^{-V^2\cos^2\theta\,\frac{R^2}{r^2}}\right)\right] & \text{for } 0 \leqslant \theta \leqslant 90^o \\ n_i^s = n_o & \text{for } 90^o \leqslant \theta \leqslant 180^o. \end{cases} \quad (6)$$

The approximate density distribution as expressed in equation (6) is particularly accurate when V is much bigger than 1. The approximation is poor at $\theta = 90^o$ for $r = R$. This approximate expression has an advantage of simplicity in form and is very convenient for the further development of the theory later.

B. Density Distribution Due to the Diffuse Reflection of the Ions.

As the main stream of the ions hits the surface of the sphere, the ions are supposed to be reflected diffusely. No specular reflection is assumed to take place. The boundary condition on the surface of the sphere is as follows: at equilibrium, the number of the ions hitting the surface of the sphere must be equal to the number of the ions reflected diffusely from it per unit time. Using this boundary condition the velocity distribution function of the diffusely reflected ions on the surface of the sphere is easily determined. After this, the density distribution of the ions due to the diffuse reflection from the sphere at any point in space away from the surface of the sphere can be obtained.

This assumes the velocity distribution function of the diffusely reflected ions on the surface of the sphere as

$$f_i^d = A_R \frac{n_o}{\pi^{3/2}} e^{-c^2} \tag{7}$$

where A_R is a coefficient to be determined. Applying the boundary condition on the surface, a relation is found as follows:

$$A_R \frac{n_o}{\pi^{3/2}} \int_0^{\infty} dc_x \int_{-\infty}^{\infty} dc_y \int_{-\infty}^{\infty} dc_z \, c_x \, e^{-c^2} = \frac{-n_o}{\pi^{3/2}} \int_{-\infty}^{0} dc_x \int_{-\infty}^{\infty} dc_y \int_{-\infty}^{\infty} dc_z \, c_x \, e^{-(\vec{c}-\vec{V})^2}$$

Again the rectangular coordinate system is used and $\vec{r}$ is made to coincide with the z-axis. The integrations can be carried out and a solution for A_R is then obtained as

$$A_R = -\sqrt{\pi}\ V\cos\theta\ \text{erfc}\,(V\cos\theta) + e^{-V^2\cos^2\theta}$$

$$\doteq -\sqrt{\pi}\ V\cos\theta\ \text{erfc}\,(V\cos\theta) \qquad \text{for } \theta \nsim 90^{o}\ . \qquad (8)$$

Therefore,

$$\left.\begin{aligned} f_i^d &= -\frac{n_o}{\pi}\ V\cos\theta\ \text{erfc}\,(V\cos\theta)\ e^{-c^2} \\ \text{or} \quad &= -\frac{n_o}{\pi}\ (\vec{V}\cdot\vec{n})\ \text{erfc}\,(\vec{V}\cdot\vec{n})\ e^{-c^2}\ . \end{aligned}\right\} \qquad (9)$$

Equation (9) shows that the density of the diffusely reflected ions on the surface of the sphere is a function of θ.

With the help of equation (9), the density distribution contributed by the diffusely reflected ions at any point in space away from the surface of the sphere can be formulated as

$$n_i^d = -\frac{n_o}{\pi}\int_0^{\infty} e^{-c^2}\ c^2\,dc \int_0^{2\pi} d\phi_o \int_0^{\theta_1} \sin\theta_o\,d\theta_o\,(\vec{V}\cdot\vec{n}')\ \text{erfc}\,(\vec{V}\cdot\vec{n}') \qquad (10)$$

Equation (10) means that only those reflected ions whose velocity vector points away from the sphere and lies within the solid angle subtended by the sphere at point P can reach there and contribute to the density. $\vec{n}'$ is the unit normal vector on the surface of the sphere at the point where the ion is reflected. And the factor $(\vec{V}\cdot\vec{n}')$ erfc $(\vec{V}\cdot\vec{n}')$ takes into account the density of the reflected ions on the surface of the sphere at that particular point where the ion is reflected.

To evaluate the integral a new spherical coordinate system is assigned and $\vec{r}$ is made its polar axis. Furthermore, some approximations are made to make the integration possible. Since V is about 10, it is a fair approximation to state that

$$\text{erfc}\,(\vec{V}\cdot\vec{n}') \doteq \text{erfc}\,(\vec{V}\cdot\vec{n}) = \text{erfc}\,(V\cos\theta)\,.$$

For $\vec{V}\cdot\vec{n}'$ an exact expression is found as follows: $\vec{n}'$ can be expressed in terms of $\vec{r}$ and $\vec{c}$ or in terms of the new spherical coordinates as

$$\vec{n}' = \frac{\vec{r}}{R} - \frac{\vec{r}\cdot\vec{c}}{Rc^2}\,\vec{c} + r\sqrt{\frac{(\vec{r}\cdot\vec{c})^2}{r^2c^2} - \left(1 - \frac{R^2}{r^2}\right)}\;\frac{\vec{c}}{Rc}$$

$$= \frac{r}{R}\left(-\cos\theta_o + \sqrt{\frac{R^2}{r^2} - \sin^2\theta_o}\right)\sin\theta_o\cos\phi_o\,\hat{x}$$

$$+ \frac{r}{R}\left(-\cos\theta_o + \sqrt{\frac{R^2}{r^2} - \sin^2\theta_o}\right)\sin\theta_o\sin\phi_o\,\hat{y}$$

