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Connections Between Three-Dimensional Inverse Scattering and Linear Least-Squares Estimation of Random Fields

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Abstract. The three-dimensional Schrödinger equation inverse scattering problem with a nonspherically-symmetric potential is related to the filtering problem of computing the linear least-squares estimate of the three-dimensional random field on the surface of a sphere from noisy observations inside the sphere. The relation consists of associating an estimation problem with the inverse scattering problem, and vice-versa. This association allows equations and quantities for one problem to be given interpretations in terms of the other problem. A new fast algorithm is obtained for the estimation of random fields using this association. The present work is an extension of the connections between estimation and inverse scattering already known to exist for stationary random processes and one-dimensional scattering potentials, and isotropic random fields and radial scattering potentials.

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1. Introduction

The objective of this paper is to demonstrate that the problems of inverse scattering and linear least-squares estimation are closely connected. This connection is well established for one-dimensional inverse scattering problems and stationary random processes [1–4]. A connection between two-dimensional inverse scattering problems with a radial scattering potential (i.e., one which varies only with radial distance from the origin) and isotropic random field estimation, has been noted in [5] and [24]. The present paper extends this connection to the more general case of three-dimensional inverse scattering with a nonspherically symmetric potential, and three-dimensional random field estimation.

The significance of such a connection is three-fold. First, it sets up a cross-fertilization between techniques known for these problems, so that methods applied to one problem can now be applied to the other. Second, it enhances the understanding of each problem, since equations in one domain can be given

interpretations in the other domain. Several examples of this are given in Section 4. This is particularly important for the general three-dimensional inverse scattering problem, which is still not well understood. Finally, it allows a new fast algorithm for the estimation of random fields to be developed from existing fast algorithms [6, 7] for three-dimensional inverse scattering. This is done in Section 5.

The specifics of the connection are as follows. The inverse scattering problem for the three-dimensional Schrödinger equation with a nonspherically symmetric potential has been treated in [8–11] and [19–20]. The solution procedure consists of solving either a generalized Gel'fand–Levitan integral equation (Equation (2.17) below) or a generalized Marchenko integral equation (Equation (2.11) below) for the scattered field, and then recovering the potential from the scattered field using a so-called *miracle* equation [9] (Equation (2.13) below). We show that the generalized Gel'fand–Levitan equation is equivalent to a three-dimensional Wiener–Hopf integral equation, which computes the linear least-squares filter for estimating a three-dimensional random field on the surface of a sphere, from noisy observations inside and on the sphere.

Conversely, starting with this estimation problem, with the covariance function of the random field required to have special structure (Equation (3.4) below), we associate with it a Schrödinger equation. The potential for this equation may be *nonlocal*; if the covariance has additional structure, the potential is local, and an inverse scattering problem can be associated with the estimation problem. It should be noted that the characterization of admissible scattering data associated with a local potential (i.e., for which the miracle equation holds) is still an open problem in inverse scattering theory. The solution to this problem may well be found by considering it in an estimation context.

The paper is organized as follows. In Section 2, some three-dimensional inverse scattering theory is briefly reviewed, and a random field least-squares estimation problem is associated with the generalized Gel'fand–Levitan procedure of [8] and [19–20]. Section 3 begins with the random field least-squares estimation problem, in which the covariance function has structure, and associates with it a Schrödinger equation with a possibly nonlocal potential; if the covariance has some additional structure, the potential is local. In this case, an inverse scattering problem is associated with the estimation problem. Section 4 provides interpretations of quantities and equations, including generalized Krein functions and the regular solution, whitening filters and Jost function, prewhitening and far-field behavior, and matched filtering and the inverse Born approximation. Each of these connects an estimation concept with an inverse scattering concept. Section 5 develops a new fast algorithm for computing the least-squares filter for estimating a random field, using existing fast inverse scattering algorithms. Section 6 concludes by summarizing the paper and noting directions for future research.

2. Three-Dimensional Inverse Scattering

In this section, we briefly review some pertinent facts and equations of inverse scattering theory for the Schrödinger equation in three dimensions with a nonspherically symmetric potential. Most of this material first appeared in [8]; it is also covered in [7, 10, 11, 19, 20]. We then associate an estimation problem with this inverse scattering problem.

2.1. REVIEW OF THREE-DIMENSIONAL INVERSE SCATTERING THEORY

The following inverse scattering problem is considered. The wave field $u(x, k)$ satisfies the Schrödinger equation

$$(\Delta + k^2 - V(x))u(x, k) = 0, \tag{2.1}$$

where $x \in R^3$ and the scattering potential $V(x)$ is real-valued, smooth, and has compact support. It is also assumed that $V(x)$ does not induce bound states; a sufficient condition for this is for $V(x)$ to be nonnegative. Two different sets of boundary conditions are specified, resulting in two different solutions.

Scattering Solution. The *scattering* solution $\psi(x, k, e_i)$ has the boundary condition

$$\psi(x, k, e_i) = e^{-ike_i \cdot x} + (e^{-ik|x|}/4\pi|x|)A(k, e_s, e_i) + O(|x|^{-2}) \quad \text{as } |x| \to \infty \tag{2.2}$$

where the *scattering amplitude* $A(k, e_s, e_i)$ is defined as

$$A(k, e_s, e_i) = -\int e^{ike_s \cdot y} V(y)\psi(y, k, e_i)\, dy \tag{2.3}$$

and e_i and e_s are unit vectors. In the time domain, this corresponds to an incident impulsive plane wave in the direction e_i being used to probe the potential, and being scattered in all directions. The scattering amplitude specifies the far-field behavior of the wave field, and constitutes the scattering data.

Regular Solution. The *regular* solution $\phi(x, k, e_i)$ is defined as being the solution to (2.1) that is an entire analytic function of k and is of exponential order $|x|$. It should be noted that, subject to mild assumptions [20], this solution generically exists and $\phi(x, k, e_i) - e^{-ike_i \cdot x}$ is square-integrable in k. Using the Paley–Wiener theorem, the inverse Fourier transform $\check{\phi}(x, t, e_i) = \mathcal{F}^{-1}\{\phi(x, k, e_i)\}$ has support in t on the interval $[-|x|, |x|]$. Thus, it has the Povsner–Levitan representation

$$\phi(x, k, e_i) = e^{-ike_i \cdot x} - \int_{-|x|}^{|x|} m(x, t, e_i)\, e^{-ikt}\, dt \tag{2.4}$$

so that $m(x, t, e_i)$ is the nonimpulsive part of $\check{\phi}(x, t, e_i)$.

Jost Operator. The regular and scattering solutions are related by a *Jost operator* $J(k)$. This is an operator on the space $L^2(S^2)$ (S^2 is the unit sphere) with kernel $J(k, e_1, e_2)$. Specifically,

$$\phi(x, k, e_i) = \int_{S^2} \psi(x, k, e_s) J(k, e_s, e_i)\, de_s, \tag{2.5a}$$

$$\psi(x, k, e_i) = \int_{S^2} \phi(x, k, e_s) J^{-1}(k, e_s, e_i)\, de_s, \tag{2.5b}$$

where J^{-1} is the inverse Jost operator. Both $J(k)$ and $J^{-1}(k)$ are analytic in the lower half-plane, which corresponds to *causality* in the time domain.

