Annals of Mathematics and Artificial Intelligence 20 (1997) 227–265

Nonmonotonic consequences in default domain theory*

Guo-Qiang Zhang

Department of Computer Science, University of Georgia, Athens, Georgia 30602, USA
E-mail: gqz@cs.uga.edu

William C. Rounds

Artificial Intelligence Laboratory, University of Michigan, Ann Arbor, Michigan 48109, USA
E-mail: rounds@engin.umich.edu

Default domain theory is a framework for representing and reasoning about commonsense knowledge. Although this theory is motivated by ideas in Reiter's work on default logic, it is in some sense a dual framework. We make Reiter's default extension operator into a constructive method of building *models*, not theories. Domain theory, which is a well established tool for representing partial information in the semantics of programming languages, is adopted as the basis for constructing *partial* models. This paper considers some of the laws of nonmonotonic consequence, due to Gabbay and to Kraus, Lehmann, and Magidor, in the light of default domain theory. We remark that in some cases Gabbay's law of cautious monotony is open to question. We consider an axiomatization of the nonmonotonic consequence relation on prime open sets in the Scott topology – the natural logic – of a domain, which omits this law. We prove a representation theorem showing that such relations are in one to one correspondence with the consequence relations determined by extensions in Scott domains augmented with default sets. This means that defaults are very expressive: they can, in a sense, represent any reasonable nonmonotonic entailment. Results about what kind of defaults determine cautious monotony are also discussed. In particular, we show that the property of unique extensions guarantees cautious monotony, and we give several classes of default structures which determine unique extensions.

1. Introduction: Defaults in domains versus default logic

The purpose of this paper is to explore a new direction in the study of nonmonotonicity, suggested by making default systems – as in Reiter's work [18] – into constructions for building partial models for various kinds of logic. Our reasons for undertaking this task are not just to propose another nonmonotonic formalism, but to

* An earlier version was presented at the Third Bar-Ilan Symposium on the Foundations of Artificial Intelligence, Bar-Ilan University, Israel, July 1993.

understand the properties of default systems in a far wider setting than just first-order logic. We use the techniques of *domain theory* for this purpose; as we go we shall introduce various notions from this subject; for an overview see [20] or [1, 7, 22, 24]. We presume a basic familiarity with default logic on the reader's part; for a general overview consult [4].

The main new result of the paper is a representation theorem which characterizes any "reasonable" nonmonotonic entailment relation between compact elements of an arbitrary Scott domain (bounded-complete partial order) as an entailment relation defined by a system of default rules defined on a Scott domain which *embeds* the given one. The principal technique used to prove this result is that of *information systems* [20], which themselves are representational devices for making Scott domains "concrete". In our case, these information systems allow us to make Reiter's default systems generalize smoothly to the abstract case. However, we also give a simplified presentation of the result which, although losing some of its technical properties, shows just how general it is.

We should also explain at this point what we mean by "reasonable" notions of nonmonotonic entailment. We use the ideas of Kraus et al. [10]. Their inference rules (for example, Cut, Cautious Monotony, and so on) are intended as part of an axiomatization of an entailment relation $\mid\!\sim$ between propositional formulas. We replace "propositional formulas" by "compact elements of a Scott domain", or equivalently, "prime open sets" in the Scott topology of a domain. Some of the inference rules (for example, the Or law) do not make sense in this general setting. We focus on only the most basic of the laws, and in particular on the law of Cautious Monotony, which we note is slightly more questionable than the very most basic ones. Cautious monotony fails for our notion of nonmonotonic entailment; but our result shows that the other laws of Kraus, Lehmann, and Magidor, when generalized to "prime open" properties of Scott domains, are always representable by our default entailment. Thus, we answer a question implicitly raised by KLM:

> (Cumulative entailment) embodies what we think, at this moment, . . . , are the rockbottom properties without which a system should not be considered a logical system. This appreciation probably only reflects the fact that, so far, we do not know anything interesting about weaker systems [10, p. 176].

Even though the Cautious Monotony law fails for us, it does hold in many reasonable settings, including the case when default systems determine unique extensions. The last sections of the paper explore conditions under which domains and default systems have the unique extension property.

1.1. Some historical reasons for default domain theory

Our study of defaults was originally motivated by a desire to understand how defaults work in the space of *feature structures* and *feature logic* [9]. Various authors, including the architects of the linguistic theory known as *generalized phrase structure grammar* [6], propose default constraints on the form of syntactic analyses embodied in feature structures. We do not wish to review these topics here, except to remark that defaults seem to us like a way of augmenting partial analyses of linguistic objects with extra "soft" information, which can be assumed present until overruled by new "hard" information, perhaps gathered as a result of continued parsing of a sentence. An example might be the sentence

The horse raced past the barn fell

in which we assume by default, halfway through a parse, that "raced" is an active verb in the past tense, rather than a participle, as it actually is.

However, trying to explain default phenomena of this kind using default logic turned out to be rather difficult. Again we do not wish to go into all of the reasons, but we ran into both the well-known standard difficulties with default logic, and also a more fundamental mismatch, which we now explain.

Grammars can be viewed as logical constraint systems on the form of linguistic analyses. The rules of the grammar are often expressed as logical formulas (for example, [17]), which are satisfied by analyses (for example, trees, or feature structures). If we wanted to express default constraints, then we would be justified in considering a default or nonmonotonic version of the logic given by a grammar. However, what is wanted as the result of a parse is default information in a *model* (analysis) of a formula. To see the distinction, we might have a "default constraint" which said that if a form ending in "ed" followed a noun, then by default it was a verb in the past tense. But in the analysis, we would mark in the model (feature structure, say) that "raced" is a past tense verb, by default.

In fact, feature structures are well-suited for this purpose. They have a natural information-theoretic ordering defined (the *subsumption* ordering) on them, which makes it natural to mark a feature structure as subsuming a more informative one containing extra default information. The subsumption relation makes the space of feature structures into a Scott domain. See [23] for more on this topic.

For these reasons, we decided to embark on a general study of defaults acting in model-theoretic space; but instead of starting from scratch, we decided to incorporate the many insights of work in nonmonotonic logic in AI. The work of Reiter [18] particularly appealed to us, as we could see a way of generalizing his nonmonotonic rules to a much more general and abstract setting. The same idea appeared in the work of Marek et al. [14], where *extensions*, the central concept of Reiter's work, were shown to have all sorts of mathematical incarnations. Our first paper [19] is in the spirit of [14], but, in an aspect not treated by those authors, brings the concept of *partial information* into sharper focus by formalizing abstract default systems in the context of domain theory, which in essence is a constructive partial model theory appropriate for giving the semantics of programs. Since we were viewing the construction of feature structures in a dynamic way, this tool seemed to us to be particularly appropriate.

Certainly, incomplete information is a key aspect of nonmonotonic reasoning. For example, you conclude that a specific bird flies *in absence of the information* that it is a penguin. You expect to meet a flight at the airport at its scheduled arrival time *in absence of the information* that the flight is delayed. You base your decisions (in the second case especially) not only on properties assumed about airports, but on the partiality inherent in a real-world situation.

Many current approaches to default reasoning are based on the total models used in classical first order logic. [1] This presents another mismatch, in a sense alluded to above, between the theoretical tool on the one hand, and the phenomena we want to describe on the other. The basic view of classical logic is total: every issue is settled as either *true* or *false*, and truth values never change. In commonsense reasoning, our conclusions cannot be supported by proofs in the mathematical sense, and we need to take action in situations of partial information. The only place such information resides, in traditional default logic, is in the incompleteness of default theories. However, the lack of information about an airplane's arrival time need not be captured by an incomplete scientific theory of its particular flight, but may perhaps be more efficiently represented by a partial model, one which does not settle all issues.

We also remark here that the idea to use partial information in studying nonmonotonicity is not new. See the book [15] for many ways to use partial models in this respect. Barwise [3] has more on partial worlds and situations considered model-theoretically, including a treatment of nonmonotonic conditionals.

1.2. Nonmonotonic entailment

We now focus on the main topic of the paper: the laws that can govern nonmonotonic reasoning. Since Gabbay [5], a considerable amount of work has been done on axiomatizations of nonmonotonic consequence relations, here denoted by $\mid\!\sim$.

What properties should be required of $\mid\!\sim$? Many authors, in particular Kraus et al. [10], have considered Gabbay's axiom of *cautious monotony:*

$$X \mid\!\sim a \;\&\; X \mid\!\sim b \Rightarrow X, a \mid\!\sim b.$$

This axiom is often interpreted as (for example, in [11]): the learning of a new fact, the truth of which was expected, cannot invalidate a previous conclusion.

There are various interpretations of nonmonotonic entailment which do not confirm this axiom, including ours. We first present an example which seems to invalidate the axiom in a completely intuitive way. The example is perhaps originally attributable to Makinson [13], in the demonstration that default logic is not cumulative. We translate it in terms of birds, penguins and flying. Any notion of nonmonotonic consequence will have to deal with this example, in such a way that

$$\mathtt{bird} \;\mid\!\sim\; \mathtt{fly} \tag{1}$$

[1] Of course, probabilistic tools can and have been used for this purpose. But in many cases, probability distributions and/or statistical information is unavailable.

holds in the system. By a standard logical weakening, we should also have

$$\texttt{bird} \mid\!\sim \texttt{fly} \vee \texttt{penguin}. \tag{2}$$

Applying cautious monotony, we get

$$\texttt{bird}, \ (\texttt{fly} \vee \texttt{penguin}) \mid\!\sim \texttt{fly}. \tag{3}$$

But it is questionable to accept (3) as a reasonable instance of nonmonotonic entailment. The natural language reading of (3) is something like this: "a bird flies, even if it is a penguin", which is counter-intuitive.

This, of course, does not mean that there is no intuitive or formal interpretation which could justify the cautious monotony law. In fact, probabilistic analyses of defaults, such as those in Adams [2], and subsequently Pearl [16], among many others, have included cautious monotony as one of the 'core laws' which should hold in any reasonable calculus of approximate reasoning. However, the ε-semantics of Adams involves modelling "Birds fly" as saying that the conditional probablility of flying given birdhood can be made arbitrarily close to 1 within a space of admissible probability distributions. A more simple-minded interpretation – that the conditional probability of flying given birdhood is greater than a fixed constant close to 1 – does not verify the law. So there is some room for arguing that the law need not be universal.

We now begin a discussion of our interpretation of entailment. Before we get into technical details, we give an intuitive description of our version of $\mid\!\sim$.

Definition 1.1 (informal). Let φ and ψ be certain formulas, propositional or otherwise. With respect to a given default structure, a formula φ nonmonotonically entails a formula ψ, written as

$$\varphi \mid\!\sim \psi,$$

iff ψ is satisfied in every extension of any informationally minimal partial model of φ.

Several points should be made about this informal definition. First, it uses extensions as models, not theories. Thus it makes sense in default domain theory to say that a formula is satisfied in an extension. Second, the intuitive interpretation of our entailment is "if I can confirm only the information given by φ, then I can believe (skeptically) ψ". This idea is related to, but is not the same as, the notion of "minimal knowledge", or "only knowing" [8, 12]. These authors initiated the use of *maximal models* to capture "minimal knowledge", because for them partiality of information is captured by the partiality of a theory. We capture the notion in a dual sense: by the use of minimal partial models. We regard partial models as sets of tokens, each of which carries a "unit" of information. Minimality of a partial model is measured by set inclusion. If, for example, we are given the formula $Bird(tweety)$, then the minimal information conveyed by the formula is a partial model of the formula simply consisting of the one "first-order tuple" $\langle\!\langle bird, tw; 1\rangle\!\rangle$. Given a theory I, we look at all the minimal partial models of the formulas in I. We call this the minimal *information* conveyed by the theory I. This is really the approach taken by circumscription as well, since circumscription would rule out the possibility that Tweety is a penguin by minimizing the class of abnormal birds. (A formal comparison between our method and circumscription is made difficult by the fact that we work with partial models and circumscription works with total ones.)

To sum up this discussion, we might say that minimizing knowledge is done by maximizing models. For us, minimizing information is done by minimizing models. Then we use default systems to recreate belief spaces from minimal information.

We now turn to a brief comparison of our information-based semantics with the cumulative model semantics of Kraus, Lehmann, and Magidor. This would explain, in particular, why cautious monotony holds in their framework, but not ours. Intuitively, a cumulative model is made of a set of states, a possible worlds function, which assigns to each state a set of plausible worlds, and a well-founded preference relation between states. With respect to a cumulative model, $\varphi \mid\!\sim \psi$ iff for any most (maximally) preferred state all of whose possible worlds satisfy φ, the plausible worlds in that state also satisfy ψ. Cautious monotony now follows fairly easily.

