On the Problem of Reconstructing a Tournament from Subtournaments[1]

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With 7 Figures

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Introduction

No one has ever specified an elegant complete set of invariants for a graph and indeed it is evidently an extraordinarily difficult task. It has been verified that none of the usual invariants of graphs such as the number of points and lines, connectivity, chromatic number, degree sequence or group of automorphisms place much of a restriction on the graph. In fact, *Izbicki* [6] has shown how to construct an infinite number of regular graphs having the same group, chromatic number and degree of regularity. Since such properties as these seem to fall far short of determining a graph, it is natural to ask the question: For a given graph G and a given set S of subgraphs of G, does the set S determine G? A specialization of this question is becoming quite a well known conjecture in graph theory:

Ulam's Conjecture [8]. If G and H are two graphs with p points v_i and u_i respectively ($p \geqslant 3$) such that for all i, $G - v_i$ is isomorphic with $H - u_i$, then G and H are themselves isomorphic.

Consider a graph G with points $v_1, \ldots, v_p$. The graph $G - v_i$ obtained from G on removing point v_i is denoted G_i; it consists of all the points of G except v_i and all the lines of G except those incident with v_i. *Kelly* [7] solved Ulam's problem for trees. *Harary* [2] reformulated it as a problem of reconstructing G from its subgraphs G_i and derived several of the invariants of G from the set G_i.

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The conjecture is, of course, not true for $p = 2$ but *Kelly* has verified by exhaustion that it holds for all graphs with p points, $3 \leqslant p \leqslant 6$. We have also verified the conjecture for $p = 7$.

In the rest of this article we assume that $p \geqslant 3$.

We will use constructive methods to show that the conjecture holds for all tournaments which are not strong and have $p \geqslant 5$ points. We also show that the "line version" of the conjecture holds for all tournaments. The concluding section provides a method for obtaining information about a graph from its subgraphs, in particular for obtaining the score sequence of a tournament from its subtournaments. Definitions which do not appear here be found in [1, 3, 4, 5].

Line Version for Tournaments

By definition, a tournament is an oriented complete graph. The *score* s_i of point v_i is the outdegree of v_i. In Figure 1, all the tournaments

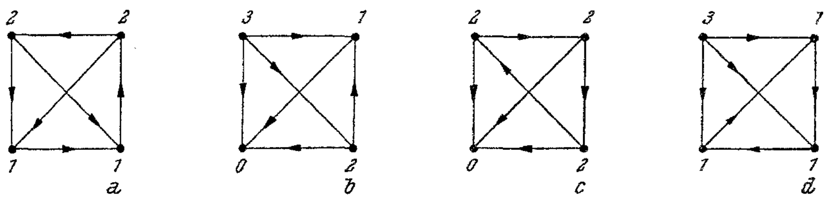


Fig. 1

with $p = 4$ points are shown with the score marked at each point. It is customary to order the points v_i so the scores satisfy $s_1 \leqslant s_2 \leqslant \ldots \leqslant s_p$.

Theorem 1. Let T be a tournament with p points whose lines are $x_1, \ldots, x_q$. Let $T - x_i$ for $i = 1, \ldots, q$ be the subgraphs obtained from T by deleting the line x_i. Then T can be reconstructed from the $T - x_i$.

Proof. Clearly T has a receiver if and only if some $T - x_i$ has a point v with id $v = p - 1$. If T does not have a receiver, then $s_1 \geqslant 1$ and we can choose $T - x_i$ with a point u such that od $u = s_1 - 1$. Let v be the other point of $T - x_i$ with total degree $p - 2$. Then T is obtained by adding the directed line uv to $T - x_i$. If T does have a receiver, choose $T - x_i$ with no points w such that id $w = p - 1$.

Then $T - x_i$ was obtained from T by deleting a line which is incident with the receiver. Let v be a point of $T - x_i$ with id $v = p - 2$ and od $v = 0$. Let u be the other point of $T - x_i$ with total degree $p - 2$. Without loss of generality we can assume that v was the receiver of T and so T is obtained by adding the directed line uv to $T - x_i$.

Point Version for Tournaments

Again consider a tournament T with points $v_1, \ldots, v_p$. Let $T_i = T - v_i$; $i = 1, \ldots, p$ be the tournaments obtained by deleting v_i from T. It is clear that a digraph D is a tournament if and only if $D - v$ is a tournament for each point v of D. We consider the following problem: If the collection $\{T_i\}$ of subtournaments is given, can T be reconstructed and if so, how?

