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A GENERALIZATION OF THE HAUSDORFF–YOUNG THEOREM

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Abstract. Considering mixed-norm sequence spaces $\ell^{p,q}$, $p,q \geq 1$, C. N. Kellogg proved the following theorem: if $1 < p \leqq 2$ then $\widehat{L^p} \subset \ell^{p',2}$ and $\ell^{p,2} \subset \widehat{L^{p'}}$, where $1/p + 1/p' = 1$. This result extends the Hausdorff–Young Theorem.

We introduce here multiple mixed-norm sequence spaces $\ell^{p,q_1,q_2,\dots,q_n}$, examine their properties and characterize the multipliers of spaces of the form $\ell^{p,[s;n],q}$, with the index s repeated n times. By an interpolation-type argument we prove that $\widehat{L^p} \subset \left(\ell^{\infty,[2;n],2}, \ell^{p',[1;n],1}\right)$ for $1 < p \leqq 2$. Using these results we obtain a further generalization of the Hausdorff–Young Theorem: if $1 < p \leqq 2$ then $\widehat{L^p} \subset \ell^{p',[2;n]}$ and $\ell^{p,[2;n]} \subset \widehat{L^{p'}}$ for each $n = 0, 1, 2, \dots$. The spaces $\ell^{p',[2;n]}$ decrease and $\ell^{p,[2;n]}$ increase properly with n for $1 < p < 2$ and $1/p + 1/p' = 1$. We also extend a theorem of J. H. Hedlund on multiplers of Hardy spaces $(\widehat{H^p}, \widehat{H^2})$ and deduce other results.

1. Introduction

For $p \geqq 1$, L^p denotes the Banach space of all 2π-periodic real or complex valued integrable functions with norm $\|f\|_{L^p} := \left(\frac{1}{2\pi}\int |f|^p\right)$, where the integral is taken over any interval of length 2π. L^∞ is the Banach space of all 2π-periodic real or complex valued essentially bounded functions with the essential supremum norm. If the series $\sum_{k\in\mathbf{Z}} c_k\, e^{ikx}$ is the Fourier series of a function $f \in L^1$ we write $\widehat{f} = \left(\widehat{f}(k)\right)_{k\in\mathbf{Z}}$ for the sequence of coefficients $(c_k)_{k\in\mathbf{Z}}$ and $s_n f$ for the partial sums s_n, $n = 0, 1, \dots$. Let $\widehat{L^p} := \{\widehat{f} : f \in L^p\}$. Clearly, $\widehat{L^p}$ is a Banach space of two-way sequences, under the induced norm $\|\widehat{f}\|_{\widehat{L^p}} := \|f\|_{L^p}$. For $p \geqq 1$, Hardy spaces and the spaces of their Fourier coefficients are denoted by H^p and $\widehat{H^p}$ respectively, i.e., $H^p = \{f \in L^p : \widehat{f}(k) = 0 \text{ for } k \in \mathbf{Z}^-\}$ and $\widehat{H^p} = \{\widehat{f} : f \in H^p\}$.

We refer to the following standard sequence spaces of two-way, respectively, one-way sequences: $\mathbf{c}_0$ — the space of all null sequences and ℓ^∞ — the space of all bounded sequences, under the norm $\|c\|_{\ell^\infty} := \sup_k |c_k|$; ℓ^p, $1 \leqq p < \infty$, — the space of all sequences (c_k) such that $\|c\|_{\ell^p} := \left(\sum_{k\in\mathbf{Z}} |c_k|^p\right)^{1/p}$ is finite.

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One of the unfortunate facts in Fourier analysis is that there are no suitable characterizations of the spaces $\widehat{L^p}$, $p \geqq 1$, except for $p = 2$, in which case it is well known that $\widehat{L^2} = \ell^2$. All available descriptions of these spaces of Fourier coefficients, in terms of sequences alone, involve only one-way inclusions. A classical result is the Hausdorff–Young Theorem: for $1 \leqq p \leqq 2$, $\widehat{L^p} \subset \ell^{p'}$ and $\ell^p \subset \widehat{L^{p'}}$.