It seems natural in our case to take the inclusion relation on partial models as the preference relation, with minimality of information corresponding to maximally preferred states. (This gets at the idea of "all else being equal".) Then the worlds plausible in a situation correspond to the extensions of that situation; this is exactly the tradition of Kripke semantics, except that the extensions are explicitly constructed using defaults. With respect to a default model, $\varphi \mid\!\sim \psi$ iff given any state x with only the minimal information to support φ, all the extensions for x satisfy ψ. We have, for example, $\mathtt{bird} \mid\!\sim \mathtt{fly}$, because given a state where you only have the information that something is a bird (and without the extra information that it is a penguin), the most plausible states support that it flies. This matches our intuition: to conclude a bird flies in absence of the information that it does not.

It is now clear that our definition disallows

$$\mathtt{bird},\ \ (\mathtt{fly}\ \vee\ \mathtt{penguin})\ \mid\!\sim\ \mathtt{fly}$$

as a valid instance of nonmonotonic entailment. There are two minimal information states which support $\mathtt{bird} \wedge (\mathtt{fly} \vee \mathtt{penguin})$, one of which is the state with only the information that the bird is a penguin. Of course, its extended states do not support that the penguin flies.

One cause for the seeming failure of cautious monotony is the disjunction used in the setting where pieces of information have propositional structure (call this the logic setting). This may lead us to believe that a similar failure would not occur in a setting where nothing is assumed about internal structure of pieces of information (call this the abstract setting). However, things are not that simple. As noted earlier, the set of axioms satisfied by the induced nonmonotonic consequence relation crucially depends on the method used for building the partial worlds. Although Reiter's notion of an extension is an important one, we have found that using it as a construction method violates cautious monotony even in the abstract setting. (Examples are given in section 4.)

1.3. *Summary of results and plan of the paper*

The rest of this paper will focus on the abstract setting of nonmonotonic entailment. We summarize the main results of this paper.

- We recast the issues involved in the study of abstract nonmonotonic entailment relations within the bigger picture of default domain theory. The main tool used to do this is the notion of a "nonmonotonic information system". This is a generalization of Scott's information systems.
- We use information systems to show that the axiomatic notion of entailment on a Scott domain can be represented as a notion of default entailment on an embedding domain. We also provide a simplified version of the theorem stated for abstract domains, with a simplified proof.
- The abstract nonmonotonic entailment relation derived by using the construction of extensions does not in general satisfy the cautious monotony law. Examples and discussions are provided to explain this.
- Subclasses of defaults are identified for which cautious monotony does hold. In particular, unique extension guarantees cautious monotony, and we provide sufficient conditions for defaults to determine unique extensions. In the case of *coherent* Scott domains we give a necessary and sufficient condition on a default set for uniqueness of extensions.
- Finally, a distinguished aspect of our general methodology is that it is *semantics-based*, or *model-driven*. This means that we do not start from a pre-selected set of axioms for nonmonotonic entailment, and then think about what kind of models are appropriate. Instead, we have started from the default models right at the beginning, and then have derived a set of valid axioms for the models. This methodology is complementary, but definitely not at odds with, that of Kraus, Lehmann, and Magidor:

  > We consider, in this paper, the axiomatic systems to be the main objects of interest... The different families of models considered in this paper are not considered to be an ontological justification for our interest in the formal systems, but only as a technical tool to study these systems... [10, p. 170].

  Our representation theorem uses the connective-free laws of KLM (minus the cautious monotony law and their "loop" law). This bears witness to the insight these authors had in stating the laws to begin with. Our result is, of course, of a different flavor, as it shows that systems of *defaults* are sufficiently general to represent reasonable entailments in Scott domains; and given Makinson's counterexample, it places precise limits on such representational capability.

In the sequel, we give a brief introduction to Scott domains in section 2, mostly via the notion of information systems. The basics of default domain theory are presented in section 3, and our representation result is given in section 4, for non-cumulative systems. Section 5 studies subclasses of defaults for which cautious monotony holds. Section 6 discusses nonmonotonic entailment in a general topological setting, a natural step from the domain theory point of view. The last section discusses further work.

2. Domain theory, information systems, and cpos

This section gives a brief introduction to domain theory for readers who may not be familiar with this area.

Domain theory is a branch of theoretical computer science developed by Scott and others for the semantics of programming languages. The development of domain theory, in the late 1960's, started with the observation that there were no satisfactory mathematical models for the lambda calculus, although the calculus was taken as a formalism in which to interpret programming languages such as Algol 60. To have a sound foundation for the semantics of programming languages, a mathematical model for the lambda calculus was desirable. However, it was not easy to come up with such a model, because of the higher-order functional characteristic of the lambda calculus: a term can take another term as its argument. There had to be a way to solve equations like $(D \to D) = D$, where $(D \to D)$ is a set of certain functions from D to D. A naive set theoretical construction, taking $(D \to D)$ to be that set of *all* functions from D to D, does not work, because $(D \to D)$ always has a larger cardinality than D for nonempty D. Scott's idea was to work with partial objects, instead of total objects. He used, in fact, *continuous* functions on domains, rather than the set of all functions. This had the effect both of cutting down on the cardinality of the function space, and providing a constructive *basis* for the continuous function space consisting of finitary *step functions*. The ideas in a general theory of this kind, are those of *approximation* of one object by another, and that of a basis of finitary partial elements used for approximations. (We intend to model defaults using these basis elements.)

The basic structures of domain theory are complete partial orders (cpos). A complete partial order is a partial order $(D, \sqsubseteq)$ with a least element $\perp$, and least upper bounds of increasing chains

$$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots .$$

When we write $x \sqsubseteq y$, we mean that x is an approximation to y, or y contains more information than x. A subset $X \subseteq D$ is bounded (or compatible, consistent, $X \uparrow$) if it has an upper bound in D. Thus for a bounded set X all the elements in X can be thought of as approximations to a single element. A compact (or finite) element a of D is one such that whenever $a \sqsubseteq \bigsqcup_i x_i$ with

$$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots ,$$

we also have $a \sqsubseteq x_k$ for some k. We write $\kappa(D)$ for the set of compact elements (the basis) of D, and let a, b, etc. range over compact elements. A cpo is algebraic if each element of which is the least upper bound of an increasing chain of compact elements. A Scott domain is an algebraic cpo in which every compatible subset has a least upper bound. By convention, we write $x \uparrow y$ if the set $\{x, y\}$ is bounded.

There is much more to domain theory than can be summarized here. Readers who are interested should consult the references mentioned at the beginning of the paper. In the remainder of this section we review a concrete representation of Scott domains, called information systems, which will be used from now on.

An information system consists of a set A of tokens, a subset Con of the set of finite subsets of A, denoted as $\mathrm{Fin}(A)$, and a relation $\vdash$ between Con and A. The subset Con on A is often called the consistency predicate, or the coherent (compatible) sets. The relation $\vdash$ is called the entailment relation, or the background constraint. The intended usage of the consistency predicate and the entailment relation suggests that we put some reasonable conditions on them. This results in the following definition.

Definition 2.1. An information system is a structure $\underline{A} = (A, Con, \vdash)$, where

- A is a set of tokens,
- $Con \subseteq \mathrm{Fin}\,(A),$ the consistent sets,
- $\vdash \subseteq Con \times A,$ the entailment relation,

which satisfy

1. $X \subseteq Y$ & $Y \in Con \Rightarrow X \in Con,$
2. $a \in A \Rightarrow \{\,a\,\} \in Con,$
3. $X \vdash a$ & $X \in Con \Rightarrow X \cup \{\,a\,\} \in Con,$
4. $a \in X$ & $X \in Con \Rightarrow X \vdash a,$
5. $(\forall b \in Y.\, X \vdash b$ & $Y \vdash c\,) \Rightarrow X \vdash c.$

Let us explain these conditions by taking the token set to be $\{0,1,2,3,4,5,6,7,8,9\}$. The first condition says that any subset of a coherent set must be coherent. So, if $\{1,2,3\}$ is a coherent set for an information system, then $\{1,2\}$ must also be a coherent set, and similarly for $\{2,3\}$, $\{1,3\}$, etc. The second condition requires that each single token must be coherent – otherwise we can do without it. The third condition says any token entailed by a finite set must be coherent with the set. For example, if we have $\{1,2,3\} \vdash 4$, then $\{1,2,3,4\} \in Con$. The fourth condition is a form of reflexive property for $\vdash$. The last condition is a kind of transitive property for $\vdash$. It makes most sense if we consider a finite coherent set X as a conjunction of tokens in X. That is, when we write $\{1,2,3\} \vdash 4$, we can imagine $1 \wedge 2 \wedge 3 \vdash 4$. It is then reasonable to insist that if $1 \wedge 2 \wedge 3 \vdash 4$, $1 \wedge 2 \wedge 3 \vdash 5$, and $4 \wedge 5 \vdash 6$, then $1 \wedge 2 \wedge 3 \vdash 6$.

Although the consistency predicate Con and the entailment $\vdash$ can be regarded as a generalization of those derived from the first order logic, it is important to note that they are *primitive* for an information system. Therefore, the decidability of Con and $\vdash$ are often *built in* to typical information systems.

It should also be pointed out that the user has full control of what Con and $\vdash$ are going to be, as long as the properties required for an information system are not violated. For the same token set A, a change of relations Con and $\vdash$ merely results in a different information system. Whether one information system is better than another depends on the application at hand.

Example. This information system is for approximating real numbers. For tokens, take the set A to be pairs of rationals $\langle q, r\rangle$, with $q \leqslant r$. The idea is that a pair of rationals stands for the information that a yet to be determined *real number* is in the interval $[q, r]$ whose endpoints are given by the pair.

Define a finite set X of "intervals" to be in Con if X is empty, or if the intersection of the 'intervals' in X is nonempty. Then say that a set $X \vdash \langle q, r\rangle$ iff the intersection of all 'intervals' in X is contained in the interval $[q, r]$. Note that there is only atomic structure to these tokens. We do not negate them or disjoin them, *by our choice*. It is straightforward to verify that the five properties for information systems hold for this case.

The notion of consistency can be easily extended to arbitrary token sets by enforcing the compactness property, i.e., a set is consistent if every finite subset of it is consistent. Overloading notation a little bit, we still write $y \in Con$, even for infinite y.

Definition 2.2. The collection $|\underline{A}|$ of *ideal elements* of an information system consists of subsets x of propositions which are

1. consistent: $x \in Con$,
2. closed under entailment: $X \subseteq x$ & $X \vdash a \Rightarrow a \in x$.

Information systems provide a way to express knowledge about the world in terms of coherence Con and constraints $\vdash$. Elements of an information system are made of tokens which are coherent, and which respect the constraint. They describe situations, or states of knowledge, which are "partial models" in the sense that situations usually do not settle the truth of every issue. One can also regard an information system as a set of rules from which to construct a Kripke structure. The ideal elements are the possible worlds. The accessibility relation is set inclusion, corresponding to the direction of information increase. Moreover, an atomic token is supported in a world just in case that the world contains the token. In this setting, genuine negative information has to be encoded in tokens as well – we regard falsity as positive information. So in the case where tokens do encode falsity – which they need not always do – truth assignment is a partial function, and we should probably call these structures partial Kripke structures.

Example. The ideal elements in our approximate real system are in 1–1 correspondence with the collection of closed real intervals $[x, y]$ with $x \leqslant y$. Although the collection of ideal elements is partially ordered by inclusion, the domain being described – intervals of reals – is partially ordered by reverse interval inclusion. The total or maximal elements in the domain correspond to "perfect" reals $[x, x]$. The bottom element is the empty set, which can be regarded as a special interval $(-\infty, \infty)$.

Theorem 2.1 (Scott). For any information system, the collection of ideal elements ordered by set inclusion forms a Scott domain. Conversely, every Scott domain is isomorphic to a domain of such ideal elements.

This result is basic in domain theory. We mention it here to convince the reader that information systems are thought of as (generators of) semantic structures, not syntactic entities. (More precisely, our purpose is not to study the syntax of information systems, which seems straightforward.) Of course, one cannot make the distinction between syntax and semantics absolute. Any "constructive" mathematical model must by nature be built from primitive pieces of "syntax".

3. Normal default structures

Now we come to the main definitions of default domain theory. Normal default structures are information systems extended with a set of 'defaults' of the form

$$\frac{X : a}{a},$$

with X a finite consistent set, and a a single token of the underlying information system. The idea of the defaults is that one can generate models (states of belief) by finding X as a subset of tokens in a state under construction, checking that a is consistent with the current state, and then adding the token a.

