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Divisibility Structure and Finitely Generated Ideals in the Disc Algebra

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Abstract

The main purpose of this paper is the following algebraic generalization of the corona theorem for the disc algebra $A(\bar{D})$: If d is a greatest common divisor of the functions $f_1, \ldots, f_n \in A(\bar{D})$, then there exist functions $g_1, \ldots, g_n \in A(\bar{D})$ with $d = f_1 g_1 + \ldots + f_n g_n$. This generalization is false for many algebras of holomorphic functions, e. g. in case of the Banach algebra H^∞. Under the assumption that a greatest common divisor d exists, also a description of d is given.

1. Introduction

Let $A(\bar{D})$ be the disc algebra, that is the Banach algebra of continuous functions on the closed unit disc $\bar{D}$, which are analytic on the open unit disc D, under the usual pointwise operations and the sup-norm.

The purely algebraic structure of $A(\bar{D})$ and subalgebras seems not very much investigated in the literature. The only newer consideration we know is Dietrich [1].

The structure of the closed (and therefore maximal) ideals in $A(\bar{D})$ are well known (see Hoffman [3], p. 82ff.). Unfortunately this gives us not too much information about the structure of the finitely generated ideals.

The only fact we can wring out is the so called baby corona theorem (corona theorem for the algebra $A(\bar{D})$), which says that the finitely generated ideal $(f_1, \ldots, f_n)$ is the whole algebra $A(\bar{D})$, if and only if the functions f_i have no common zero on $\bar{D}$. For a more convenient discription we introduce the ideal

$$W(f_1, \ldots, f_n) := \{f \in A(\bar{D}) : \text{There exists a constant } C > 0 \text{ s. t. } |f(z)| \leqslant C \sum_{i=1}^{n} |f_i(z)| \text{ for all } z \in \bar{D}\}.$$

Now the baby corona theorem can be stated in the following terms:

Theorem 1.1. *For the algebra $A(\bar{D})$ we have:*

$$1 \in (f_1, \ldots, f_n) \Leftrightarrow 1 \in W(f_1, \ldots, f_n).$$

$(f_1, \ldots, f_n)$ is clearly contained in $W(f_1, \ldots, f_n)$, but in general properly, even in the case $n = 1$.

Although the algebra $A(\bar{D})$ is commonly known as much nicer than e. g. the Banach algebra H^∞, the following fact aggravates the theory of finitely generated ideals:

Two functions may not have a greatest common divisor (GCD) as the example with $f_1(z) = 1 - z$ and $f_2(z) = (1-z)\exp((z+1)/(z-1))$ shows.

If $1 \in W(f_1, \ldots, f_n)$ then 1 is of course a GCD of $f_1, \ldots, f_n$ and the corona theorem for $A(\bar{D})$ says that the GCD is a linear combination of the f_i. We show that this result holds whenever the functions $f_1, \ldots, f_n$ have a GCD.

2. Algebraic Generalization of the Corona Theorem for the Disc Algebra

Lemma 2.1. *Let $k(e^{it}) \geqslant 0$ be a continuous function on ∂D and on $P := \{e^{it}: k(e^{it}) \neq 0\}$ continuously differentiable. If*

$$\int_0^{2\pi} \log k(e^{it})\, dt > -\infty,$$

then it follows that

$$f(z) := \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log k(e^{it})\, dt\right\}$$

is an outer function in $A(\bar{D})$.

Proof. That f is an outer function in H^∞ follows from $\log k(e^{it}) \in L^1(D\partial)$ and $k(e^{it})$ continuous on ∂D (see DUREN [2], p. 24). Since k is continuously differentiable on P $(k(e^{it}) \neq 0!)$, so is $\log k(e^{it})$. With the corollary in HOFFMAN [3], p. 79 it follows that f is continuous on P. Since $f(e^{it})$ tends to zero when e^{it} tends to a point of $\partial D \setminus P$, f is actually continuous on ∂D, therefore $f \in A(\bar{D})$.

Lemma 2.2. *Let f, g be two outer functions in H^∞. Then $|f(e^{it})| \leqslant |g(e^{it})|$ a. e. on ∂D if and only if $|f(z)| \leqslant |g(z)|$ on D.*

Proof. Since one direction is clear, let $|f(e^{it})| \leqslant |g(e^{it})|$ a. e. on ∂D, or in other words $\log|f(e^{it})| - \log|g(e^{it})| \leqslant 0$ a. e..