Here, and throughout the paper we will assume that for $p \geqq 1$, $1/p + 1/p' = 1$ where p' is interpreted to be ∞ whenever $p = 1$.

The inclusions of the Hausdorff–Young Theorem are only interesting for $1 < p < 2$. Namely, for $p = 2$ they clearly reduce to the above mentioned equality $\widehat{L^2} = \ell^2$, and for $p = 1$ to the inclusions $\widehat{L^1} \subset \ell^\infty$ and $\ell^1 \subset \widehat{L^\infty}$. Compared to the well known facts that $\widehat{L^1} \subset \mathbf{c}_0$ and $\ell^1 \subset \widehat{\mathcal{A}}$, *the latter inclusions* are too weak. Here $\mathcal{A}$ denotes the space of absolutely convergent Fourier series. It is also well known that the analogue of the Hausdorff–Young Theorem for $p > 2$ does not hold, see for example [6, 14.4].

C. N. Kellogg gave a first proper generalization of the Hausdorff–Young Theorem by proving the following result [10, Theorems 3 and 4]:

THEOREM K. *If* $1 < p \leqq 2$ *then* $\widehat{L^p} \subset \ell^{p',2}$ *and* $\ell^{p,2} \subset \widehat{L^{p'}}$.

Here, $\ell^{p,q}$, $p, q \geqq 1$, is a mixed-norm space of all sequences $c = (c_k)_{k\in\mathbf{Z}}$ such that the sequence of ℓ^p-norms of the dyadic sections of c, $d^j c = \sum_{2^j \leqq |k| < 2^{j+1}} c_k e^k$, belongs to the space ℓ^q, i.e., $\left(\|d^j c\|_{\ell^p}\right)_{j=0}^{\infty} \in \ell^q$. Since for $1 < p < 2$, $\ell^{p',2} \subset \ell^{p'}$ and $\ell^p \subset \ell^{p,2}$ hold properly, Kellogg's theorem is a proper extension of the Hausdorff–Young Theorem. For $p = 2$ the statements of Theorem K also reduce to the equality $\widehat{L^2} = \ell^2$ because clearly $\ell^{2,2} = \ell^2$.

REMARK 1. The inclusions of Theorem K are not valid for $p = 1$.

Kellogg remarked in [10, p. 125] that the restriction $p > 1$ is necessary for the inclusion $\widehat{L^p} \subset \ell^{p',2}$ proving that there exists a function $f \in L^1$ such that $\widehat{f} \notin \ell^{\infty,2}$.

We supplement Kellogg's remark by showing that the restriction $p > 1$ is also necessary for the second inclusion $\ell^{p,2} \subset \widehat{L^{p'}}$, i.e., there exists $c \in \ell^{1,2}$ such that $c \notin \widehat{L^\infty}$. Namely, let $c = (c_k)_{k\in\mathbf{Z}}$ be an even sequence with $c_k = c_{-k} = 1/(k \log k)$ for $k = 2, 3, \dots$ and 0 otherwise. Then $\|d^j c\|_{\ell^1} \leqq 1/j$ and therefore $c \in \ell^{1,2}$. The corresponding trigonometric series with coefficients $c = (c_k)_{k\in\mathbf{Z}}$ is the cosine series $\sum_{k=2}^{\infty} \cos kx/(k \log k)$. It converges pointwise for all $x \neq 0 \pmod{2\pi}$ to a function $f \in L^1$ and is therefore the Fourier series of that function [6, 7.3.1]. However, as explained in [6, 12.8.3] $f \notin L^\infty$. Consequently $c = \widehat{f} \in \ell^{1,2}$ and $c = \widehat{f} \notin \widehat{L^\infty}$.

Kellogg proved Theorem K by giving a characterization of the multipliers $(\ell^{r,s}, \ell^{u,v})$ [10, Theorem 1] and by appealing to the following result of J. H. Hedlund on multipliers of Hardy spaces [8, Theorem 1]:

THEOREM H. *If* $1 \leqq p < 2$ *and* $q = 2p/(2-p)$ *then* $\ell^{q,\infty} \subset (\widehat{H^p}, \widehat{H^2})$.

