2.11. NECESSARY CONDITIONS FOR INTERPOLATION BY ENTIRE FUNCTIONS*

Let p be a subharmonic function on $\mathbb{C}$ such that $\log(1+|z|) = O(p(z))$ and let A_p denote the algebra of entire functions f such that $|f(z)| \leqslant A\exp(Bp(z))$ for some $A, B > 0$. Let V denote a discrete sequence of points $\{a_n\}$ of $\mathbb{C}$ together with a sequence of positive integers $\{p_n\}$ (the multiplicities of $\{a_n\}$). If $f \in A_p$, $f \not\equiv 0$, then V(f) denotes the sequence $\{a_n\}$ of zeros of f and p is the order of zero of f at a_n.

In this situation, there are three natural problems to study.

I. Zero Set Problem. Given p, describe the sets V(f), $f \in A_p$

II. Interpolation Problem. If $\{a_n, p_n\} = V \subset V(f)$ for some $f, f \in A_p$, describe all sequences $\{\lambda_{n,k}\}$ which are of the form

$$\lambda_{n,k} = \frac{g^{(k)}(a_n)}{k!}, 0 \leqslant k < p_n, n = 1, 2, \ldots \text{ for some } g, g \in A_p. \tag{1}$$

III. Universal Interpolation Problem. If $V \subset V(f)$ for some $f, f \in A_p$, under what conditions on V is it true that for every sequence $\{\lambda_{n,k}\}$, such that $|\lambda_{n,k}| \leqslant A\exp(Bp(a_n))$, there exists g, $g \in A_p$, satisfying (1).

In case $p(z) = p(|z|)$ (and satisfies some mild, technical conditions), quite good solutions to problems I-III are known. This work has been carried out by Leont'ev and others (see, e.g., [1] for a survey). However, when p is not a function of $|z|$, the general solutions are not known.

The purpose of this note is to call attention to an interesting special case of III. Consider the case $p(z) = |\operatorname{Im} z| + \log(1+|z|^2)$. Then $A_p = \hat{\varepsilon}'$, the space of all entire functions of exponential type with polynomial growth on the real axis. The space $\hat{\varepsilon}'$ is of special interest because, by the Paley–Wiener–Schwartz theorem, it is the space of Fourier transforms of distributions on $\mathbb{R}$ with compact support. The problems I-III are then dual to some problems about convolution operators on the space $\mathcal{E} = C^\infty(\mathbb{R})$ (see, e.g., [1-3]).

Specifically, suppose for some $\varepsilon > 0$, $c > 0$, $f \in \hat{\varepsilon}'$, we have

$$V = \{a_n, p_n\} \subset V(f), \text{ where } \frac{|f^{(p_n)}(a_n)|}{p_n!} \geqslant \varepsilon \exp\left(-\frac{p(a_n)}{p_n}\right) \tag{2}$$
$$(p(z) = |\operatorname{Im} z| + \log(1+|z|^2)).$$

Then it is not hard to show that (2) is a sufficient condition that V has the universal interpolation property III. We wish to pose the converse problem.

Problem. Suppose that $V \subset V(F)$ for some F, $F \in \hat{\varepsilon}'$, and that V is a universal interpolating sequence, i.e., III holds. Is it true that (2) must hold for some f, $f \in \hat{\varepsilon}'(\mathbb{R})$?

In all the cases known to the author where the problem has answer yes, it is also true that the range of the multiplication operator $M_F: A_p \to A_p$ given by $M_F(f) = Ff$ is closed. Is the fact that M_F has closed range necessary for a "yes" answer? (In the case $A_p = \hat{\varepsilon}'$, if M_F has closed range, then the problem has answer yes, as can be shown by the techniques of [4].) However, the main interest in the problem is to find if (2) must hold with no additional assumptions on F.

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