If $p = 3$, T is either a cyclic triple or a transitive triple and the collection $\{T_i\}$ is the same in both cases. Hence T is not determined by the T_i.

If $p = 4$, there are four tournaments (see Figure 1 above) and only two of them can be reconstructed. One is strong, (see Figure 1a), has score sequence (1, 1, 2, 2) and $\{T_i\}$ has exactly two cyclic triples. One is transitive, (Figure 1b), has score sequence (0, 1, 2, 3) and of course $\{T_i\}$ has no cyclic triples. The other two have score sequences (0, 2, 2, 2) and (1, 1, 1, 3) and each collection $\{T_i\}$ has exactly one cyclic triple. Hence in the latter two cases T is not determined by the T_i.

The next theorem is not quite in the literature [4, 5] and will be useful in determining whether a tournament T is strong, given the T_i.

Theorem 2. A tournament T with at least four points is strong if and only if it has neither a transmitter nor a receiver and for some point v, $T - v$ is strong.

Proof. Suppose T is strong. Clearly T does not have a transmitter or a receiver. By Theorem 7 of [4], T has a cycle of length $p - 1$, say $Z = v_1 v_2 \ldots v_{p-1} v_1$. Therefore $T_p = T - v_p$ contains Z and by Corollary 7a of [4] it is therefore strong.

Conversely, suppose $T_p = T - v_p$ is strong. Let a complete cycle of T_p be $Z = v_1 v_2 \ldots v_{p-1} v_1$. If v_p is neither a transmitter nor a receiver, v_p is adjacent to some point of Z and adjacent from another. Hence by the same reasoning as in case (1) of Theorem 7 in [4], T has a complete cycle and hence is strong.

The strong tournament T on three points is, of course, a cyclic triple. Each of the three subtournaments T_i has a transmitter. That this is not the case for strong tournaments with more than three points is shown by the next theorem.

Theorem 3. If T is a strong tournament with at least four points, then at most two of the subtournaments T_i have transmitters.

Proof. Let T be a strong tournament with $p \geqslant 4$ points and suppose v_1 is a transmitter in $T_2 = T - v_2$. Then the score of v_1 in T is at least $p - 2$. Since T is strong the score of v_1 is exactly $p - 2$. If T has two subtournaments T_i with transmitters, there must be two points v_1 and v_2 with score $p - 2$ and we can assume that v_2 is adjacent to v_1.

Since v_2 has score $p - 2$, there is another point v_3 which is adjacent to v_2 and adjacent from v_1. The other $p - 3$ points, v_i with $i \geqslant 4$, are adjacent from both v_1 and v_2 (see Figure 2). Hence for $i \geqslant 4$ the score

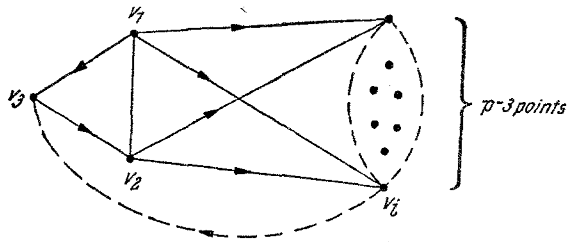


Fig. 2

of v_i is less than $p - 2$ and therefore none of these v_i can be a transmitter in any subtournament $T - v$. Further no T_i with $i \geqslant 4$ has a transmitter. As in Figure 2, T_2 and T_3 have transmitters v_1 and v_2 respectively. The only other possibility is that v_3 is a transmitter in T_1. But since T is strong, v_3 must be adjacent from some v_i with $i \geqslant 4$. So the score of v_3 is less than $p - 2$ and T_1 does not have a transmitter.

Theorem 4. A tournament T with at least five points has a transmitter if and only if at least four of the T_i have a transmitter.

Proof. Assume that v_1 is a transmitter of T. Then for $i \geqslant 2$, v_1 is also a transmitter in T_i. Since T has at least five points, there are at least four such T_i.

In the converse at least four of the T_i, say T_1, T_2, T_3, T_4, have a transmitter by hypothesis. Suppose T does not have a transmitter.

Let S be the strong component of T which is the transmitter of T^*, the condensation of T. Then S contains at least three points. So if v is any point which is not in S then $T - v$ cannot have a transmitter. But then the four points v_1, v_2, v_3, v_4 such that T_i has a transmitter for $i = 1$ to 4 must all be in S. Therefore S has at least four points. Further the transmitters of the T_i are also in S. Thus $S - v_i$ has a transmitter for each i. Since S is itself a strong tournament, this contradicts Theorem 3. Hence T has a transmitter.