Our aim is to extend these results to multiple mixed-norm sequence spaces of type $\ell^{p,q_1,q_2,\dots,q_n}$ leading to a generalization of Kellogg's, and therefore of the Hausdorff–Young Theorem for $1 < p < 2$, with $\ell^{p,2}$, respectively ℓ^p, replaced by multiple mixed-norm spaces $\ell^{p,[2;n]}$ where $[2;n]$ stands for the index 2 repeated n times. The spaces $\ell^{p,q_1,q_2,\dots,q_n}$ are defined inductively by requiring that $c \in \ell^{p,q_1,q_2,\dots,q_n}$ if and only if the sequence of $\ell^{p,q_1,q_2,\dots,q_{n-1}}[d_j]$-norms of the dyadic sections $d^j c$ belongs to the space ℓ^{q_n}. Precise definitions of these spaces are given in Section 2.

Characterizing the multipler spaces $\left(\ell^{r,[t;n],s}, \ell^{u,[w;n],v}\right)$ and extending some of the ideas used in the proof of the Marcinkievicz's theorem for operators of weak type to such operators on multiple mixed-norm spaces, we derive a new generalization of the Hausdorff–Young Theorem. We prove that for $1 < p \leqq 2$ and for each $n = 1, 2, \dots$, $\widehat{L^p} \subset \ell^{p',[2;n]}$ and $\ell^{p,[2;n]} \subset \widehat{L^{p'}}$, where $\ell^{p',[2;n]}$ decrease and $\ell^{p,[2;n]}$ increase properly with n. Consequently, $\widehat{L^p} \subset \bigcap_{n=0}^{\infty} \ell^{p',[2;n]}$ and $\bigcup_{n=0}^{\infty} \ell^{p,[2;n]} \subset \widehat{L^{p'}}$.

We also deduce that the above statement on multipliers of Hardy spaces, i.e., Theorem H, can be properly extended for $1 < p < 2$ to the inclusion $\ell^{q,[\infty;n]} \subset (\widehat{H^p}, \widehat{H^2})$ valid for all $n = 1, 2, \dots$ with a remark that the spaces $\ell^{q,[\infty;n]}$ increase properly with n. We observe that for $p = 1$ not only does the above generalization of Theorem K fail if $n \geqq 1$, but also that the corresponding generalization of Theorem H fails for $p = 1$ and $n > 1$. Hence, our approach turns out to be quite different from that of Kellogg [10], which relies on properties of operators of strong type, i.e., in this case on the corresponding properties of Hardy spaces for $p = 1$ and $p = 2$, and a result of Hedlund on operators of strong type [8, Theorem 2].

We discuss also other properties of the multiple mixed-norm spaces, in particular the inclusion relations, observe that they are BK spaces, but that the above intersections and unions are not, except for $p = 2$, and deduce several implications of our results.

2. Multiple mixed-norm spaces

In this section we give the basic definitions and discuss some of the essential properties of multiple mixed-norm sequence spaces.

Let $d_0 = \{-1, 0, 1\}$ and for $j = 1, 2, \dots$ let $d_j = \left\{k \in \mathbf{Z} : 2^j \leqq |k| < 2^{j+1}\right\}$ be the j-th dyadic block of $\mathbf{Z}$. Clearly $(d_j)_{j=0}^{\infty}$ forms a partition of the integers, i.e., $\mathbf{Z} = \bigcup_{j=0}^{\infty} d_j$. This allows breaking any sequence $c = (c_k)_{k \in \mathbf{Z}}$ into a sequence of dyadic sections $(d^j c)_{j=0}^{\infty}$, where $d^j c = \sum_{k \in d_j} c_k e^k$ and e^k denotes the sequence with 1 at the kth place and 0 elsewhere.

