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On a Linear Combination of Some Expressions in the Theory of the Univalent Functions

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Abstract

Let $H(\alpha)$ denote the class of regular functions $f(z)$ normalized so that $f(0)=0$ and $f'(0)=1$ and satisfying in the unit disc E the condition

$$\operatorname{Re}\{(1-\alpha)f'(z)+\alpha(1+zf''(z)/f'(z))\}>0$$

for fixed α. It is known that $H(0)$ is a particular class NW of close-to-convex univalent functions. The authors show the following results: *Theorem 1.* Let $f(z)\in H(\alpha)$. Then $f(z)\in \mathrm{NW}$ if $\alpha\leqslant 0$ and $z\in E$. *Theorem 2.* Let $f(z)\in\mathrm{NW}$. Then $f(z)\in H(\alpha)$ in $|z|=r<r_\alpha$ where i) $r_\alpha=(1+\sqrt{2\alpha})^{-1/2}$, $\alpha\geqslant 0$, and ii) $r_\alpha=\sqrt{\dfrac{1-\alpha-\sqrt{\alpha(\alpha-1)}}{1-\alpha}}$, $\alpha<0$. All results are sharp. *Theorem 3.* If $f(z)=z+a_2z^2+a_3z^3+\ldots$ is in $H(\alpha)$ and if μ is an arbitrary complex number, then

$$|1+\alpha|\,|a_3-\mu a_2^2|\leqslant(2/3)\max[1,|1+2\alpha-3/2\mu(1+\alpha)|].$$

1. Introduction. We consider functions $f(z)$ which are regular in the unit disc E: $|z|<1$ and normalized so that $f(0)=0$ and $f'(0)=1$, and we let

$$I(\alpha,f(z))\equiv(1-\alpha)f'(z)+\alpha\left(1+\left(zf''(z)/f'(z)\right)\right), \tag{1}$$

where α is real number. We denote by $H(\alpha)$ the class of functions satisfying $\operatorname{Re}\{I(\alpha,f(z))\}>0$, for fixed α and for all $z\in E$. It is known that $H(0)$ is a particular class NW of close-to-convex univalent functions as demonstrated by K. Noshiro [2] and S. Warschawski [4].

In section 2 we show that every $f\in H(\alpha)$ is in NW for $\alpha\leqslant 0$. *We are unable to determine the univalency of $H(\alpha)$ if $\alpha>0$.* However, in section 3 we obtain the radius of the largest disc r_α such that if $f\in\mathrm{NW}$ and $z\in E$, then $f\in H(\alpha)$ for $|z|<r_\alpha$. In section 4 some estimates for the coefficients of functions in $H(\alpha)$ are obtained.

2. The univalency of $H(\alpha)$. We show the following theorem about the univalency of the class $H(\alpha)$.

Theorem 1. *Let $f(z) \in H(\alpha)$. Then $f(z)$ is univalent with* $\text{Re}\{f'(z)\} > 0$, $z \in E$, *if* $\alpha \leqslant 0$ (*i. e.* $f \in \text{NW}$).

Proof. Let

$$f'(z) = p(z). \tag{2}$$

To prove Theorem 1 we need to show that if $f \in H(\alpha)$, $\alpha \leqslant 0$, then $\text{Re}\{p(\alpha)\} > 0$ in E. The case $\alpha = 0$ is trivial. If $\alpha < 0$, then we assume that $\text{Re}\{p(z)\} \not> 0$ in E. Hence, since by (2), $p(0) = 1$, then there exists a first r_0, and θ_0 so that $\text{Re}\{p(z)\} > 0$ for $|z| < r_0$ and

$$\text{Re}\{p(z_0)\} = \text{Re}\{p(r_0 e^{i\theta_0})\} = 0 \tag{3}$$

where $0 < r_0 < 1$. The condition in (3) implies that $\partial \arg p(z)/\partial\theta = 0$ at $z = r_0 e^{i\theta_0}$. Consequently, since