Corollary 4a. When $p \geqslant 5$, T has a transmitter if and only if at least $p - 1$ of the T_i have a transmitter.

Note that Theorem 4 does not hold for tournaments with four points because the tournament of Figure 1d has a transmitter but only three of its subtournaments T_i have one. Further the theorem does not hold if we only require that three of the T_i have transmitters. If S, the strong component of T which is the transmitter of T^* is a cyclic triple, then three of the T_i have transmitters but T does not.

If G and H are two graphs with no points in common, the new graph $G + H$ is obtained by joining each point of G with each point of H by a line. When these additional lines are all to be directed from G to H we indicate this by writing $G + \rightarrow H$.

Fig. 3

Let K_p be the complete graph on p points and $\overline{K}_p$ its complement with p points and no lines.

Here are some examples:

Theorem 5. If T is a tournament with at least five points such that one of the T_i, say $T_1 = T - v_1$, does not have a transmitter and at least four of the T_i do have a transmitter, then $T = v_1 + \rightarrow T_1$.

Proof. By Theorem 4, T has a transmitter, say v. Then $T - u$ has a transmitter whenever $u \neq v$. Hence $v = v_1$ and $T = v_1 + \rightarrow T_1$.

Theorem 6. If T is a tournament with at least five points and each T_i has a transmitter, then T can be reconstructed from the T_i.

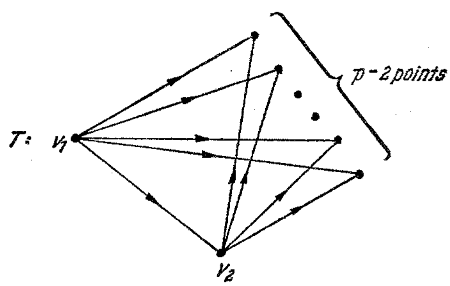
Proof. We shall show that there is a largest integer m with $2 \leqslant m \leqslant p$ such that for suitable labeling of the T_i the following conditions hold:

(1) each T_i has points of score $p-2, \ldots, p-m$.

(2) $T_1, \ldots, T_m$ do not have a point of score $p-(m+1)$ but $T_{m+1}, \ldots, T_p$ do have such a point.

(3) $T_1, \ldots, T_m$ are all isomorphic and $T = v_1 + \rightarrow T_1$.

By Theorem 4, T has a transmitter, say v_1. Since each T_i has a transmitter, T must have a point, say v_2, of score $p-2$ (see Figure 4).



Fig, 4

Now there are two possibilities:

Case 1. None of the other points v_i with $i \geqslant 3$ has score $p-3$.

Case 2. Some point, say v_3, has score $p-3$.

In Case 1, T_1 and T_2 do not have points of score $p-3$ but for each $i \geqslant 3$, T_i does (namely v_2). Each T_i has a point of score $p-2$. Clearly T_1 and T_2 are isomorphic, $T = v_1 + \rightarrow T_1$, and $m = 2$.

In Case 2, $m \geqslant 3$ and again there are two possibilities. If none of the v_i with $i \geqslant 4$ has score $p-4$, then T_1, T_2 and T_3 do not have points of score $p-4$ but for $i \geqslant 4$ each T_i does (namely v_3). Each T_i has points of score $p-2$ and $p-3$. Clearly T_1, T_2 and T_3 are isomorphic, $T = v_1 + \rightarrow T_1$ and $m = 3$. Otherwise some point, say v_4, has score $p-4$ and $m \geqslant 4$. Continuing in this way we obtain (1), (2) and (3) and hence T can be reconstructed. Note that if $m = p$, T is the transitive tournament on p points.

Every concept in directed graph theory has a "converse concept". For example the outdegree and indegree of a point are converse concepts of each other. A valuable principle mentioned in [5] is that of directional duality: For each theorem about digraphs, there is a corresponding theorem which is obtained by replacing each concept by its converse.

Each of Theorems 3, 4, and 5 has a directional dual. The dual theorems are obtained by replacing the word *transmitter* by *receiver*.

Theorem 7. If T is a tournament with at least five points and T is not strong, then T can be reconstructed from the T_i.

Proof. Using Theorem 4 and its dual, we can tell from the T_i whether or not T has a transmitter or a receiver. Then using Theorem 2 we can tell from the T_i whether or not T is strong. If T is not strong and has a transmitter or a receiver, T can be reconstructed by Theorems 5 or 6 or their directional duals.

