On Taylor Series of Functions Regular in Gaier Regions

To Alexander Ostrowski for his 60th birthday

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1. Introduction

This paper deals with the Taylor coefficients of functions that are regular inside the unit circle C and have only one singularity on C. Much of its motivation comes from the following lemma of Gaier [3, pp. 327, 328]:

If for some $a > 0$ the function $f(z) = \Sigma a_n z^n$ is regular and bounded in the disc $|z + a| < 1 + a$, then $a_n = O(n^{-1/2})$.

We shall apply the term Gaier *region* to any open region which contains the unit disc $|z| < 1$ and whose boundary does not meet the unit circle C except at $z = 1$. In particular, if a Gaier region is one of the circular discs in the lemma above, we shall call it a Gaier *disc*.

In § 2, we show that Gaier's lemma cannot be improved, in the sense that the O cannot be replaced by o, and we state our Theorem 2, of which Gaier's lemma is a special case. Three sections are devoted to the proof of this theorem.

While Theorem 2 provides bounds for individual Taylor coefficients of functions satisfying certain restrictions in a Gaier disc, Theorem 3 (§ 6) gives a bound on the sum of the moduli of coefficients in certain blocks of coefficients. On the one hand, this bound cannot be deduced from Theorem 2; on the other hand, certain results of Fejér show that the bound is the best possible.

§ 7 uses the technique of § 6 to obtain a theorem on the series $\Sigma n \mid a_n \mid^2$; § 8 deals with the convergence of Σa_n and with the uniform convergence on the unit circle of $\Sigma a_n z^n$; and § 9 is devoted to a partial analogue of Theorem 2 for Gaier regions other than Gaier discs.

2. On Gaier's lemma

Since Gaier's lemma is proved by means of the equation

$$a_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z^{n+1}}\, dz ,$$

where Γ is a suitable contour, it is reasonably regarded as a generalization of CAUCHY's inequality $|a_n| \leqq M$ on the TAYLOR coefficients of a bounded function. Since CAUCHY's inequality can be sharpened (for large n) to the relation $a_n = o(1)$ by GUTZMER's relation

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\Theta})|^2 \, d\Theta ,$$

the question arises whether GAIER's lemma can be improved in the same way. If the hypotheses are slightly strengthened, this is indeed the case: in a private communication, GAIER has pointed out that if $f(z)$ is continuous in the closure of a GAIER disc, then $a_n = o(n^{-1/2})$. But GUTZMER's proof of his theorém [5] is based on the relation $\cos(\Theta + \pi) = -\cos\Theta$; and the customary modern proof of his theorem relies heavily on the fact that the set of functions $\{z^n\}$ $(n = 0, 1, 2, \ldots)$ form an orthogonal set on C. In the case of GAIER's lemma, the trigonometric relation cannot be used, and the functions $\{z^n\}$ do not form an orthogonal set on GAIER's contour; therefore the following result should not come as a surprise.

Theorem 1. *There exists a function* $f(z) = \Sigma\, a_n z^n$, *regular and bounded in a* GAIER *disc, for which* $\limsup(|a_n| \, n^{1/2}) > 0$.

To prove this theorem, we show first that for any fixed number $b (0 < b < 1)$ the function

$$g(z) = \sum_{j=0}^{\infty} (-1)^j z^{m_j}$$

is bounded in the disc $|z - b| < 1 - b$, provided the sequence of positive integers m_j increases fast enough.

We denote by C_b the circle $|z - b| = 1 - b$, and we choose a sequence $\{\varepsilon_j\}$ with $\varepsilon_j > 0$ and $\Sigma\, \varepsilon_j < \infty$. The integers m_0 and m_1 can be chosen arbitrarily. There then exists an open arc A_0 on C_b, containing the point $z = 1$, and such that

$$|z^{m_0} - z^{m_1}| < \varepsilon_0$$

on A_0. We choose m_2 and m_3 large enough so that

$$|z|^{m_3} < |z|^{m_2} < \varepsilon_1$$

on the complement of A_0 relative to C_b. The arc A_0 has an open subarc A_1, containing $z = 1$, such that

$$|z^{m_2} - z^{m_3}| < \varepsilon_1$$

on A_1. We choose m_4 and m_5 large enough so that

$$|z|^{m_5} < |z|^{m_4} < \varepsilon_2$$

on the complement of A_1. If the construction is continued in this manner, then

$$\Big| \sum_{j=0}^{J} (z^{m_{2j}} - z^{m_{2j+1}}) \Big| < 2 + 2 \sum_{j=0}^{J} \varepsilon_j$$

on C_b and consequently $g(z)$ is bounded inside of C_b.