From [8], the Jost operator satisfies

$$J(-k) = QS(k)J(k)Q \tag{2.6}$$

where $S(k)$ is the scattering operator with kernel

$$S(k, e_1, e_2) = \delta(e_1 - e_2) + (k/2\pi i)A(k, e_1, e_2) \tag{2.7}$$

and Q is the operator such that $QA(k, e_1, e_2) = A(k, -e_1, e_2)$. The one-dimensional versions of (2.6) and (2.7) may be found in [12]. For the one-dimensional problem, the scattering and Jost operators become the scattering and Jost matrices, relating scattering solutions for e_i being the $\pm x$ directions.

Orthonormality. It is well known [8, 11] that in the absence of bound states the scattering solutions $\{\psi(x, k, e_i)\}$ from a complete set. Thus they are orthonormal, in that

$$(2\pi)^{-3} \int_0^\infty \int_{S^2} \psi(x, k, e) \psi^*(y, k, e) k^2\, de\, dk = \delta(x - y). \tag{2.8}$$

It follows that the solutions $\{\phi(x, k, e_i)\}$ are orthonormal with respect to the positive definite *spectral function* $(J^H J)^{-1}$, i.e.,

$$(2\pi)^{-3} \int_0^\infty \int_{S^2} \int_{S^2} \int_{S^2} \phi(x, k, e_1) J^{-1}(k, e_1, e_3) J^{-1}(k, e_3, e_2)^* \times$$
$$\times \phi^*(y, k, e_2) k^2\, de_1\, de_2\, de_3\, dk = \delta(x - y). \tag{2.9}$$

The orthonormality of the solutions $\{\phi(x, k, e_i)\}$ with respect to $(J^H J)^{-1}$ was used in [10] to interpret three-dimensional inverse scattering as an orthogonalization procedure. It will be shown to have an interpretation as the orthonormality of the innovations in an estimation problem.

2.2. INTEGRAL EQUATIONS

Generalized Marchenko Procedure. The inverse scattering problem is solved as follows. Given far-field scattering data in the form of the scattering amplitude

$A(k, e_s, e_i)$ compute

$$G(t, e_s, e_i, x) = -(2\pi)^{-2} \int_{-\infty}^{\infty} e^{ik(t-(e_s-e_i)\cdot x)} ikA(k, e_s, e_i) \tag{2.10}$$

and then solve the *generalized Marchenko integral equation*

$$v(x, t, e_i) = \int_{S^2} G(t, e_s, e_i, x)\, de_s + \int_0^{\infty} \int_{S^2} G(t+\tau, e', e_i, x) v(x, t, -e')\, de'\, d\tau \tag{2.11}$$

for the delayed scattered field

$$v(x, t, e_i) = \check{\psi}(x, t - e_i \cdot x, e_i) - \delta(t - e_i \cdot x). \tag{2.12}$$

The scattering potential $V(x)$ is then recovered from the *delayed scattered field* $v(x, t, e_i)$ using the *miracle* equation [8–11]

$$V(x) = 2e_i \cdot \nabla v(x, t = 0, e_i). \tag{2.13}$$

Note that the right side of (2.13) must be independent of the directon of incidence e_i. This is the 'miracle' [8], and it imposes a constraint on possible $\psi(x, k, e)$ and, hence, on possible scattering amplitudes $A(k, e_s, e_i)$. Since the scattering data $A(k, e_s, e_i)$ has five degrees of freedom and the potential $V(x)$ generating it has only three degrees of freedom, this constraint is not surprising. There is no known simple test to determine which functions $A(k, e_s, e_i)$ result in a miraculous solution $\psi(x, k, e_i)$.

Generalized Gel'fand–Levitan Procedure. As an alternative to the above procedure we may do the following. First, obtain the inverse Jost operator kernel $J^{-1}(k, e_1, e_2)$ from the scattering data $A(k, e_s, e_i)$ by solving the integral equation

$$L(t, e_s, e_i) = G(t, -e_s, e_i, 0) + \int_0^{\infty} \int_{S^2} L(\tau, -e_s, e') G(t+e', e_i, 0)\, de'\, d\tau, \tag{2.14}$$

where $G(t, e_1, e_2, x)$ is defined in (2.10). Although (2.14) looks like (2.11), note that (2.14) need only be solved for $x = 0$, whereas (2.11) must be solved for all x. We then have

$$J^{-1}(k, e_1, e_2) = 1 + \int_0^{\infty} L(t, e_1, e_2)\, e^{-ikt}\, dt. \tag{2.15}$$

Equations (2.14) and (2.15) implement (2.6) with the constraint that $J^{-1}(k)$ be causal in the time domain.

Following [20], let

$$M_{|x|}(t, e_1, e_2) = \mathcal{F}^{-1}\{[(J^H J)^{-1} - \delta(e_1 - e_2)]\Pi_{|x|}(t, k, e_1 - e_2)\} \tag{2.16}$$

be the inverse spatial Fourier transform of the perturbation of the spectral

function away from its free-space representation. Here $\Pi_{|x|}(t, k, e_1 - e_2)$ is a spatial filter in $k(e_1 - e_2)$ that rejects $|x| < t$. Equation (2.16) is equivalent to Equation (6.11) of [20], which involves a *partial Radon transform* [20]. Then the nonimpulsive part $m(x, t, e_i)$ of the *regular* solution $\check{\phi}(x, t, e_i)$ may be obtained by solving the *generalized Gel'fand–Levitan integral equation*

$$m(x, t, e_i) = \int_{S^2} M_{|x|}(t + e_s \cdot x, e_s, e_i)\, de_s - \int_{S^2} \int_{-|x|}^{|x|} m(x, \tau, e_s) M_{|x|}(t + \tau, e_s, e_i)\, d\tau\, de_s. \tag{2.17}$$

An incorrect generalized Gel'fand–Levitan equation was first given in [8]; the corrected equation (2.17) was given in [20].

The potential is then recovered from the miracle equation for the regular solution

$$V(x) = 2e_i \cdot \nabla[m(x, t = e_i \cdot x^+, e_i) - m(x, t = e_i \cdot x^-, e_i)]. \tag{2.18}$$

Note that since the regular solution has support in t in the interval $[-|x|, |x|]$, the potential $V(x)$ is found from the *jump* in the gradient of the regular solution at $t = e_i \cdot x$.

Although the generalized Gel'fand–Levitan procedure requires that the integral equation (2.14) be solved in addition to (2.17), it should be noted that (2.17) has only a finite range of integration, while the generalized Marchenko equation (2.11) has an infinite range of integration. Simpler derivations of the generalized Marchenko equation are given in [7] and [11], using a generalized Radon transform and a representation theorem, respectively. A simpler derivation of the generalized Gel'fand–Levitan equation is given in [10], using an orthogonalization argument.

2.3. CONNECTION WITH ESTIMATION OF RANDOM FIELDS

We now show that the solution of an inverse scattering problem using the generalized Gel'fand–Levitan procedure is equivalent to solving a random field linear least-squares estimation problem. This is accomplished by transforming the integral equation (2.17) into a three-dimensional Wiener–Hopf integral equation, with a positive definite covariance function. Since both equations are expressions of *orthogonality*, this transformation is not surprising.

The mechanism for the transformation is the *Radon transform*

$$\mathcal{R}\{f(x)\} = \hat{f}(\tau, e) = \int_{R^3} f(x)\delta(\tau - e \cdot x)\, dx, \tag{2.19a}$$

$$\mathcal{R}^{-1}\{\hat{f}(\tau, e)\} = f(x) = \frac{-1}{8\pi^2} \int_{S^2} \int_0^\infty \frac{\partial^2 \hat{f}(\tau, e)}{\partial \tau^2} \delta(\tau - e \cdot x)\, d\tau\, de. \tag{2.19b}$$

A good treatment of the Radon transform is [13].