Since g is outer, the function f/g is in D holomorphic. The estimation of the modulus of f/g yields

$$\left|\frac{f(z)}{g(z)}\right| = \left|\exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} [\log|f(e^{it})| - \log|g(e^{it})|]\, dt\right\}\right| \leqslant 1 .$$

Lemma 2.3. *Suppose that the functions $f_1,\ldots,f_n \in A(\bar{D})$ have a GCD d and a common zero $z_0 \in \bar{D}$. Then d is zero on z_0.*

Proof. Let $f_i = B_i S_i F_i$ be the canonical factorizations with Blaschke products B_i, singular inner parts S_i, outer parts $F_i (i = 1,\ldots,n)$. If $z_0 \in D$, then $B_i(z_0) = 0$ for all $i \in \{1,\ldots,n\}$ and since d is a GCD, the Blaschke product B of d is the one formed with the common zeros of the functions B_i, therefore $d(z_0) = 0$.

Now let $z_0 \in \partial D$. All we have to do is to construct an outer function $G \in A(\bar{D})$ which fulfills $|G(e^{it})| \geqslant \sum_{i=1}^{n} |f_i(e^{it})|$ on ∂D and G has the same zeros as $\sum_{i=1}^{n} |f_i|$ on the boundary ∂D. From this by Lemma 2.2 it follows $|G(z)| \geqslant |F_i(z)|$ for every $i \in \{1,\ldots n\}$ in $\bar{D}$. Let $F = \sqrt{G}$, then F is also an outer function in $A(\bar{D})$.

We show that F is moreover a divisor of every f_i. Since $f_i/F \in A(\bar{D})$ it is enough to show that $F_i/F \in A(\bar{D})$ and $(F_i/F)(e^{it}) = 0$ on the singularities of B_i or S_i.

If B_i or S_i has a singularity on e^{it_0} then $F_i(e^{it_0}) = 0$. But this is fulfilled by construction if and only if $(F_i/F)(e^{it_0}) = 0$. The continuity of F_i/F in $\bar{D}$ follows from $|F_i(z)/F(z)| \leqslant |G(z)/F(z)| = |F(z)|$ $(z \in \bar{D})$. Since d is a GCD, F must divide d, therefore we have $d(z_0) = 0$.

Now let us construct such a function G. For short we write t instead of e^{it}. $P := \{t : \sum_{i=1}^{n} |f_i(t)| > 0\}$ is an open subset of ∂D of Lebesgue measure 2π, therefore we can decompose P in a (at most) countable number of disjoint open arcs (a_k, b_k). Now we construct a function G_k on $[a_k, b_k]$ with the following properties: G_k is continuous in $[a_k, b_k]$ and continuously differentiable in (a_k, b_k), $G_k(a_k) = G_k(b_k) = 0$ and $G_k(t) \geqslant \sum_{i=1}^{n} |f_i(t)|$.

Only small neighbourhoods of a_k and b_k are critical for such a construction. Therefore we construct G_k only in the neighbourhood $[a_k, a_k + \varepsilon)$ with $0 < \varepsilon < \frac{1}{2}|a_k - b_k|$ and choose w. l. o. g. $a_k = 0$. In $[0, \varepsilon)$ the function

$$h(\theta) := \theta + \max_{t \in [0,\theta]} \sum_{i=1}^{n} |f_i(t)|$$

is properly monotone increasing, therefore the inverse function $h^{-1}(\theta)$ exists and is also properly monotone increasing.

The above problem is equivalent to construct a continuously differentiable nonnegative minorant to h^{-1} and this is equivalent to construct a continuously differentiable minorant $f(\theta)$ to $\log[h^{-1}(\theta)]$ with $\lim_{\theta \to 0+} f(\theta) = -\infty$. But this construction is obvious.

Now let

$$k(\theta) := \begin{cases} G_k(\theta) & \text{if } \theta \in (a_k, b_k) \\ 0 & \text{otherwise} \end{cases}.$$

The hypotheses of Lemma 2.1. are fulfilled. Therefore

$$G(z) := \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log k(e^{it})\, dt\right\}$$

is an outer function in $A(\bar{D})$ with $|G(e^{it})| = k(e^{it}) \geqslant \sum_{i=1}^{n} |f_i(e^{it})|$.

Theorem 2.4. *If the functions* $f_1, \ldots, f_n \in A(\bar{D})$ *have a GCD* d, *then there exist functions* $g_1, \ldots, g_n \in A(\bar{D})$ *such that*

$$d = \sum_{i=1}^{n} f_i g_i.$$

Proof. From the hypothesis it follows that the functions $h_i := f_i/d$ have the GCD 1. Lemma 2.3 gives $\sum_{i=1}^{n} |h_i| > 0$ in $\bar{D}$. By theorem 1.1 we have $1 \in (h_1, \ldots, h_n)$ and this implies $d \in (f_1, \ldots, f_n)$.