We turn now to the function

$$f(z) = g\left(\frac{z+1}{2}\right) = \sum_{j=0}^{\infty} (-1)^j \left(\frac{z+1}{2}\right)^{m_j} = \sum_{n=0}^{\infty} a_n z^n .$$

If $m_j \to \infty$ fast enough, $f(z)$ is bounded in the GAIER disc $|z + 1/2| < 3/2$. Also, a slight computation shows that if $m_j \to \infty$ fast enough, then

$$|a_n| \sim (\pi n)^{-1/2}$$

for $n = [m_j/2]$, $j = 0, 1, 2, \ldots$. This proves the theorem.

The following theorem differs from GAIER's lemma in that it replaces the boundedness of $f(z)$ by the boundedness of $(1-z)^k f(z)$, where k is a real constant.

Theorem 2. *Let $f(z) = \Sigma a_n z^n$ be regular in some* GAIER *disc $|z + a| < 1 + a$, and let k be a real number such that $(1-z)^k f(z)$ is bounded in this disc. Then*

$$a_n = O(n^{k-1}) \qquad \text{if } k > 1 ,$$
$$a_n = O(\log n) \qquad \text{if } k = 1 ,$$
$$a_n = O(n^{(k-1)/2}) \qquad \text{if } k < 1 .$$

In this estimate, the O cannot generally be replaced by o; the replacement is permissible if $k \leqq 1$ and $(1-z)^k f(z)$ approaches a limit whenever $z \to 1$ from the interior of the GAIER *disc.*

3. The case $k > 1$

Here the estimate is well known. It can be obtained from CAUCHY's formula by integration along the circle $|z| = 1 - 1/n$. That the O cannot be replaced by o is seen from the example $f(z) = (1-z)^{-k}$. It is noteworthy that, in the case $k > 1$, the hypothesis that $(1-z)^k f(z)$ is regular in a GAIER disc and continuous on the closure of this disc does not yield a better estimate on a_n than does the hypothesis that $(1-z)^k f(z)$ is regular and bounded in the unit disc.

4. The case $k = 1$

Again, integration along the circle $|z| = 1 - 1/n$ gives the estimate $a_n = O(\log n)$. For a precise discussion of the situation where $(1-z) f(z)$ is merely assumed to be regular and bounded in the unit disc, the reader is referred to NEDER [7]. Here we shall only show that

i) $a_n = o(\log n)$ if $(1-z) f(z)$ is regular and bounded in the unit disc and approaches a limit as $z \to 1$ from the interior of the unit disc;

ii) regularity and boundedness of $(1-z) f(z)$ in a GAIER disc does not imply that $a_n = o(\log n)$.

To prove the first of these propositions, suppose that $(1-z)\,f(z)$ is regular and bounded in $|z| < 1$, and that $\lim_{z \to 1} (1-z)\,f(z) = A$. Then

$$f(z) = \frac{A}{1-z} + \frac{\Phi(z)}{1-z},$$

where $\Phi(z)$ is bounded in $|z| < 1$ and $\Phi(z) \to 0$ as $z \to 1$ in the unit disc. The TAYLOR coefficients of $A/(1-z)$ are all equal to A and cause no trouble. Let

$$\frac{\Phi(z)}{1-z} = \sum_{n=0}^{\infty} b_n z^n .$$

Then

$$|b_n| \leqq \frac{1}{2\pi} \int_{\Gamma_n} \frac{|\Phi(z)|\,|dz|}{|z^{n+1}|\,|1-z|},$$

where Γ_n is the contour $|z| = 1 - 1/n$. Since the value of $|z|^{n+1}$ on Γ_n approaches $1/e$ as $n \to \infty$, it suffices to prove that

$$\int_{\Gamma_n} \frac{|\Phi(z)|\,|dz|}{|1-z|} = o(\log n),$$

and geometrical considerations reduce the problem to the task of showing that, with $z = (1-1/n)\,e^{i\Theta}$,

$$\int_{-\pi}^{\pi} \frac{|\Phi(z)|\,d\Theta}{\sqrt{\Theta^2 + n^{-2}}} = o(\log n).$$

Now, if $0 < \Theta_n \leqq \pi$ and $M_n = \max |\Phi(z)|$ for $z = (1-1/n)\,e^{i\Theta}$, $-\Theta_n \leqq \Theta \leqq \Theta_n$, then

$$\int_{-\Theta_n}^{\Theta_n} \frac{|\Phi(z)|\,d\Theta}{\sqrt{\Theta^2 + n^{-2}}} \leqq 2\,M_n \int_0^{\Theta_n} \frac{d\Theta}{\sqrt{\Theta^2 + n^{-2}}} = 2\,M_n \log\left(n\,\Theta_n + \sqrt{n^2\,\Theta_n^2 + 1}\right)$$
$$< 2\,M_n \log(1 + 2n\,\Theta_n) \leqq 2\,M_n \log(1 + 2\pi n).$$