$$\frac{\partial}{\partial\theta} \arg p(z) = \text{Im}\left\{\frac{\partial}{\partial\theta} \ln p(z)\right\} = \text{Im}\left\{iz\frac{p'(z)}{p(z)}\right\} = \text{Re}\left\{\frac{z\,p'(z)}{p(z)}\right\},$$

then

$$\text{Re}\{z\,p'(z)/p(z)\} = 0 \tag{4}$$

at $z = r_0 e^{i\theta_0}$. We use (1), (2), (3) and (4) to get

$$\text{Re}\{I(\alpha, f(z_0))\} = \text{Re}\left\{(1-\alpha)p(z_0) + \alpha\left(1 + \frac{z_0 p'(z_0)}{p(z_0)}\right)\right\} = \alpha\,. \tag{5}$$

Thus if $\alpha < 0$, then (5) shows that $f \notin H(\alpha)$ which is a contradiction. This completes the proof of Theorem 1.

3. The radius r_α. Let P denote the class of regular functions $p(z)$, $p(0) = 1$, with positive real part, $\text{Re}\{p(z)\} > 0$, $z \in E$. If $f \in \text{NW}$, then there is a $p(z) \in P$ such that $f'(z) = p(z)$. By substitution (1) becomes $I(\alpha, f(z)) = \varphi_\alpha(p, zp') = (1-\alpha)p(z) + \alpha(1 + zp'(z)/p(z))$. Let

$$Q_\alpha(r) = \min_{p \in P}\ \min_{|z| = r < 1} \text{Re}\{(1-\alpha)p(z) + \alpha(1 + zp'(z)/p(z))\}\,. \tag{6}$$

Hence, the problem of finding the largest r_α, for fixed α, such that for each $f \in \text{NW}$ and for each z, $|z| < r_\alpha$ we have $\text{Re}\{I(\alpha, f(z))\} > 0$ is equivalent to finding the smallest positive root of $Q_\alpha(r) = 0$, where $Q_\alpha(r)$ is given by (6). To find r_α, we make use of a theorem due to V. A. Zmorovic [5].

Theorem A (ZMOROVIC). *Let* $\Psi(w, W) = M(w) + N(w)W$, *where* $M(w)$ *and* $N(w)$ *are defined and are finite in the half plane* $\operatorname{Re}\{w\} > 0$. *We set*

$$w = \lambda_1 (1 + z_1)/(1 - z_1) + \lambda_2 (1 + z_2)/(1 - z_2),$$
$$W = \lambda_1 2 z_1/(1 - z_1)^2 + \lambda_2 2 z_2/(1 - z_2)^2,$$

where z_1 *and* z_2 *are points on the circumference* $|z| = r < 1$, $\lambda_k \geq 0$ $(k = 1, 2)$, $\lambda_1 + \lambda_2 = 1$. *Then the function* $\Psi(w, W)$ *can be put in the form*

$$\Psi(w, W) = M(w) + \tfrac{1}{2}(w^2 - 1) N(w) + \tfrac{1}{2}(\varrho^2 - \varrho_0^2) N(w) e^{2i\psi},$$

where $\qquad (1 + z_k)/(1 - z_k) = a + \varrho \exp i\psi_k \, (k = 1, 2),$

$$w = a + \varrho_0 \exp i\psi_0 \quad (0 \leq \varrho_0 \leq \varrho), \quad |z_1| = |z_2| = r,$$

$$a = (1 + r^2)/(1 - r^2), \quad \varrho = 2r/(1 - r^2), \quad \exp i\psi = i \exp[\tfrac{1}{2}(\psi_1 + \psi_2)].$$