Assume T is not strong and has neither a transmitter nor a receiver. Thus T must contain at least six points. Let the components of T be $S_1, \ldots, S_n$ with S_1 the transmitter and S_2 the receiver in T^*. The number of points in the component S_i is denoted by $|S_i|$. Since T does not have a transmitter, $|S_1| \geqslant 3$ and since T does not have a receiver, $|S_2| \geqslant 3$. For each $i = 1, \ldots, p$ let $S_1^i, \ldots, S_{n_i}^i$ be the components of $T_i = T - v_i$ with S_1^i and S_2^i the transmitter and receiver respectively of T_i^*.

Choose the notation so that $|S_1^1| \geqslant |S_1^i|$ and $|S_2^2| \geqslant |S_2^i|$ for all i. Then S_1 and S_1^1 are isomorphic, and S_2 and S_2^2 are isomorphic. If $|S_1^1| + |S_2^2| = p$. Then $T = S_1^1 + \rightarrow S_2^2$. Otherwise the number of components of T is greater than two.

If $|S_1| \geqslant 4$, then by Theorem 7 of [4] there is a cycle of length $|S_1| - 1$ in S_1. Therefore there is a point v in S_1 such that $s_1 - v$ is a strong tournament. Then we can choose $T_3 = T - v_3$ with $|S_1^3| = |S_1| - 1$. Now delete all of the points of S_1^3 from T_3 to obtain $T_3 - S_1^3$, a subtournament of T_3. It is clear that

$$T = S_1 + \rightarrow (T_3 - S_1^3).$$

Similarly T can be reconstructed if $|S_2| \geqslant 4$.

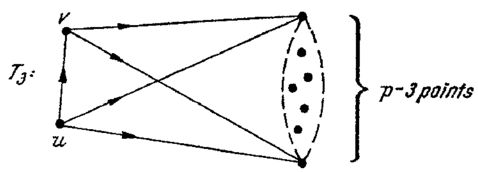


Fig. 5

If $|S_1| = |S_2| = 3$, then both S_1 and S_2 are cyclic triples. Therefore if v is a point of S_1, then $T - v$ has a transmitter. Choose $T_3 = T - v_3$ so that T_3 has a transmitter. Let u be the transmitter of T_3 and let v be the point of T_3 which is adjacent to every point of T_3 except u (see Figure 5).

Then T is obtained by adding a point v_3 to T_3 with v_3 adjacent from v and v_3 adjacent to all other points of T_3 (see Figure 6).

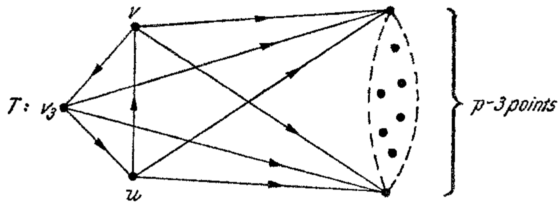


Fig. 6

Remarks on Ordinary Graphs

We now turn to ordinary graphs and prove a theorem which enables us to obtain much information about a graph G from its subgraphs G_i. The following theorem is a generalization of a result which appears in [2, p. 48]. Note that it holds for *any* graph, i.e. connected or unconnected, with multiple lines or with loops, directed or undirected, etc.

Theorem 8. Let G be a graph with $p > 2$ points and H be a connected subgraph with $n < p$ points. Let $\mu = \mu(H, G)$ be the number of subgraphs of G which are isomorphic to H. Let $\mu_i = \mu(H, G_i)$. Then

$$\mu = \frac{\sum_{i=1}^{p} \mu_i}{p - n}.$$

Proof. Obviously $\mu_i = \mu -$ (the number of subgraphs of G which are isomorphic to H and contain v_i). Summing over all the G_i, we obtain

$$\sum_{i=1}^{p} \mu_i = p\,\mu - n\,\mu.$$

Corollary 8a. Let T be a tournament with points $v_1, \ldots, v_p$ $(p > 3)$ and $T_i = T - v_i$, where the number of transitive triples in T is $b = b(T)$ and the number in T_i is $b_i = b(T_i)$. Then

$$b = \frac{\Sigma\, b_i}{p - 3}.$$

Corollary 8b. Let G be a connected graph with $e = e(G)$ triangles and $p > 3$ points. Let each G_i have $e_i = e(G_i)$ triangles. Then

$$e = \frac{\Sigma\, e_i}{p - 3}.$$

Given the collection of subgraphs G_i the following properties of G can be determined using Theorem 8.