$$+ \frac{r}{R}\left[1 - \left(\cos\theta_o + \sqrt{\frac{R^2}{r^2} - \sin^2\theta_o}\right)\cos\theta_o\right]\hat{z}$$

and

$$\vec{V} = V\,(\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}).$$

The substitution of these expressions in equation (10) gives

$$n_i^d = -\frac{n_o}{\pi} \operatorname{erfc}(V\cos\theta) \int_0^{\infty} e^{-c^2} c^2 dc \int_0^{\theta_1} \sin\theta_o d\theta_o \int_0^{2\pi} d\phi_o \cdot$$

$$\cdot \frac{Vr}{R} \left[\left(-\cos\theta_o + \sqrt{\frac{R^2}{r^2} - \sin^2\theta_o}\right) \sin\theta_o \sin\theta \cos(\phi_o - \phi) \right.$$

$$\left. + \cos\theta + \left(-\cos\theta_o + \sqrt{\frac{R^2}{r^2} - \sin^2\theta_o}\right) \cos\theta_o \cos\theta \right]$$

$$= \frac{-n_o\sqrt{\pi}}{2} V\cos\theta \operatorname{erfc}(V\cos\theta) \frac{r}{R} \left[\frac{2}{3} - \sqrt{1 - \frac{R^2}{r^2}} + \frac{1}{3}\left(1 - \frac{R^2}{r^2}\right)^{3/2} + \frac{1}{3}\frac{R^3}{r^3} \right] \tag{11}$$

Equation (11) expresses the density distribution contributed by the ions reflected diffusely from the sphere. It shows a pile-up of ions in front of the sphere and this density dies away in the radial direction. These points are of physical plausibility.

C. The Final Density Distribution of the Ions in a Plasma Medium.

If the sphere travels in a region where there exist only positive ions, the final density of the ions must be $n_i = n_i^s + n_i^d$. However, in a plasma medium this is not the case. Suppose the sphere is made of conducting material and charged negatively as it moves in the plasma region, those ions which hit the sphere are neutralized immediately by the electrons on the conducting surface of the sphere. Therefore, those reflected ions are already neutralized and n_i^d actually means a pile-up of neutralized particles in front of the sphere. A more complicated model can be established on the basis of the argument that the process of neutralization of the positive ions by the electrons on the surface of a sphere which is not conducting is probably not complete. In this case, part of n_i^d is neutralized and the result is a pile-up of ions and a pile-up of neutralized particles in front of the sphere.

It is concluded that

(1) If the sphere is conducting

$$n_i = n_i^s$$

(2) If the sphere is not conducting

$$n_i = n_i^s + p\ n_i^d$$

p is a fraction which represents the percentage of the reflected ions left unneutralized. p may be a function of the property of the material used on the surface of the sphere and the density of the ions and so forth.

In this paper, the sphere is assumed to be conducting and n_i is assumed to be equal to n_i^s .

III

DENSITY DISTRIBUTION OF ELECTRONS

In the previous section, the density distribution of the positive ions is found to be

$$\left.\begin{aligned} n_i &= n_o \left[1 - e^{-V^2 \sin^2\theta} \left(e^{V^2 \sin^2\theta \frac{R}{r}} - \sqrt{1 - \frac{R^2}{r^2}}\; e^{-V^2 \cos^2\theta \frac{R^2}{r^2}} \right) \right] \\ &\qquad\qquad \text{for } 0 \leqslant \theta \leqslant 90^o \\ &= n_o \qquad \text{for } 90^o \leqslant \theta \leqslant 180^o . \end{aligned}\right\} \tag{6}$$

It is now possible to proceed to find the density distribution of the electrons. In a highly ionized plasma, the relaxation time for the electrons is so short that the electrons obey Boltzmann's distribution at equilibrium. That is

$$n_e = n_o\, e^{\frac{e\phi}{KT_e}} . \tag{12}$$

Where ϕ is the static potential at any point in the space, e is the charge of an electron (magnitude), K is Boltzmann's constant and T_e is the temperature of the electrons. It is reasonable to set $T_e = T_i = T$ at the altitude of 500 Km.

One more important condition is to assign a potential for the sphere. Since the sphere is conducting, the sphere itself is an equipotential body.

So let the potential of the sphere be

$$\phi = \phi_o \text{ at } r = R. \tag{13}$$

With equations (6), (12) and (13), it is sufficient to seek a solution for ϕ and then n_e.

To determine ϕ, a Poisson's equation as follows must be solved.

$$\nabla^2 \phi = - \frac{e}{\epsilon_o} (n_i - n_e)$$

$$= \frac{en_o}{\epsilon_o} \left(e^{\frac{e\phi}{KT}} - \frac{n_i}{n_o} \right) \tag{14}$$

A conventional way of solving equation (14) is based on the assumption that $\left| \frac{e\phi}{KT} \right|$ is much less than unity and the exponential term is then expanded in series. This converts equation (14) to an inhomogeneous Helmholtz equation. However, this method breaks down when $\left| \frac{e\phi}{KT} \right|$ is not much less than unity. A modified method is presented here to solve equation (14) more generally.