Also for a fixed w *in the circle* $|w - a| < \varrho$, *the angle* 2ψ *in the above formula can take all values from* $[0, 2\pi]$, *and hence*

$$\begin{aligned} &\min \operatorname{Re}\{\Psi(w, W) \equiv \Psi_\varrho(w) = \\ &= \operatorname{Re}\{M(w) + \tfrac{1}{2}(w^2 - 1) N(w)\} - \tfrac{1}{2}|N(w)|(\varrho^2 - \varrho_0^2). \end{aligned} \tag{7}$$

This minimum is reached when

$$\exp[i(2\psi + \arg N(w))] = -1. \tag{8}$$

We shall need the following result:

Lemma 1. *Let*

$$\begin{aligned} &\min \operatorname{Re}\{\Psi(w, W)\} \equiv \Psi_\varrho(w) = \\ &= \operatorname{Re}\{(1 - \alpha) w + \alpha + (\alpha/2)(w^2 - 1)/w\} - (|\alpha|/2)((\varrho^2 - \varrho_0^2)/|w|), \end{aligned} \tag{9}$$

where $\operatorname{Re}\{w\} > 0$. *Then the* $\min \Psi_\varrho(w)$ *in the circle* $|w - a| = \varrho_0 \leq \varrho$ *is reached* i) *on the diameter if* $\alpha \geq 0$ *and* $r \in (0, 1)$, *and* ii) *on the diameter if* $\alpha < 0$ *and* $r \in (0, \tilde{r})$, *where*

$$\tilde{r} = \left(\frac{1 + \alpha + (\alpha(\alpha - 2))^{1/2}}{1 - 3\alpha + (\alpha(\alpha - 2))^{1/2}}\right)^{1/2}. \tag{10}$$

Proof: (i) For $\alpha \geq 0$. Let $w = a + \xi + i\eta$, $R^2 = |w|^2 = (a + \xi)^2 + \eta^2$, where a, ϱ, and ϱ_0 are defined in Theorem A. Then from (9) we get

$$\Psi_\varrho(w) \equiv \psi_\varrho(\xi, \eta) = (a + \xi)\left(1 + \frac{\alpha}{2} - \frac{\alpha}{2R^2}\right) - \frac{\alpha}{2} \cdot \frac{\varrho^2 - \xi^2 - \eta^2}{R} + \alpha.$$

The above yields

$$\partial \psi_\varrho(\xi,\eta)/\partial\eta = \eta\,\alpha\,R^{-4}/2\,[2(a+\xi)+(\varrho^2-\xi^2-\eta^2)R+2R^3].$$

The expression in the square brackets is positive; consequently for each fixed ξ, the nonnegative minimum of $\psi_\varrho(\xi,\eta)$ is achieved at $\eta=0$. It follows that the minimum in the circle $\xi^2+\eta^2\leqslant\varrho^2$ is also reached on the diamenter $\eta=0$.

(ii) For $\alpha<0$. Let $w=R\exp i\varphi$. Then (9) becomes

$$\psi_\varrho(w)\equiv L(R,\varphi)=[(1-\alpha/2)R-(\alpha/2)R^{-1}-|\alpha|a]\cos\varphi+\alpha+ + (|\alpha|/2)(R+R^{-1}). \tag{11}$$

From (11) one can conclude only that the minimum of $L(R,\varphi)$ on any arc $R=$ constant inside the circle $|w-a|\leqslant\varrho$ is reached either when $\varphi=0$ or at the end points of this arc which are located on the circumference $\varrho=\varrho_0$. But by setting $\varrho=\varrho_0$ in (9) we get

$$L(R,\varphi)=[(1-\alpha/2)R-\alpha/2\,R^{-1}]\cos\varphi+\alpha, \tag{12}$$

where

$$R^2-2aR\cos\varphi+1=0,\quad R\in[a-\varrho,a+\varrho]. \tag{13}$$

Eliminating φ between (12) and (13) we get

$$L(R)=(1/2a)[R^2+1]-(\alpha/4a)[R+R^{-1}]^2+\alpha. \tag{14}$$

We now show the minimum of $L(R)$ cannot be zero in $[a-\varrho,a+\varrho]$ if $r<\tilde r$ where $\tilde r$ is given by (10). Consequently, the minimum of $\Psi_\varrho(w)$ may vanish on the diameter $\varphi=0$, if $r<\tilde r$, which is our lemma. With $a^2=\varrho^2+1$, (14) yields