Define

$$-h(x, y) = \mathcal{F}^{-1}_{ke_i \to y}\{\phi(x, k, e_i) - e^{-ike_i \cdot x}\} = \mathcal{R}^{-1}_{te_i \to y}\{m(x, t, e_i)\} \tag{2.20}$$

where the projection-slice property $\mathcal{F}_x = \mathcal{F}_t \mathcal{R}$ [13] of the Radon transform has been used in the second equality. Since $m(x, t, e_i)$ has support in t on the interval $[-|x|, |x|]$, $h(x, y)$ has support inside the sphere $|y| \leq |x|$, as long as $m(x, t, e_i)$ satisfies some mild conditions [13]. (In fact, the triangularity of $h(x, y)$ has been established generically in [20].) Note that we also have

$$\phi(x, k, e_i) = e^{-ike_i \cdot x} - \int h(x, y)\, e^{-ike_i \cdot y}\, dy = \mathcal{F}_{y \to ke_i}\{\delta(x - y) - h(x, y)\}. \tag{2.21}$$

Taking the partial inverse Radon transform ([20], p. 601) of the generalized Gel'fand–Levitan Equation (2.17) and using the projection-slice property results in [20]

$$k(x, y) = h(x, y) + \int_{|z| \leq |x|} h(x, z) k(z, y)\, dz, \quad |y| \leq |x|. \tag{2.22}$$

Here

$$k(x, y) = (2\pi)^{-3} \int_0^\infty \int_{S^2} \int_{S^2} M(k, e_1, e_2)\, e^{-ik(e_1 \cdot x - e_2 \cdot y)} k^2\, de_1\, de_2\, dk, \tag{2.23}$$

where $M(k, e_1, e_2) = ((J^H J)^{-1} - \delta(e_1 - e_2))(k, e_1, e_2)$ is the kernel of the perturbation of the spectral function away from its free-space value.

Equation (2.22) also appears in [8] and [10] in the course of deriving generalized Gel'fand–Levitan integral equations. However, these latter equations require that the regular solution $\check{\phi}(x, t, e_i)$ have support in t in the interval $[-e_i \cdot x, e_i \cdot x]$, and the existence of such a solution cannot be guaranteed. Also, (2.23) is a much cleaner version of the Gel'fand–Levitan equation, since it avoids the partial Radon transform used in [20]. Here we recognize it as a *three-dimensional Wiener–Hopf equation* for computing the linear least-squares filter $h(x, y)$ that estimates a random field from noisy observations inside and on the sphere $|y| \leq |x|$.

The Filtering Problem. More specifically, let

$$w(x) = z(x) + v(x) \tag{2.24}$$

with $x \in R^3$ be some noisy observations of a zero-mean real-valued random field $z(x)$ having covariance

$$E[z(x)z(y)] = k(x, y). \tag{2.25}$$

The noise field $v(x)$ is a zero-mean white noise field with unit intensity

$$E[v(x)v(y)] = \delta(x - y) \tag{2.26}$$

and it is uncorrelated with the field $z(x)$. Let $h(x, y)$ be the linear least-squares filter for estimating $z(x)$ on the surface of a sphere of radius $|x|$ from observations $\{w(y): |y| \leq |x|\}$ inside and on the sphere, so that

$$\hat{z}(x) = \int_{|y| \leq |x|} h(x, y) w(y) \, dy. \tag{2.27}$$

Then, by the orthogonality principle of Wiener filtering, the filter $h(x, y)$ satisfies the Wiener–Hopf equation (2.22).

It has been demonstrated that the construction of the regular solution $\phi(x, k, e_i)$ from the inverse Jost function $J^{-1}(k, e_1, e_2)$ is the same problem as that of constructing the optimal filter $h(x, y)$ from the covariance function $k(x, y)$. The association is given explicitly by Equation (2.20), relating $\phi(x, k, e_i)$ and $h(x, y)$, and Equation (2.23) relating $J^{-1}(k, e_1, e_2)$ and $k(x, y)$. A similar association for the one-dimensional case was given in [12]. The only difference is that in the estimation problem the filter $h(x, y)$ is the desired quantity, while in the inverse scattering problem the regular solution $\phi(x, k, e_i)$ is only an intermediate quantity for computing the potential $V(x)$. In Section 5 it will be shown that the potential $V(x)$ completely characterizes the filter $h(x, y)$, just as the reflection coefficients in the Levinson algorithm in one dimension completely characterize the forward and backward prediction filters.

Structure of the Covariance Function. Note from (2.23) that

$$\begin{aligned} E[w(x)w(y)] &= \delta(x - y) + k(x, y) \\ &= \mathcal{F}^{-1}_{k_1 e_1 \to x} \mathcal{F}^{-1}_{k_2 e_2 \to y} \{(J^H J)^{-1}(k_1, e_1, e_2) \delta(|k_1|^2 - |k_2|^2). \end{aligned} \tag{2.28}$$

This shows that $\delta(x - y) + k(x, y)$ is indeed a positive definite function, as it must be for the estimation problem to be well defined. However, it also shows that the 2-dimensional Fourier transform of $k(x, y)$ is zero except for the *on-shell* values for which $|k_1| = |k_2|$. This structure in the wavenumber domain implies that the covariance function $k(x, y)$ has *structure*, in that

$$(\Delta_x - \Delta_y) k(x, y) = 0, \tag{2.29}$$

where Δ_x is the Laplacian with respect to x. This property is a direct generalization of the *displacement properties*

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) k(x, y) = 0, \tag{2.30a}$$

$$\left\{ \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) - \left(\frac{\partial^2}{\partial s^2} + \frac{1}{s} \frac{\partial}{\partial s} \right) \right\} k(r, s) = 0, \tag{2.30b}$$

which were imposed in [4] and [5], respectively, on the covariance functions of one-dimensional and isotropic random fields, respectively. Both of these properties resulted in an inverse scattering interpretation being associated with an

estimation problem in [4, 5, 24]. The implications of the structure (2.29) are discussed in Section 3 below.

3. Three-Dimensional Random Field Estimation

In this section, we go the other way. Starting with a random field estimation problem, we associate with this problem an inverse scattering problem. However, the potential associated with this problem may be *nonlocal*. The problem of determining necessary and sufficient conditions for the covariance function $k(x, y)$ to guarantee a local potential is related to the problem of determining admissible scattering data for a local potential, and is still unsolved. The miracle Equation (2.18) links these two problems.

3.1. THE FILTERING PROBLEM

The estimation problem considered is a *filtering* problem of computing the linear least-squares estimate of a zero-mean real-valued random field $z(x)$ on the surface of a sphere of radius $|x|$, from noisy observations

$$w(x) = z(x) + v(x) \tag{3.1}$$

inside and on the sphere. As before, $v(x)$ is a zero-mean real-valued white noise field with covariance

$$E[v(x)v(y)] = \delta(x - y) \tag{3.2}$$

while $z(x)$ is uncorrelated with $v(x)$ and has covariance function

$$E[z(x)z(y)] = k(x, y) \tag{3.3}$$

where the function $k(x, y)$ is positive definite and has the *generalized displacement property*

$$(\Delta_x - \Delta_y)k(x, y) = 0. \tag{3.4}$$

The structure of $k(x, y)$ implied by (3.4) reduces the number of degrees of freedom in the function $k(x, y)$ from six to five. This is still a far more general case than the case of a *homogeneous* random field having covariance $k(x - y)$ (three degrees of freedom) treated in [14], or the case of an *isotropic* random field having covariance $k(|x - y|)$ (one degree of freedom) treated in [5]. Note that both homogeneous and isotropic random fields are included as special cases of the property (3.4).

