On a Family of Line-Critical Graphs[1]

By

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With 1 Figure

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1. Introduction

A set of points M of a graph G is a *point-cover* if each line of G is incident with at least one point of M. A *minimum cover*, abbreviated m. c., for G is a point cover with a minimum number of points. The *point covering number* of G, $\alpha(G)$, is the number of points in any minimum cover of G. If x is a line in G, we denote by $G—x$ the graph obtained by deleting x. A line x is said to be *critical* (with respect to point-cover) if $\alpha(G - x) < \alpha(G)$. Ore [4] mentions such critical lines and in [3], we consider the case where the graph involved is a tree. In particular, if each line of the graph G is critical, G is said to be *line-critical.* It is obvious that all odd cycles are line-critical graphs as are all complete graphs. *Erdös* and *Gallai* [2] obtain a bound on the number of lines in such a graph in terms of the point covering number. A structural characterization of this family of graphs is, however, presently unknown. In an earlier paper [1], we show that if two adjacent lines of a graph are both critical, then they must lie on a common odd cycle. Hence, in particular, a line-critical graph is a block in which every pair of adjacent lines lie on a common odd cycle.

In this paper we shall develop an infinite family of line-critical graphs. This family includes all graphs known by the author to be line-critical and in particular it includes all those line-critical graphs with fewer than eight points.

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2. Additional Terminology

A *graph* G consists of a finite set of *points* $V(G)$ together with a collection of *lines* $E(G)$ each of which is an unordered pair of points. If x is the line containing points u and v, then we write $s = uv$. If two points (lines) are joined by a line (point), we say that the points (lines) are *adjacent*. If a point and a line meet, we say they are *incident*. The *degree* of a point v, $d(v)$, is the number of lines incident with it. If $d(v) = 0$, we say v is an *isolated* point. A *path* joining points u and v is an alternating sequence of distinct points and lines beginning with u and ending with v so that each line is incident with the point before it and the point after it.

A *cycle* is a path containing more than one line together with an additional line joining the first and last points of the path. The *length* of a path or a cycle is the number of lines in it and a cycle is said to be *even* (*odd*) if its length is even (odd). A graph G is said to be *connected* if every two distinct points in G are joined by a path. A point v is a *cutpoint* of a connected graph G if the graph obtained from G by deleting v is disconnected. A subgraph of G is a *block* of G if B is a maximal connected subgraph of G having no cutpoints. Finally, let $| A |$ denote the number of elements in the set A.

3. The Construction

Let v_0 be a point of a connected graph G with $d(v_0) > 1$. For any $p \geqslant 3$, let $K_p - x$ denote the complete graph on p points with any one line x deleted, where $x = u_1 u_2$. We construct a new graph as follows. Split G at v_0 forming two new points v_1 and v_2, retaining all lines from G and adding no new lines. Furthermore, we make the restriction that neither v_1 nor v_2 is an isolated point in the new graph. Beyond this, however, no restriction is made on how the set of lines incident with v_0 is divided between v_1 and v_2. There are of course $2^{d(v_0)} - 2$ graphs obtainable from G in this fashion, some of which may be isomorphic. If $d(v_1) = m$ and $d(v_2) = n$, we denote any of the graphs obtained above by $S(G, v_0; m, n)$. Next, we attach $K_p - x$ to $S(G, v_0; m, n)$ by identifying u_1 with v_1 and u_2 with v_2. The resulting graph is denoted by $S(p; G, v_0; m, n)$. We illustrate this construction in Figure 1.

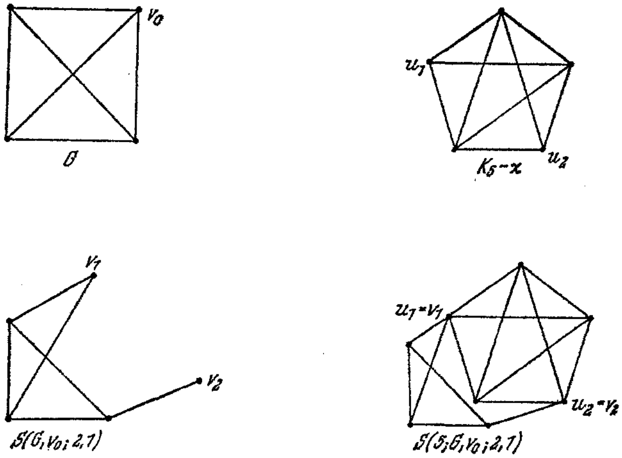


Fig. 1

We now proceed to show that if one performs the above construction, a resulting graph $S(p; G, v_0; m, n)$ is line-critical if and only if G is line-critical.