On the other hand,

$$\int_{\Theta_n}^{2\pi-\Theta_n} \frac{|\Phi(z)|\,d\Theta}{\sqrt{\Theta^2 + n^{-2}}} \leqq 2\,M \int_{\Theta_n}^{\pi} \frac{d\Theta}{\Theta} = 2\,M \log(\pi/\Theta_n),$$

where M is a bound for $|\Phi(z)|$ in $|z| < 1$. If we choose $\Theta_n = 1/\log n$, then $M_n \to 0$, and it follows that $b_n = o(\log n)$.

To prove that regularity and boundedness of $(1-z)\,f(z)$ in a GAIER disc does not imply that $a_n = o(\log n)$, we use the polynomials

$$P_n(z) = \frac{1}{n} + \frac{z}{n-1} + \cdots + \frac{z^{n-1}}{1} - \frac{z^n}{1} - \frac{z^{n+1}}{2} - \cdots - \frac{z^{2n-1}}{n}.$$

FEJÉR [2, pp. 74—76] proved that on the unit circle C these polynomials have a bound which is independent of n. We believe that the following new proof of this proposition is of interest because of its simple and elementary character.

At $z = e^{i\Theta}$, the sum of the $2r$ middle terms of $P_n(z)$ has modulus

$$\left| \sum_{j=1}^{r} \frac{z^{n-j}(1-z^{2j-1})}{j} \right| \leqq |\Theta| \sum_{j=1}^{r} \frac{2j-1}{j} \leqq 2r|\Theta|.$$

Also, by ABEL's summation, the sum of the first $n-r$ terms is

$$\sum_{j=r+1}^{n} \frac{z^{n-j}}{j} = \frac{1}{r+1} \sum_{h=0}^{n-r-1} z^h - \sum_{j=r+1}^{n-1} \frac{1}{j(j+1)} \sum_{h=0}^{n-j-1} z^h,$$

and, for $0 < |\Theta| \leqq \pi$, this has modulus less than $2\pi/(r+1)|\Theta|$. The modulus of the last $n-r$ terms has the same bound, and therefore

$$|P_n(e^{i\Theta})| \leqq 2r|\Theta| + \frac{4\pi}{(r+1)|\Theta|}$$

for $0 < |\Theta| \leqq \pi$. The choice $r = \min(n, [\pi/|\Theta|])$ then gives the desired result.

We now write

$$Q_n(z) = z^{n^2} P_n(z^{n^2})$$
$$= \frac{z^{n^2}}{n} + \frac{z^{2n^2}}{n-1} + \dots + \frac{z^{n^3}}{1} - \frac{z^{n^3+n^2}}{1} - \frac{z^{n^3+2n^2}}{2} - \dots - \frac{z^{2n^3}}{n},$$

and we form the function

$$F(z) = \sum_{i=1}^{\infty} Q_{n_i}(z).$$

We assume that the sequence $\{n_i\}$ is chosen in such a way that $2n_i^3 < n_{i+1}^2$ for $i = 1, 2, \dots$. Since the TAYLOR series of $F(z)$ has infinitely many terms with coefficient one, $F(z)$ is not bounded in $|z| < 1$. However, it follows from considerations similar to those in § 2 that $F(z)$ is bounded in the disc $|z - 1/4| < 3/4$, provided $n_i \to \infty$ fast enough.

Let

$$G(z) = F\left(\frac{z+1}{2}\right) = \sum_{n=0}^{\infty} b_n z^n.$$

Then $G(z)$ is bounded in the GAIER disc $|z + 1/2| < 3/2$. On the other hand, it is easily verified that, if $n_i \to \infty$ fast enough,

$$\sum_{j=0}^{[n_i^3/2]} b_j > \tfrac{1}{2} \log n_i - O(n_i^{-1}),$$

where the second term on the right-hand side is obtained by a slight modification of Problem 145 of PÓLYA and SZEGÖ [10, vol. I, pp. 66 and 230]. It follows that, if

$$f(z) = \frac{G(z)}{1-z} = \sum_{n=0}^{\infty} a_n z^n ,$$

then $(1-z) f(z)$ is bounded in a GAIER disc, while $a_n/\log n$ remains greater than a positive constant for $n = [n_i^3/2]$.