For $1 \leqq p, q \leqq \infty$ the mixed-norm space $\ell^{p,q}$ is defined as the set of all $c = (c_k)_{k \in \mathbf{Z}}$ such that $\left(\|d^j c\|_{\ell^p}\right)_{j=0}^{\infty} \in \ell^q$ with the norm

$$\|c\|_{\ell^{p,q}} := \left(\sum_{j=0}^{\infty} \|d^j c\|_{\ell^p}^q \right)^{1/q}, \quad 1 \leqq q < \infty; \quad \|c\|_{\ell^{p,\infty}} := \sup_j \|d^j c\|_{\ell^p}.$$

The multiple mixed-norm spaces introduced here are defined using further dyadic partitions of the blocks d_j and multiple refinements of the dyadic sections of sequences $c = (c_k)_{k \in \mathbf{Z}}$. More precisely, we consider n-fold dyadic subblocks of d_j defined for $j = 1, 2, \dots$ as follows:

$$d_{j00\cdots0} := \left\{ k \in \mathbf{Z} : |k| = 2^j, 2^j + 1 \right\},$$

$$d_{ji_1\cdots i_m 0\cdots 0} := \left\{ k \in \mathbf{Z} : |k| = 2^j + 2^{i_1} + \cdots + 2^{i_m}, 2^j + 2^{i_1} + \cdots + 2^{i_m} + 1 \right\}$$

for $1 \leqq i_m < i_{m-1} < \cdots < i_1 < j$, $m = 1, 2, \dots, n-1$ and

$$d_{ji_1\cdots i_n} := \left\{ k \in \mathbf{Z} : 2^j + 2^{i_1} + \cdots + 2^{i_{n-1}} + 2^{i_n} \leqq |k| < 2^j + 2^{i_1} + \cdots + 2^{i_{n-1}} + 2^{i_n+1} \right\}$$

for $1 \leqq i_n < i_{n-1} < \cdots < i_1 < j$. For $j = 0$ we define $d_{0\cdots0} := d_0$. The $(n+1)$-fold refinements of the form $d_{ji_1\cdots i_m 0\cdots 0}$ can be termed 'degenerate'. We may observe that each block d_j of index $j \leqq n$ is partitioned precisely into these degenerate subblocks. This of course is not the case if $j > n$.

Since for a fixed $n = 1, 2, \dots$ and for each j the blocks $d_{ji_1\cdots i_n}$ form a partition of the d_j block we can write $d_{ji_1\cdots i_{n-1}} = \bigcup_{i_n=0}^{i_{n-1}-1} d_{ji_1\cdots i_n}, \dots$ and

$$d_j = \bigcup_{i_1=0}^{j-1} \bigcup_{i_2=0}^{i_1-1} \cdots \bigcup_{i_n=0}^{i_{n-1}-1} d_{ji_1\cdots i_n}. \tag{2.1}$$

Here, if $i_m = 0$ for some $m = 1, 2, \dots, n-1$, the unions over the indices $i_m, i_{m+1}, \dots, i_n$ contain only the corresponding 0 members, i.e., degenerate blocks. Clearly,

$$\mathbf{Z} = \bigcup_{j=0}^{\infty} \bigcup_{i_1=0}^{j-1} \bigcup_{i_2=0}^{i_1-1} \cdots \bigcup_{i_n=0}^{i_{n-1}-1} d_{ji_1\cdots i_n},$$

i.e., the set of all integers $\mathbf{Z}$ is partitioned into $(n+1)$-fold blocks.

For a sequence $c = (c_k)_{k \in \mathbf{Z}}$ we define the $(n+1)$-fold dyadic section of c by

$$d^{j i_1 \cdots i_n} c := \sum_{k \in d_{j i_1 \cdots i_n}} c_k \, e^k. \tag{2.2}$$