Assume the potential at any point in the space is a superposition of two potentials, namely: (1) a potential maintained by the charges on the sphere; (2) a potential maintained by the positive ions and the electrons in space. The potential at any point in space maintained by the charges on the sphere can be written as $\frac{R}{r} \phi_o$, because the sphere has a potential of ϕ_o and this potential must decay as $\frac{1}{r}$. The potential at any point in the space as maintained by the positive ions and the electrons is denoted as ϕ_1 and is to be determined. Usually $\frac{R}{r}\phi_o$ will be much greater than ϕ_1 in the immediate neighborhood of the sphere. The resultant potential at any point in space is then

$$\phi = \frac{R}{r}\phi_o + \phi_1 \tag{15}$$

The substitution of equation (15) in equation (14) gives

$$\nabla^2 \phi_1 = \frac{en_o}{\epsilon_o}\left[e^{\frac{e}{KT}\frac{R}{r}\phi_o} e^{\frac{e\phi_1}{KT}} - \frac{n_i}{n_o}\right].$$

It is now allowable to assume that $\left|\frac{e\phi_1}{KT}\right|$ is much less than unity. This is justified because in the usual case $|\phi_o|$ is much bigger than $|\phi_1|$ After this assumption is made, the above equation can be rewritten as

$$\nabla^2 \phi_1 - \left(\frac{e^2 n_o}{\epsilon_o KT} e^{\frac{e}{KT}\frac{R}{r}\phi_o}\right)\phi_1 \doteq \frac{en_o}{\epsilon_o}\left[e^{\frac{e}{KT}\frac{R}{r}\phi_o} - \frac{n_i}{n_o}\right]. \tag{16}$$

Equation (16) looks like an inhomogeneous Helmholtz equation except that the second term has a variable coefficient. The factor $e^{\frac{e}{KT}\frac{R}{r}\phi_o}$ approaches unity as r increases. This factor may deviate appreciably from unity only in the region where r is around R or in the immediate neighborhood of the sphere. Therefore, it is a fair approximation to rewrite equation (16) as

$$\nabla^2 \phi_1 - \beta^2 \phi_1 = \frac{en_o}{\epsilon_o} \left[e^{\frac{e}{KT} \frac{R}{r} \phi_o} - \frac{n_i}{n_o} \right] \tag{17}$$

with

$$\beta^2 = \frac{e^2 n_o}{\epsilon_o KT}.$$

The solution of equation (17) can be written down immediately according to the theory of Green's function as follows:

$$\phi_1(\vec{r}) = \frac{-1}{4\pi} \int_V \frac{e^{-\beta |\vec{r}' - \vec{r}|}}{|\vec{r}' - \vec{r}|} \frac{en_o}{\epsilon_o} \left[e^{\frac{e}{KT} \frac{R}{r'} \phi_o} - \frac{n_i}{n_o} \right] dV$$

$$= \frac{en_o}{4\pi \epsilon_o} \int_V \frac{e^{-\beta |\vec{r}' - \vec{r}|}}{|\vec{r}' - \vec{r}|} \left[\frac{n_i}{n_o} - e^{\frac{e}{KT} \frac{R}{r'} \phi_o} \right] dV. \tag{18}$$

Due to the property of the kernel of the integral in equation (18), it is reasonable to write

$$\phi_1(\vec{r}) = \frac{en_o}{4\pi \epsilon_o} \left[\frac{n_i}{n_o} - e^{\frac{e}{KT} \frac{R}{r'} \phi_o} \right]_{\vec{r}' = \vec{r}} \int_V \frac{e^{-\beta |\vec{r}' - \vec{r}|}}{|\vec{r}' - \vec{r}|} dV .$$

The integration can be carried out by letting $|\vec{r}' - \vec{r}| = s$ as follows:

$$\int_V \frac{e^{-\beta\,|\vec{r}\,'-\vec{r}|}}{|\vec{r}\,'-\vec{r}|}\,dV = \int_0^\infty \frac{e^{-\beta s}}{s}\,4\pi s^2\,ds = \frac{4\pi}{\beta^2}\,.$$

Therefore,

$$\phi_1(\vec{r}) = \frac{KT}{e}\left[\left(\frac{n_i}{n_o}\right)_{\vec{r}\,'=r} - e^{\frac{e}{KT}\frac{R}{r}\phi_o}\right] \tag{19}$$

The electron density is obtained after the substitution of equation (19) in equation (12)

$$n_e = n_o\, e^{\frac{e}{KT}\frac{R}{r}\phi_o}\, e^{\frac{e\phi_1}{KT}} = n_o\, e^{\frac{e}{KT}\frac{R}{r}\phi_o}\left(1 + \frac{e\phi_1}{KT} + \cdots\right)$$

$$= n_o\, e^{\frac{e}{KT}\frac{R}{r}\phi_o}\left[1 + \left(\frac{n_i}{n_o}\right)_{\vec{r}\,'=\vec{r}} - e^{\frac{e}{KT}\frac{R}{r}\phi_o}\right].$$

The substitution of equation (6) in the above expression leads to the final solution for the density distribution of the electrons as follows:

$$\begin{cases} n_e = n_o\, e^{\frac{e}{KT}\frac{R}{r}\phi_o}\left[2 - e^{\frac{e}{KT}\frac{R}{r}\phi_o} - e^{-V^2\sin^2\theta}\left(e^{V^2\sin^2\theta\frac{R}{r}} - \sqrt{1-\frac{R^2}{r^2}}\; e^{-V^2\cos^2\theta\frac{R^2}{r^2}}\right)\right] & \text{for } 0^o \leqslant \theta \leqslant 90^o \\ = n_o\, e^{\frac{e}{KT}\frac{R}{r}\phi_o}\left[2 - e^{\frac{e}{KT}\frac{R}{r}\phi_o}\right] & \text{for } 90^o \leqslant \theta \leqslant 180^o\,. \end{cases} \tag{20}$$

Equation (20) gives the complete solution of the electron density distribution in space except that the potential of the sphere ϕ_o is left undetermined. It is noted that there is a discontinuity at $\theta = 90^o$ for small r as expressed by two different formulas for two separate spaces. This discontinuity is not important because it is smoothed down as soon as r is increased. The next section is devoted to the determination of ϕ_o. After that the electron density distribution is completely determined.