Definition 3.1. A normal default information structure is a tuple

$$\underline{A} = (A, \mathit{Con}, \Delta, \vdash),$$

where $(A, \mathit{Con}, \vdash)$ is an information system, Δ is a set of defaults, each element of which is written as $\frac{X:a}{a}$, with $X \in \mathit{Con}$, $a \in A$. If each default is of the form $\frac{:a}{a}$, we call the default structure precondition free.

The notion of deductive closure associated with standard information systems will be often used. Let $(A, \mathit{Con}, \vdash)$ be an information system, and G a coherent subset of A. The *deductive closure* of G is the set

$$\overline{G} := \{a \mid \exists X \subseteq^{fin} G.\ X \vdash a\},$$

where $\subseteq^{fin}$ stands for "finite subset of".

Extensions are a key notion related to a default structure. An extension of a situation x is a partial world y extending x, constructed in such a way that everything in y faithfully reflects an agent's belief expressed by defaults. If the current situation is x, then because it is a partial model, it may not contain enough information to settle an issue (either positively or negatively). Extensions of x are partial models containing at least as much information as x, but the extra information in an extension is only plausible, not factual.

The following definition is just a reformulation, in information-theoretic terms, of Reiter's own notion of extension in default logic.

Definition 3.2. Let $\underline{A} = (A, Con, \Delta, \vdash)$ be a default information structure, and x a member of $|\underline{A}|$. For any subset S, define $\Phi(x, S)$ to be the union $\bigcup_{i\in\omega} \phi(x, S, i)$, where

$$\phi(x, S, 0) = x,$$
$$\phi(x, S, i+1) = \overline{\phi(x, S, i)} \cup \left\{ a \mid \frac{X : a}{a} \in \Delta \ \& \ X \subseteq \phi(x, S, i) \ \& \ \{a\} \cup S \in Con \right\}.$$

Call y an extension of x if $\Phi(x, y) = y$. In this case we also write $x\varepsilon_{\underline{A}}y$, with the subscript omitted from time to time.

Example. Using the 'approximating real' system described earlier, we might like to say that 'by default, a real number is either between 0 and 1, or is the number π'. We could express this by letting Δ consist of the rules $\frac{:a}{a}$, where a ranges over rational pairs $\langle p, q\rangle$ such that $p \leqslant 0$ and $q \geqslant 1$, together with those pairs $\langle r, s\rangle$ such that $r < \pi$ and $s > \pi$.

In the ideal domain, we refer to elements by the real intervals to which they are isomorphic. The only extension of $[-1, 2]$ would be $[0, 1]$; the interval $[-2, 0.5]$ would have $[0, 0.5]$ as an extension, and there would be 2 extensions of $[-2, 4]$, namely $[0, 1]$ and $[\pi, \pi]$.

Another example: the eight queens problem. We have in mind an 8×8 chessboard, so let $8 = \{0, 1, \ldots, 7\}$. Our token set A will be 8×8. A subset X of A will be in Con if it corresponds to an admissible placement of up to 8 queens on the board. For defaults Δ we take

$$\left\{ \frac{:\langle i, j\rangle}{\langle i, j\rangle} \mid \langle i, j\rangle \in 8 \times 8 \right\}.$$

We may take $\vdash$ to be trivial: $X \vdash \langle i, j\rangle$ iff $\langle i, j\rangle \in X$. Extensions are those placements of queens which do not violate any constraints of the rules of chess, and which cannot be augmented without causing a violation. The eight queens problem can be rephrased as finding all the extensions of size 8 for $\emptyset$.

We now look at ways extensions can be constructed for normal defaults. Again, our construction is a generalization of Reiter's construction of an extension for a normal default theory.

For sets T, U of tokens with $U \subseteq T$, let $\mathsf{M}(T, U)$ be the set of maximal consistent subsets of T which happen to include U. For example, if T is itself consistent, then $\mathsf{M}(T, U) = \{T\}$ for any subset U of T. Given a normal default information structure $(A, Con, \vdash, \Delta)$, extensions can be constructed in the following way for a given x.

Let $x_0 = x$. For each $i > 0$, let

$$x_i \in \mathsf{M}\Big(\overline{x_{i-1}} \cup \Big\{a \mid \frac{X:a}{a} \in \Delta \;\&\; X \subseteq x_{i-1}\Big\}, \overline{x_{i-1}}\Big).$$

It is easy to see that x_i is an increasing chain. Let $M := \bigcup_{i \in \omega} x_i$. We show that

$$M = \bigcup_{i \in \omega} \phi(x, M, i).$$

We prove by mathematical induction that

$$x_i = \phi(x, M, i)$$

for each $i \geqslant 0$.

The base case is clear. Assume $x_i = \phi(x, M, i)$ for some i. Since

$$\begin{aligned} \overline{x_i} \subseteq x_{i+1} &\in \mathsf{M}\Big(\overline{x_i} \cup \Big\{a \mid \frac{X:a}{a} \in \Delta \;\&\; X \subseteq x_i\Big\}, \overline{x_i}\Big) \\ &= \mathsf{M}\Big(\overline{\phi(x, M, i)} \cup \Big\{a \mid \frac{X:a}{a} \in \Delta \;\&\; X \subseteq \phi(x, M, i)\Big\}, \overline{x_i}\Big), \end{aligned}$$

we have

$$\overline{\phi(x, M, i)} \cup \Big\{a \mid \frac{X:a}{a} \in \Delta \;\&\; X \subseteq \phi(x, M, i) \;\&\; \{a\} \cup M \in Con\Big\} \subseteq x_{i+1}.$$

On the other hand, if $a \in x_{i+1}$, then either $a \in \overline{x_i}$ or $a \in x_{i+1} - \overline{x_i}$. For the latter, we have $\frac{X:a}{a} \in \Delta$, $X \subseteq \phi(x, M, i)$, and $\{a\} \cup M \in Con$. Therefore $a \in \phi(x, M, i+1)$.

Thus $x_{i+1} = \phi(x, M, i+1)$. Note, however, it is not necessary to take a maximal consistent set at each step. As long as maximal consistent sets are taken infinitely often, we will get an extension.

We have, in effect, proved the following generalized Reiter theorem.

Theorem 3.1. Extensions always exist for normal default structures.

The question arises as to whether every extension for x can be constructed in this way. Although this is true for precondition free defaults, it is not true in general.

Example. Not all extensions can be constructed using the method described above. Consider the following default structure with defaults

$$\frac{:a}{a}, \; \frac{:b}{b}, \; \frac{a:c}{c},$$

but $\{a, b, c\}$ inconsistent. $\{a, b\}$ is an extension for $\emptyset$ constructed by this method. However, $\{a, c\}$, which is also an extension for $\emptyset$, can not be constructed in this way.

Example. This example shows that there are extensions that cannot be constructed by taking any maximal consistent sets. This infinite default structure has defaults

$$\frac{:a_1}{a_1}, \ \frac{a_1 : a_2}{a_2}, \ \frac{a_2 : a_3}{a_3}, \cdots$$

and

$$\frac{:b_1}{b_1}, \ \frac{:b_2}{b_2}, \ \frac{:b_3}{b_3}, \cdots,$$

such that $\{b_i, a_{i+1}\}$ is inconsistent for each $i > 0$. One can check that $y = \{a_i \mid i > 0\}$ is an extension of $\emptyset$, and $\phi(x, y, j) = \{a_i \mid j \geqslant i > 0\}$. None of the $\phi(x, y, j)$'s is maximal consistent in

$$\overline{\phi(x, y, j-1)} \cup \left\{ a \mid \frac{X : a}{a} \in \Delta \ \& \ X \subseteq \phi(x, y, j-1) \right\},$$

because $\{b_j\} \cup \{a_i \mid j \geqslant i > 0\}$ is such a maximal consistent set.

We now summarize properties of extensions for normal default structures. (Recall that $x \,\varepsilon\, y$ means y is an extension of x.)

- Extensions always exist.
- If $x \,\varepsilon\, y$ then $y \supseteq x$.
- If $x \,\varepsilon\, y$, then $y \,\varepsilon\, z$ if and only if $y = z$.
- If $x \,\varepsilon\, y$ and $x \,\varepsilon\, y'$, then either $y = y'$ or $y \cup y' \notin \mathit{Con}$.
- If $x \,\varepsilon\, z$ and $y \subseteq z$, then $\overline{x \cup y} \,\varepsilon\, z$.

The proofs of these facts again follow those of Reiter, so we omit them.

4. Defaults and nonmonotonic consequences

We are interested in the relation between default structures and nonmonotonic entailment relations.

In our opinion, axiomatizing nonmonotonic entailment $\mid\!\sim$ without referring to a consistency predicate Con does not seem to fully capture our intuition. One thing we would like to rule out, in particular, is an example like

$$a \mid\!\sim \overline{a},$$

where $\overline{a}$ is in some sense the negation of the token a. It does not seem reasonable to entail a piece of information freely from an inconsistent piece. At first sight, this assertion seems to be directly at odds with the intuition expressed by Kraus, Lehmann, and Magidor, in which the authors argue for a statement such as

if I'm the Queen of England then anything goes.

Here, "anything goes" may include "I am not the Queen of England". Our notion is not really at odds with this, though. It is important to remember that our notion of "entailment" is going to be that the consequent is a piece of information supported by all plausible worlds extending a current situation itself supporting the antecedent. We make no claims if there are no situations supporting the antecedent; our definition does not in general refer to inconsistent situations. Further, it is not reasonable to have the extending worlds contain pieces of information which contradict facts already known in the current world. Perhaps better said: we will never entertain worlds which contain contradictory facts like a person who is not the Queen of England being the Queen of England.

In a slight variant of the notation in [10], we consider the following Gentzen-style inference rules (laws). Here, $X \mid\!\sim Y$ is an abbreviation for $\forall b \in Y\ X \mid\!\sim b$.

Identity: $a \in X \Rightarrow X \mid\!\sim a$.
Cautious Monotony: $X \mid\!\sim a$ & $X \mid\!\sim b \Rightarrow X, a \mid\!\sim b$.
Cut: $X \mid\!\sim T$ & $T, Y \mid\!\sim b \Rightarrow X, Y \mid\!\sim b$.
Cautious Cut: $X \mid\!\sim T$ & $T, X \mid\!\sim b \Rightarrow X \mid\!\sim b$.

For these axioms, X, Y, and T range over finite sets of formulas, and a and b are single formulas, and a set X is thought of conjunctively. (The notation X, T stands for $X \cup T$.) We use the form of these axioms as a guide for our abstract formulation.

We next introduce the notion of a nonmonotonic system. Now, in contrast to the above axioms, we let X, Y, and T range over finite subsets of an *arbitrary set* A.

Definition 4.1. A nonmonotonic system is a triple

$$(A, Con, \mid\!\sim),$$

where Con is a collection of finite subsets X of A, called the consistent or coherent sets; and $\mid\!\sim$ is a subset of $Con \times A$, called the relation of nonmonotonic entailment, which satisfies the following axioms:

1. $X \subseteq Y \in Con \Rightarrow X \in Con$,
2. $a \in A \Rightarrow \{a\} \in Con$,
3. $X \mid\!\sim T \Rightarrow X \cup T \in Con$,
4. $a \in X$ & $X \in Con \Rightarrow X \mid\!\sim a$,
5. $X \mid\!\sim T$ & $T \cup X \mid\!\sim b \Rightarrow X \mid\!\sim b$.

Note that $X \mid\!\sim T$ is an abbreviation for $(\forall a \in T)(X \mid\!\sim a)$, where T is a finite subset of A.

It is clear that the concept of a nonmonotonic system is a generalization of the concept of information system mentioned in section 2. Axioms 1, 2, and 4 are adopted here without change.

Axiom 3 is derivable for information systems, but bears some consideration for nonmonotonic entailment. It can be regarded as a very weak form of the cautious monotony axiom, and it expresses as well a kind of skeptical property: some coherence is required of the conjunction of tokens entailed by the same information. The axiom says precisely that if

$$X \mid\!\sim a_1,\ X \mid\!\sim a_2,\ \ldots,\ X \mid\!\sim a_n,$$

then the set $X \cup \{a_1, a_2, \ldots, a_n\}$ must be coherent. (Note that this is much weaker than monotonicity, which says that if $X \mid\!\sim a$ and $Y \supseteq X$, then $Y \mid\!\sim a$.)

Is Axiom 3 a bit too strong? Should we never allow instances such as $X \mid\!\sim a$ and $X \mid\!\sim \neg a$ to hold at the same time? Although one can argue for not taking Axiom 3, it all depends on the kind of nonmonotonic entailment to be captured. Here, our skeptical version of entailment makes it reasonable to require Axiom 3; this will be confirmed later by our representation theorem. It is important to note that nonmonotonic entailment is different from default rules, where one does allow

$$\frac{X : a}{a}, \quad \frac{X : \neg a}{\neg a}$$

to appear in the same default set. However, defaults are lower level objects compared to nonmonotonic entailment. They are primitive construction rules for building models, and do not express any entailments on their own.

