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Some Remarks on Vinogradov's Mean Value Theorem and Tarry's Problem

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Abstract. Let $W(k,2)$ denote the least number s for which the system of equations $\sum_{i=1}^{s} x_i^j = \sum_{i=1}^{s} y_i^j (1 \leqslant j \leqslant k)$ has a solution with $\sum_{i=1}^{s} x_i^{k+1} \neq \sum_{i=1}^{s} y_i^{k+1}$. We show that for large k one has $W(k,2) \leqslant \frac{1}{2}k^2(\log k + \log\log k + O(1))$, and moreover that when K is large, one has $W(k,2) \leqslant \frac{1}{2}k(k+1)+1$ for at least one value k in the interval $[K, K^{4/3+\varepsilon}]$. We show also that the least s for which the expected asymptotic formula holds for the number of solutions of the above system of equations, inside a box, satisfies $s \leqslant k^2(\log k + O(\log\log k))$.

1. Introduction

The new efficient differencing methods recently brought into play within the Hardy-Littlewood method have improved substantially many estimates in problems of additive number theory (see, in particular, [7, 8, 9]). In this note we examine the consequences of such improvements in Vinogradov's mean value theorem for the Prouhet-Tarry-Escott problem, which surprisingly has seen little progress in nearly half a century. Along the way we improve the bound for the number of variables required to establish the asymptotic formula in Vinogradov's mean value theorem.

In order to set the scene, when j, k and s are positive integers with $s \geqslant 2$, consider the non-trivial solutions of the simultaneous diophantine equations

$$\sum_{i=1}^{j} x_{i1}^h = \sum_{i=1}^{j} x_{i2}^h = \cdots = \sum_{i=1}^{j} x_{is}^h \quad (1 \leqslant h \leqslant k). \tag{1}$$

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Let $P(k,s)$ denote the least j for which the system (1) has a solution $\mathbf{x}$ in which the sets $\{x_{1u},\ldots,x_{ju}\}(1\leqslant u\leqslant s)$ are distinct. Further, let $W(k,s)$ denote the least j such that (1) has a solution $\mathbf{x}$ with $\sum_{i=1}^{j}x_{iu}^{k+1}\neq\sum_{i=1}^{j}x_{iv}^{k+1}(u\neq v)$.

The problem of estimating $P(k,s)$ was investigated by Prouhet in 1851, and subsequently re-discovered by Escott and Tarry (see [14] for some historical notes). By using counting arguments, WRIGHT [12, 13] has shown that $P(k,2)\leqslant\frac{1}{2}(k^2+4)$, and in general,

$$k+1\leqslant P(k,s)\leqslant\tfrac{1}{2}k(k+1)+1.$$

Meanwhile, numerical examples show that $P(k,2)=k+1$ for $2\leqslant k\leqslant 9$ (see [2, Chapter XXI, notes]), and indeed it is plausible that $P(k,2)=k+1$ for every k. Wright also considered the harder problem of estimating $W(k,s)$. Later, motivated by features of Vinogradov's mean value theorem and diminishing ranges arguments, HUA [3] constructed an ingenious elementary method, which, after generalisations of WRIGHT [14] and HUA [4], yields the bound

$$W(k,s)\leqslant(k+1)\left(\left[\frac{\log\frac{1}{2}(k+2)}{\log(1+1/k)}\right]+1\right)\sim k^2\log k. \tag{2}$$

(Here, $[x]$ denotes the least integer not exceeding x). Moreover, when k is odd, a simple trick of HUA [3] enables one to essentially halve the latter bound.

In Section 2 we employ the latest developments in Vinogradov's mean value theorem to obtain improved bounds for $W(k,2)$.

Theorem 1. $W(k,2)\leqslant\frac{1}{2}k^2(\log k+\log\log k+O(1))$.

The latter estimate is superior to (2), but for odd k does not supersede Hua's bound. However, it is possible to do rather better infinitely often.