The estimate $\hat{z}(x)$ of $z(x)$ has the form

$$\hat{z}(x) = \int_{|y| \leq |x|} h(x, y)w(y)\,dy. \tag{3.5}$$

By the orthogonality principle the filter $h(x, y)$ is determined by the three-

dimensional Wiener–Hopf integral equation

$$k(x, y) = h(x, y) + \int_{|z| \leq |x|} h(x, z) k(z, y)\, \mathrm{d}z, \quad |y| \leq |x|. \tag{3.6}$$

Without loss of generality, we define $h(x, y) = 0$ for $|y| > |x|$.

Applying the operator $(\Delta_x - \Delta_y)$ to the integral Equation (3.6) and using the generalized displacement property (3.4), Green's theorem, and the unicity of solution to (3.6) when $k(x, y)$ is positive definite yields, after some algebra (see [14]),

$$(\Delta_x - \Delta_y) h(x, y) = \int_{S^2} V_f(x, e) h(|x|e, y)\, \mathrm{d}e, \tag{3.7}$$

where the *nonlocal filter potential* $V_f(x, e)$ is defined as

$$V_f(x, e) = -\frac{2}{|x|^2} \frac{\mathrm{d}}{\mathrm{d}|x|} |x|^2 h(x, |x|e). \tag{3.8}$$

3.2. CONNECTION WITH INVERSE SCATTERING

Define

$$m(x, t, e_i) = -\mathcal{R}_{y \to t, e_i}\{h(x, y)\}. \tag{3.9}$$

Then a Radon transform of (3.7) taking y into t and e_i yields

$$\left(\Delta - \frac{\partial^2}{\partial t^2}\right) m(x, t, e_i) = \int_{S^2} V_f(x, e) m(|x|e, t, e_i)\, \mathrm{d}e \tag{3.10}$$

which is the inverse Fourier transform of a Schrödinger equation with a *nonlocal potential* $V_f(x, e)$. Note that the interaction at x is determined by the field $m(y, t, e_i)$ at all $\{|y| = |x|\}$, i.e., on the surface of the sphere of radius $|x|$. The inverse scattering problem for this particular type of non-local potential is considered in [21–23].

It is not surprising that the potential is nonlocal. Since $k(x, y)$ has five degrees of freedom, the set of potentials that characterize it must also have five degrees of freedom. A local potential $V_f(x)$ only has three degrees of freedom, whereas the nonlocal potential $V_f(x, e)$ has five, as required. Also note that nonlocal potentials do not arise in the one-dimensional and isotropic cases considered in [4] and [5], since in these cases the distribution over extreme points (endpoints of an interval, or surface of a sphere) cannot arise due to symmetry. For the homogeneous case considered in [14], this symmetry no longer holds, and the potential is nonlocal.

The structure (3.4) of $k(x, y)$ implies that its double Fourier transform is zero except for its on-shell values. More specifically,

$$\mathcal{F}_{x\to k_1 e_1}\mathcal{F}_{y\to k_2 e_2}\{\delta(x-y)+k(x,y)\} = M(k, e_1, e_2)\delta(|k_1|^2-|k_2|^2) \tag{3.11}$$

for some function $M(k, e_1, e_2)$. Equation (3.11) should be compared to (2.23).

The covariance function $\delta(x-y)+k(x,y)$ is positive definite. We assume that its spectral density $M(k, e_1, e_2)$ can be spectrally factored into

$$M(k, e_1, e_2) = \int_{S^2} J^{-1}(k, e_1, e_3)J^{-1}(k, e_2, e_3)^* \, de_3, \tag{3.12}$$

where $J^{-1}(k, e_1, e_2)$ is analytic in k in the lower half-plane and has the form (2.15). It is evident that $J^{-1}(k, e_1, e_2)$ plays the same role in (3.12) as the kernel of the inverse Jost operator played in (2.16).

The association of an inverse scattering problem with the random field estimation problem specified by Equations (3.1)–(3.6) proceeds as follows. Given the covariance function $k(x, y)$ satisfying (3.4), compute its double Fourier transform (3.11) and perform the spectral factorization (3.12), yielding the inverse Jost operator $J^{-1}(k)$. Then use Equation (2.6) (modified for $J^{-1}(k)$ instead of $J(k)$) to synthesize a scattering amplitude $A(k, e_s, e_i)$ using (2.7). The smooth part $m(x, t, e_i)$ of the regular solution to this inverse scattering problem is minus the Radon transform of the filter $h(x, y)$.

The major difference between the association made here and that made in Section 2 is that in Section 2 we started with an inverse scattering problem with a local potential $V(x)$ and associated with it an estimation problem in which the covariance function $k(x, y)$ was derived from the scattering data, so that $k(x, y)$ was restricted to functions that, in addition to satisfying (2.29), gave rise to local potentials only. In Section 3, this entire process is reversed. This shows the equivalence of the inverse scattering problem with a local potential with estimation problems in which the covariance, in addition to satisfying (3.4), has additional structure resulting in a local filtering potential $V_f(x)$. This additional structure seems to be related to the structure required of the scattering amplitude in order to give rise to a local scattering potential.

4. Innovations Processes and Inverse Scattering

In this section details of the connection between inverse scattering and estimation are explored by attaching estimation interpretations to inverse scattering quantities and equations, and vice-versa.

4.1. GENERALIZED KREIN FUNCTIONS AND THE REGULAR SOLUTION

Recall the relation (2.21) between the regular solution and the optimal filter:

$$\phi(x, k, e_i) = \mathcal{F}_{y\to ke_i}\{\delta(x-y)-h(x,y)\}. \tag{4.1}$$

Note that the filter $\delta(x-y)-h(x,y)$ applied to the data $\{w(y): |y| \leq |x|\}$ results in

a white noise field $\{i(x)\}$:

$$i(x) = \int_{|y| \leq |x|} (\delta(x-y) - h(x, y)) w(y)\, dy. \tag{4.2}$$

The field $\{i(x)\}$ is white since all of the information in the data $\{w(y): |y| \leq |x|\}$ relevant to estimating $z(x)$ has been subtracted out of it. The field $\{i(x)\}$ is also causal and causally invertible, where causality is defined in increasing $|x|$, with equal ordering of all points on the surface of a sphere of any radius. The white field $\{i(x)\}$ can be recursively undone back into the observations in decreasing $|x|$ (this would require the solution of simultaneous equations over the surface of the sphere, which could be done in principle). Hence, $\{i(x)\}$ is an innovations field.

This shows that the regular solution is the Fourier transform of the *innovations filter*. Thus the regular solution is a direct generalization of the *Krein functions* [16], which are the Fourier transforms of the innovation filters for the one-dimensional problem [1]. The only difference is that in the present problem the Fourier transform is three-dimensional.

In the two-dimensional isotropic case treated in [5], the solutions $\gamma(r, \lambda)$ to the associated radial inverse scattering problem were the Hankel transforms of the innovations filters. The Hankel transform appeared in [5] since the two-dimensional Fourier transform of an isotropic function becomes a Hankel transform. The appearance of extra transforms is not surprising; in the one-dimensional *discrete* problem the Szego polynomials are the z-transforms of the innovations filters, while in the present problem the regular solution in the time domain is the Radon transform of the innovations filter (see (3.17)). Indeed, the Hankel and Radon transforms both have the interpretation of performing a decomposition into plane waves, for an evident scattering interpretation.