5. The case $k < 1$

In proving that $a_n = O(n^{(k-1)/2})$, we shall essentially follow GAIER and estimate a_n by CAUCHY's formula, with the contour of integration composed of the arc

$$\Gamma: \qquad z = (1 + c_1 \Phi^2)\, e^{i\Phi}, \qquad -\pi \leqq \Phi \leqq \pi ,$$

where c_1 is a positive constant small enough so that the curve Γ lies in the GAIER disc $|z + a| < 1 + a$, except for the point $z = 1$. (A moment's consideration shows that this path of integration is permissible even when $0 < k < 1$.) Then

$$2\pi\, |a_n| = \left| \int_{\Gamma} \frac{f(z)}{z^{n+1}}\, dz \right| \leqq c_2 \int_0^{\pi} \frac{1}{\Phi^k (1 + c_1 \Phi^2)^n} \left(1 + \frac{2 c_1 \Phi}{1 + c_1 \Phi^2}\right) d\Phi$$

$$\leqq c_3 \int_0^{\pi} \frac{d\Phi}{\Phi^k (1 + c_1 \Phi^2)^n} = c_3 (c_1 n)^{(k-1)/2} \int_0^{\pi\sqrt{c_1 n}} \frac{du}{u^k (1 + u^2/n)^n} .$$

Now, when $u \geqq 0$ and p is a positive integer,

$$\left(1 + \frac{u^2}{n}\right)^n \geqq 1 + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{p-1}{n}\right) \frac{u^{2p}}{p!} ;$$

any choice of p greater than $(1-k)/2$ shows that the last integral has a bound independent of n, and it follows that $a_n = O(n^{(k-1)/2})$.

If the function $(1-z)^k f(z)$ is not only bounded in a GAIER disc but is continuous in the closure of a GAIER disc, then

$$f(z) = \frac{A}{(1-z)^k} + \frac{\Phi(z)}{(1-z)^k} ,$$

where $\Phi(z)$ is bounded in the GAIER disc and $\Phi(z) \to 0$, as $z \to 1$. The contribution to a_n from the n-th TAYLOR coefficient of the first function on the right is $O(n^{k-1}) = o(n^{(k-1)/2})$. The contribution from the second term on the right can be treated much as in the discussion of the case $k = 1$ (see § 4); we omit the details.

We will now show that for every $k < 1$ there exists a function $f(z) = \Sigma\, a_n z^n$ such that $(1-z)^k f(z)$ is regular and bounded in the GAIER disc $|z + 1/2| < 3/2$, and such that $\limsup (|a_n|\, n^{(1-k)/2}) > 0$.

For any fixed k, we choose an integer h such that $h + k > 0$. We begin with the function

$$G(z) = \sum_{j=1}^{\infty} g(z, p_j),$$

where $\{p_j\}$ is an increasing sequence of positive integers and

$$g(z,p) = p^k z^{2p^2}(1-z^p)^h.$$

We choose a positive constant c_4 small enough so that the curve

$$K: \qquad z = (1-c_4\Phi^2)\, e^{i\Phi}, \qquad -\pi \leqq \Phi \leqq \pi$$

encloses the disc $|z - 1/4| < 3/4$. On K,

$$|z|^{2p^2} = (1-c_4\Phi^2)^{2p^2} \leqq e^{-2c_4p^2\Phi^2}.$$

Also on K,

$$|1-z^p| = |1-(1-c_4\Phi^2)^p e^{ip\Phi}|$$

$$= \left\{ [1-(1-c_4\Phi^2)^p]^2 + 4(1-c_4\Phi^2)^p \sin^2\frac{p\Phi}{2} \right\}^{1/2}.$$

Since $1 - x^p \leqq p(1-x)$ for $0 \leqq x \leqq 1$, the first term in the braces is not greater than $(c_4 p \Phi^2)^2$, and it follows that $|1-z^p| \leqq c_5 |p\Phi|$. Therefore, for z on K,

$$|(1-z)^k g(z,p)| \leqq c_6 |p\Phi|^{h+k} e^{-2c_4(p\Phi)^2},$$

and since $h + k > 0$, this has an upper bound independent of p and Φ. Moreover, $(1-z)^k g(z,p) \to 0$, as $z \to 1$ along K; and on any closed arc of K that does not pass through $z = 1$, the function $(1-z)^k g(z,p)$ can be made arbitrarily small by choosing the integer p sufficiently large. It follows that we can apply the method used in the proof of Theorem 1 to choose the sequence $\{p_j\}$ in such a way that $(1-z)^k G(z)$ is regular and bounded in the interior of K.

Finally, we consider the function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = G\left(\frac{z+1}{2}\right) = \sum_{j=1}^{\infty} g\left(\frac{z+1}{2}, p_j\right).$$

From our discussion of $G(z)$ it follows that $(1-z)^k f(z)$ is regular and bounded in the GAIER disc $|z + 1/2| < 3/2$. It remains only to examine the coefficients a_n.