$$L(a-\varrho)=a-\varrho+\alpha(1-a)$$

and

$$L(a+\varrho)=a+\varrho+\alpha(1-a).$$

Since $\alpha<0$, $a>1$ and $a-\varrho>0$, then $L(a-\varrho)>0$ and $L(a+\varrho)>0$. On the other hand, from (14) we obtain

$$dL(R)/dR=R/a-(\alpha/2a)[R+R^{-1}][1-R^2]=0$$

if $R=[\alpha/(\alpha-2)]^{\frac14}$. However, $d^2L(R)/dR^2>0$ at $R=[\alpha/(\alpha-2)]^{\frac14}$. This shows that the minimum of the continuous function $L(R)$, if it is not attended at an end point, it must be attended in $(a-\varrho,a+\varrho)$ at $R=[\alpha/(\alpha-2)]^{\frac14}$. Direct computations show

$$L\left(\left(\frac{\alpha}{\alpha-2}\right)^{\frac{1}{2}}\right)=$$

$$=\frac{1}{4a}\left[[4\alpha a-2\alpha+2+2\sqrt{\frac{\alpha}{\alpha-2}}-\alpha\sqrt{\frac{\alpha-2}{\alpha}}-\alpha\sqrt{\frac{\alpha}{\alpha-2}}\right]$$

$$=\frac{1}{4a}\left[4\alpha a-2\alpha+2+(2-\alpha)\sqrt{\frac{\alpha}{\alpha-2}}+\sqrt{\alpha(a-2)}\right]$$

$$=(1/4a)\left[4\alpha a-2\alpha+2+2\sqrt{\alpha(\alpha-2)}\right].$$

Hence $L([\alpha/(\alpha-2)]^{\frac{1}{2}})>0$ provided

$$a<\left(\alpha-1-\sqrt{\alpha(\alpha-2)}\right)/2\alpha .$$

Since $a=(1+r^2)/(1-r^2)$, then the above condition is equivalent to (10). Thus $L(R)>0$ in $[a-\varrho, a+\varrho]$ provided $r<\tilde{r}$ and $\tilde{r}$ is as given by (10). This completes the proof of Lemma 1.

The following theorem describes r_α.

Theorem 2. *Let $f(z)$ be in the class of normalized regular functions with* $\mathrm{Re}\{f'(z)\}>0$ *for $z\in E$. Then $f\in H(\alpha)$ in $|z|=r<r_\alpha$ where*

i) $r_\alpha=(1+\sqrt{2\mathrm{a}})^{-\frac{1}{2}}, \quad \alpha\geqslant 0,$

ii) $r_\alpha=\sqrt{\left(1-\alpha-\sqrt{\alpha(\alpha-1)}\right)/(1-\alpha)}, \quad \alpha<0.$

All results are sharp.

Proof: In (3) it is shown that the minimum in (6) is attained by a function of the form

$$P(z)=\lambda_1\frac{1+ze^{-i\theta_1}}{1-ze^{-i\theta_1}}+\lambda_2\frac{1+ze^{-i\theta_2}}{1-ze^{-i\theta_2}}, \tag{15}$$

where θ_1, θ_2 are arbitrary real constants in $[0, 2\pi]$ and where λ_1, λ_2 are nonnegative numbers satisfying $\lambda_1+\lambda_2=1$. We may, therefore, apply (7) to (6) with $p(z)=w(z)$, $M(w)=(1-\alpha)w(z)+\alpha$, and $N(w)=\alpha/w(z)$ to get

$$\min \mathrm{Re}\{\psi(w, W)\}\equiv\psi_\varrho(w)= \tag{9'}$$
$$=\mathrm{Re}\left\{(1-\alpha)w+\alpha+(\alpha/2)(w^2-1/w)\right\}-(|\alpha|/2)(\varrho^2-\varrho_0^2/|w|).$$