Axiom 5 is the *cautious cut*. We call it 'cautious cut' because sometimes the cut axiom takes the following form:

$$X \mid\!\sim T \;\&\; T, Y \mid\!\sim b \Rightarrow X, Y \mid\!\sim b.$$

This axiom is equivalent to the more restricted version

$$X \mid\!\sim T \;\&\; T, X \mid\!\sim b \Rightarrow X \mid\!\sim b$$

in the monotonic case, but definitely not in the nonmonotonic case.

Axiom 5 departs most from the corresponding axiom for information systems. By the fifth axiom for an information system, together with the other axioms, one can derive the following monotonicity property:

$$\left[X \vdash a \;\&\; (X \supseteq Y \in \mathit{Con})\right] \Rightarrow Y \vdash a.$$

But we do not even require cautious monotony to hold for a nonmonotonic system.

Given a default information structure $(A, \mathit{Con}, \vdash, \Delta)$, what is the appropriate entailment relation associated with it? Would it be reasonable to let $X \mid\!\sim a$ if $\frac{X:a}{a}$ is a default rule? Although this may seem to be a reasonable translation at first glance, it turns out to be too simple-minded. More often than not, no extra conditions are imposed on defaults, as in the case of the conflicting default rules above. Therefore, we rule out this definition.

The intuitive interpretation of a nonmonotonic entailment $X \mid\!\sim a$ in default domain theory is that, if X is our current state of knowledge, then a is believed. If we take extensions as plausible worlds, then $X \mid\!\sim a$ can also be interpreted as a is true in all plausible superworlds of (the monotonic closure) of X. We point out that this interpretation is a special case of the general definition we introduced in section 1.2. We mentioned in that section that

> With respect to a given default structure, a formula φ nonmonotonically entails a formula ψ, written as
> $$\varphi \mid\!\sim \psi,$$
> iff ψ is satisfied in every extension of any minimal partial model of φ.

In the case that φ is a finite consistent set of tokens X, the minimal partial model for X is just $\overline{X}$, the deductive closure of X. We arrive at the following definition.

Definition 4.2. Let $(A, \mathit{Con}, \vdash, \Delta)$ be a normal default structure. We define

$$X \mid\!\sim_A a$$

if

$$\forall y \big[\, \overline{X} \varepsilon_A y \Rightarrow a \in y \big],$$

where ε_A is the extension relation for $\underline{A}$.

Since extensions exist in normal default structures, $X \mid\!\sim a$ is equivalent to

$$a \in \bigcap \{ y \mid \overline{X}\, \varepsilon\, y \},$$

where subscripts are omitted.

First observe that the nonmonotonic entailment relation determined by a default structure does not in general have the cautious monotony property.

Example. Consider the following normal default structure $(A, \mathit{Con}, \vdash, \Delta)$, where

$$A = \{a, b, c\},$$
$$\Delta = \left\{ \frac{: b}{b}, \frac{b : a}{a},\ \frac{a : c}{c} \right\},$$
$$\{a, b, c\} \notin \mathit{Con}.$$

There is a unique extension for $\emptyset$: $\{a, b\}$. There are, however, two extensions for $\{a\}$:

$$\{a, b\},\ \{a, c\}.$$

We have, therefore, $\emptyset \mid\!\sim a$, $\emptyset \mid\!\sim b$, but $\{a\} \not\mid\!\sim b$.

A more convincing example shows that we can have $\emptyset \mid\!\sim a$, $\emptyset \mid\!\sim b$, but neither $\{a\} \mid\!\sim b$, nor $\{b\} \mid\!\sim a$.

Example. Consider the normal default structure $(A, Con, \vdash, \Delta)$, where

$$A = \{a, b, a_1, b_1, \overline{a}_1, \overline{b}_1\},$$

$$\Delta = \left\{\frac{:a_1}{a_1}, \frac{a_1 : a}{a}, \frac{:b_1}{b_1}, \frac{b_1 : b}{b}, \frac{a : \overline{a}_1}{\overline{a}_1}, \frac{b : \overline{b}_1}{\overline{b}_1}\right\},$$

$$\{a_1, \overline{a}_1\} \notin Con, \quad \{a_1, \overline{b}_1\} \notin Con, \quad \{b_1, \overline{b}_1\} \notin Con, \quad \{b_1, \overline{a}_1\} \notin Con,$$

$$\{b, \overline{a}_1\} \notin Con, \quad \{a, \overline{b}_1\} \notin Con.$$

There is a unique extension for $\emptyset$: $\{a, b, a_1, b_1\}$. However, $\{a, \overline{a}_1\}$ is an extension for $\{a\}$, and $\{b, \overline{b}_1\}$ is an extension for $\{b\}$. Therefore, $\emptyset \mid\!\sim b$, $\emptyset \mid\!\sim a$, but neither $b \mid\!\sim a$, nor $a \mid\!\sim b$.

What we are going to show next is a general result: the $\mid\!\sim$ relation associated with every default information structure forms a nonmonotonic system. The following lemma will be needed to show the cautious cut axiom holds.

Lemma 4.1. If $\overline{Z}\,\varepsilon\, t$ and $P \subseteq t$, then $\overline{Z \cup P}\,\varepsilon\, t$.

Proof. Recall that $\overline{Z}\,\varepsilon\, t$ means

$$t = \bigcup_{i \in \omega} \phi(\overline{Z}, t, i).$$

We prove by mathematical induction that

$$t = \bigcup_{i \in \omega} \phi(\overline{Z \cup P}, t, i).$$

$\subseteq$: Clearly $\phi(\overline{Z \cup P},\ t, 0) \subseteq t$ by the assumption $P \subseteq t$. Suppose $\phi(\overline{Z \cup P}, t, i) \subseteq t$. Then

$$\begin{aligned}
&\phi(\overline{Z \cup P}, t, i+1) \\
&\quad = \overline{\phi(\overline{Z \cup P}, t, i)} \cup \left\{a \mid \frac{X : Y}{a} \in \Delta \;\&\; X \subseteq \phi(\overline{Z \cup P},\ t, i) \;\&\; Y \cup t \in Con\right\} \\
&\quad \subseteq \overline{\phi(\overline{Z \cup P}, t, i)} \cup \left\{a \mid \frac{X : Y}{a} \in \Delta \;\&\; X \subseteq t \;\&\; Y \cup t \in Con\right\} \\
&\quad \subseteq t.
\end{aligned}$$

Therefore, $\phi(\overline{Z \cup P}, t, i) \subseteq t$ for every $i \geqslant 0$.

$\supseteq$: Obviously $\phi(\overline{Z \cup P}, t, 0) \supseteq \phi(\overline{Z}, t, 0)$. Assume $\phi(\overline{Z \cup P}, t, i) \supseteq \phi(\overline{Z}, t, i)$. Then

$$\begin{aligned}
&\phi\bigl(\overline{Z \cup P}, t, i+1\bigr) \\
&\quad = \overline{\phi(\overline{Z \cup P}, t, i)} \cup \left\{a \mid \frac{X : Y}{a} \in \Delta \;\&\; X \subseteq \phi(\overline{Z \cup P}, t, i) \;\&\; Y \cup t \in \mathit{Con}\right\} \\
&\quad \supseteq \overline{\phi(\overline{Z}, t, i)} \cup \left\{a \mid \frac{X : Y}{a} \in \Delta \;\&\; X \subseteq \phi(\overline{Z}, t, i) \;\&\; Y \cup t \in \mathit{Con}\right\} \\
&\quad = \phi\bigl(\overline{Z}, t, i+1\bigr).
\end{aligned}$$

Therefore $\phi(\overline{Z \cup P}, t, i) \supseteq \phi(\overline{Z}, t, i)$ for every $i \geqslant 0$. $\square$

Theorem 4.1. Let $(A, \mathit{Con}, \vdash, \Delta)$ be a normal default structure. Define the triple

$$(A, \mathit{Con}, \mid\!\sim_A),$$

with $X \mid\!\sim_A a$ iff $X \in \mathit{Con}$ and

$$\forall y.\bigl[\,\overline{X} \varepsilon y \Rightarrow a \in y\bigr].$$

Then $(A, \mathit{Con}, \mid\!\sim_A)$ is a nonmonotonic system.

Proof. The nontrivial part is the cautious cut. Let $X \mid\!\sim_A T$, $T, X \mid\!\sim_A a$. This means

$$T \subseteq \bigcap \bigl\{y \mid \overline{X} \varepsilon_A y\bigr\}$$

and

$$a \in \bigcap \bigl\{y \mid \overline{X \cup T} \varepsilon_A y\bigr\}.$$

Let t be an extension of $\overline{X}$ with $T \subseteq t$. By Lemma 4.1, t is an extension of $\overline{X \cup T}$. However, a belongs to every extension of $\overline{X \cup T}$. Therefore, $a \in t$ and

$$a \in \bigcap \bigl\{y \mid \overline{X} \varepsilon_A y\bigr\}. \qquad \square$$

Now we come to the main result of the paper – the converse of Theorem 4.1. *Every* nonmonotonic system is determined by some normal default structure.

When engaged in the preliminary work on this paper, we thought that attempting the converse of Theorem 4.1 was a difficult goal, for several reasons aside from the fact that no one has attempted it.

- A simple minded approach, which translates each instance of nonmonotonic entailment $X \mid\!\sim a$ into a default rule $\frac{X:a}{a}$, does not work. For example, suppose the nonmonotonic entailment contains two instances: $\emptyset \mid\!\sim a$, $\{b\} \mid\!\sim c$, but *not* $\{b\} \mid\!\sim a$. The simple minded approach would result in a default set

  $$\Delta = \left\{\frac{\emptyset : a}{a},\ \frac{\{b\} : c}{c}\right\}.$$

  However, the nonmonotonic entailment determined by the default set would include the instance $\{b\} \mid\!\sim_\Delta a$, which is not in the original nonmonotonic entailment relation.

- In fact, *the exact converse of Theorem 4.1 is not true*. Suppose we have a nonmonotonic system with $\emptyset \mathrel{|\!\sim} a$, $\emptyset \mathrel{|\!\sim} b$, but $\{a\} \not\mathrel{|\!\sim} b$. There are no default structures with exactly the tokens a and b, which determine the nonmonotonic system. This is because without new tokens, the default system will have at least two defaults
$$\frac{:a}{a}, \ \frac{:b}{b},$$
and a, b are consistent. However, that also means $\{a\} \mathrel{|\!\sim} b$ in the default structure. Although this example seems to suggest the use of cautious monotony, we have already pointed out that the nonmonotonic entailment relation determined by a default information structure does not in general have this property.
- The best thing one can achieve is to find a default structure so that the nonmonotonic system can be faithfully embedded into the one determined by the default structure. New tokens are introduced for this purpose. The idea is for each instance $X \mathrel{|\!\sim} a$, one introduces a new token (X, a), distinct from the existing ones. It remains to specify the roles the new tokens will play; in particular, how the consistency predicate is going to be extended to them. This turns out to be a fairly complicated task: the reader is encouraged to think about the problem prior to reading the proof below.

  Here is the ‘converse’ of Theorem 4.1.

Theorem 4.2. Let $(A, \mathit{Con}, \mathrel{|\!\sim})$ be a nonmonotonic system. There is a normal default structure
$$\underline{B} = (B, \mathit{Con}', \vdash, \Delta)$$
with $B \supseteq A$, and
$$X \in \mathit{Con} \quad \text{iff} \quad X \subseteq A \ \&\ X \in \mathit{Con}',$$
which determines the nonmonotonic entailment $\mathrel{|\!\sim}$, i.e., for $X \subseteq A$ and $a \in A$,
$$X \mathrel{|\!\sim} a \quad \text{iff} \quad X \mathrel{|\!\sim}_B a.$$

Example. An example will be helpful to illustrate the idea of the proof. We would like to construct a default structure $\underline{B}$, which determines the nonmonotonic entailment generated by
$$\emptyset \mathrel{|\!\sim} a, \ \emptyset \mathrel{|\!\sim} b,$$
but
$$\{a\} \not\mathrel{|\!\sim} b, \{b\} \not\mathrel{|\!\sim} a.$$
The defaults are
$$\frac{:(\emptyset, a)}{(\emptyset, a)}, \frac{(\emptyset, a) : a}{a},$$

$$\frac{:(\emptyset,b)}{(\emptyset,b)}, \frac{(\emptyset,b):b}{b},$$
$$\frac{a:(\{a\},a)}{(\{a\},a)}, \frac{b:(\{b\},b)}{(\{b\},b)},$$
$$\frac{\{a,b\}:(\{a,b\},a)}{(\{a,b\},a)}, \frac{\{a,b\}:(\{a,b\},b)}{(\{a,b\},b)}.$$

For the consistency predicate, we let

$$\{(\emptyset,a),(\{a\},a)\} \notin \mathit{Con}, \{(\emptyset,b),(\{b\},b)\} \notin \mathit{Con},$$
$$\{(\{a\},a),(\{b\},b)\} \notin \mathit{Con},$$
$$\{(\{a,b\},t),(Y,s)\} \notin \mathit{Con} \quad \text{if} \quad Y \neq \{a,b\},$$
$$\{b,(\{a\},a)\}, \{a,(\{b\},b)\} \text{ are also inconsistent.}$$

It is routine to check that $\mathrel{|\!\sim}_B a$, $\mathrel{|\!\sim}_B b$, but neither $b \mathrel{|\!\sim}_B a$, nor $a \mathrel{|\!\sim}_B b$.