The coefficient of z^{p^2} in the polynomial $g\left(\frac{z+1}{2}, p\right)$ is

$$p^k 2^{-2p^2} \sum_{\lambda=0}^{h} (-1)^\lambda \binom{h}{\lambda} 2^{-\lambda p} \binom{2p^2+\lambda p}{p^2} = p^k 2^{-2p^2} \binom{2p^2}{p^2} \left[1 - \sum_{\lambda=1}^{h} \prod_{r=1}^{\lambda p} \frac{2p^2+r}{2p^2+2r}\right].$$

Each product in the last expression is of the form

$$\prod_{r=1}^{\lambda p} \left(1 - \frac{r}{2p^2+2r}\right),$$

and its logarithm approaches $-\lambda^2/4$ as $p \to \infty$.

It follows that the coefficient of z^{p^2} in $g\left(\frac{z+1}{2}, p\right)$ is asymptotically equal to

$$p^k(\pi p^2)^{-1/2} \sum_{\lambda=0}^{h} (-1)^\lambda \binom{h}{\lambda} e^{-\lambda^2/4}.$$

Since $e^{-1/4}$ is a transcendental number, the sum in the last expression is different from zero. Now let $n = p_i^2$. Because the contributions to a_n from the terms $g\left(\frac{z+1}{2}, p_j\right)$ are zero for $j < i$ and $o(p_i^{k-1})$ for $j > i$, provided $p_j \to \infty$ fast enough, it follows that, as $n \to \infty$ through the values p_i^2,

$$a_n \sim \beta p_i^{k-1} = \beta n^{(k-1)/2},$$

where β is a constant different from zero. This concludes the proof of Theorem 2.

6. On blocks of terms from the series $\sum |a_j|$

We now turn our attention from individual coefficients to sums of the form

$$S_n = \sum_{j=n}^{2n} |a_j|.$$

If $f(z)$ satisfies the conditions of Theorem 2, then

$$\begin{aligned} S_n &= O(n^k) && \text{if } k > 1, \\ S_n &= O(n \log n) && \text{if } k = 1, \\ S_n &= O(n^{(k+1)/2}) && \text{if } k < 1. \end{aligned}$$

For the case $k > 1$, this estimate cannot be improved, as is shown by the example $f(z) = (1-z)^{-k}$. The following theorem improves the estimate for the remaining values of k.

Theorem 3. *If $f(z)$ satisfies the condition of Theorem 2, then*

$$\begin{aligned} S_n &= O(n^k) && \textit{if } k > 1/2, \\ S_n &= O(\sqrt{n} \log n) && \textit{if } k = 1/2, \\ S_n &= O(n^{k/2+1/4}) && \textit{if } k < 1/2; \end{aligned}$$

if $k \neq 1/2$ the O in this estimate cannot be replaced by o.

In the proof we will need the following auxiliary result.

Lemma. *Let $f(z) = \Sigma a_j z^j$ be regular in $|z| < 1$, and let m be a non-negative integer such that*

$$\int_{\Gamma_n} |f^{(m)}(z)|^2 |dz| < n^\alpha, \qquad n = 2, 3, \ldots,$$

where Γ_n is the circle $|z| = 1 - 1/n$. Then

$$S_n < c\, n^{(\alpha - 2m + 1)/2},$$

where c depends only on m and on α.

By GUTZMER's relation (see § 2), the hypothesis of the lemma implies that (with $z=(1-1/n)\,e^{i\Theta}$)

$$\sum_{j=n}^{2n}[j(j-1)\,\ldots\,(j-m+1)\mid a_j\mid]^2\left(1-\frac{1}{n}\right)^{2j-2m}\leqq\frac{1}{2\pi}\int_{-\pi}^{\pi}\mid f^{(m)}(z)\mid^2 d\Theta<n^{\alpha}\,.$$

But for $j\leqq 2\,n$,

$$\left(1-\frac{1}{n}\right)^{2j-2m}\geqq\left(1-\frac{1}{n}\right)^{4n-2m}>c_1>0\,,$$

where c_1 is independent of n; also, for $j>m$,

$$j(j-1)\,\ldots\,(j-m+1)>c_2\,j^m\,,$$

where $c_2>0$. Therefore, the last inequality implies that

$$S_n^2\leqq\sum_{j=n}^{2n}j^{2m}\mid a_j\mid^2\cdot\sum_{j=n}^{2n}j^{-2m}<c_3\,n^{\alpha}\,n^{-2m+1}\,,$$

and the lemma follows.