IV

POTENTIAL OF SATELLITE

It is an important aspect to determine the potential of the sphere which moves in a plasma medium. Since the root mean square velocity of the electrons is much higher than that of the positive ions, many more electrons than positive ions may hit the sphere per unit time. At equilibrium, equal quantities of electrons and positive ions should hit the sphere per unit time. To achieve this equilibrium, the sphere must be charged negatively so that the number of electrons hitting the sphere is cut down, because only those electrons having high enough energy to overcome the potential barrier at the surface of the sphere can reach the sphere. The photoelectric effect is ignored as it has been proved to be small.

The velocity distribution function for the electrons is

$$f_e^s = n_e \left(\frac{m_e}{2\pi KT_e} \right)^{3/2} e^{-\frac{m_e}{2KT_e}(\vec{c} - \vec{V})^2}$$

After the normalization as before, or after division of the velocity by $\sqrt{\frac{2KT_i}{m_i}}$, a new function is expressed as follows:

$$f_e^s = \frac{n_e}{\pi^{3/2}} \alpha^3 e^{-\alpha^2(\vec{c} - \vec{V})^2} \qquad (21)$$

where $\alpha = \sqrt{\frac{m_e}{m_i}} \ll 1$, and T_e is assumed to be equal to T_i.

The density distribution of the electrons at the surface of the sphere is obtained from equation (20) as follows:

At r = R

$$\left.\begin{aligned} n_e &= n_o\, e^{\frac{e\phi_o}{KT}} \left[1 - e^{\frac{e\phi_o}{KT}}\right] \quad \text{for } 0 \leqslant \theta < 90^o \\ &= n_o\, e^{\frac{e\phi_o}{KT}} \left[2 - e^{\frac{e\phi_o}{KT}}\right] \quad \text{for } 90^o < \theta \leqslant 180^o. \end{aligned}\right\} \qquad (22)$$

The critical velocity c_e^o is defined as

$$\frac{1}{2} m_e \left(\sqrt{\frac{2KT}{m_i}}\, c_e^o\right)^2 = |e\phi_o|$$

or

$$c_e^o = \frac{1}{\alpha} \sqrt{\frac{|e\phi_o|}{KT}} \qquad (23)$$

where ϕ_o is assumed to be negative. Thus only those electrons having velocity higher than c_o^e can overcome the potential barrier and reach the surface of the sphere.

The number of electrons hitting a unit area of the surface of the sphere per unit time is found as follows:

$$\frac{N_e^s}{m^2} = - \int_{-\infty}^{-c_o^e} dc_x \int_{-\infty}^{\infty} dc_y \int_{-\infty}^{\infty} dc_z\, c_x \frac{n_e}{\pi^{3/2}} \alpha^3 e^{-\alpha^2(\vec{c}-\vec{V})^2}$$

$$= \frac{n_e}{2} \left[\frac{1}{\sqrt{\pi}\,\alpha} e^{-\alpha^2 (c_e^o + V\cos\theta)^2} - V\cos\theta \,\mathrm{erfc}\left[\alpha(c_e^o + V\cos\theta)\right]\right]$$

It is noted that the above integration is performed by using rectangular coordinates and $\vec{r}$ is made the z-axis. The retardation of the electrons caused by the negative potential of the sphere is neglected.

From equation (23) it follows that

$$c_e^o \gg V$$

because $\frac{1}{\alpha}$ is of the order of 10^2 and V is around 10. Using this inequality, it follows that

$$\frac{N_e^s}{m^2} = \frac{n_e}{2} \left[\frac{1}{\sqrt{\pi}\,\alpha} e^{-(\alpha c_e^o)^2} - V \cos\theta \operatorname{erfc}(\alpha c_e^o) \right] \qquad (24)$$

Therefore, the total number of electrons hitting the surface of the sphere per unit time is

$$N_e^s = 2\pi R^2 \int_0^{\pi} \sin\theta \, d\theta \frac{n_e}{2} \left[\frac{1}{\pi\alpha} e^{-(\alpha c_e^o)^2} - V \cos\theta \operatorname{erfc}(\alpha c_e^o) \right]$$

$$= \pi R^2 n_o e^{\frac{e\phi_o}{KT}} \left[1 - e^{\frac{e\phi_o}{KT}} \right] \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta \left[\frac{1}{\pi\alpha} e^{-(\alpha c_e^o)^2} - V \cos\theta \operatorname{erfc}(\alpha c_e^o) \right]$$

$$+ \pi R^2 n_o e^{\frac{e\phi_o}{KT}} \left[2 - e^{\frac{e\phi_o}{KT}} \right] \int_{\frac{\pi}{2}}^{\pi} \sin\theta \, d\theta \left[\frac{1}{\pi\alpha} e^{-(\alpha c_e^o)^2} - V \cos\theta \operatorname{erfc}(\alpha c_e^o) \right]$$