Theorem 2. *For each $\varepsilon>0$, there is a real number $K(\varepsilon)$ with the property that for each $K\geqslant K(\varepsilon)$, there exists a k in the interval $[K,K^{4/3+\varepsilon}]$ with*

$$W(k,2)\leqslant\tfrac{1}{2}k(k+1)+1.$$

The latter theorem may be refined so that the expression $K^{4/3+\varepsilon}$ is replaced by $((4e)^{1/3}+\varepsilon)K^{4/3}(\log K)^{1/3}$. We note that while Theorem 2 implies that $W(k,2)\leqslant\frac{1}{2}k(k+1)+1$ infinitely often, a trivial argument leads from Wright's bound $P(k,2) \leqslant \frac{1}{2}(k^2+4)$ to the conclusion that $W(k,2) \leqslant \frac{1}{2}(k^2+4)$ infinitely often.

In Section 3 we turn our attention to the problem of establishing the asymptotic formula in Vinogradov's mean value theorem, which, as observed by HUA [5, §X.3], is closely related to estimating the number of solutions of Tarry's problem inside a box. In order to describe our conclusion, we must record some notation. Let $J_{t,k}(P)$ denote the number of solutions of the system of diophantine equations

$$\sum_{i=1}^{t} (x_i^j - y_i^j) = 0 \quad (1 \leqslant j \leqslant k), \tag{3}$$

with $1 \leqslant x_i, y_i \leqslant P (1 \leqslant i \leqslant t)$. We write $e(z)$ for $e^{2\pi i z}$, and define $S(q,\mathbf{a}) = S(q, a_1, \ldots, a_k)$ by

$$S(q,\mathbf{a}) = \sum_{x=1}^{q} e((a_1 x + a_2 x^2 + \cdots + a_k x^k)/q). \tag{4}$$

We define the *singular series*, $\mathfrak{S}(s,k)$, and *singular integral*, $\mathfrak{J}(s,k)$, by

$$\mathfrak{S}(s,k) = \sum_{q=1}^{\infty} \sum_{a_1=1}^{q} \cdots \sum_{a_{k-1}=1}^{q} \sum_{\substack{a_k=1 \\ (a_1,\ldots,a_k,q)=1}}^{q} |q^{-1} S(q,\mathbf{a})|^{2s}, \tag{5}$$

and

$$\mathfrak{J}(s,k) = \int_{\mathbb{R}^k} \left| \int_0^1 e(\beta_1 \gamma + \cdots + \beta_k \gamma^k) d\gamma \right|^{2s} d\boldsymbol{\beta}. \tag{6}$$

Theorem 3. *There are positive numbers C and $\delta(k)$ such that whenever*

$$s \geqslant k^2(\log k + 2\log\log k + C),$$

one has

$$J_{s,k}(X) = \mathfrak{S}(s,k)\mathfrak{J}(s,k) X^{2s-k(k+1)/2} + O_{k,s}(X^{2s-k(k+1)/2-\delta(k)}). \tag{7}$$

We note that $\mathfrak{J}(s,k)$ and $\mathfrak{S}(s,k)$ are both positive, in view of a simple argument of VAUGHAN [6, §7.3]. Previously, HUA (see [5, Theorem 15]) had established such an asymptotic formula for s satisfying an inequality of strength $s \geqslant (3+o(1))k^2 \log k$. Moreover, WOOLEY [9, Corollary 1.4] has remarked that recent developments enable one to improve the latter bound, to the extent that $3+o(1)$

may be replaced by $5/3 + o(1)$. In each of the latter approaches (the second of which was modelled after VAUGHAN [6, §7.3]), the Hardy-Littlewood dissection employed to obtain the asymptotic formula is essentially a cartesian product of dissections of the unit interval. By using a result of R. C. BAKER (see [1, Theorem 4.4]), we develop an improved dissection which permits greater control to be exercised over the size of the relevant exponential sums. Our treatment is otherwise similar to those of Hua and Vaughan.

2. Tarry's Problem

Our proofs of Theorems 1 and 2 employ a lemma which associates estimates for $J_{t,k}(P)$ with bounds for $W(k, 2)$. In order to describe this lemma we require some notation. We shall say that an exponent $\Delta_{t,k}$ is *permissible* if for every sufficiently large real number P we have the bound

$$J_{t,k}(P) \ll_{t,k} P^{2t - \frac{1}{2}k(k+1) + \Delta_{t,k}}, \tag{8}$$

where here, and throughout, $\ll$ and $\gg$ refer to Vinogradov's well-known notation.