Remark. The theorem above is a generalization of the corona theorem for $A(\bar{D})$. The analogous generalization of the corona theorem for H^∞ is not true. Also this is false for many subalgebras of $A(\bar{D})$, e. g. for $\Lambda_\alpha := \{f \in A(\bar{D}) : f \in \mathrm{Lip}_\alpha(\partial D)\}$.

Corollary 2.5. The ideal $(f_1, \ldots, f_n)$ *is a principal ideal in* $A(\bar{D})$ *if and only if the functions* $f_1, \ldots, f_n$ *have a GCD in* $A(\bar{D})$.

Corollary 2.6. If the functions $f_1, \ldots, f_n \in A(\bar{D})$ *have a GCD* d, *then there exists a bounded holomorphic function* h *on* D, *which is* *bounded away from zero, such that $d = hBSH$ with inner part BS and*

$$H(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log \sum_{i=1}^{n} |f_i(e^{it})|\, dt\right\}.$$

Proof. By theorem 2.4 it follows $(d) = (f_1, \ldots, f_n)$, therefore there exist two positive constants c, C such that

$$c \sum_{i=1}^{n} |f_i(z)| \leqslant |d(z)| \leqslant C \sum_{i=1}^{n} |f_i(z)| \text{ for all } z \in \bar{D}.$$

H is an outer function in H^∞, since $f_i \in A(\bar{D})$ and

$$\int_0^{2\pi} \log \sum_{i=1}^{n} |f_i(e^{it})|\, dt > -\infty.$$

From the inequality above it follows

$$c\,|H(e^{it})| \leqslant |d(e^{it})| \leqslant C\,|H(e^{it})| \text{ a. e. on } \partial D.$$

Thus Lemma 2.2 implies for the outer part F of d

$$c\,|H(z)| \leqslant |F(z)| \leqslant C\,|H(z)|.$$

F/H is a unit in H^∞ means that there exists a function $h \in H^\infty$ invertible in H^∞ with $F = hH$. So we get $d = BSF = hBSH$.

Now we will give a necessary condition such that the functions $f_1, \ldots, f_n \in A(\bar{D})$ have a GCD. If f is a complex valued function on $\bar{D}$ (resp. D), we denote by $Z(f)$ the zeros of f in $\bar{D}$ (resp. D). In the later case $\overline{Z(f)}$ denotes the closure of $Z(f)$ in $\bar{D}$. For $f_i, f \in A(\bar{D})$ μ_i, μ denote the positive singular measures of the singular inner parts of f_i, f and $\operatorname{supp}(\mu_i)$, $\operatorname{supp}(\mu)$ the closed supports of these measures.

Proposition 2.7. *If the functions $f_1, \ldots, f_n \in A(\bar{D})$ have a GCD d, then for every $\zeta \in \partial D \cap Z(\sum_{i=1}^{n} |f_i|)$ and for every $i \in \{1, \ldots, n\}$ with $\zeta \in \overline{Z(B_i/B)} \cap \partial D$ or $\zeta \in \operatorname{supp}(\mu_i - \mu)$ it follows that*

$$|f_i(e^{it})| = o\left(\sum_{i=1}^{n} |f_i(e^{it})|\right) \quad \textit{for} \quad e^{it} \to \zeta.$$

Proof. Let $f_i = B_i S_i F_i$, $d = \mathrm{BSF}$ be the canonical factorizations as in the proof of Lemma 2.3. Since d is a divisor of every f_i, it follows that every F_i/F is in $A(\bar{D})$ and must be zero at such points where B_i/B or S_i/S have a singularity (on ∂D). This is the case, exactly when $\zeta \in \overline{Z(B_i/B)} \cap \partial D$ or $\zeta \in \operatorname{supp}(\mu_i - \mu)$. Hence $|(F_i/F)(e^{it})| = o(1)$ for $e^{it} \to \zeta$. By the above corollary it follows that

$$|f_i(e^{it})| = o\Big(\sum_{i=1}^{n} |f_i(e^{it})|\Big) \text{ for } e^{it} \to \zeta.$$

If the functions $f_1, \ldots, f_n \in A(\bar{D})$ have no common zero on ∂D, then there exists a GCD d and it is easy to see that d is the finite Blaschke product formed by the common zeros of $f_1, \ldots, f_n$ in the open unit disc D.

References

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