$$= \frac{\sqrt{\pi} R^2}{\alpha} n_o e^{\frac{e\phi_o}{KT}} e^{-(\alpha c_e^o)^2} \left(3 - 2 e^{\frac{e\phi_o}{KT}}\right) + \frac{\pi V R^2}{2} n_o e^{\frac{e\phi_o}{KT}} \operatorname{erfc}(\alpha c_e^o)$$

$$\doteq \frac{\sqrt{\pi} R^2}{\alpha} n_o e^{\frac{e\phi_o}{KT}} e^{-(\alpha c_e^o)^2} \left(3 - 2 e^{\frac{e\phi_o}{KT}}\right).$$

With $\alpha\, c_e^o = \sqrt{\frac{|e\phi_o|}{KT}}$, N_e^s can be expressed as follows:

$$N_e^s = \frac{\sqrt{\pi}\, R^2}{\alpha}\, n_o \left(3\, e^{\frac{2e\phi_o}{KT}} - 2\, e^{\frac{3e\phi_o}{KT}}\right) \tag{25}$$

The number of positive ions hitting the surface of the sphere can be determined in a similar way as follows:

The normalized velocity distribution function for the main stream of the ions is

$$f_i^s = \frac{n_i}{\pi^{3/2}}\, e^{-(\vec{c} - \vec{V})^2} \; .$$

The density distribution of the positive ions on the surface of the sphere is obtained from equation (5) by letting $r = R$. That is

at $r = R$

$$n_i = \frac{n_o}{2}\, \text{erfc}\, (V \cos\theta),$$

or from equation (6), at $r = R$

$$n_i \doteq \begin{cases} 0 & \text{for } 0^o \leqslant \theta < 90^o \\ n_o & \text{for } 90^o < \theta \leqslant 180^o \end{cases} \tag{26}$$

A velocity c_i^o which is defined analogously to c_e^o has the meaning that those positive ions having a velocity lower than c_i^o and pointing away from the sphere may be attracted back by the negative potential of the sphere. The value of c_i^o is defined as

$$\frac{1}{2} m_i \left(\sqrt{\frac{2KT}{m_i}}\, c_i^o\right)^2 = |e\phi_o|$$

or

$$c_i^o = \sqrt{\frac{|e\phi_o|}{KT}} \quad . \tag{27}$$

As c_i^o is much smaller than V, and as the motion of those ions mentioned above caused by the negative potential of the sphere is unknown, the effect of c_i^o is neglected in the following calculation.

The number of positive ions hitting a unit area of the surface of the sphere per unit time is

$$\frac{N_i^s}{m^2} = -\int_{-\infty}^{o} dc_x \int_{-\infty}^{\infty} dc_y \int_{-\infty}^{\infty} dc_z\, c_x \frac{n_i}{\pi^{3/2}} e^{-(\vec{c} - \vec{V})^2}$$

$$= \frac{n_i}{2} \left[\frac{1}{\sqrt{\pi}} e^{-V^2 \cos^2\theta} - V\cos\theta \,\mathrm{erfc}\,(V\cos\theta)\right] . \tag{28}$$

Therefore, the total number of the positive ions hitting the surface of the sphere per unit time is

$$N_i^s = 2\pi R^2 \int_0^{\pi} \sin\theta\, d\theta \frac{n_i}{2} \left[\frac{1}{\sqrt{\pi}} e^{-V^2\cos^2\theta} - V\cos\theta \,\mathrm{erfc}\,(V\cos\theta)\right]$$

$$= \pi R^2 n_o \int_{\frac{\pi}{2}}^{\pi} \sin\theta\, d\theta \left[\frac{1}{\sqrt{\pi}} e^{-V^2\cos^2\theta} - V\cos\theta \,\mathrm{erfc}\,(V\cos\theta)\right]$$

$$= \pi R^2 n_o V + \frac{\sqrt{\pi} R^2}{V} n_o \,\mathrm{erfc}\,(V)$$

$$\doteq \pi R^2 n_o V \, . \tag{29}$$

At equilibrium, the boundary condition $N_e^s = N_i^s$, leads to an equation as follows:

$$3\, e^{\frac{2e\phi_o}{KT}} - 2\, e^{\frac{3e\phi_o}{KT}} = \sqrt{\pi}\, \alpha\, V \; . \tag{30}$$

The potential of the sphere, ϕ_o, can be determined from equation (30). The numerical calculation shows the result in good agreement with the experimental data. It is noted that equation (30) mathematically gives two solutions for ϕ_o, one negative the other positive when V is around 10. The negative solution for ϕ_o is recognized as it agrees with the original assumption. The positive solution for ϕ_o lacks physical justification and no attempt is made to interpret it.

V

CHANGE OF SCATTERING CROSS SECTION OF SATELLITE

A uniform plasma is disturbed by the flight of the sphere. The electron density in the region surrounding the sphere has been obtained in a previous section. The result shows that the electron density deviates significantly from the unperturbed value in the immediate neighborhood of the sphere. This deviation dies down gradually in the direction away from the sphere. If the density of the electrons is not very high, the attenuation of the wave as it penetrates into the region and the secondary scattering between the electrons can be neglected. The scattering due to the more massive ions is also ignored. Therefore, the radar return from the disturbed region is obtained by integrating the individual Compton scattering from the electrons.