For $1 \leqq p, q_1, q_2, \ldots, q_n \leqq \infty$ the n-fold mixed-norm space $\ell^{p, q_1, q_2, \ldots, q_n}$ is defined as the set of all sequences $c = (c_k)_{k \in \mathbf{Z}}$ such that

$$\|c\|_{\ell^{p,q_1,q_2,\ldots,q_n}} := \left(\sum_{j=0}^{\infty} \left(\sum_{i_1=0}^{j-1} \left(\sum_{i_2=0}^{i_1-1} \cdots \left(\sum_{i_{n-1}=0}^{i_{n-2}-1} \|d^{j i_1 \cdots i_{n-1}} c\|_{\ell^p}^{q_1} \right)^{q_2/q_1} \cdots \right)^{q_{n-1}/q_{n-2}} \right)^{q_n/q_{n-1}} \right)^{1/q_n} < \infty. \tag{2.3}$$

Here, if $j = 0$ or $i_m = 0$ for some m, then all lower indices range only over 0. The corresponding norms and summations are to be properly modified if any of the parameters $p, q_1, \ldots, q_n$ is ∞.

Restricting the domain of a sequence c to a fixed finite block of integers d_j, we define $\ell^{p,q_1,q_2,\ldots,q_n}[d_j]$ as the space of $c = (c_k)_{k \in d_j}$ with the norm

$$\|c\|_{\ell^{p,q_1,q_2,\ldots,q_n}[d_j]} := \left(\sum_{i_1=0}^{j-1} \left(\sum_{i_2=0}^{i_1-1} \cdots \left(\sum_{i_n=0}^{i_{n-1}-1} \|d^{j i_1 \cdots i_n} c\|_{\ell^p}^{q_1} \right)^{q_2/q_1} \cdots \right)^{q_n/q_{n-1}} \right)^{1/q_n} < \infty. \tag{2.4}$$

We observe that for $c = (c_k)_{k \in \mathbf{Z}}$, $\|d^j c\|_{\ell^{p,q_1,q_2,\ldots,q_n}[d_j]} \neq \|d^j c\|_{\ell^{p,q_1,q_2,\ldots,q_n}}$, since the former norm is obtained through the n-fold dyadic partition of d_j, which is the $(n+1)$-fold dyadic partition of $\mathbf{Z}$, and the latter through the $(n-1)$-fold partition of d^j, i.e., the n-fold partition of Z. However, $\|d^j c\|_{\ell^{p,q_1,q_2,\ldots,q_n}[d_j]} = \|d^j c\|_{\ell^{p,q_1,q_2,\ldots,q_n,r}}$ for any $r \geqq 1$.

REMARK 2. From the above definitions it follows that $c \in \ell^{p,q_1,q_2,\ldots,q_n}$ if and only if the sequence of $\ell^{p,q_1,q_2,\ldots,q_{n-1}}[d_j]$-norms of $d^j c$ belongs to the space ℓ^{q_n}, i.e., $\left(\|d^j c\|_{\ell^{p,q_1,q_2,\ldots,q_{n-1}}[d_j]} \right)_{j=0}^{\infty} \in \ell^{q_n}$. Consequently, $\ell^{p,q_1,q_2,\ldots,q_n}$ is a composed space in the terminology of Jakimovski and Russell [9], composed from ℓ^{q_n} and the sequence of spaces $\left(\ell^{p,q_1,q_2,\ldots,q_{n-1}}[d_j] \right)_{j=0}^{\infty}$, and we can write $\ell^{p,q_1,q_2,\ldots,q_n} = \ell^{q_n}\left(\ell^{p,q_1,q_2,\ldots,q_{n-1}}[d_j] \right)$.

The last remark exhibits the inductive nature of our definition of n-fold mixed-norm spaces. It allows applications of rather general results proved in [9], from which we can immediately deduce that these are BK spaces and obtain their duals.

A sequence space E is a BK space if E is a Banach space with continuous coordinatewise functionals and E contains all finite sequences. We say that a normed sequence space E has the property AK if for each $c \in E$ we have $\|s^n c - c\|_E \to 0$ as $n \to \infty$, where $s^n c = \sum_{k=0}^{n} c_k e^k$. We say that E is *solid* if $c \in E$, $x \in \omega$ and $|x_k| \leqq |c_k|$ for each k imply that $x \in E$ and $\|x\|_E \leqq \|c\|_E$. The corresponding definitions are similar for spaces of two-way sequences. As usual by E^* we will denote the functional dual to E.