A *Kolmogorov isometry* can also be identified between linear combinations of the observations $\{w(x)\}$ and of the functions $e^{-ike_i \cdot x}$. The inner product for the random processes is the usual expectation-of-product, while the inner product for the functions is with respect to the spectral function $(J^H J)^{-1}(k, e_1, e_2)$. Using this isometry, the regular solutions $\phi(x, k, e_i)$ can be identified with the innovations field $\{i(x)\}$ using (4.1). It is then evident that the orthonormality (2.9) of the regular solutions $\phi(x, k, e_i)$ with respect to the spectral function $(J^H J)^{-1}(k, e_1, e_2)$ is equivalent to a statement of orthonormality of the innovations fields $\{i(x)\}$ of the associated estimation problem. A similar result was noted for the one-dimensional problem in [4], and for the two-dimensional isotropic problem in [5].

4.2. WHITENING AND MODELLING FILTERS AND JOST OPERATORS

Since $(J^H J)^{-1}(k, e_1, e_2)$ is the spectral density of the observations $\{w(x)\}$ (see (3.11) and (3.12)), the Jost operator $J(k)$ can be interpreted as a *whitening filter* for $\{w(x)\}$, and the inverse Jost operator $J^{-1}(k)$ can be interpreted as a *modelling*

filter for $\{w(x)\}$. This follows since the Jost operator kernel $J(k, e_1, e_2)$ and the inverse Jost operator kernel $J^{-1}(k, e_1, e_2)$ are both analytic in the lower half-plane, so that the computation of the kernels from the covariance function $k(x, y)$ is equivalent to solving a *spectral factorization* problem. Specifically, we associate with $\{w(x)\}$ the *whitened* random field $\{\nu(x)\}$ obtained from $\{w(x)\}$ using the invertible (but *not* causal) transformation with $J^{-1}(k, e_1, e_2)$ as its kernel.

More specifically, let $\{\hat{w}(k, e)\}$ and $\{\check{w}(\tau, e)\}$ be the formal Fourier and Radon transforms of $\{w(x)\}$, and define the whitened field $\{\hat{\nu}(k, e)\}$ defined from $\{\hat{w}(k, e)\}$ by

$$\hat{\nu}(k, e_1) = \int_{S^2} J(k, e_1, e_2)\hat{w}(k, e_2)\,\mathrm{d}e_2. \tag{4.3}$$

In the Radon transform domain (4.3) corresponds to passing $\{\check{w}(\tau, e)\}$ through a filter

$$\check{\nu}(\tau_2, e_2) = \int_{S^2}\int J(\tau_1 - \tau_2, e_1, e_2)\check{w}(\tau_1, e_1)\,\mathrm{d}\tau_1\,\mathrm{d}e_1 \tag{4.4}$$

where

$$\check{J}(\tau_1 - \tau_2, e_1, e_2) = \mathscr{F}^{-1}\mathscr{F}^{-1}\{k_1^2 J(k, e_1, e_2)\delta(|k_1|^2 - |k_2|^2)\} \tag{4.5}$$

and Parseval's theorem has been used twice. Note that $\check{J}(\tau_1 - \tau_2, e_1, e_2)$ is zero for $\tau_1 - \tau_2 < 0$ by the analyticity of $J(k, e_1, e_2)$ in the lower half-plane. This shows that the whitening takes place in increasing $|\tau|$. However, the double inverse Radon transform $\hat{J}(x, y) = \mathscr{R}^{-1}\mathscr{R}^{-1}\{J(\tau_1 - \tau_2, e_1, e_2)\}$ is *not* causal in $|x| - |y|$, so the transformation (4.3) is not causal. Thus $J(k, e_1, e_2)$ is *not* an innovations filter, although it is a whitening filter.

The scattering solution $\psi(x, k, e_i)$ to the inverse scattering problem can now be given an estimation interpretation. Recall that the regular solution $\phi(x, k, e_i)$ is the Fourier transform of the innovations filter. Equations (2.5) relating $\psi(x, k, e_i)$ and $\phi(x, k, e_i)$ can now be interpreted as cascades of two filters. Specifically,

$$\psi(x, k, e_2) = \int_{S^2} \phi(x, k, e_1)J^{-1}(k, e_1, e_2)\,\mathrm{d}e_1 \tag{2.5b}$$

can now be interpreted as the cascade of a modelling filter $J^{-1}(k)$ and an innovations filter $\phi(x, k, e_1)$, where both filters are of course specified in the wavenumber domain. This is illustrated in Figure 1a.

Figure 1 and (2.5b) show that the action of $\psi(x, k, e_1)$ is to transform the whitened observations field $\{\hat{\nu}(k, e)\}$ into the innovations field $\{\hat{\imath}(k, e)\}$. Of course, both fields are white, so that the filter $\psi(x, k, e_1)$ should not affect power spectral densities; this interprets the orthonormality (2.8) of the $\psi(x, k, e_1)$. Also, if we start with the observations field $\{w(x)\}$, whiten it with the filter $J(k, e_1, e_2)$, and then process it with the filter $\psi(x, k, e_1)$, the end result should be the innovations field $\{i(x)\}$. The overall effect of the cascade of $J(k, e_1, e_2)$ and

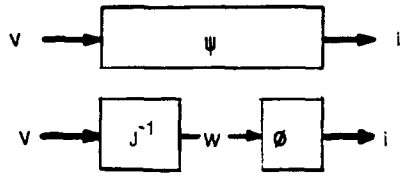

Fig. 1a. Depiction of Equation (2.5a). w = observation field, v = whitened observation field, i = innovations field.

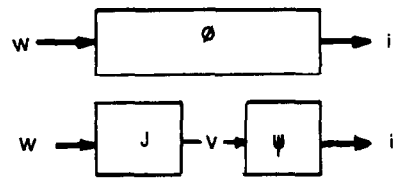

Fig. 1b. Depiction of Equation (2.5b). w = observation field, v = whitened observation field, i = innovations field.

$\psi(x, k, e_1)$ is the same as application of the innovations filter $\phi(x, k, e_1)$, which interprets (2.5a). This is illustrated in Figure 1b.

The one-dimensional version of (2.6), which defines the Jost operator, appeared in an estimation context as Equation (2.151) of [17], with the inverse spectral factor $J(k, e_1, e_2)$ interpreted as a whitening filter. A similar equation appeared in [12].

4.3. PRE-WHITENING INTERPRETATION FOR LARGE $|x|$

There is no pre-whitening approach to solving the Wiener–Hopf Equation (3.6), since the observations $\{w(y): |y| \leq |x|\}$ are given over a compact region for a given $|x|$. Heuristically, the field does not have the infinite past required for such an approach, since the innovations filter in general does not have compact support. However, it is interesting to note that in the limit $|x| \to \infty$, corresponding to an infinite sphere of observations, a pre-whitening interpretation can be attached to the limiting solution of (3.6).