We now give a uniform procedure to construct the required default structure

$$\underline{B} = (B, \vdash_B, \mathit{Con}_B, \Delta)$$

from a nonmonotonic entailment $(A, \mathit{Con}, \mathrel{|\!\sim})$.

The token set B is

$$A \cup \{(X,a) \mid X \mathrel{|\!\sim} a\},$$

where (X,a) are distinguished new tokens. The idea is to introduce a new token for each instance of the entailment.

Let $\vdash_B$ be flat, i.e., $X \vdash_B a$ iff $a \in X$. This means $\vdash_B$ does not play a key role here; it is the trivial entailment.

The default set Δ is

$$\bigcup_{(Y,b)\in B} \left\{ \frac{Y:(Y,b)}{(Y,b)}, \frac{\{(Y,b)\}:b}{b} \right\}.$$

That is, each instance of nonmonotonic entailment $Y \mathrel{|\!\sim} b$ will induce two default rules: one is $\frac{Y:(Y,b)}{(Y,b)}$, and the other is $\frac{\{(Y,b)\}:b}{b}$. Note that new tokens are used in both rules.

The consistency predicate Con_B plays an important role in ensuring the desired effect on defaults. It is specified by extending the consistency predicate Con to new tokens in the following way.

1. For $(X,a), (Y,b) \in B$,

$$\{(X,a),(Y,b)\} \in \mathit{Con}_B \Leftrightarrow X = Y.$$

   This means two new tokens are consistent just in case they are talking about what can be nonmonotonically entailed by the same set.

2. For $(Y, b) \in B$,
$$X \cup \{(Y, b)\} \in Con_B \Leftrightarrow X \cap A \subseteq \widetilde{Y}$$
where
$$\widetilde{Y} := \{c \mid Y \mathrel{|\!\!\sim} c\}.$$
By Axiom 3, $\widetilde{Y}$ is consistent. This specification means that when old and new tokens are mixed, a set is consistent if and only if all the old tokens are nonmonotonic consequences of the same set the new tokens are talking about. Note that for a set of old tokens, consistency remains the same.

We show first that the construction gives us a default structure. We check that the consistency predicate defined in this way has the required properties. First, each individual token is indeed consistent. Suppose $Z \in Con_B$ and $W \subseteq Z$. Items (1) and (2) above can be checked and we conclude that $W \in Con_B$.

Next, it has to be shown that the default structure $\underline{B}$ has the required properties. We will show this using several lemmas.

Lemma 4.2. Let $(A, Con, \mathrel{|\!\!\sim})$ be a nonmonotonic system. We have, for any consistent set X,
$$\bigcap \{\widetilde{Y} \mid Y \subseteq X \subseteq \widetilde{Y}\} = \widetilde{X}.$$

Proof. $\subseteq$: We clearly have $X \subseteq X \subseteq \widetilde{X}$. The required inclusion easily follows.

$\supseteq$: Suppose $Y \subseteq X \subseteq \widetilde{Y}$. Because $X \subseteq \widetilde{Y}$, we have $Y \mathrel{|\!\!\sim} X - Y$. If $X \mathrel{|\!\!\sim} a$, we can rewrite this as $X - Y, Y \mathrel{|\!\!\sim} a$. Now, applying cut, we get $Y \mathrel{|\!\!\sim} a$. Therefore $\widetilde{X} \subseteq \widetilde{Y}$. □

Lemma 4.3. Let W be a consistent set in the nonmonotonic system. Put
$$\rho(W) = \widetilde{W} \cup \{(W, b) \mid W \mathrel{|\!\!\sim} b\}.$$
Then $\rho(W)$ is consistent in the default structure $\underline{B}$.

Proof. This is straightforward from the specification of the consistency predicate Con_B. □

Lemma 4.4. Given $X \in Con$, suppose that X' is such that $X' \subseteq X \subseteq \widetilde{X}'$. Then $\rho(X')$ is an extension of X in the default structure $\underline{B}$.

Proof. Since $\vdash_B$ is flat, any consistent set, in particular $\rho(X')$, is an ideal element. To show $\rho(X')$ is an extension of X, it is enough to check that
$$\rho(X') = X \cup \left\{t \mid \frac{Y : t}{t} \in \Delta \ \&\ Y \subseteq \rho(X') \ \&\ \rho(X') \cup \{t\} \in Con_B\right\}.$$

Since

$$\frac{X : (X, b)}{(X, b)}, \frac{\{(X, b)\} : b}{b}$$

are in the default set Δ for each b such that $X \mid\!\sim b$, $\rho(X')$ clearly is a subset of the right-hand side.

On the other hand, suppose

$$\frac{Y : t}{t} \in \Delta \ \&\ Y \subseteq \rho(X') \ \&\ \rho(X') \cup \{t\} \in \mathit{Con}_B.$$

If $t = (Z, b)$ for some $b \in A$, then by item 2 for Con_B, $X' = Z$. Thus t is already in $\rho(X')$. If $t = b$ for some $b \in A$, then from the style of defaults we know that $Y = \{(Z, b)\}$ for some Z, and this means $Z = X'$ which again implies $t \in \rho(X')$. $\square$

Lemma 4.5. Fix $X \in \mathit{Con}$. Every extension of X is of the form $\rho(X')$ with $X' \subseteq X \subseteq \widetilde{X}'$.

Proof. Suppose y is an extension of X. By definition,

$$y = \bigcup_{i \in \omega} \phi(X, y, i),$$

where $\phi(X, y, 0) = X$, and

$$\phi(X, y, i+1) = \phi(X, y, i) \cup \left\{ b \mid \frac{Y : b}{b} \in \Delta \ \&\ Y \subseteq \phi(X, y, i) \ \&\ y \cup \{b\} \in \mathit{Con}_B \right\}.$$

The default rules are designed in such a way that if $b \in A \cap \phi(X, y, i)$ but $b \notin X$, then there must be some (X', b) already in $\phi(X, y, i)$, with the property that $X' \subseteq X$, and $X \subseteq \widetilde{X}'$. The latter is required by the consistency condition.

If no (X', b)'s are in $\phi(X, y, 1)$, then $\phi(X, y, 1) \subseteq X$, and we consider $\phi(X, y, 2)$, and so on. Eventually, $(X', b) \in \phi(X, y, k)$ for the first k, with the properties $X' \subseteq X$ (implied by the applicability of the default rule) and $X \subseteq \widetilde{X}'$ (implied by the consistency requirement). For each $i > k$, we have

$$\phi(X, y, i) \subseteq \widetilde{X}' \cup \{(X', b) \mid (X', b) \in B\},$$

again by the consistency requirement. Since $y \cup \{(X', b)\}$ is consistent for some $b \in A$,

$$y \cup \{(X', c) \mid (X', c) \in B\}$$

is also consistent, and, moreover, $y \cap A \subseteq \widetilde{X}'$. This means we have $y = \rho(X')$. $\square$

Proof of Theorem 4.2. Let X, a be in A such that $X \mid\!\sim a$. Lemmas 4.4 and 4.5 say that the $\rho(X')$, with $X' \subseteq X \subseteq \widetilde{X}'$ are exactly the extensions of X. However, Lemma 4.2 implies that

$$A \cap \bigcap \{\rho(X') \mid X' \subseteq X \subseteq \widetilde{X}'\} = \widetilde{X}.$$

We have, therefore, $X \mid\!\sim_B a$. On the other hand, if $X \not\mid\!\sim a$, then $X \not\mid\!\sim_B a$ because the propositions from A which belong to all the extensions of X are exactly $\widetilde{X}$, the set of nonmonotonic consequences of X. This proves Theorem 4.2. $\square$

It is worth noting that the construction used in the proof of Theorem 4.2 also tells us that there is a one to one correspondence between those subsets X' of X such that $X \subseteq \widetilde{X}'$, and extensions for X in the default structure $\underline{B}$.

5. Cautious monotonic systems

This section treats the cautious monotony axiom in default structures. We show that precondition free default structures give rise to nonmonotonic entailment relations satisfying this axiom. We also remark that uniqueness of extensions implies cautious monotony (the converse is not true). To better present the material, we give a name to the collection of nonmonotonic systems satisfying cautious monotony: *cautious monotonic systems*.

Definition 5.1. A cautious monotonic system is a triple

$$(A, Con, \mid\!\sim),$$

where $(A, Con, \mid\!\sim)$ is a nonmonotonic system which satisfies the additional axiom of cautious monotony:

$$6.\ X \mid\!\sim a \ \&\ X \mid\!\sim b \Rightarrow X, a \mid\!\sim b.$$

Our first result in this section is the observation that precondition free structures give rise to an extension relation supporting cautious monotony. We now consider only precondition free structures.

For the next lemma, some terminology will be useful: given a precondition free default structure $\underline{A} = (A, Con, \vdash, \Delta)$, the set of default conclusions of $\underline{A}$ is the set $\{a \mid \frac{:a}{a} \in \Delta\}$. Further, we say that a set B is compatible with a set x if $x \cup B \in Con$.

Lemma 5.1. Let $(A, Con, \vdash, \Delta)$ be a precondition free default structure. Then $y \in |A|$ is an extension of $x \in |A|$ if and only if there is a subset B of the default conclusions of $\underline{A}$ which is (i) maximal with the property that x is compatible with B, and (ii) $y = \overline{x \cup B}$.

Proof. The proof is straightforward from the definitions. $\square$

Lemma 5.1 is the key to the following theorem.

Theorem 5.1. Suppose $(A, Con, \vdash, \Delta)$ is a precondition free default structure. Define the triple $(A, Con, \mid\!\sim_A)$, with $X \mid\!\sim_A a$ iff $X \in Con$ and

$$\forall y. \big[\, \overline{X} \,\varepsilon\, y \Rightarrow a \in y \big].$$

Then $(A, Con, \mid\!\sim_A)$ is a cautious monotonic system.

Proof. We need only verify that cautious monotony is satisfied because Theorem 4.1 concludes that $(A, Con, \mid\!\sim_A)$ is a nonmonotonic system. Let $X \mid\!\sim_A T$ and $X \mid\!\sim_A b$. We want $T, X \mid\!\sim_A b$. Suppose that y is an extension of (the monotonic closure of) $T \cup X$. Then there is a maximal set B of default conclusions compatible with $T \cup X$, such that $y = \overline{T \cup X \cup B}$. We want to show that $b \in y$. Define

$$z = \overline{X \cup B}.$$

We claim that z is an extension of $\overline{X}$. Clearly $\overline{X} \subseteq z$ and z is consistent. If z is not an extension of $\overline{X}$, it is because B is not maximal in the sense of Lemma 5.1. That is, there is some maximal C, a proper superset of B, compatible with X, and such that $w = \overline{X \cup C}$ is an extension of $\overline{X}$. By hypothesis, $T \subseteq w$, and we already have $X \subseteq w$. So C is a larger set of default conclusions than B, but compatible with $T \cup X$, violating the maximality of B. This contradiction proves that z is an extension of $\overline{X}$. Thus $b \in z$, and since $z \subseteq y$, we have $b \in y$ as desired. □

It is worth noting the relation between uniqueness of extensions and the cumulative property. The following proposition says that if there is only one extension for x, then the extension can be constructed in a cumulative way, in the sense that it is enough to check consistency with the current situation (rather than with the yet-to-be found extension) when applying a default rule.