We now begin with the proof of the theorem for the case $k>1/2$. Here we will use our lemma with $m=0$. To estimate $\mid f(z)\mid$ at the point $z=(1-1/n)\,e^{i\Theta}$, we apply CAUCHY's formula, with the circular contour $\mid\zeta-z\mid=1/2n$. On this contour, $\mid 1-\zeta\mid>c_4\,(\mid\Theta\mid+n^{-1})$; therefore, $\mid f(z)\mid<c_5\,(\mid\Theta\mid+n^{-1})^{-k}$; and consequently

$$\int_{\Gamma_n}\mid f(z)\mid^2\mid dz\mid<2\,c_5\int_0^{\pi}(\Theta+n^{-1})^{-2k}\,d\Theta<c_6\,n^{2k-1}\,.$$

The desired estimate now follows from the lemma. That the O cannot be replaced by o is seen at once from the example $f(z)=(1-z)^{-k}$.

The case $k=1/2$ is treated in the same way. The only difference is this, that

$$\int_{\Gamma_n}\mid f(z)\mid^2\mid dz\mid<2\,c_5\int_0^{\pi}(\Theta+n^{-1})^{-1}\,d\Theta<c_7\log n\,.$$

A slight modification in the computations in the proof of the lemma gives the estimate $S_n=O\left(\sqrt{n\log n}\right)$.

We point out that in the case where $k\geqq 1/2$, we have only used the fact that $(1-z)^k f(z)$ is bounded in the unit circle, rather than in a GAIER disc.

For $k<1/2$, we apply our lemma, with the integer m chosen in such a way that $2\,m+k-1/2>0$. To estimate $\mid f^{(m)}(z)\mid$ at the point $z=(1-1/n)\,e^{i\Theta}$, we use the circle

$$\mid\zeta-z\mid=1/2n\qquad\text{if }\mid\Theta\mid\leqq n^{-1/2}\,,$$

and the circle

$$\mid\zeta-z\mid=c_8\,\Theta^2\qquad\text{if }n^{-1/2}<\mid\Theta\mid\leqq\pi\,;$$

here c_8 denotes a positive constant small enough so that $(1-\zeta)^k f(\zeta)$ is regular and bounded inside of the curve $\zeta = (1+c_8 \Phi^2)\, e^{i\Phi}$, $-\pi \leqq \Phi \leqq \pi$.

We observe that in either case the relations

$$c_9(|\Theta| + n^{-1}) < |1-\zeta| < c_{10}(|\Theta| + n^{-1})$$

hold on the circle associated with the point $z = (1-1/n)\, e^{i\Theta}$. It follows that, for $|\Theta| \leqq n^{-1/2}$,

$$|f^{(m)}(z)| < c_{11}\, n^m (|\Theta| + n^{-1})^{-k},$$

and

$$\int_{-n^{-1/2}}^{n^{-1/2}} |f^{(m)}|^2\, d\Theta < c_{12}\, n^{2m+k-1/2}.$$

Similarly, for $n^{-1/2} < |\Theta| \leqq \pi$,

$$|f^{(m)}(z)| < c_{13}\, \Theta^{-2m} (|\Theta| + n^{-1})^{-k};$$

since $|\Theta| < |\Theta| + n^{-1} < 2|\Theta|$, this leads to the estimates

$$|f^{(m)}(z)| < c_{14}\, |\Theta|^{-2m-k},$$

$$\int_{-\pi}^{-n^{-1/2}} + \int_{n^{-1/2}}^{\pi} |f^{(m)}|^2\, d\Theta < c_{15}\, n^{2m+k-1/2},$$

and it follows from the lemma that $S_n = O(n^{k/2+1/4})$.

FEJÉR [1] (see also PERRON [8] and [9, § 5]) has shown that if $f(z) = \Sigma\, a_n z^n = (1-z)^{-k} e^{1/(z-1)}$, then as $n \to \infty$, while k is a fixed real number,

$$a_n = \frac{1}{\sqrt{\pi e}}\, n^{k/2-3/4} \left\{ \sin\left[2\sqrt{n} - \left(\frac{k}{2} - \frac{3}{4}\right)\pi\right] + O\left(\frac{1}{\sqrt[4]{n}}\right) \right\}.$$

Consequently, the O in the estimate above cannot be replaced by o.

The following result is an immediate consequence of Theorem 3.

Theorem 4. *If $f(z) = \Sigma\, a_n z^n$ satisfies the condition of Theorem 2, with $k < -1/2$, then $\Sigma |a_n| < \infty$.*

Again, FEJÉR's example shows that the theorem becomes false for $k = -1/2$.