The problem is still the filtering problem of estimating $\{z(x)\}$ on the surface of a sphere of observations of radius $|x|$. In the limit as $|x| \to \infty$, the filter $\psi(x, k, e_i)$ becomes $e^{-ike_1 \cdot x}$ (see (2.2)), and using (2.5a) yields

$$\phi(x, k, e_2) = \int_{S^2} \psi(x, k, e_1) J(k, e_1, e_2)\, de_1 \simeq \int_{S^2} J(k, e_1, e_2)\, e^{-ike_1 \cdot x}\, de_1 \tag{4.6}$$

and an inverse Fourier transform with respect to ke_2 gives

$$\begin{aligned} h(x, y) &\simeq \delta(x - y) - \mathcal{F}^{-1}\mathcal{F}^{-1}\{J(k, e_1, e_2)\delta(|k_1|^2 - |k_2|^2)\} \\ &= \mathcal{R}^{-1}\mathcal{R}^{-1}\{K(\tau_1 - \tau_2, e_1, e_2)\}. \end{aligned} \tag{4.7}$$

where $K(t, e_1, e_2)$ is the causal function defined from $J(k, e_1, e_2)$ in the same way that $L(t, e_1, e_2)$ was defined from $J^{-1}(k, e_1, e_2)$ in (2.15). This can be interpreted as pre-whitening the observation field with $K(t, e_1, e_2)$, after which the optimal filter is just $\delta(x - y)$. Note that an infinite sphere of observations is necessary to constitute an infinite 'past' in $|x|$. The one-dimensional version of (4.6) appeared as Equation (2.156) of [17] and in [26].

4.4. MATCHED FILTER AND THE INVERSE BORN APPROXIMATION

The Born approximation is a single-scattering approximation in which the scattered field is assumed to arise solely from interactions of the incident plane wave with the scattering potential. In other words, multiple scattering events are ignored; interactions between the *scattered* field and the potential are neglected. The Born approximation is made by neglecting the scattered field under the integral in the integral equation version of the Schrodinger equation (the Lippman–Schwinger equation; see [6–11]). The *inverse* Born approximation, which to first order is equivalent to the Born approximation, is made by neglecting the scattered field in the integral Equations (2.11) and (2.17). An estimation interpretation of the inverse Born approximation is now made.

Neglecting the scattered field in the generalized Gel'fand–Levitan Equation (2.17) amounts to neglecting the double integral term, since this term is dominated by the first term. This leaves

$$m(x, t, e_i) = \int_{S^2} M_{|x|}(t + e_s \cdot x, e_s, e_i)\, de_s \tag{4.8}$$

and an inverse partial Radon transform and some algebra results in

$$h(x, y) = k(x, y), \tag{4.9}$$

which is also the first term in an iteration solution of (3.6).

Equation (4.9) states that a first approximation to the optimal filter is simply to use the covariance function $k(x, y)$. This *matched filter* solution seems reasonable, since the stronger the correlation between $z(x)$ and $z(y)$, the more useful an observation at y will be to estimation at x. But this is simply weighting each observation as though there are no other observations; there is no *joint* use of observations. This is analogous to reconstructing a scattering potential by treating every instant of the scattered field separately, without recognizing that the field at one moment may be influenced by the field at an earlier moment being scattered again, and that this can be helpful in reconstructing the potential.

4.5. MEAN-SQUARE ERROR AND THE SCATTERING POTENTIAL

The potential $V_f(x, e)$ can be interpreted for $e = x/|x|$ as twice the rate of decrease of the normalized mean-square error in the outward radial direction. To see this, let $e(x) = z(x) - \hat{z}(x)$ and note that

$$E[e^2(x)] = E[z(x)e(x)] = k(x, x) - \int_{|z| \le |x|} h(x, z)k(z, x)\, dz = h(x, x), \tag{4.10}$$

where the orthogonality principle and (3.5) have been used. Then the definition (3.8) of $V_f(x, e)$ gives the desired result. A similar observation was made in [4] and [5] for the one-dimensional and two-dimensional isotropic cases. Note that

even for a distributed potential, only the local weight matters in computing the mean-square error at a point.

5. Fast Algorithms for Inverse Scattering and Estimation

In this section the relation between these problems is used to derive a new fast algorithm for the estimation problem from existing fast algorithms for inverse scattering. The new algorithm is much faster computationally than the algorithm of [14].

5.1. REVIEW OF EXISTING FAST ALGORITHM

In [14] a fast algorithm for computing the optimal filter $h(x, y)$ for estimating a homogeneous random field on the surface of a sphere was derived. This algorithm was obtained by writing the Laplacian as

$$\Delta_x = \frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} + \Delta_x^T, \tag{5.1}$$

where

$$\Delta_x^T = \frac{1}{x^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{x^2\sin^2\phi}\frac{\partial^2}{\partial\phi^2} \tag{5.2}$$

is the transverse radial Laplacian operator in spherical coordinates. Here and in the sequel, we use x to represent both position x and its magnitude $|x|$; the choice will be obvious from context. Equation (3.7) may then be written as

$$\left\{\left(\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x}\right) - \left(\frac{\partial^2}{\partial y^2} + \frac{2}{y}\frac{\partial}{\partial y}\right)\right\} h(x, y)$$

$$= (\Delta_y^T - \Delta_x^T)h(x, y) + \int_{S^2} V(x, e)h(|x|e, y)\,de = H(x, y), \tag{5.3}$$

which in turn can be written as the coupled system of first-order equations

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)|x||y|h(x, y) = |x||y|Q(x, y), \tag{5.4a}$$

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)|x||y|Q(x, y) = |x||y|H(x, y), \tag{5.4b}$$

where $Q(x, y)$ is an auxiliary quantity defined in (5.4a) and $H(x, y)$ is defined in (5.3). Note that (3.8) now becomes

$$V(x, e) = -2Q(x, |x|e). \tag{5.5}$$

The basic idea of the algorithm of [14] was to propagate Equations (5.4) recursively in $|x|$ and $|y|$ for $\{|y| \leq |x|\}$. If $h(x, y)$, $Q(x, y)$, and $H(x, y)$ are known

on the surface of a sphere of radius $|x_0|$ and for all $\{|y| \leq |x_0|\}$, then they can be propagated to the surface of a sphere of radius $|x_0| + \Delta$ and all $\{|y| \leq |x_0| + \Delta\}$. The recursion patterns for $h(x, y)$ and $Q(x, y)$ are illustrated in Figures 2a and b; $H(x, y)$ is then computed from $h(x, y)$ using (5.3). It can be seen from Figures 2 that the two missing values at each recursion are $h(x, 0)$ and $Q(x, |x|e) = -V(x, e)/2$. The former missing value is determined in [14] by splitting the estimation problem up into even and odd problems; this was also done for the simpler problems treated in [4] and [5].

Note that the optimal filters $h(x, y)$ are indeed completely specified by the potential $V(x, e)$, just as the optimal forwards and backwards filters in the Levinson algorithm are completely specified by the reflection coefficients. Indeed, in the one-dimensional and isotropic cases this algorithm can easily be transformed into the Levinson algorithm (see [4] and [5]); however, there seem to be no multi-dimensional counterpart to the reflection coefficients.

Nonetheless, this algorithm can be considered to be a generalized, three-dimensional split Levinson algorithm. This follows since the split Levinson algorithm recursively computes the regular solution to the discrete Schrödinger equation in one dimension, just as the present algorithm recursively computes the regular solution in three dimensions.

This algorithm is much faster computationally than solving a discretized version of (3.6) by Gaussian elimination: If each spatial coordinate is discretized to N values, the reduction in computation is from $O(N^{12})$ to $O(N^8)$. This is due to the fact that the algorithm is taking advantage of the structure (3.4) of the covariance function $k(x, y)$, just as the Levinson algorithm takes advantage of the Toeplitz structure of the autocovariance of a stationary random process.