Proposition 5.1. Let $(A, Con, \vdash, \Delta)$ be a normal default structure for which extensions are always unique. Then $y \in |A|$ is an extension of $x \in |A|$ if and only if

$$y = \bigcup_{i \in \omega} \psi(x, i),$$

where

$$\psi(x, 0) = x,$$

$$\psi(x, i+1) = \overline{\psi(x, i)} \cup \left\{ a \mid \frac{X : a}{a} \in \Delta,\ X \subseteq \psi(x, i),\ \{a\} \cup \psi(x, i) \in Con \right\}.$$

Proof. The proof of Theorem 4.1 illustrates a canonical way to construct an extension. However, by choosing different maximal consistent subsets, one gets different extensions. The uniqueness of extensions imply that there must be only one maximal subset to choose at each step. By Definition 4.4, for each $i \geqslant 0$, this maximal subset must be

$$\overline{\phi(x, y, i)} \cup \left\{ a \mid \frac{X : a}{a} \in \Delta,\ X \subseteq \phi(x, y, i),\ \{a\} \cup \phi(x, y, i) \in Con \right\}.$$

This proves the proposition. □

The following theorem specifies another class of normal default structures which give rise to cautious monotonic systems.

Theorem 5.2. Suppose $(A, Con, \vdash, \Delta)$ is a normal default structure for which extensions are unique. Then the induced nonmonotonic entailment $\mid\!\sim_A$ satisfies the cautious monotony law.

Proof. Apply Proposition 5.1 and the unique-extension property. $\square$

What can we say towards categorizing cautious monotonic systems by default structures? First, it is easy to see that precondition free structures are not enough. Second, default structures with unique extensions are not enough either. Third, Theorem 4.2 implies that each cautious monotonic system can be represented by a normal default structure. We do not have at present a characterization of those normal default structures which induce cautious monotonic entailment relations under our interpretation of that entailment using extensions. We leave this as an open problem.

We also remark that if the axiom of cautious monotony is assumed, then the construction in Theorem 4.2 can be simplified slightly. We close the section with this construction.

Let's look at a concrete example first. We want to construct a default structure $\underline{B}$, which determines the nonmonotonic entailment generated by

$$\emptyset \mid\!\sim a,\ \emptyset \mid\!\sim b,$$

and

$$\{a\} \mid\!\sim b, \{b\} \mid\!\sim a.$$

The defaults are

$$\frac{:(\emptyset, a)}{(\emptyset, a)}, \frac{(\emptyset, a) : a}{a},$$
$$\frac{:(\emptyset, b)}{(\emptyset, b)}, \frac{(\emptyset, b) : b}{b}.$$

The consistency relation is trivial in the sense that everything is consistent.

It is routine to check that $\mid\!\sim_B b$, $\mid\!\sim_B a$, and further, $b \mid\!\sim_B a$, $a \mid\!\sim_B b$.

The key difference from the general construction for Theorem 4.2 is that we do not need to introduce new tokens for trivial instances of nonmonotonic entailment such as $X \mid\!\sim a$, where $a \in X$.

In general, the default structure

$$\underline{B} = (B, \vdash_B, Con_B, \Delta)$$

is constructed from a cautious monotonic system $(A, Con, \mid\!\sim)$ as follows.

The token set B is

$$A \cup \big\{(X, a) \mid X \mid\!\sim a \ \&\ a \notin X\big\},$$

where (X, a) are distinguished new tokens. Note the difference from the construction in the previous section: here we require $a \notin X$ for (X, a) to be a token of B.

Let $\vdash_B$ be flat, i.e., $X \vdash_B a$ iff $a \in X$.

The default set Δ is

$$\bigcup_{(Y,b)\in B} \left\{ \frac{Y : (Y, b)}{(Y, b)}, \frac{\{(Y, b)\} : b}{b} \right\};$$

almost the same as before, except that we do not allow default rules where b is already a member of Y.

The consistency predicate Con_B extends Con with the following clauses.

1. For $(X, a), (Y, b) \in B$,

$$\{(X, a), (Y, b)\} \in Con_B \Leftrightarrow X = Y.$$

2. For $(Y, b) \in B$,

$$X \cup \{(Y, b)\} \in Con_B \Leftrightarrow X \cap A \subseteq \widetilde{Y}.$$

This gives us a default structure $\underline{B}$ with the required properties. The proof follows the same structure as that of Theorem 4.2, with a couple of simplifications. Under the assumption of cautious monotony, we can deduce the stronger result $\widetilde{X} = \widetilde{Y}$ from $X \subseteq Y \subseteq \widetilde{X}$. The extensions of $\overline{X}$ are exactly sets of the form

$$\widetilde{X} \cup \{(Y, b) \mid (Y, b) \in B \;\&\; b \notin X\}$$

with $X \subseteq Y$ and $\widetilde{X} = \widetilde{Y}$.

6. Nonmonotonic entailment in Scott topology

This section introduces nonmonotonic entailment in a more general topological setting. Our purpose is twofold: one is to represent nonmonotonic entailment in a more abstract form so that ideas from domain theory may be directly applied. The other is to prove two new results characterizing conditions on defaults which ensure cautious monotony. These results are more illuminating in this general setting.

6.1. Abstract defaults and extensions

Recall (see section 2) that a Scott domain $(D, \sqsubseteq)$ is a cpo which is consistently complete: every bounded set has a least upper bound. The set of compact elements of a cpo D is written as $\kappa(D)$.

Definition 6.1. Let $(D, \sqsubseteq)$ be a Scott domain. A default set in D is a subset Λ of $\kappa(D) \times \kappa(D)$. We call a pair $(a, b) \in \Lambda$ a default and think of it as a rule $\frac{a:b}{b}$, though this is an abuse of notation.

A notion of extension can be introduced on Scott domains. By standard convention, $\sqcup$ stands for least upper bound, and $\sqcap$ stands for greatest lower bound.

Definition 6.2. Let $(D, \sqsubseteq)$ be a Scott domain. Let Λ be a default set in D. Write $x \,\varepsilon\, y$ for $x, y \in D$ if

$$y = \bigsqcup_{i \geqslant 0} \phi(x, y, i),$$

where $\phi(x, y, 0) = x$, and

$$\phi(x, y, i+1) = \phi(x, y, i) \sqcup \bigsqcup \{b \mid (a, b) \in \Lambda \;\&\; a \sqsubseteq \phi(x, y, i) \;\&\; b \uparrow y\}.$$

When $x \,\varepsilon\, y$, we call y an (abstract) extension of x.

The following is a characterization of abstract extensions. A discussion of a slightly different, but equally useful view on extensions follow the proof.

Theorem 6.1. For a Scott domain $(D, \sqsubseteq)$ and a subset $\Lambda \subseteq \kappa(D) \times \kappa(D)$, we have $x \,\varepsilon\, y$ if and only if

$$y = \bigsqcap \Big\{t \mid t = x \sqcup \bigsqcup \{b \mid (a, b) \in \Lambda \;\&\; a \sqsubseteq t \;\&\; b \uparrow y\}\Big\}.$$

Proof. We prove a stronger result: for any y,

$$\bigsqcup_{i \geqslant 0} \phi(x, y, i) = \bigsqcap \Big\{t \mid t = x \sqcup \bigsqcup \{b \mid (a, b) \in \Lambda \;\&\; a \sqsubseteq t \;\&\; b \uparrow y\}\Big\}.$$

We first show that

$$\bigsqcup_{i \geqslant 0} \phi(x, y, i) \sqsubseteq \bigsqcap \Big\{t \mid t = x \sqcup \bigsqcup \{b \mid (a, b) \in \Lambda \;\&\; a \sqsubseteq t \;\&\; b \uparrow y\}\Big\}.$$

This is done by mathematical induction on i, to show that whenever

$$t = x \sqcup \bigsqcup \{b \mid (a, b) \in \Lambda \;\&\; a \sqsubseteq t \;\&\; b \uparrow y\},$$

we have $\phi(x, y, i) \sqsubseteq t$ for all i. Clearly

$$\phi(x, y, 0) \sqsubseteq t.$$

Suppose

$$\phi(x, y, i) \sqsubseteq t.$$

It is enough to show that

$$\begin{aligned} &\bigsqcup \{b \mid (a, b) \in \Lambda \;\&\; a \sqsubseteq \phi(x, y, i) \;\&\; b \uparrow y\} \\ &\quad \sqsubseteq x \sqcup \bigsqcup \{b \mid (a, b) \in \Lambda \;\&\; a \sqsubseteq t \;\&\; b \uparrow y\}. \end{aligned}$$

But this is clear from the assumption that $\phi(x,y,i) \sqsubseteq t$.

We now show that

$$\bigsqcup_{i\geqslant 0} \phi(x,y,i) \sqsupseteq \bigsqcap \Bigl\{t \mid t = x \sqcup \bigsqcup \bigl\{b \mid (a,b) \in \Lambda \ \&\ a \sqsubseteq t \ \&\ b \uparrow y\bigr\}\Bigr\}.$$

This is done by demonstrating that $\bigsqcup_{i\geqslant 0} \phi(x,y,i)$ is one of the t's, that is,

$$\bigsqcup_{i\geqslant 0} \phi(x,y,i) = x \sqcup \bigsqcup \Bigl\{b \mid (a,b) \in \Lambda \ \&\ a \sqsubseteq \bigsqcup_{i\geqslant 0} \phi(x,y,i) \ \&\ b \uparrow y\Bigr\}.$$

However, the above follows from the fact that a's are isolated elements and

$$\bigl\{\phi(x,y,i) \mid i \geqslant 0\bigr\}$$

is an ω-increasing chain. □

Theorem 6.1 suggests the use of fixed point theorems in domain theory (or lattice theory). Recall that a function $f : D \to D$ is continuous if it is monotonic and it preserves least upper bounds of ω-increasing chains. The least fixed point of any such continuous function can be constructed by taking the least upper bound of the ω-chain

$$\bot \sqsubseteq f(\bot) \sqsubseteq f(f(\bot)) \sqsubseteq \cdots \ .$$

For a fixed domain D and default set Λ, let

$$\xi(x,u,v) = x \sqcup \bigsqcup \bigl\{b \mid (a,b) \in \Lambda \ \&\ a \sqsubseteq u \ \&\ b \uparrow v\bigr\},$$

$$\eta(x,v) = \bigsqcap \bigl\{t \mid t = \xi(x,t,v)\bigr\}.$$

It is easy to check that for fixed x and v, $\xi(x,u,v)$ is a continuous function in u. Therefore, $\xi(x,u,v)$ has a least fix point, such that $\xi(x,l,v) = l$, which is exactly what $\eta(x,v)$ trying to express. Indeed, the proof of Theorem 6.1 also confirms that

$$\xi\bigl(x, \eta(x,v), v\bigr) = \eta(x,v).$$

We have the following representation theorem, which makes use of Scott's representation theorem for information systems. The proof is straightforward, hence omitted.

Theorem 6.2. Every default information system determines an extension relation isomorphic to the abstract extension relation on the Scott domain corresponding to the underlying information system, via the correspondence sending a default $\frac{X:Y}{Y}$ to the pair (x,y) of compact elements determined by (X,Y); and conversely via the 'inverse' correspondence from Scott domains to information systems.

Note that in this representation theorem a slightly more general form of defaults $\frac{X:Y}{Y}$ is used, where Y is a finite set instead of a singleton. However, all previous results directly generalize to this case.

6.2. Nonmonotonic entailment between open sets

Let $(D, \sqsubseteq)$ be a Scott domain. A subset $U \subseteq D$ is said to be *Scott open* if (i) U is upward closed: $x \in U$ and $x \sqsubseteq y$ imply $y \in U$; and (ii) for any ω-increasing chain

$$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots,$$

$\bigsqcup_i x_i \in U$ implies $x_k \in U$ for some k.

One checks readily that under this definition of "open": the collection of open subsets of a Scott domain form a topological space. Such a space must contain $\emptyset$ and D, and be closed under finite intersections and arbitrary unions. Furthermore, we can regard open sets as being 'properties' of domain elements. The definition says that if an element has a certain property, then we can discover that the property holds by testing a sequence of finite elements which 'converges' to the given element. After a finite time, we find that the element does indeed have the property. Such properties are sometimes called 'affirmable' [21].

It is straightforward to prove the following in any Scott domain D.

Theorem 6.3 (compactness in the Scott topology). 1. For each finite element $x \in D$, the set $\uparrow x = \{u \mid x \sqsubseteq u\}$ is open. We call it *prime open.*

2. Every open set U is the union of the prime opens generated by the compact elements of U.

3. Every compact open set X is a *finite* union of such prime opens. (Compact here means the topological usage: every covering of X by open sets has a finite subcovering.)