The backscattered power per unit solid angle per electron for unit incident power density is given by

$$\sigma_e = \left[\frac{e^2}{4\pi \epsilon_o m c^2} \right]^2 . \qquad (31)$$

Thus the change in the radar cross section of the sphere caused by the disturbed region surrounding the sphere in a plasma medium, or in other words, caused by the plasma sheath, is represented as

$$\sigma = 4\pi \sigma_e n_o^2 \left| \int_V \left(\frac{n_e}{n_o} - 1 \right) e^{2ikd} dV \right|^2 . \qquad (32)$$

n_e has been found in equation (20), and d is the distance from the observation point to the individual electron. k is the propagation constant of the electromagnetic wave and is defined as $\frac{2\pi}{\lambda}$ with λ as the wavelength. The factor e^{2ikd} takes into account the phase relation of the wave as it passes from the transmitting antenna to the region and back again to the receiving antenna.

The distance d as shown in Figure 2 can be expressed as a function of r, θ, ϕ (electron coordinates) and R_o, Θ, Φ, (observation point coordinates) as follows:

$$d = (R_o^2 + r^2 - 2 \cdot \vec{R}_o \cdot \vec{r})^{\frac{1}{2}}$$

$$= R_o \left[1 + \frac{r^2}{R_o^2} - \frac{2r}{R_o} (\cos\theta \cos\Theta + \sin\theta \sin\Theta \cos(\phi - \Phi)) \right]^{\frac{1}{2}}$$

$$\doteq R_o + \frac{r^2}{2R_o} - r(\cos\theta \cos\Theta + \sin\theta \sin\Theta \cos(\phi - \Phi)) .$$

The square of the magnitude of the integral appearing in equation (32) can be transformed into the following form:

$$|I|^2 = \left| \int_V \left(\frac{n_e}{n_o} - 1\right) e^{2ikd} \, dV \right|^2$$

$$= \left| \int_V \left(\frac{n_e}{n_o} - 1\right) e^{2ik\left[\frac{r^2}{2R_o} - r(\cos\theta\cos\Theta + \sin\theta \sin\Theta \cos(\phi - \Phi))\right]} dV \right|^2 .$$

Substitution of the expression for n_e in the above integral gives

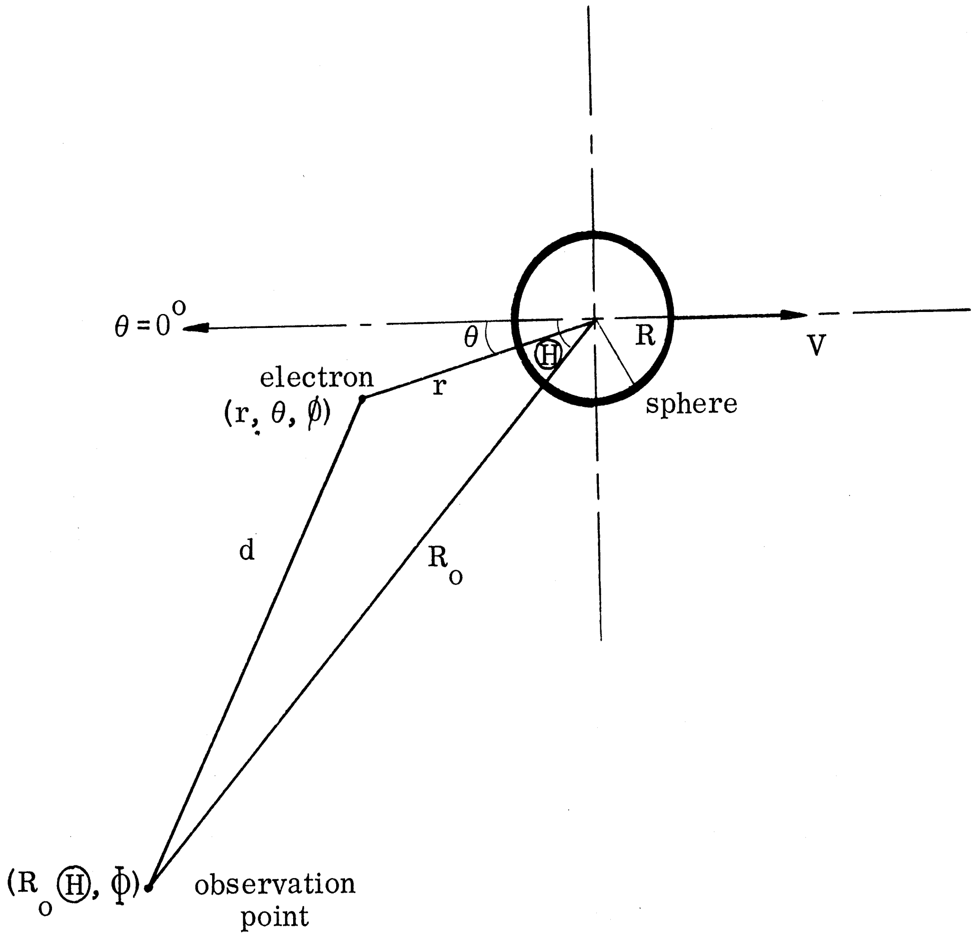

FIG. 2. COORDINATE SYSTEM FOR THE DETERMINATION OF SCATTERING CROSS SECTION

$$I = -\int_R^\infty dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \; r^2 \sin\theta \left[1 - e^{\frac{e\phi_o}{KT}\frac{R}{r}}\right]^2 .$$