(1) The cycle type $(c_3, c_4, \ldots, c_n)$, where $c_k =$ the number of cycles of length k, provided $p > n$.

(2) The clique type $(f_2, f_3, \ldots, f_n)$, where $f_k =$ the number of complete k-subgraphs, provided $p > n$.

(3) The degree sequence $(d_1, d_2, \ldots, d_p)$, where $d_i \leqslant d_{i+1}$ and $d_i = \deg v_i$.

(4) The score sequence $(s_1, s_2, \ldots, s_p)$ for tournaments, where $s_i \leqslant s_{i+1}$, $s_i = \text{od } v_i$ and $p - 1 > s_p$.

We outline computation of the degree sequence of a graph G. Let $H = K_2$ and let q_i be the number of lines in G_i. Now we can apply Theorem 8 to obtain $q = (\Sigma q_i)/(p - 2)$, the number of lines of G. But since $d_i = q - q_i$, for suitable relabeling we have the degree sequence $(d_1, \ldots, d_p)$ with $d_i \leqslant d_{i+1}$.

To compute the score sequence of a tournament, let T have subtournaments T_i as defined earlier. Theorem 4 gives a necessary and sufficient condition for T to have a transmitter in terms of the T_i. Therefore when $p \geqslant 5$, we can tell from the subtournaments T_i whether or not T has a point of score $p - 1$. If not, then we can choose a point v in some T_i such that the score of v is greater than of equal to the score of any point of any T_i. Therefore the score of v is s_p and $s_p < p - 1$.

Let n_k be the number of points in T of score k. Since the number of points in the subgraph $K_1 + \rightarrow K_{s_p}$ is less than p, we have

$$n_{s_p} = \mu(K_1 + \rightarrow \overline{K}_{s_p}, T),$$

$$n_{s_p n-1} = \mu(K_1 + \rightarrow \overline{K}_{s_p - 1}, T) - (s_p^{s_p} - 1)\, n_{s_p}.$$

In general:

$$n_h = \mu(K_1 + \rightarrow \overline{K}_k, T) - \sum_{r=k+1}^{s_p} \binom{r}{k} n_r.$$

In particular, when T is strong, $s_p \leqslant p - 2$ so we can always compute score sequence of T from the T_i.

We conclude by observing that the score sequence and the number of cyclic triples in a strong tournament do not determine the tournament.

In Figure 7 the two tournaments T_1 and T_2 on five points are different but each has score sequence (1, 2, 2, 2, 3) and each has four cyclic triples. Clearly T_1 has a path of length two from the point of minimum score to the point of maximum score and T_2 does not.

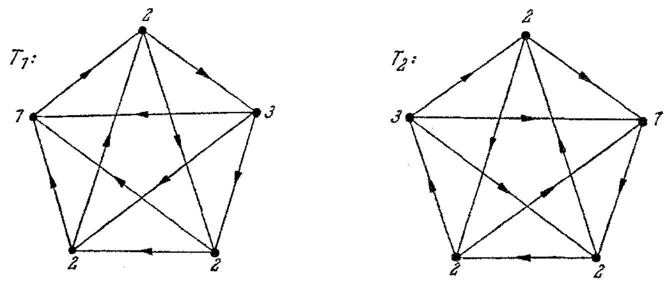


Fig. 7

It is evidently much more difficult to reconstruct the strong tournaments, but we believe that the conjecture will still be valid.

References

[1] *L. Beineke* and *F. Harary:* The maximum number of strongly connected subtournaments. Canad. Math. Bull., **8** (1965), 491—498.

[2] *F. Harary:* On the reconstruction of a graph from a collection of subgraphs. A chapter in Graph Theory and its Applications, Prague, 1964, pp. 47—52.

[3] *F. Harary:* The theory of tournaments: a miniature mathematical system. Colloq. Math., **15** (1966), 159—165.

[4] *F. Harary* and *L. Moser:* The theory of round robin tournaments. Amer. Math. Monthly **73** (1966), 231—246.

[5] *F. Harary, R. Norman,* and *D. Cartwright:* Structural Models: an introduction to the theory of directed graphs. New York, 1965, esp. Chapter 11.

[6] *H. Izbicki:* Reguläre Graphen beliebigen Grades mit vorgegebenen Eigenschaften. Monatsh. für Math., **64** (1960), 15—21.

[7] *P. J. Kelly:* A congruence theorem for trees. Pacific J. Math. **7** (1957), 961—968.

[8] *S. M. Ulam:* A collection of mathematical problems. New York, 1960, p. 29.

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