7. On the series $\sum_j j\, |a_j|^2$

If $f(z)$ satisfies the hypothesis of Theorem 2, with $k < -1$, then the conclusion of Theorem 2 implies that $\sum_j j\, |a_j|^2 < \infty$. Again this result can be improved by the method of the preceding section.

Theorem 5. *If $f(z) = \Sigma\, a_n z^n$ satisfies the hypothesis of Theorem 2, with $k < -1/2$, then $\sum_j j\, |a_j|^2 < \infty$.*

It suffices to deal with the case $-1 \leqq k < -1/2$, so that we may apply the procedure of § 6, with $m = 1$. The theorem then follows from the inequalities

$$\sum_{j=n}^{2n} j \mid a_j \mid^2 \leqq n^{-1} \sum_{j=n}^{2n} j^2 \mid a_j \mid^2 \leqq c_1 n^{-1} \int_{\Gamma_n} \mid f'(z) \mid^2 \mid dz \mid$$

and the fact that the integral in the last member is less than $c_2 n^{k+3/2}$.

If $k = -1/2$ the conclusion need not hold, as is easily seen from FEJÉR's example.

8. Convergence and uniform convergence on the unit circle

Theorem 6. *If $f(z) = \Sigma a_j z^j$ satisfies the condition of Theorem 2, with $k < 0$, then Σa_j converges.*

We apply Theorem 2 to the function

$$g(z) = \frac{f(z)}{1-z} = \sum_{j=0}^{\infty} s_j z^j ,$$

where $s_j = a_0 + a_1 + \cdots + a_j$. Since $(1-z)^{k+1} g(z)$ is bounded in a GAIER disc, $s_n = O(n^{k/2})$, and the proof is complete.

It follows from a theorem of GAIER [3, Zusatz, p. 331] that mere continuity of $f(z)$ in the closure of a GAIER disc does not imply convergence of Σa_j. This is also seen from the following example:

Let

$$F(z) = \sum_{i=1}^{\infty} b_i Q_{n_i}(z) ,$$

where the $Q_{n_i}(z)$ are the functions constructed in § 4. If the sequence $\{n_i\}$ is chosen as in § 4, and if $b_i \to 0$, then $\Sigma b_i Q_{n_i}(z)$ converges uniformly in the disc $|z - 1/4| \leqq 3/4$ and thus $F(z)$ is continuous in this disc. Hence the function $f(z) = F\left(\frac{z+1}{2}\right)$ is continuous in the disc $\mid z + 1/2 \mid \leqq 3/2$. Since, for $n = [n_i^3/2]$,

$$\mid \sum_{j=0}^{n} a_j \mid > c \mid b_i \mid \log n_i ,$$

where $c > 0$, the partial sums of the series Σa_j are not even bounded if $b_i \to 0$ slowly enough.

We turn now to the problem of uniform convergence. If $k < 1$ and $(1-z)^k f(z)$ is regular and bounded in a GAIER disc, the TAYLOR series $\Sigma a_n z^n$ of $f(z)$ converges uniformly on every arc of the unit circle that does not contain the point $z = 1$; this follows from Theorem 2 in conjunction with a well-known theorem of M. RIESZ [12] (see LANDAU [6, p. 73]). On the other hand, we know from Theorem 6 that the TAYLOR series converges at the point $z = 1$ if $k < 0$, and from Theorem 4 that the

TAYLOR series converges uniformly on the entire unit circle if $k < -1/2$. The question remains as to whether this last statement can be extended to the case $k < 0$. The answer is in the affirmative.

Theorem 7. *If $f(z) = \Sigma a_n z^n$ satisfies the condition of Theorem 2, with $k < 0$, then $\Sigma a_n z^n$ converges uniformly on $|z| = 1$.*

By the preceding remarks, it suffices to assume that $-1/2 \leqq k < 0$ and to prove uniform convergence of $\Sigma a_n e^{in\alpha}$ for $0 < |\alpha| < \pi/4$. For the sake of convenience, we shall restrict ourselves to the interval $0 < \alpha < \pi/4$; the proof for the interval $-\pi/4 < \alpha < 0$ is analogous.