A set Λ of defaults on a Scott domain $(D, \sqsubseteq)$ induces a nonmonotonic entailment relation between open sets. For any open set U, let μU stands for the set of minimal elements of U. Because U is an open set, elements in μU are isolated (compact). Suppose U, V are open sets of D. With respect to Λ, define $U \mid\!\sim V$ iff given any element x in μU, every extension y of x is a member of V. This, of course, corresponds to the more concrete description given in section 1.3. Unsurprisingly, cautious monotony fails in this general setting: from $U \mid\!\sim V$ and $U \mid\!\sim W$, we cannot conclude $U \cap V \mid\!\sim W$. This is because general open sets allow disjunctions, in the form of unions of prime opens. Our representation theorems are really about nonmonotonic entailment relations on prime opens. But because prime opens are determined as the upward closure of compact elements, we can also regard this entailment as a relation between compact elements. This is what we called the "abstract setting" for entailment relations in the introductory section. We reserve the notation $\mid\!\sim$ for entailment between open sets, and use $\rightsquigarrow$ as a relation on $\kappa(D)$. (Think of this as the entailment relation on compact opens.) If now Λ is a default set on D, we define the relation $\rightsquigarrow_\Lambda$ in analogy with our default entailment relation: we let $a \rightsquigarrow_\Lambda b$ iff for every extension e of a, we have $b \sqsubseteq e$.

Our representation theorem for nonmonotonic systems is now the following.

Theorem 6.4. For any nonmonotonic system $(A, Con, \mid\!\sim)$ we can find a Scott domain D, a default set Λ, and an injective mapping $\theta : Con \to \kappa(D)$ satisfying the following: For any X and Y in Con, $X \mid\!\sim Y$ if and only if $\theta(X) \rightsquigarrow_\Lambda \theta(Y)$.

This result is just a corollary of Theorem 4.2 and the representation of monotonic information systems as Scott domains.

We now reformulate our original laws for nonmonotonic information systems using the relation $\rightsquigarrow_\Lambda$. Observe that it satisfies the following laws (we suppress the subscript):

- **Reflexivity:** $a \rightsquigarrow a$ for all compact a.
- **Right Weakening:** if c is compact and $c \sqsubseteq a$ then $a \rightsquigarrow c$.
- **Consistency:** if $a \rightsquigarrow b$ then $a \uparrow b$.
- **Right Conjunction:** if F is a finite subset of $\kappa(D)$ and $a \rightsquigarrow b$ for all $b \in F$ then $a \rightsquigarrow \bigsqcup F$ (note that in particular F is consistent).
- **Cautious Cut:** if $a \rightsquigarrow b$ and $a \sqcup b \rightsquigarrow c$ then $a \rightsquigarrow c$.

We now present an abstract version of our representation theorem, and a simplified proof. We start with a Scott domain, and an entailment relation satisfying the above laws, and then represent it by means of abstract defaults in an embedding domain. First we summarize our notion of entailment.

Definition 6.3. An abstract nonmonotonic entailment is a triple $(D, \sqsubseteq, \rightsquigarrow)$, where $(D, \sqsubseteq)$ be a Scott domain, and $\rightsquigarrow$ is a relation on $\kappa(D)$ which satisfies Reflexivity, Right Weakening, Consistency, Right Conjunction, and Cautious Cut.

Theorem 6.5. Let $(D, \sqsubseteq, \rightsquigarrow)$ be an abstract nonmonotonic entailment. There is a default domain $(D', \sqsubseteq', \Lambda)$ with $(D, \sqsubseteq)$ a subdomain of $(D', \sqsubseteq')$, such that

$$a \rightsquigarrow b \quad \text{iff} \quad a \rightsquigarrow_\Lambda b$$

for $a, b \in \kappa(D)$.

To describe the construction needed in the proof, we introduce an auxiliary notion (analogous to $\widetilde{Y}$ in the previous proof) called the *nonmonotonic consequence bound*, which is defined as:

$$\widetilde{a} := \bigsqcup \{b \mid a \rightsquigarrow b\}.$$

Note that this slight overload of notation will not cause confusion with the earlier notion of nonmonotonic closure used in the proof of Theorem 4.2.

Nonmonotonic consequence bounds always exist, because the law of Right Conjunction implies immediately that the set $\{b \mid a \rightsquigarrow b\}$ is directed.[2]

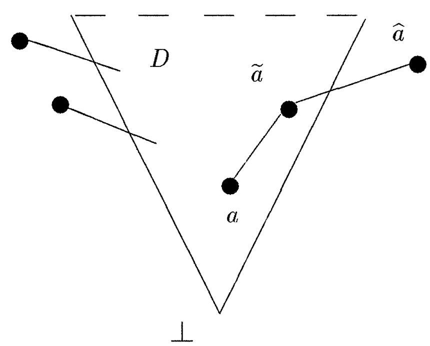
[2] A directed set in a domain is a generalization of a chain: A nonempty set $A \subseteq D$ is directed if for any $a, b \in A$ there is $c \in A$ with $a, b \sqsubseteq c$. Directed sets can be used in the definition of cpos more or less interchangeably with increasing chains; in particular when there is a countable basis.

We now describe the construction. Start with an abstract nonmonotonic entailment $(D, \sqsubseteq, \rightsquigarrow)$. We construct a default domain $(D', \sqsubseteq', \Lambda)$, where

- $D' = D \cup \{\widehat{x} \mid x \in D\}$, with $\widehat{x}$'s the new elements added to D in such a way that $\widehat{x}$ is the unique maximal element immediately above x.
- $\Lambda = \{(a, \widehat{\widetilde{a}}) \mid a \in \kappa(D)\}$.

Several remarks are in order. First, notice that all the new elements are compact. So, although $\widetilde{a}$ need not be compact, $\widehat{\widetilde{a}}$ always is, for compact a. This makes the set Λ a legitimate candidate for a default set. Second, all the new elements are incompatible with each other, and the only elements in D which are compatible with $\widehat{x}$ are in the set $\downarrow x$. Third, D is clearly a subdomain of D', in whatever reasonable sense one could make out of the word 'subdomain'.

The following picture helps us to visualize the construction. The domain D' looks like a new domain with lots of "hair" growing out of the old D.



The proof that the above construction works for arbitrary default domains is again achieved via several lemmas.

Lemma 6.1. Let $(D, \sqsubseteq, \rightsquigarrow)$ be an abstract nonmonotonic entailment. For any element $a \in \kappa(D)$, we have

$$\bigsqcap \{\widetilde{b} \mid b \in \kappa(D) \ \&\ b \sqsubseteq a \sqsubseteq \widetilde{b}\} = \widetilde{a}.$$

Proof. We clearly have $a \sqsubseteq a \sqsubseteq \widetilde{a}$. Therefore,

$$\bigsqcap \{\widetilde{b} \mid b \in \kappa(D) \ \&\ b \sqsubseteq a \sqsubseteq \widetilde{b}\} \sqsubseteq \widetilde{a}.$$

On the other hand, suppose $b \sqsubseteq a \sqsubseteq \widetilde{b}$. By Right Conjunction, $\{x \mid b \rightsquigarrow x\}$ is a directed set whose least upper bound is $\widetilde{b}$. Since $a \in \kappa(D)$, we have $a \sqsubseteq b_0$ for some b_0 such that $b \rightsquigarrow b_0$. This implies, by Right Weakening, $b \rightsquigarrow a$. For any $x \in D$, if $a \rightsquigarrow x$ then $a \sqcup b \rightsquigarrow x$, as $b \sqsubseteq a$. Therefore, $b \rightsquigarrow x$, by applying Cautious Cut. This proves $\widetilde{a} \sqsubseteq \widetilde{b}$. Hence,

$$\bigsqcap \{\widetilde{b} \mid b \in \kappa(D) \ \&\ b \sqsubseteq a \sqsubseteq \widetilde{b}\} \sqsupseteq \widetilde{a}. \qquad \square$$

Lemma 6.2. Given $a, b \in \kappa(D)$, suppose that $b \sqsubseteq a \sqsubseteq \widetilde{b}$. Then $\widehat{\widetilde{b}}$ is an extension of a in $(D', \sqsubseteq', \Lambda)$.

Proof. This is because $(b, \widehat{\widetilde{b}})$ is a member of Λ, and from the given assumption we have

$$a \sqsubseteq \widetilde{b} \sqsubseteq \widehat{\widetilde{b}}.$$

So, $\widehat{\widetilde{b}}$ is an extension of a, since $\widehat{\widetilde{b}}$ is a maximal element. $\square$

The previous lemma shows that if $b \sqsubseteq a \sqsubseteq \widetilde{b}$, then $\widehat{\widetilde{b}}$ is an extension of a. The next lemma shows that all extensions of a are of this form.

Lemma 6.3. Fix $a \in \kappa(D)$. Every extension of a is of the form $\widehat{\widetilde{b}}$ with

$$b \in \kappa(D) \quad \text{and} \quad b \sqsubseteq a \sqsubseteq \widetilde{b}.$$

Proof. Clearly every extension of a must be some $\widehat{\widetilde{b}}$, because of the special kind of pairs of elements in Λ. We have to explain why such b's must have the properties mentioned in the lemma. It is easy to see that we have $b \sqsubseteq a$; but we must also have $a \sqsubseteq \widetilde{b}$, because otherwise a and $\widehat{\widetilde{b}}$ will be incompatible. $\square$

These lemmas lead to the proof for Theorem 6.5, as follows.

Proof. Let $(D, \sqsubseteq, \rightsquigarrow)$ be an abstract nonmonotonic entailment, and let the default domain $(D', \sqsubseteq', \Lambda)$ be the one described earlier.

Suppose $a \rightsquigarrow b$, and suppose y is an extension of a in D'. By the previous lemma, $y = \widehat{\widetilde{x}}$ for some $x \in \kappa(D)$ such that

$$x \sqsubseteq a \sqsubseteq \widetilde{x}.$$

However, by Lemma 6.1, we have $\widetilde{a} \sqsubseteq \widetilde{x}$. Hence,

$$b \sqsubseteq \widetilde{a} \sqsubseteq \widetilde{x} \sqsubset y,$$

which shows that $a \rightsquigarrow_\Lambda b$.

On the other hand, suppose $a \rightsquigarrow_\Lambda b$ for $a, b \in \kappa(D)$. By the previous lemmas again, $\widehat{\widetilde{a}}$ is an extension for a in D'. Therefore, $b \sqsubseteq \widehat{\widetilde{a}}$, which in turn implies that $b \sqsubseteq \widetilde{a}$ since $b \in D$. This means $a \rightsquigarrow b$, by Right Weakening, the directedness of $\{q \mid a \rightsquigarrow q\}$, as well as the compactness of b. $\square$

What is the difference between Theorems 6.5 and 4.2? In the construction of the default set Λ, we used pairs like $(a, \widehat{\widetilde{a}})$, where $\widetilde{a}$ need not be a compact element. So, the construction of D' may transform a finitary domain D (in the sense that any compact element dominates only finitely many elements) to a non-finitary domain. This means that although $\widehat{\widetilde{a}}$ is compact, it may be required to code an 'infinite amount' of information. An inspection of the construction in Theorem 4.2 shows that it avoids this problem, but at the cost of considerable complication in defining the default set.

6.3. Trace sets ensure cautious monotony

It has been mentioned in section 5 that cautious monotony can be regained when extensions are unique, as long as we look only at prime open sets. However, the condition of 'unique extensions' is not very useful, because it is not always realistic to verify. In this subsection we provide a sufficient condition for default sets to determine unique extensions. This gives a concrete and efficient way to verify cautious monotony by checking a property of the default set.

Our result is motivated by results in domain theory. It is well-known in domain theory that a function $f : D \to D$, where D is a Scott domain, is continuous iff for every compact element $b \in \kappa(D)$, $b \sqsubseteq f(x)$ implies $b \sqsubseteq f(a)$ for some $a \sqsubseteq x$, with a compact. This means, in a sense, that continuous functions are determined by pairs of finite elements. But what kind of properties must a set of such pairs have to ensure that it corresponds to a continuous function?

We have:

Theorem 6.6. A set $\{(a_i, b_i) \mid i \in I\} \subseteq \kappa(D) \times \kappa(D)$ determines a continuous function if for all finite set $J \subseteq I$, $\{a_i \mid i \in J\}$ bounded above implies $\{b_i \mid i \in J\}$ is bounded above.

The proof of Theorem 6.6 is routine, and we omit it except to point out that the function determined by the set $\{(a_i, b_i) \mid i \in I\}$ is the following:

$$f(x) = \bigsqcup \bigl\{ b_k \mid a_k \sqsubseteq x \;\&\; k \in I \bigr\}.$$

We call a set with the property mentioned in the previous theorem a *trace set*.

Definition 6.4. Let $F \subseteq \kappa(D) \times \kappa(D)$, where D is a Scott domain. F is a trace set if for every finite subset G of F, the consistency of G's first components implies the consistency of G's second components, i.e.,

$$\pi_1 G\!\uparrow \;\Rightarrow\; \pi_2 G\!\uparrow.$$

Of course, here π_i's are projections to the ith component, and $X\uparrow$ means that X is bounded above in D.