$$\cdot\, e^{2\,ik\left[\frac{r^2}{2R_o} - r(\cos\theta\cos Ⓗ + \sin\theta\sin Ⓗ \cos(\phi - \Phi))\right]}$$

$$- \int_R^\infty dr \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} d\theta \; r^2 \sin\theta \, e^{\frac{e\phi_o}{KT}\frac{R}{r}} e^{-V^2 \sin^2\theta}$$

$$\cdot \left(e^{V^2 \sin\theta \frac{R}{r}} - \sqrt{1 - \frac{R^2}{r^2}} \; e^{-V^2\cos^2\theta \frac{R^2}{r^2}} \right)$$

$$\cdot\, e^{2ik\left[\frac{r^2}{2R_o} - r(\cos\theta\cos Ⓗ + \sin\theta\sin Ⓗ \cos(\phi - \Phi))\right]} . \tag{33}$$

The first integral of equation (33) is independent of Ⓗ , so it can be simplified to the following form by letting Ⓗ = 0^o.

$$I_1 = -\int_R^\infty dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \; r^2 \sin\theta \left[1 - e^{\frac{e\phi_o}{KT}\frac{R}{r}}\right]^2 e^{2ik\left[\frac{r^2}{2R_o} - r\cos\theta\right]}$$

$$= -\frac{2\pi}{k} \int_R^\infty \left[1 - e^{\frac{e\phi_o}{KT}\frac{R}{r}}\right]^2 e^{ik\frac{r^2}{2R_o}} \sin 2kr \; r \, dr \quad . \tag{34}$$

The second integral of equation (33) can be simplified a little by assuming the the main contribution of the integral is from the region where θ is small. This leads to

$$I_2 = - \int_R^{\infty} dr \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} d\theta \; r^2 \sin\theta \; e^{\frac{e\phi_o}{KT}\frac{R}{r}} \; e^{-V^2 \sin^2\theta}$$

$$\cdot \left(e^{V^2 \sin^2\theta \frac{R}{r}} - \sqrt{1 - \frac{R^2}{r^2}} \; e^{-V^2 \cos^2\theta \frac{R^2}{r^2}} \right) \cdot$$

$$\cdot \; e^{2ik\left[\frac{r^2}{2R_o} - r(\cos\theta \cos \text{Ⓗ} + \sin\theta \sin \text{Ⓗ} \cos(\phi - \Phi))\right]}$$

$$\doteq -2\pi \int_R^{\infty} dr \; r^2 \; e^{ik\frac{r^2}{R_o}} \; e^{\frac{e\phi_o}{KT}\frac{R}{r}} \int_0^{\frac{\pi}{2}} d\theta \sin\theta \; e^{-2ikr\cos(\theta - \text{Ⓗ})} \; .$$

$$\cdot \left[e^{-V^2 \sin^2\theta (1 - \frac{R}{r})} - \sqrt{1 - \frac{R^2}{r^2}} \; e^{-V^2(\sin^2\theta + \cos^2\theta \frac{R^2}{r^2})} \right] . \quad (35)$$

I_1 and I_2 can be numerically evaluated and after that the change of the radar cross section of the sphere is obtained from equation (32) as

$$\sigma = 4\pi \sigma_e n_o^2 \left| I_1 + I_2 \right|^2 \; . \quad (36)$$

VI

NUMERICAL RESULTS

The theoretical formulas developed in the previous sections are used in the numerical calculation for a typical case of a spherical satellite. In this section, the potential of the satellite and the change of its radar cross section due to the plasma sheath are calculated.

A spherical satellite of 1 m radius moving with a velocity of 8 Km/sec at altitude of 500 Km is considered. For this case the following numerical data can be assigned:

$$KT \doteq 0.1 \text{ electron-volt} \qquad (\text{or } T \doteq 1160^{\circ}K)$$

$$n_o = 10^{12} \quad 1/m^3$$

$$V_i \sim 1 \text{ Km/sec} \ , \qquad V_e \sim 200 \text{ Km/sec}$$

$$\sqrt{\frac{2KT}{m_i}} \doteq 1 \text{ Km/sec}$$

$$\alpha = \sqrt{\frac{m_e}{m_i}} = \frac{1}{166} \qquad (\text{assuming } O^+ \text{ and } N^+ \text{ ions}) \ .$$

The normalized velocity $V \doteq 8$.

(1) The potential of the Satellite.

From equation (30)

$$3e^{\frac{2e\phi_o}{KT}} - 2e^{\frac{3e\phi_o}{KT}} = \sqrt{\pi}\,\alpha V \doteq 0.085 \ .$$

So

$$e^{\frac{e\phi_o}{KT}} \doteq 0.18$$

$$\frac{e\phi_o}{KT} = -\ln 5.55 = -1.71$$

with

$$KT = 0.1 \text{ e.v.}$$

$$\underline{\phi_o = -0.171 \text{ volt.}}$$

The Russian experimental data shows that the potential of a satellite at night is of the order of 0 ± 1 volt. The theoretical prediction is of the right order.

(2) The change of the scattering cross section of the satellite.