Theorem 1. If v is a point incident with a critical line of a graph G, then G has minimum covers M_1, M_2 such that $v \in M_1 - M_2$.

Proof. Let M_x be an m.c. for $G - x$ and suppose $x = uv$ is a critical line of G. Now since x is critical, neither u nor v is in M_x. Let $M_1 = M_x \cup \{v\}$ and $M_2 = M_x \cup \{u\}$. Then M_1 and M_2 each cover G and since $| M_1 | = | M_x | + 1 = \alpha(G) = | M_2 |$, each is a minimum cover. This completes the proof.

Our next theorem relates the point covering numbers of G and of $S(p; G, v_0; m, n)$.

Theorem 2. If v_0 is a point of graph G and $d(v_0) > 1$ and if $p \geqslant 3$, then $\alpha[S(p; G, v_0; m, n)] = \alpha(G) + p - 2$.

Proof. Suppose G has an m.c. M which contains v_0. Let W be any m.c. for K_p which includes u_1 and u_2. Then $[M - \{v_0\}] \cup W$ covers $S(p; G, v_0; m, n)$ and thus $\alpha[S(p; G, v_0; m, n)] \leqslant | M - \{v_0\} | + | W | = | M | - 1 + (p - 1) = \alpha(G) + p - 2$.

Now suppose G has an m.c. M' which does not contain v_0. Let $W' = V(K_p) - \{u_1, u_2\}$. Clearly, W' is the m.c. for $K_p - x$. Hence $M' \cup W'$ covers $S(p; G, v_0; m, n)$ and we have

$$\alpha[S(p; G, v_0; m, n)] \leqslant |M' \cup W'| = |M'| + |W'| = \\ = \alpha(G) + \alpha(K_p - x) = \alpha(G) + p - 2.$$

Thus in either case, one obtains the inequality:

(1) $\alpha[S(p; G, v_0; m, n)] \leqslant \alpha(G) + p - 2.$

Now let M be an m.c. for $S(p; G, v_0; m, n)$. We have three cases to consider.

(i) Suppose $\{u_1, u_2\} \cap M = \phi$. Then all points of K_p other than u_1 and u_2 are in M. Also, if $u_1 w_i$, $i = 1, \ldots, m$ are the lines of G incident with u_1 and if $u_2 w_i$, $i = m+1, \ldots, m+n$, are those lines incident with u_2, then $\{w_1, \ldots, w_m, w_{m+1}, \ldots, w_{m+n}\} \subset M$. Hence $M - [V(K_p) - \{u_1, u_2\}]$ covers G. Hence $\alpha(G) \leqslant |M - [V(K_p) - \{u_1, u_2\}]| = |M| - (p-2) = \alpha[S(p; G, v_0; m, n)] - p + 2$, and thus $\alpha[S(p; G, v_0; m, n)] \geqslant \alpha(G) + p - 2$.

(ii) Next suppose $\{u_1, u_2\} \subset M$. Then clearly $M \cap [V(G) - \{v_0\}]$ is an m.c. for $G - v_0$. Also in this case, $|M \cap V(K_p - x)| = p - 1$. Thus $\alpha[S(p; G, v_0; m, n)] = |M| = |M \cap V(K_p - x)| + |M \cap V(G - v_0)| = = p - 1 + \alpha(G - v_0) \geqslant p - 2 + \alpha(G)$ and thus

$$\alpha[S(p; G, v_0; m, n)] \geqslant \alpha(G) + p - 2.$$

(iii) Finally, suppose $u_1 \in M$, $u_2 \notin M$. Now every point of $K_p - x$, except u_2, is in M, since all such points are adjacent to u_2. Let w be a point of $K_p - x$, $w \neq u_1$. Then $w \in M$ and $M_0 = [M - \{w\}] \cup \{u_2\}$ covers $S(p; G, v_0; m, n)$. Since $|M_0| = |M|$, M_0 is a minimum cover for $S(p; G, v_0; m, n)$ and $u_1, u_2 \in M_0$. Hence by (ii), we again obtain $\alpha[S(p; G, v_0; m, n)] \geqslant \alpha(G) + p - 2$. Thus all three cases give rise to the inequality:

(2) $\alpha[S(p; G, v_0; m, n)] \geqslant \alpha(G)+p-2.$

This, together with inequality (1) yields $\alpha[S(p; G, v_0; m, n)] = \alpha(G) + p - 2$ and the theorem is proved.