Lemma 1. *Let $t, H, K \in \mathbb{N}$, and suppose that $\Delta_{t,K+H}$ is a permissible exponent satisfying*

$$\Delta_{t,K+H} < \tfrac{1}{2}((K+H)(K+H+1) - K(K+1)). \tag{9}$$

Then $W(k, 2) \leqslant t$ for some k in the interval $[K, K+H-1]$.

Proof. Suppose that $W(k, 2) > t$ for each $k \in [K, K+H-1]$. Then each solution $\mathbf{x}, \mathbf{y}$ of the equations (3) with $k = K$ is also a solution of the equations (3) with $k = K + H$, and consequently for each positive P we have

$$J_{t,K+H}(P) = J_{t,K}(P). \tag{10}$$

But in view of (8) and the hypothesis (9), it follows from the well-known lower bound,

$$J_{t,K}(P) \gg (2t)^{-K} P^{2t - K(K+1)/2},$$

(see, for example, [10, Theorem 2]), that when P is sufficiently large in terms of t, K and H, we have $J_{t,K+H}(P) < J_{t,K}(P)$. The latter inequality contradicts equation (10), and thus the proof of the lemma is completed.

Proof of Theorem 1. We suppose that k is sufficiently large, and apply Lemma 1 with $K=k$, $H=1$ and $t=(k+1)t_{k+1}$, where for each positive integer h we write

$$t_h=[\tfrac{1}{2}h(\log h+\log\log h+3)].$$

It follows from [9, Theorem 1.2] that $\Delta_{t,k+1}$ is a permissible exponent, where

$$\Delta_{t,k+1}=(k+1)^2\log(k+1)\left(1-\frac{2}{k+1}(1-1/\log(k+1))\right)^{t_{k+1}}.$$

Moreover a simple estimation reveals that $\Delta_{t,k+1}<\frac{1}{2}(k+1)$, so that the hypothesis (9) of Lemma 1 is satisfied. Then we may conclude from that lemma that $W(k,2)\leqslant(k+1)t_{k+1}$, which suffices to prove Theorem 1.

Proof of Theorem 2. We suppose that ε is a small positive number, and that K is sufficiently large in terms of ε. We apply Lemma 1 with

$$H=[((4e)^{1/3}+\varepsilon)K^{4/3}(\log K)^{1/3}-K]$$

and $t=\frac{1}{2}K(K+1)+1$. It follows from [11, Corollary 1.1] that $\Delta_{t,K+H}$ is a permissible exponent, where

$$\Delta_{t,K+H}=\tfrac{1}{2}(K+H)(K+H+1)-t+\delta_{t,K+H},$$

and for each s and k the number $\delta_{s,k}$ satisfies

$$\delta_{s,k}\ll sk^{3/2}\exp\left(-\frac{k^3}{4es^2}(1+O(k^2/s^2))\right).$$

A little calculation reveals that our choice of H ensures that $\delta_{t,K+H}\ll K^{-\varepsilon}$, and hence, since K is assumed to be sufficiently large in terms of ε, that the hypothesis (9) of Lemma 1 is satisfied. Then we may conclude from Lemma 1 that $W(k,2)\leqslant\frac{1}{2}K(K+1)+1$ for some $k\in[K,K+H-1]$, which suffices to prove Theorem 2.

3. The Asymptotic Formula

Our proof of Theorem 3 is a fairly standard application of the Hardy-Littlewood method. The new ingredient in our proof is the following weak consequence of Theorem 4.4 of R. C. Baker [1].

Lemma 2. *Let k be an integer with $k \geqslant 4$, and define $\sigma(k)$ by*

$$\sigma(k)^{-1} = 8k^2(\log k + \tfrac{1}{2}\log\log k + 2). \qquad (11)$$

Define also the exponential sum $f(\boldsymbol{\alpha}; Q)$ by

$$f(\boldsymbol{\alpha}; Q) = \sum_{1 \leqslant x \leqslant Q} e(\alpha_1 x + \cdots + \alpha_k x^k). \qquad (12)$$

Suppose that P is sufficiently large in terms of k, and that $|f(\boldsymbol{\alpha}; P)| \geqslant P^{1-\sigma(k)}$. Then there exist integers $q, a_1, \ldots, a_k$ such that

$$1 \leqslant q \leqslant P^{1/k} \quad \text{and} \quad |q\alpha_j - a_j| \leqslant P^{1/k-j} \quad (1 \leqslant j \leqslant k).$$

Proof. The lemma follows immediately from the case $M = 1$ of [1, Theorem 4.4].