As at the beginning of § 5, let Γ be the curve $z = (1 + c_1 \Phi^2) e^{i\Phi}$, $-\pi \leqq \Phi \leqq \pi$; and let $z_0 = e^{i\alpha}$, $0 < \alpha < \pi/4$. We note first that for any z on Γ,

$$|z - z_0| = |(1 + c_1 \Phi^2) e^{i\Phi} - e^{i\alpha}| > c_2(\Phi^2 + |\Phi - \alpha|),$$

where $c_2 > 0$. Hence we obtain, for positive integral n and p,

$$\left|\sum_{j=n+1}^{n+p} a_j z_0^j\right| = \left|\frac{1}{2\pi i}\int_\Gamma f(z) \frac{z_0^{n+1}}{z^{n+2}} \frac{1-(z_0/z)^p}{1-z_0/z}\, dz\right|$$

$$\leqq \frac{1}{\pi}\int_\Gamma \frac{|f(z)|\,|dz|}{|z|^{n+1}\,|z - z_0|} < c_3 \int_{-\pi}^{\pi} \frac{|\Phi|^{-k}\, d\Phi}{(1 + c_1\Phi^2)^n (\Phi^2 + |\Phi - \alpha|)}.$$

Let the parts of the last integral that correspond to the intervals $-\pi \leqq \Phi \leqq \alpha/2$, $\alpha/2 \leqq \Phi \leqq 2\alpha$, and $2\alpha \leqq \Phi \leqq \pi$ be denoted by I_1, I_2, and I_3, respectively.

In I_1 we use the estimate $\Phi^2 + |\Phi - \alpha| \geqq \alpha - \Phi \geqq |\Phi|$; in I_3 we use the estimate $\Phi^2 + |\Phi - \alpha| \geqq \Phi - \alpha \geqq \Phi/2$. From these and from the method used in § 5 we get the inequality

$$I_1 + I_3 \leqq \int_{-\pi}^{\pi} \frac{2|\Phi|^{-k-1}\, d\Phi}{(1 + c_1\Phi^2)^n} < c_4 n^{k/2}.$$

In I_2, we use the estimate $\Phi^2 + |\Phi - \alpha| \geqq \alpha^2/4 + |\Phi - \alpha|$ and obtain

$$I_2 \leqq \frac{(2\alpha)^{-k}}{(1 + c_1\alpha^2/4)^n} \int_{\alpha/2}^{2\alpha} \frac{d\Phi}{\alpha^2/4 + |\Phi - \alpha|}.$$

The integral in the last member can be evaluated directly and is equal to

$$\log\frac{\alpha + 2}{\alpha} + \log\frac{\alpha + 4}{\alpha} < -c_5 \log\alpha < c_6\, \alpha^{k/2}.$$

Thus

$$I_2 < c_7 \frac{\alpha^{-k/2}}{1 + c_1 n \alpha^2/4} = c_7 n^{k/4} \frac{(n\alpha^2)^{-k/4}}{1 + c_1 n \alpha^2/4} < c_8 n^{k/4}$$

(note that $0 < -k/4 \leqq 1/8$, and that therefore $t^{-k/4}/(1 + c_1 t/4) < M(k, c_1)$ for $t > 0$).

The preceding estimates lead to the inequality

$$\left|\sum_{j=n+1}^{n+p} a_j z_0^j\right| \leqq c_3(I_1 + I_2 + I_3) < c_9 n^{k/4},$$

and the proof is complete.

9. Functions regular in Gaier regions

We shall call a Gaier region (see § 1) a Gaier *region of order* $p\,(p > 0)$ if it contains the interior of the curve

$$z(\Phi) = (1 + c|\Phi|^p)\, e^{i\Phi}, \qquad -\pi \leqq \Phi \leqq \pi,$$

for some positive value of c.

Theorem 8. *Let $f(z) = \Sigma a_n z^n$ be regular in some* Gaier *region of order p ($p \geqq 1$), and let k be a real number ($k < 1$) such that $(1-z)^k f(z)$ is bounded in this region. Then*

$$a_n = O(n^{(k-1)/p}).$$

The proof proceeds as in § 5. We choose an appropriate curve

$$\Gamma: \qquad z = (1 + c_1|\Phi|^p)\, e^{i\Phi}, \qquad -\pi \leqq \Phi \leqq \pi,$$

and we use the fact that

$$2\pi |a_n| = \left|\int_\Gamma \frac{f(z)}{z^{n+1}} dz\right| \leqq c_2 \int_0^\pi \frac{1}{\Phi^k (1 + c_1 \Phi^p)^n} \left[1 + \frac{c_1 p \Phi^{p-1}}{1 + c_1 \Phi^p}\right] d\Phi$$

$$\leqq c_3 \int_0^\pi \frac{d\Phi}{\Phi^k (1 + c_1 \Phi^p)^n} < c_4 n^{(k-1)/p}.$$

We note that if $(1-z)^k f(z)$ is continuous in the closure of a Gaier region of order p, the O in the statement of Theorem 8 can be replaced by o. Moreover, for functions continuous in the closure of a Gaier region of order 1, our result reduces to a theorem of M. Riesz [11] (see also Landau [6, p. 64]). For a similar result closely related to Riesz's theorem, see Gaier [4, Theorems 1 and 2].

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