We are now prepared to prove the main theorem concerning this construction.

Theorem 3. If v_0 is a point of degree at least 2 of a graph G and if $p \geqslant 3$, then G is line-critical if and only if $S(p; G, v_0; m, n)$ is line-critical.

Proof. Suppose that G is a line-critical graph. Let $x = u_1 u_2$ be the line deleted from K_p. Let y be any line of $S(p; G, v_0; m, n)$. We shall consider four cases.

(i) Suppose y is a line of G incident with v_0. Let M_y be an m.c. for $G-y$. In this proof we shall denote the set of lines in G incident with v_0 by $v_0 w_i$, $i = 1, \ldots, r$. Hence we may assume without loss of generality in this case that $y = v_0 w_1$. Then $v_0, w_1 \notin M_y$ and hence $w_2, \ldots, w_r \in M_y$. Then $M_y \cup [V(K_p) - \{u_1, u_2\}]$ covers $S(p; G, v_0; m, n) - y$. Thus

$$\begin{aligned}\alpha[S(p; G, v_0; m, n) - y] &\leqslant |M_y \cup [V(K_p) - \{u_1, u_2\}]| = \\ &= |M_y| + |V(K_p) - \{u_1, u_2\}| = \alpha(G - y) + p - 2 = \\ &= \alpha(G) + p - 3 = \alpha[S(p; G, v_0; m, n)] - 1\end{aligned}$$

by Theorem 2. Hence y is critical in $S(p; G, v_0; m, n)$.

(ii) Suppose y is a line of G which is not incident with v_0. Again let M_y be an m.c. for $G-y$. If $v_0 \notin M_y$, then $\{w_2, \ldots, w_r\} \subset M_y$ and $M_y \cup [V(K_p) - \{u_1, u_2\}]$ again covers $S(p; G, v_0; m, n) - y$ and as in case (i) y is critical in $S(p; G, v_0; m, n)$. If $v_0 \in M_y$, then $[M_y - \{v_0\}] \cup W$ covers $S(p; G, v_0; m, n) - y$, where W is any m.c. for K_p which contains u_1 and u_2. Thus $\alpha[S(p; G, v_0; m, n) - y] \leqslant |M_y - \{v_0\}| + |W| = |M_y| - 1 + p - 1 = \alpha(G - y) + p - 2 = \alpha(G) + p - 3 = \alpha[S(p; G, v_0; m, n)] - 1$ by Theorem 2 and hence again y is critical in $S(p; G, v_0; m, n)$.

(iii) In this case we assume that y is a line in $K_p - x$ incident with u_1 or u_2. Without loss of generality, let $y = u_1 w_0$. Let z be any line of G, $z = v_0 w_k$, incident with v_0 in G and with u_2 in $S(p; G, v_0; m, n)$. Now z is critical in G, hence there is an m.c. M_z for $G-z$, such that $v_0, w_k \notin M_z$, and $w_j \in M_z$ for $j \neq k$. Let $M_0 = M_z \cup [V(K_p) - \{u_1, w_0\}]$. Then M_0 covers $S(p; G, v_0; m, n) - y$. Hence

$\alpha[S(p; G, v_0; m, n) - y] \leqslant |M_0| = |M_z| + |V(K_p) - \{u_1, w_0\}| = \alpha(G - z) + p - 2 = \alpha(G) + p - 3 = \alpha[S(p; G, v_0; m, n)] - 1$ and y is critical in $S(p, G, v_0; m, n)$ again using Theorem 2.

(iv) Finally let us suppose that $y = v_3 v_4$ is a line in $K_p - x$ incident with neither u_1 nor u_2. By Theorem 1, there is an m.c. M for G which contains v_0. Let $W' = V(K_p) - \{v_3, v_4\}$. Then W' contains u_1 and u_2 and hence it covers $(K_p - x) - y$. Hence $[M - \{v_0\}] \cup W'$ covers $S(p; G, v_0; m, n) - y$. Thus $\alpha[S(p; G, v_0; m, n) - y] \leqslant |M - \{v_0\}| + |W'| = |M| - 1 + (p-2) = \alpha(G) + p - 3 = \alpha[S(p; G, v_0; m, n)] - 1$ by Theorem 2 and hence y is critical in $S(p; G, v_0; m, n)$.

Now assume that $S(p; G, v_0; m, n)$ is line-critical.