Note that (9′) is (9) of Lemma 2. Let $w=R\exp i\varphi$. Then (9′) becomes

$$\psi_\varrho(w)\equiv L(R,\varphi)= \tag{11'}$$
$$=\left[\left(1-(\alpha/2)\right)R-(\alpha/2)R^{-1}-|\alpha|a\right]\cos\varphi+\alpha+(|\alpha|/2)(R+R^{-1}).$$

(i) Let $\alpha > 0$. By Lemma 1, part (i) the minimum of $\psi_\varrho(w)$ in the circle $|w - a| < \varrho$ is reached on the diameter of this circle $\varphi = 0$. In view of this, put $\varphi = 0$ in (11′)

$$L(R, 0) \equiv L(R) = R - \alpha a + \alpha .$$

The minimum of $L(R)$ is at the end of the diameter $R = a - \varrho$. If we set

$$Q_\alpha(r) = L(a - \varrho) = a - \varrho - \alpha a + \alpha$$

with $a = (1 + r^2)/(1 - r^2)$, then the least positive root of $Q_\alpha(r) = 0$ is $r_\alpha = 1/(1 + \sqrt{2\alpha})$ this completes our proof of part (i) of the present theorem.

(ii) Let $\alpha < 0$. Again Lemma 1, part (ii) shows that the minimum of $\psi_\varrho(w)$ is on the diameter $\varphi = 0$ for $r < \tilde{r}$, and $\tilde{r}$ is given by (10). However, direct calculations show that $r_\alpha < \tilde{r}$ if r_α is given by part (ii) of Theorem 2. Therefore, if we set $\varphi = 0$ in (11′), $\alpha < 0$ we get

$$l(R) \equiv L(R, 0) = (1 - \alpha) R - \alpha R^{-1} + \alpha(1 + a), \quad R \in [a - \varrho, a + \varrho] .$$

Then it follows that

$$dl(R)/dR = (1 - \alpha) + \alpha R^{-2} = 0$$

for $R^2 = \alpha/(\alpha - 1)$, or $R = \sqrt{\alpha/(\alpha - 1)} \equiv R_0$. It is clear that $R_0 = \sqrt{\alpha/(\alpha - 1)} < 1 < a + \varrho$ but R_0 is not necessarily greater than $a - \varrho$. Hence the minimum is either attained at $R_0 = \sqrt{\alpha/(\alpha - 1)}$ or at $R_1 \equiv a - \varrho$. For the latter case, we find

$$l(R_1) = (1 - \alpha)(a - \varrho) - \alpha(a - \varrho)^{-1} + \alpha(1 + a)$$

does not vanish for real r. The other alternative is

$$Q_\alpha(r) \equiv l(R_0) = 2\sqrt{\alpha(\alpha - 1)} + \alpha(1 + a)$$

whose smallest positive zero is $r_\alpha = \sqrt{(1 - \alpha - \sqrt{\alpha(\alpha - 1)})/(1 - \alpha)}$. Our proof of the theorem is now complete.