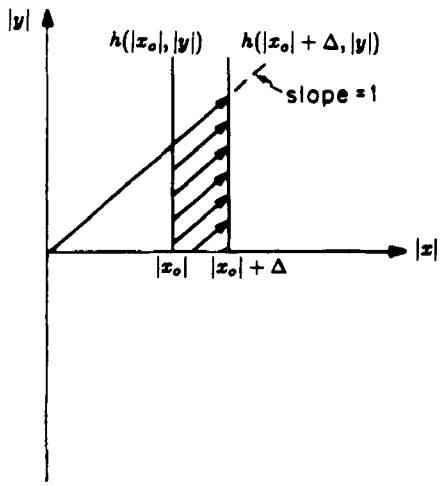

Fig. 2a. Recursion pattern for updating $h(x, y)$ in the fast algorithm of [14] for computing $h(x, y)$.

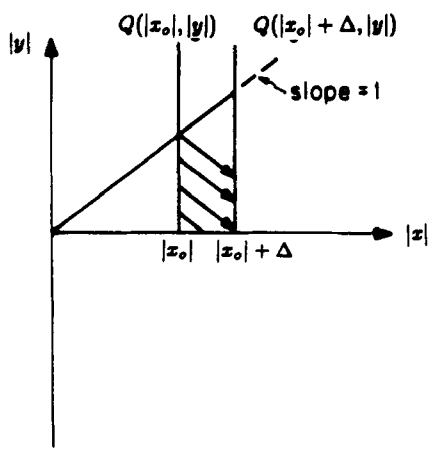

Fig. 2b. Recursion pattern for updating $Q(x, y)$ in the fast algorithm of [14] for computing $h(x, y)$.

Compare the reduction of computation $O(N^{12})$ to $O(N^8)$ to the reduction in computation $O(N^3)$ to $O(N^2)$ attained by the Levinson algorithm.

The problem with this algorithm lies in the computation of the boundary value $Q(x, |x|e)$. In [4, 5, 14], this boundary value is computed using the integral equation (3.6) and all of the other values of $h(x, y)$ and $Q(x, y)$ at a given $|x|$. This is a large, non-parallelizable computation that accounts for a significant amount of the total computation required by the algorithm. The corresponding computation in the one-dimensional Levinson algorithm accounts for roughly one-third of all the computation required by the algorithm. It would be extremely desirable to avoid this computation.

5.2. A NEW FAST ALGORITHM FOR THE FILTERING PROBLEM

We now consider the more general filtering problem posed at the beginning of Section 3. The random field $z(x)$ is no longer required to be homogeneous; its covariance function need only satisfy (3.4) and give rise to a local potential $V(x)$. It is possible to derive fast algorithms for the general case of a nonlocal potential, but these algorithms are more complex [21, 25]. Although the recursions of the algorithm of [14] apply to this more general problem, there is no way to initialize them. The new algorithm avoids this problem.

The identification of an inverse scattering problem with this estimation problem, derived in Section 3, makes it possible to derive a new fast algorithm that avoids the computation of $V(x)$ using (3.6). It also avoids splitting the estimation problem into even and odd parts, as required by the Levinson-like algorithms of [4, 5, 14]. This new algorithm can be considered to be a generalized *split Schur* or *split fast Cholesky* algorithm; a similar approach was used for the one-dimensional case in [1] and [2]. The potential $V(x)$ is obtained *directly* from the Schur variables, and then the filters $h(x, y)$ are computed from the potentials using (5.4).

The splitting into even and odd problems is avoided by the equivalent operation of extending the range of the recursions (5.4) to $-|x| \leq |y| \leq |x|$, in the sense of $|y|e = (-|y|)(-e)$. This saves no computation but simplifies the bookkeeping in the algorithm.

The Scattered Field $u(x, t, e_i)$ Initialized Using $L(t, e_1, e_2)$. Recall that the regular solutions $\phi(x, k, e_i)$ are related to the filters $h(x, y)$ by a Radon transform. Indeed, the action of the Radon transform is to map the interior $\{|y| \leq |x|\}$ of the sphere of radius $|x|$ to the interval $-|x| \leq t \leq |x|$. Therefore the regular solutions should not be of much help. The key idea is to use the scattering solutions $\psi(x, k, e_i)$ rather than the regular solutions.

The two solutions are related by (2.5b), which we repeat here as

$$\psi(x, k, e_2) = \int_{S^2} \phi(x, k, e_1) J^{-1}(k, e_1, e_2)\, de_1 . \tag{5.6}$$

From (2.4), the regular solution $\phi(x, k, e_i)$ equals one at the origin. Therefore, setting $x = 0$ in (5.6) results in

$$\psi(0, k, e_2) = \int_{S^2} J^{-1}(k, e_1, e_2)\, de_1. \tag{5.7}$$

Define the *scattered field* $u(x, t, e_i)$ in the time domain as

$$u(x, t, e_i) = \mathcal{F}_{k\to t}^{-1}\{\psi(x, k, e_i) - e^{-ike_i \cdot x}\}. \tag{5.8}$$

An inverse Fourier transform of (5.7) along with (5.8) and (2.15) then yields

$$u(0, t, e_2) = \int_{S^2} L(t, e_1, e_2)\, de_1, \tag{5.9}$$

where $L(t, e_1, e_2)$ is the smooth part of the inverse Fourier transform of $J^{-1}(k, e_1, e_2)$, as defined in (2.15).

Equation (5.9) shows that the scattered field at the origin for all angles of incidence e_2 is determined by the spectral factor $J^{-1}(k, e_1, e_2)$. This is used to initialize the new algorithm at the origin $x = 0$. Note also that

$$\mathcal{F}^{-1}\{\psi(x, k, e_i)\} = \delta(t - e_i \cdot x) + u(x, t, e_i), \tag{5.10}$$

which shows that $u(x, t, e_i)$ is indeed the scattered field resulting from a probing plane wave in the direction e_i. Hence $u(x, t, e_i) = 0$ for $t < e_i \cdot x$ by time causality.

In [6] a fast algorithm that reconstructs a scattering potential $V(x)$ from knowledge of the solution $\psi(x, k, e_i)$ on a plane is given. This algorithm is a generalized split Schur or split fast Cholesky algorithm, rather than a generalized split Levinson algorithm; hence it avoids the computation of $V(x)$ from an integral equation. This algorithm cannot be used as is for the estimation problem, since the initial data is now specified only at the origin $x = 0$. However, we can revise the derivation of the algorithm of [6] as follows.

Derivation of the Algorithm. Both the regular and scattering solutions satisfy the Schrödinger Equation (2.1). Its inverse temporal Fourier transform has the same form as (3.14), except that the potential $V(x)$ is now assumed to be local. Writing this as a coupled set of first-order equations similar to (5.4), and substituting (5.10) results in

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)|x|u(x, t, e_i) = |x|P(x, t, e_i), \tag{5.11a}$$

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)|x|P(x, t, e_i) = |x|N(x, t, e_i), \tag{5.11b}$$

$$V(x) = -2P(x, t = e_i \cdot x, e_i), \tag{5.11c}$$

where $P(x, t, e_i)$ is an auxiliary quantity defined in (5.11a) and $N(x, t, e_i)$ is

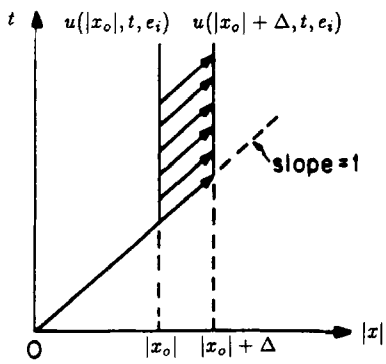

Fig. 3a. Recursion pattern for updating $u(x, t, e_i)$ in the fast algorithm for computing $V(x)$.