We note that the value of $\sigma(k)$ in the statement of Lemma 2 could be improved, essentially by a factor of 2, by using the bounds of [9]. Such an improvement would affect only the second order terms in the bound for s contained in the statement of Theorem 3.

Proof of Theorem 3. Let k be a large positive integer, and P be a real number sufficiently large in terms of k. We define the integers $r_1(k)$, t_k and u_k by

$$r_1(k) = [k(\log k - \log\log k)] + 1, \quad t_k = [3k \log\log k] + 1,$$
$$u_k = 5k^2 + 1, \qquad (13)$$

and write

$$t = k(r_1(k) + t_k) \quad \text{and} \quad s = t + u_k. \qquad (14)$$

We aim to obtain an asymptotic formula for $J_{s,k}(P)$ by applying the Hardy-Littlewood method, noting that by orthogonality

$$J_{s,k}(P) = \int_{[0,1)^k} |f(\boldsymbol{\alpha}; P)|^{2s} d\boldsymbol{\alpha}, \qquad (15)$$

where $f(\boldsymbol{\alpha}; P)$ is defined by (12). We first define the dissection which forms the basis of our application of the circle method. Write $\mathscr{U}_k^*$ for the cartesian product of the intervals $(P^{1/k-j}, 1 + P^{1/k-j})(1 \leqslant j \leqslant k)$. When $q \leqslant P^{1/k}$, $1 \leqslant a_j \leqslant q$ $(1 \leqslant j \leqslant k)$ and $(q, a_1, \ldots, a_k) = 1$, define the major arc $\mathfrak{M}(q, \mathbf{a})$ by

$$\mathfrak{M}(q, \mathbf{a}) = \{\boldsymbol{\alpha} \in \mathscr{U}_k^* : |q\alpha_j - a_j| \leqslant P^{1/k-j}(1 \leqslant j \leqslant k)\}. \qquad (16)$$

Notice that the $\mathfrak{M}(q, \mathbf{a})$ are disjoint. Let $\mathfrak{M}$ denote the union of the major arcs $\mathfrak{M}(q, \mathbf{a})$, and define the minor arcs $\mathfrak{m}$ by $\mathfrak{m} = \mathscr{U}_k^* \setminus \mathfrak{M}$. Thus from (15),

$$J_{s,k}(P) = \int_{\mathfrak{M}} |f(\boldsymbol{\alpha}; P)|^{2s} d\boldsymbol{\alpha} + \int_{\mathfrak{m}} |f(\boldsymbol{\alpha}; P)|^{2s} d\boldsymbol{\alpha}. \tag{17}$$

In order to estimate the contribution of the minor arcs in (17), we first bound $f(\boldsymbol{\alpha}; P)$ when $\boldsymbol{\alpha} \in \mathfrak{m}$. Suppose that there exists $\boldsymbol{\alpha} \in \mathfrak{m}$ such that $|f(\boldsymbol{\alpha}; P)| \geqslant P^{1-\sigma(k)}$, with $\sigma(k)$ defined by (11). Then Lemma 2 implies that there exist integers $q, a_1, \ldots, a_k$ such that $1 \leqslant q \leqslant P^{1/k}$ and $|q\alpha_j - a_j| \leqslant P^{1/k-j} (1 \leqslant j \leqslant k)$. Dividing through by the common factor $(q, a_1, \ldots, a_k)$, we find from (16) that $\boldsymbol{\alpha} \in \mathfrak{M}$, contradicting the assumption that $\boldsymbol{\alpha} \in \mathfrak{m}$. Thus we conclude that

$$\sup_{\boldsymbol{\alpha} \in \mathfrak{m}} |f(\boldsymbol{\alpha}; P)| \leqslant P^{1-\sigma(k)}. \tag{18}$$