We now determine the extremal functions $f_0(z)$. We remark that as a consequence of (8) the minimum of (9′) is reached when the point $w(|w - a| < \varrho)$ is fixed, and the chord passing through it and through the points $a + \varrho \exp \psi_k$ $(k = 1, 2)$ is perpendicular to the vector $\exp(i\varphi/2)$, where $w = R \exp i\varphi$. Taking this into account, as well as the fact that the minimum of $\psi_\varrho(w)$ is realized at an end point of the diameter when $\alpha \geqslant 0$, we conclude that $p(z)$ of (15) should be taken in the form $p(z)=(1+z)/(1-z)$. Hence for $\alpha \geqslant 0$, the extremal function is

$$f_0(z)=-\ln(1-z)-z$$

which realizes part (i) of Theorem 2 at $z=-r$. For $\alpha<0$, the minimum is reached at a point of the diameter (not an end point) and thus $p(z)$ should in this case be taken in the form

$$p(z)=\frac{1}{2}\,\frac{1+z\,e^{-i\theta}}{1-z\,e^{-i\theta}}+\frac{1}{2}\,\frac{1+z\,e^{i\theta}}{1-z\,e^{i\theta}},$$

where θ is given by the relation

$$R_0=\sqrt{\alpha/(\alpha-1)}=\operatorname{Re}\{p(z)\}=(1-r_\alpha^2)(1-2r_\alpha\cos\theta+r_\alpha^2)^{-1} \quad (16)$$

and r_α is given by (ii) of the Theorem 2. This shows

$$f_0(z)=-[e^{i\theta}\ln(1-z\,e^{-i\theta})+e^{-i\theta}\ln(1-z\,e^{i\theta})+z].$$

4. A coefficient inequality for functions in $H(\alpha)$. In this section we obtain some coefficient properties for functions in $H(\alpha)$. We show the following theorem.

Theorem 3. *If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ is in $H(\alpha)$ and if μ is an arbitrary complex number, then*

$$|1+\alpha|\,|a_3-\mu a_2^2|\leqslant\tfrac{2}{3}\max[1,|1+2\alpha-\tfrac{3}{2}\mu(1+\alpha)|]. \quad (17)$$

Proof: If $f(z)\in H(\alpha)$, then there exists a regular function $w(z)=\sum_{n=1}^{\infty}c_nz^n$ such that $|w(z)|<1$ in E and

$$(1-\alpha)f'(z)+\alpha\big(1+zf''(z)/f'(z)\big)=\big(1+w(z)\big)/\big(1-w(z)\big). \quad (18)$$

Now by expanding (18) and equating coefficients we have

$$a_2=c_1 \quad (19)$$

and

$$3(1+\alpha)a_3-4\alpha a_2^2=2(c_2+c_1^2). \quad (20)$$

We may assume $\alpha\neq-1$. From (19) and (20) we get

$$|a_3-\mu a_2^2|=(2/3\,|1+\alpha|)\,c_2-\big(1+2\alpha-\tfrac{3}{2}\mu(1+\alpha)\big)c_1^2|. \quad (21)$$

A result due to Keogh and Merkes [1] shows that $|c_2-\nu c_1|\leqslant$ $\leqslant\max[1,|\nu|]$ for arbitrary complex number ν. We apply this result to the right hand side of (21) with $\nu = 1 + 2\alpha - \frac{3}{2}\mu(1+\alpha)$ to obtain (17). This completes Theorem 3.

Remarks. i) For $\alpha = 1$, (17) reduces to

$$|a_3 - \mu a_2^2| \leqslant \max\left[\tfrac{1}{3}, |\mu - 1|\right]$$

which is a result of KEOGH and MERKES [1].

ii) For $\alpha = 0$, (17) reduces to

$$|a_3 - \mu a_2^2| \leqslant \max\left[\tfrac{2}{3}, |\mu - \tfrac{2}{3}|\right].$$

Corollary. If $f(z) \in H(\alpha)$, then

$$|a_2| \leqslant 1 \tag{22}$$

and

$$|1+\alpha|\,|a_3| \leqslant \begin{cases} \frac{2}{3} & \text{if } -1 \leqslant \alpha \leqslant 0 \\ \frac{2}{3}|1+2\alpha| & \text{if } |1+2\alpha| > 1. \end{cases} \tag{23}$$

Proof: The inequalities in (22) and (23) follow directly from (19) and (21), respectively.

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