Let's define, for a function $f : D \to D$, the set $\mathsf{tr}(f)$ to be a set of pairs (a, b) in $\kappa(D) \times \kappa(D)$ such that $b \sqsubseteq f(a)$. Clearly $\mathsf{tr}(f)$ is a trace set, and we call it the trace set of f. The set $\mathsf{tr}(f)$ has the following additional property of *saturation*:

$$\bigl[(a, b) \in \mathsf{tr}(f) \;\&\; a \sqsubseteq a' \;\&\; b' \sqsubseteq b\bigr] \Rightarrow (a', b') \in \mathsf{tr}(f).$$

However, when constructing a function from a trace set, the 'derived' pairs like (a', b') do not contribute anything.

Observe that if we start with a continuous function $f : D \to D$, construct its trace set $\mathsf{tr}(f)$, and then derive a function from it via Theorem 6.6, we get back the same function. On the other hand, if we start with a *saturated* trace set, derive a continuous function from the set, and then find the trace set of the function, we get exactly the same trace set back.

We now come to the main theorem of this subsection.

Theorem 6.7. Let Λ be an abstract default set in a Scott domain D. Then extensions are unique for Λ if

$$\Lambda \cup I_{\kappa(D)}$$

is a trace set, where $I_{\kappa(D)} = \{(a,a) \mid a \in \kappa(D)\}$.

It is easy to check if the nonmonotonic consequence induced by a default structure $(A, Con, \vdash, \Delta)$ satisfies the trace set condition. One simply augments Δ with all trivial instances $\frac{\{a\}:a}{a}$, for $a \in A$, then check that for any finite set

$$\frac{X_1 : a_1}{a_1}, \ \frac{X_2 : a_2}{a_2}, \ \ldots, \frac{X_n : a_n}{a_n}$$

from Δ, if

$$X_1 \cup X_2 \cup \cdots \cup X_n \in Con,$$

then

$$\{a_1, a_2, \ldots, a_n\} \in Con.$$

Note that adding $\frac{\{a\}:a}{a}$ into the default set seems quite innocent, but with the trace set property, it produces some extra effect, as can be seen in the following proof.

Proof of Theorem 6.7. Suppose Λ has the property mentioned in the statement of the theorem. Clearly Λ itself is a trace set and, moreover, the function determined by $\Lambda \cup I_{\kappa(D)}$ is greater than or equal to the identity function under the pointwise order.

Given any element $x \in D$, we have the following monotonic procedure for building an extension, by taking advantage of the trace set property of $\Lambda \cup I_{\kappa(D)}$.

Let f denote the continuous function determined by $\Lambda \cup I_{\kappa(D)}$. Clearly f is *inflationary*, in the sense that $f(t) \sqsupseteq t$ for every t. There is a canonical way to construct a fixed point for such an inflationary function: just take the least upper bound of

$$x \sqsubseteq f(x) \sqsubseteq f(f(x)) \sqsubseteq \cdots.$$

It is easy to show that this fixed point is the unique extension for x. $\square$

Although Theorem 6.7 captures a large class of default sets which determine unique extensions, one wonders if the condition in Theorem 6.7 is also necessary for unique extensions. The answer is, unfortunately, no. It is not hard to find finite counterexamples.

6.4. *Capturing unique extensions in coherent domains*

In the previous subsection, an effective sufficient condition is given for unique extensions. However, that condition is not necessary. We now give a very simple condition which is both sufficient and necessary for unique extensions on coherent Scott domains (an effective characterization of unique extensions on general Scott domains remains unsolved). We say that a Scott domain $(D, \sqsubseteq)$ is *coherent* if for every subset X of D, the compatibility of every pair of elements in X implies the compatibility of the whole set X. Note that in the following theorem we assume without loss of generality that if (a, b) is a pair in a default set Λ, then $a \sqsubseteq b$.

Theorem 6.8. Let Λ be an abstract default set in a coherent Scott domain D. Then extensions are unique for Λ if and only if for every pair $(a, b), (a', b')$ in Λ, if a, a' is compatible (denoted as $a \uparrow a'$), then

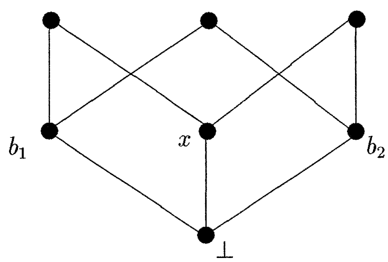
$$\big[(a \sqcup a') \uparrow b \;\&\; (a \sqcup a') \uparrow b'\big] \Rightarrow b \uparrow b'.$$

To better understand the theorem, we explain why it does not hold for non-coherent Scott domains, and why the condition cannot be replaced by a more familiar one, such as

$$a \uparrow a' \Rightarrow b \uparrow b'.$$

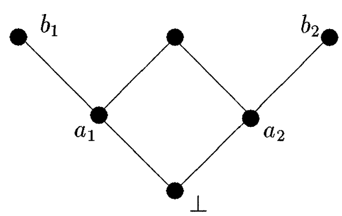
Answers to both questions can be found in the two examples below.

Example. Consider this typical non-coherent Scott domain.



The default set $\{(\perp, b_1), (\perp, b_2)\}$ clearly satisfies the condition in the theorem. However, there are two extensions for x.

Example. Consider the next Scott domain, which is coherent.



The default set $\{(a_1, b_1), (a_2, b_2)\}$ generates unique extensions in this domain. However, the condition

$$a \uparrow a' \Rightarrow b \uparrow b'$$

does not hold.

We now go back to the proof of Theorem 6.8.

If: We prove that for any d in D, there is a maximum among the subsets B of

$$\big\{b \mid (a,b) \in \Lambda \ \& \ a \sqsubseteq d\big\}$$

such that $\{d\} \cup B$ is compatible. Consider the set

$$M(d) := \big\{b \mid (a,b) \in \Lambda \ \& \ a \sqsubseteq d \ \& \ d \uparrow b\big\}.$$

For any b_1, b_2 in this set, there are a_1, a_2, both below d, such that (a_1, b_1), (a_2, b_2) are in Λ. Moreover, $(a_1 \sqcup a_2) \uparrow b_1$, and $(a_1 \sqcup a_2) \uparrow b_2$. Therefore, by the condition given in Theorem 6.8, we have $b_1 \uparrow b_2$. This means, by coherence of D, that $M(d)$ is the largest set among the subsets B of

$$\big\{b \mid (a,b) \in \Lambda \ \& \ a \sqsubseteq d\big\}$$

such that $\{d\} \cup B$ is compatible.

Now we can complete the proof of the If direction. For each $d \in D$ let $M(d)$ be the maximal set constructed in the previous paragraph. We want to show that each x has a unique extension. Fix x, and define a sequence $m_0, m_1, \ldots$ as follows. Put $m_0 = x$, and

$$m_{i+1} = m_i \sqcup \bigsqcup M(m_i).$$

This gives an increasing sequence; it is easy to show that the least upper bound m of this sequence is an extension of x. But now let e be any other extension of x; then

$$e = \bigsqcup \phi(i, x, e)$$

by the definition of extension. By induction, it is straightforward to show that for eacvh that for each $i \geqslant 0$

$$\phi(i, x, e) \sqsubseteq m_i.$$

Therefore $e \sqsubseteq m$, which implies that $e = m$.

Only If: On the other hand, suppose for some $(a_1, b_1), (a_2, b_2)$ in Λ such that $a_1 \uparrow a_2$ we have $(a_1 \sqcup a_2) \uparrow b_1$, and $(a_1 \sqcup a_2) \uparrow b_2$. We must have $b_1 \uparrow b_2$, for otherwise there are clearly at least two extensions for $a_1 \sqcup a_2$.

7. Conclusion

This paper views the issues involved in the study of nonmonotonic entailment in the light of default domain theory. We have proved a representation theorem which characterizes properties of an arbitrary nonmonotonic entailment relation on a Scott domain in terms of a relation derived from default rules and extensions. Cautious monotony fails in general for our notion of default nonmonotonic entailment; classes of defaults are identified for which cautious monotony holds. We then consider the nonmonotonic entailment relation in a topological setting. This also makes it possible to apply results in domain theory which identify other subclasses of defaults for which the induced nonmonotonic entailment relation satisfies cautious monotony. The most general characterization of such defaults remains an open question. However, it is important to point out that properties of derived nonmonotonic entailment not only depend on the kind of defaults used, but also depend on the procedure for building extended partial worlds. Although extensions are one of the key model building methods, there are other possibilities. In [19], we introduce a construction called a *dilation*, motivated from the need to ensure the existence of extended partial world for all reasonable defaults. Dilations are a robust generalization of extensions, and exist for all semi-normal default structures. It is quite possible that other nonmonotonic fixpoint operators could lead to better-behaved consequence relations, but we have no other candidates at the moment.

Another research direction would be to think about domains which encode some of the propositional connectives. For example, the *Smyth powerdomain* of a domain encodes sets of domain elements as its own elements, where the sets are thought of disjunctively. This would allow us to talk about the Or law of KLM as a possibility.

References

[1] S. Abramsky and A. Jung, Domain theory, in: *Handbook of Logic in Computer Science*, Vol. 3, eds. S. Abramsky, D.M. Gabbay and T.S.E. Maibaum (Oxford Science Publications, 1995) pp. 1–168.

[2] E. Adams, *The Logic of Conditionals* (D. Reidel, Netherlands, 1975).

[3] J. Barwise, *The Situation in Logic*, Vol. 17 (Center for Study of Language and Information, Stanford, California, 1989).

[4] P. Besnard, *An Introduction to Default Logic* (Springer-Verlag, 1989).

[5] D. Gabbay, Theoretical foundations for nonmonotonic reasoning in expert systems, in: *Proceedings of NATO Advanced Study Institute on Logics and Models of Concurrent Systems*, ed. K.R. Apt (Springer-Verlag, 1985) pp. 439–457.

[6] G. Gazdar, E. Klein, G. Pullum and I. Sag, *Generalized Phrase Structure Grammar* (Harvard University Press, 1985).

[7] C. Gunter, *Semantics of Programming Languages: Structures and Techniques* (MIT Press, Boston, 1992).

[8] J. Halpern and Y. Moses, A guide to modal logics of knowledge and belief, in: *Proc. IJCAI '85* (1985) pp. 480–490.

[9] R. Kasper and W. Rounds, The logic of unification in grammar, *Linguistics and Philosophy* 13 (1990) 33–58.

[10] S. Kraus, D. Lehmann and M. Magidor, Nonmonotonic reasoning, preferential models, and cumulative logics, *Artificial Intelligence* 44 (1990) 167–207.

[11] D. Lehmann, What does a conditional knowledge base entail?, in: *Proceedings of KR '89* (Morgan-Kaufmann, 1989) pp. 212–234.

[12] H.J. Levesque, All I know: A study in autoepistemic logic, *Artificial Intelligence* 42 (1990) 263–309.

[13] D. Makinson, General patterns in nonmonotonic reasoning, in: *Handbook of Logic in Artificial Intelligence and Logic Programming*, Vol. 3, eds. D.M. Gabbay, C.J. Hogger and J.A. Robinson (1994) pp. 35–110.

[14] W. Marek, A. Nerode and J. Remmel, A theory of nonmonotonic rule systems, in: *Proceedings of 5th IEEE Symposium on Logic in Computer Science* (1990) pp. 79–94.

[15] W. van der Hoek, J.J.Ch. Meyer, Y.H. Tan and C. Witteveen, eds., *Non-Monotonic Reasoning and Partial Semantics* (Ellis Horwood, 1992).

[16] J. Pearl, From Adams' conditionals to default expressions, causal conditionals, and counterfactuals, in: *Festschrift for Ernest Adams* (Cambridge University Press) (to appear).

[17] F. Pereira and D.H.D. Warren, Definite clause grammars for language analysis: A survey of the formalism and a comparison with augmented transition networks, *Artificial Intelligence* 13 (1980) 231–278.

[18] R. Reiter, A logic for default reasoning, *Artificial Intelligence* 13 (1980) 81–132.

[19] W. Rounds and G.-Q. Zhang, Domain theory meets default logic, *Logic and Computation* 5 (1995) 1–25.

[20] D.S. Scott, Domains for denotational semantics, in: *Lecture Notes in Computer Science* 140 (1982) pp. 577–613.

[21] S. Vickers, *Topology via Logic* (Cambridge University Press, 1989).

[22] G. Winskel, *The Formal Semantics of Programming Languages* (MIT Press, Boston, 1992).

[23] M. Young and W. Rounds, A logical semantics for nonmonotonic feature structures, in: *Proc. ACL Symp. on Computational Linguistcs* (1993) pp. 209–215.

[24] G.-Q. Zhang, *Logic of Domains* (Birkhauser, Boston, 1991).