Next, on noting (14), we deduce from (18) that

$$\int_{\mathfrak{m}} |f(\boldsymbol{\alpha}; P)|^{2s} d\boldsymbol{\alpha} \leqslant \left(\sup_{\boldsymbol{\alpha} \in \mathfrak{m}} |f(\boldsymbol{\alpha}; P)| \right)^{2u_k} \int_{[0,1)^k} |f(\boldsymbol{\alpha}; P)|^{2t} d\boldsymbol{\alpha} \leqslant$$
$$\leqslant (P^{1-\sigma(k)})^{2u_k} J_{t,k}(P). \tag{19}$$

Moreover, it follows from [9, Theorem 1.2] that $\Delta = \Delta_{t,k}$ is a permissible exponent, where

$$\Delta = 5(\log k)^3 \left(1 - \frac{3}{2k}(1 - 1/k) \right)^{t_k}.$$

A simple estimation reveals that $\Delta < 1/\log k$, whence $\Delta < 2u_k \sigma(k)$. Thus we deduce from (19) that for some positive number $\delta(k)$, we have

$$\int_{\mathfrak{m}} |f(\boldsymbol{\alpha}; P)|^{2s} d\boldsymbol{\alpha} \leqslant (P^{1-\sigma(k)})^{2u_k} P^{2t - \frac{1}{2}k(k+1) + \Delta} \leqslant P^{2s - \frac{1}{2}k(k+1) - \delta(k)}. \tag{20}$$

Next we consider the contribution from the major arcs $\mathfrak{M}$. When $\boldsymbol{\alpha} \in \mathfrak{M}(q, \mathbf{a})$, write

$$V(\boldsymbol{\alpha}; q, \mathbf{a}) = q^{-1} S(q, \mathbf{a}) I(\boldsymbol{\beta}),$$

where $S(q, \mathbf{a})$ is defined by (4),

$$I(\boldsymbol{\beta}) = \int_0^P e(\beta_1 \gamma + \cdots + \beta_k \gamma^k) d\gamma,$$

and $\beta_j = \alpha_j - a_j/q (1 \leqslant j \leqslant k)$. Further, define the function $V(\boldsymbol{\alpha})$ to be $V(\boldsymbol{\alpha}; q, \mathbf{a})$ when $\boldsymbol{\alpha} \in \mathfrak{M}(q, \mathbf{a})$, and to be zero otherwise. By VAUGHAN [6, Theorem 7.2], when $\boldsymbol{\alpha} \in \mathfrak{M}(q, \mathbf{a})$ we have

$$f(\boldsymbol{\alpha}; P) - q^{-1} S(q, \mathbf{a}) I(\boldsymbol{\beta}) \ll q(1 + |\beta_1| P + \cdots + |\beta_k| P^k).$$

Thus for each $\boldsymbol{\alpha} \in \mathfrak{M}$,

$$f(\boldsymbol{\alpha}; P) - V(\boldsymbol{\alpha}) \ll P^{2/k}.$$

Then

$$\int_{\mathfrak{M}} |f(\boldsymbol{\alpha}; P)|^{2s} - |V(\boldsymbol{\alpha})|^{2s} d\boldsymbol{\alpha} \ll$$

$$\ll P^{1+2/k} \int_{[0,1)^k} |f(\boldsymbol{\alpha}; P)|^{2s-2} + |V(\boldsymbol{\alpha})|^{2s-2} d\boldsymbol{\alpha}.$$

On imitating the argument described in VAUGHAN [6, §7.3], we therefore deduce that

$$\int_{\mathfrak{M}} |f(\boldsymbol{\alpha}; P)|^{2s} d\boldsymbol{\alpha} = \int_{\mathfrak{M}} |V(\boldsymbol{\alpha})|^{2s} d\boldsymbol{\alpha} + O(P^{2s - \frac{1}{2}k(k+1) - \delta(k)}). \quad (21)$$

A standard analysis, as outlined in VAUGHAN [6, §7.3], shows that the main term in (21) contributes the main term of (7) with an acceptable error. Thus the theorem follows on collecting together (17), (20) and (21).

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