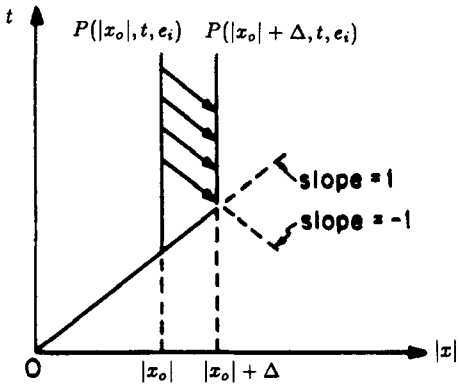

Fig. 3b. Recursion pattern for updating $P(x, t, e_i)$ in the fast algorithm for computing $V(x)$.

defined by (compare with (5.3))

$$N(x, t, e_i) = (V(x) - \Delta_x^T)u(x, t, e_i). \tag{5.12}$$

Equations (5.11) are similar in form to (5.4), but the quantities being propagated are different and have different support. The recursion patterns are illustrated in Figures 3a and b. The important point is that now $V(x) = -2P(x, t = e_i \cdot x, e_i)$ is computed during the recursions, rather than afterwards using an integral equation. This saves a considerable amount of computation and allows the algorithm to be parallelized almost completely. Note that $u(0, t, e_i)$ for all $t \geq 0$ and e_i suffices to initialize the algorithm.

It is interesting to note that $u(x, t, e_i)$ has support in t on $[e_i \cdot x, \infty]$, but the algorithm only computes it for $t \geq |x|$. This is no problem; we are not interested in $u(x, t, e_i)$, only $V(x)$, which can be recovered for each x using (5.11c) with $e_i = x/|x|$.

The physical interpretation of (5.11c) is quite interesting. This equation states that the jump in the scattered field at the wave front can, by causality, be due only to the value of the scattering potential at that point on the wave front. This allows the scattered field to be propagated recursively, reconstructing the potential on the wave front as it advances. By interpreting the estimation problem as an inverse scattering problem, we can use time causality to derive a simpler fast algorithm.

5.3. SUMMARY OF THE ALGORITHM

Initialization. Compute $J^{-1}(k)$ by a spectral factorization (2.23) of $k(x, y)$. Compute $u(0, t, e_i)$ using (5.9).

Recursions. Propagate (5.11) in increasing $|x|$ for all $t \geq |x|$, yielding $V(x)$ from (5.11c) using $e_i = x/|x|$. Recursion patterns are shown in Figures 3.

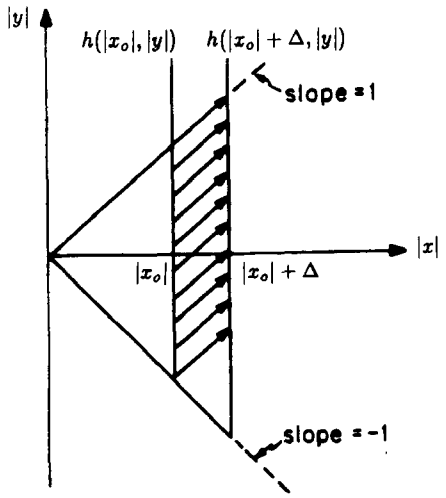

Fig. 4a. Recursion pattern for updating $h(x, y)$ in the fast algorithm for computing $h(x, y)$.

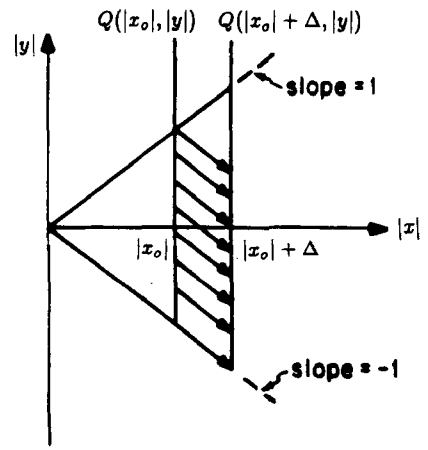

Fig. 4b. Recursion pattern for updating $Q(x, y)$ in the fast algorithm for computing $h(x, y)$.

Using $V(x)$ computed using (5.11), and in parallel with (5.11), propagate (5.4) in increasing $|x|$ and for all $-|x| \leq |y| \leq |x|$, yielding $h(x, y)$. Here

$$h(x, y) = h(x, |y|e) = h(x, (-|y|)(-e))$$

and similarly for $Q(x, y)$. These are consistent, since $h(x, y)$ is extended so as to be a Radon transform. The missing values are $Q(x, |x|e) = -V(x)/2$ and $h(x, (-|x|)e) = h(x, |x|(-e))$. Recursion patterns are shown in Figures 4a and b.

Comments. The most computationally intensive step is the spectral factorization (2.23), although in some problems $J^{-1}(k)$ may be known from the process giving rise to the random field $z(x)$. The recursions (5.4) and (5.11) can be propagated in parallel, since the only purpose of (5.11) is to compute the potential $V(x)$ to be used in (5.4) (this idea was proposed for the one-dimensional case in [18]). Note that the recursions for each point on the surface of the sphere may be performed in parallel as well. These recursions may be propagated in increasing $|x|$ indefinitely; however, if there is a desired stopping radius $|x| = T$, then (5.11) need only be computed for $2T \geq |y| \geq |x|$ (see Figures 3). The scattering interpretation of this is that the scattered field for $0 \leq t \leq 2T$ determines the scattering potential (for unit wave speed) out to a distance T.

6. Conclusion

The Schrödinger equation inverse scattering problem with a nonspherically symmetric potential has been shown to be closely related to the linear least-squares estimation problem for a random field on the surface of a sphere of noisy

observations. The relation consists of identifying each problem with a problem of the other type. This work extends results [5] for radial inverse scattering and isotropic random fields to a much more difficult and general case.

Two interesting features of this relation are the generalized displacement property satisfied by the covariance function of the random field, and the introduction of nonlocal scattering potentials. This latter feature does not appear in the simpler cases treated previously. In Section 4 interpretations were attached to a wide variety of quantities and equations. These added to the relation between the two problems by interpreting concepts in one problem in terms of the other problem.

The connection between the two problems resulted in a new fast algorithm for the random fields estimation problem that is both more general and computationally simpler than a previous fast algorithm for this problem. The algorithm was developed by interpreting the estimation problem as an inverse scattering problem, and applying a modification of a fast algorithm developed in [6] for inverse scattering.

Several issues need to be addressed. The major issue is the characterization of the set of covariance functions that give rise to only local potentials, which is similar to and closely related to the problem of characterizing admissible scattering data. The latter problem is unsolved; the present work has possibly supplied a new angle of approach. Other issues include simple ways of implementing the transverse Laplacian computation in the algorithm, an estimation interpretation of the generalized Marchenko integral equation (in terms of innovations), and some way to initialize the algorithm without performing a spectral factorization. The latter two issues are linked, since the generalized Marchenko procedure uses the scattering amplitude directly, without requiring computation of the Jost operator.

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