First suppose that y is a line in G incident with v_0. Thus y is incident with u_1 or u_2 in $S(p; G, v_0; m, n)$, say u_1. Now y is critical in $S(p; G, v_0; m, n)$ and hence there is an m.c. M' for $S(p; G, v_0; m, n) - y$ with $u_1 \notin M'$ and with $[V(K_p) - \{u_1, u_2\}] \subset M'$. Now $M' - [V(K_p) - \{u_1, u_2\}]$ covers $G - y$. Hence,

$$\alpha(G - y) \leqslant | M' - [V(K_p) - \{u_1, u_2\}] | = | M' | - (p - 2)$$
$$= \alpha[S(p; G, v_0; m, n) - y] - p + 2 = \alpha[S(p; G, v_0; m, n)] - p + 1$$
$$\leqslant \alpha(G) + p - 2 - p + 1 = \alpha(G) - 1$$ by Theorem 2 and thus y is critical in G.

Finally, suppose y is a line in G which is not incident with v_0. Let M' be an m.c. for $S(p; G, v_0; m, n) - y$. If $[V(K_p) - \{u_1, u_2\}] \subset M'$, then y is critical in G as before. If $[V(K_p) - \{u_1, u_2\}] \not\subset M'$, then let w be a point in $[V(K_p) - \{u_1, u_2\}] - M'$. But w_1 and u_1 are adjacent as are w and u_2. Hence $u_1, u_2 \in M'$. Thus $| M' \cap V(K_p) | = p - 1$. Now $[M' - V(K_p)] \cup \{v_0\}$ covers $G - y$. Hence

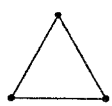
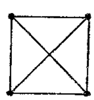
$$\alpha(G - y) \leqslant | [M' - V(K_p)] \cup \{v_0\} | = | M' - V(K_p) | + 1$$
$$= | M' | - (p - 1) + 1 = \alpha[S(p; G, v_0; m, n) - y] - p + 2$$
$$= \alpha[S(p; G, v_0; m, n)] - p + 1 \leqslant \alpha(G) - 1$$ again by Theorem 2.

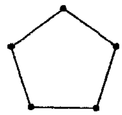
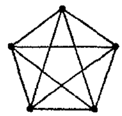
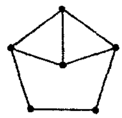
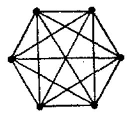
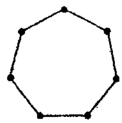
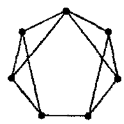
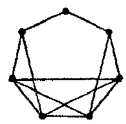
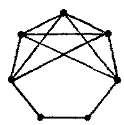
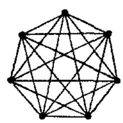
Thus y is critical in G and the theorem is proved.

4. Data on line-critical graphs

In Table 1 below, p denotes the number of points and q the number of lines of the corresponding graph.

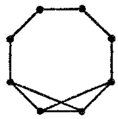
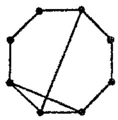
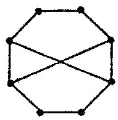
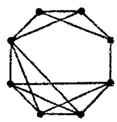
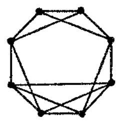
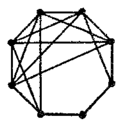
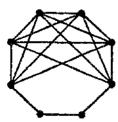
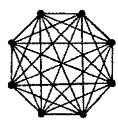
Table 1. *The line-critical graphs with fewer than eight points*

p	q	$\alpha(G)$	G
2	1	1	K_2
3	3	2	K_3 
4	6	3	K_4 

5	5	3	C_5	
	10	4	K_5	
6	8	4		
	15	5	K_6	
7	7	4	C_7	
	11	5		
	12	5		
	12	5		
	21	6	K_7	

One may obtain each of the graphs in Table 1 by starting with a complete graph and performing a sequence of constructions of the type

Table 2. *A family of line-critical graphs with eight points*

p	q	$\alpha(G)$	G
8	10	5	
	10	5	
	10	5	
	15	6	
	15	6	
	17	6	
	17	6	
	28	7	K_8 

described in Section 3. Hence this construction yields all the line-critical graphs with fewer than eight points. The large number of graphs with eight points has discouraged a direct search for those which are line-critical. In Table 2, however, we present all those line-critical graphs with eight points obtainable by starting with a complete graph and performing a sequence of constructions as in Section 3.

References

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