From equation (36)

$$\sigma = 4\pi \sigma_e n_o^2 \left| I_1 + I_2 \right|^2$$

where

$$4\pi \sigma_e \sim 10^{-28} m^2 ,$$

$$n_o^2 = 10^{24} \ 1/m^6 ,$$

$$R = 1 \text{ m} , \ R_o = 5 \times 10^5 \text{ m} ,$$

$$\frac{e\phi_o}{KT} R = -1.71$$

and

$$I_1 = -\frac{2\pi}{k}\int_1^{\infty}\left[1 - e^{-\frac{1.71}{r}}\right]^2 e^{ik\frac{r^2}{2R_o}} \sin 2kr \; r \, dr$$

$$I_2 = -2\pi \int_1^{\infty} dr \, r^2 \, e^{ik\frac{r^2}{R_o}} \, e^{-\frac{1.71}{r}} \int_0^{\frac{\pi}{2}} d\theta \, \sin\theta \, e^{-2ikr\cos(\theta - \Theta)} \, .$$

$$\cdot \left[e^{-64\sin^2\theta(1-\frac{1}{\cdot}) \quad \cdot + \cos^2\theta \frac{1}{r^2})} \right] .$$

Because t ...ed, the upper limit of r is assigned to be 30 m. ... on the fact that the main contribution to I_2 is from the region of small θ, I_2 can be approximated further. The forms of I_1 and I_2 used in the numerical calculation are as follows:

$$I_1 = -\frac{2\pi}{k}\int_1^{30}\left[1 - e^{-\frac{1.71}{r}}\right]^2 \sin 2kr \; r \, dr$$

$$I_2 = -2\pi \int_1^{30} dr \, r^2 \, e^{-\frac{1.71}{r}} \, e^{-2ikr\cos\Theta} \, .$$

$$\cdot \left[\frac{e^{-64(1-\frac{1}{r})}}{8\sqrt{1-\frac{1}{r}}} \int_0^{8\sqrt{1-\frac{1}{r}}} dx \, e^{x^2} - \frac{e^{-64}}{8}\int_0^{8\sqrt{1-\frac{1}{r^2}}} dx \, e^{x^2} \right] .$$

I_1 and I_2 are calculated for one case of k and three values of Ⓗ. Namely, $k = 2\pi/15$, and $\theta = 45^o, 60^o, 90^o$. The numerical results are shown in the following table:

TABLE I

CHANGE ON SCATTERING CROSS SECTION OF SATELLITE
(1 M RADIUS)

[illegible]	[illegible]	[illegible]
		2.29×10^2
	Ⓗ $= 45^o$	3.67×10

It is understood that as the frequency increases the change on the scattering cross section decreases. The method used in calculating the change on the scattering cross section at a frequency much higher than the plasma frequency may have to be improved.

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ADDENDA TO THE UNIVERSITY OF MICHIGAN RADIATION LABORATORY REPORT 2764-6-T, "STUDIES IN RADAR CROSS SECTIONS XLIII - PLASMA SHEATH SURROUNDING A CONDUCTING SPHERICAL SATELLITE AND THE EFFECT ON RADAR CROSS SECTION"

NUMERICAL CALCULATIONS (continued)

In connection with the calculation of the change of the scattering cross section of the satellite, it is learned that the upper limits of the integrals, I_1 and I_2, should be specified carefully. In the first place, the upper limits should be finite because: (1) The disturbance caused by the satellite in the plasma will be restored gradually through the ambipolar diffusion which was not taken into account in the analysis; (2) The beamwidth of the radar used in the satellite tracking is very sharp and only a limited space can be illuminated. Fortunately, it is found in the numerical calculation that the significant contribution to the integrals comes from the region where r is smaller than 2 Km. As r becomes bigger than this value the phase factor becomes important and the effect of cancellation takes place.

A numerical calculation is made for the case of 1 m radius satellite and the incident electromagnetic wave of 15 m wavelength. The integrals used in the calculation are

$$I_1 = -\frac{2\pi}{k}\int_1^{\sqrt{3.75}\times 10^3}\left[1-e^{-\frac{1.71}{r}}\right]^2 e^{i\frac{kr^2}{R_o}}\sin 2kr\, r\, dr$$

$$I_2 = -2\pi \int_1^{\sqrt{3.75} \times 10^3} dr\, r^2\, e^{-\frac{1.71}{r}}\, e^{-2ikr\cos \textcircled{H}}\, e^{i\frac{kr^2}{R_o}}$$

$$\cdot \left[\frac{e^{-64(1-\frac{1}{r})}}{8\sqrt{1-\frac{1}{r}}} \int_0^{8\sqrt{1-\frac{1}{r}}} e^{x^2}\, dx - \frac{e^{-64}}{8} \int_0^{8\sqrt{1-\frac{1}{r^2}}} e^{x^2}\, dx \right] .$$

The upper limit is specified in such a way that the absolute value of I_2 is a maximum. The final results are shown in the following table.

$\textcircled{H}$	σ (cross section in m^2)
90^o	3.6×10^5
60^o	5.4
45^o	2.7

The results show that the scattering of the electromagnetic wave has the broadside effect. This implies that the reflection of the electromagnetic wave is maximum when the satellite is right overhead and this reflection dies out rapidly as the satellite moves away. This effect may cause a strong pulse type reflection in the course of the satellite passage.

The accuracy of the numerical data is not very high because all computation was done on desk calculators and the theory itself is approximate. However, the orders of the results are expected to be correct.

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