PROPOSITION 1. *Let $1 \leqq p, q_1, q_2, \dots, q_{n-1} \leqq \infty$ and $1 \leqq q_n < \infty$. Then*

i) *$\ell^{p,q_1,q_2,\dots,q_n}$ and $\ell^{p,q_1,q_2,\dots,q_n}[d_j]$ with the norms (2.3) and (2.4), respectively, are BK spaces, each is solid and has the property AK.*

ii) *$(\ell^{p,q_1,q_2,\dots,q_n})^*$ and $\left(\ell^{p,q_1,q_2,\dots,q_n}[d_j]\right)^*$ can be identified with $\ell^{p',q_1',q_2',\dots,q_n'}$ and $\ell^{p',q_1',q_2',\dots,q_n'}[d_j]$, respectively, where $1/p + 1/p' = 1$ and $1/q_i + 1/q_i' = 1$ for $i = 1, 2, \dots, n$.*

PROOF. Both statements follow by induction from [9, Theorems 1 and 2].

Clearly ℓ^q is a BK space, ℓ^q is solid and has the property AK. Furthemore, by Remark 2, $\ell^{p,q_1,q_2,\dots,q_n} = \ell^{q_n}(\ell^{p,q_1,q_2,\dots,q_{n-1}}[d_j])$. Hence, in this composed space, the outer space ℓ^{q_n} satisfies the assumptions of Theorems 1 and 2 in [9]. Since $\left(\ell^p[d_j]\right)_{j=0}^{\infty}$ is a sequence of BK spaces by [9, Theorem 1], it follows that $\ell^{p,q}$ and $\ell^{p,q}[d_j]$ are BK spaces. So, suppose that $\ell^{p,q_1,q_2,\dots,q_{n-1}}$ and $\ell^{p,q_1,q_2,\dots,q_{n-1}}[d_j]$ are BK spaces. Then from the above observations and [9, Theorem 1] it follows that $\ell^{p,q_1,q_2,\dots,q_n}$ and $\ell^{p,q_1,q_2,\dots,q_n}[d_j]$ are BK spaces. The claim that each of these spaces is solid and that it has the property AK is almost obvious form (2.3) and (2.4). This verifies i).

Since $(\ell^q)^*$ can be identified with $\ell^{q'}$ and $(\ell^p)^*$ with $\ell^{p'}$, from [9, Theorem 2] it follows that $(\ell^{p,q})^*$ can be identified with $\ell^{p',q'}$. Assuming that ii) holds for $n-1$, from the above argument and [9, Theorem 2] it follows that the dual of the space $\ell^{p,q_1,q_2,\dots,q_n}$ is given by $(\ell^{p,q_1,q_2,\dots,q_n})^* = (\ell^{q_n})^*\left(\left(\ell^{p,q_1,q_2,\dots,q_{n-1}}[d_j]\right)^*\right)$, which can be identified with

$$\ell^{q_n'}\left(\ell^{p',q_1',q_2',\dots,q_{n-1}'}[d_j]\right) = \ell^{p',q_1',q_2',\dots,q_n'}.$$

Thus statement ii) is valid for all n.

REMARK 3. If $1 \leqq p, q_1, q_2, \dots, q_n \leqq \infty$, $p \leqq u$ and $q_i \leqq v_i$ for $i = 1, 2, \dots, n$ then $\ell^{p,q_1,q_2,\dots,q_n} \subset \ell^{u,v_1,v_2,\dots,v_n}$ and $\ell^{p,q_1,q_2,\dots,q_n}[d_j] \subset \ell^{u,v_1,v_2,\dots,v_n}[d_j]$. This follows immediately from (2.3) and (2.4).

That the above inclusions are also proper whenever at least one pair of indices satisfies a proper inequality can be shown by considering appropriate examples. Since we are interested in the multiple mixed-norm spaces of special type, namely the case when $q_1 = q_2 = \dots = q_{n-1} = z$ and $q_n = q$, denoted by $\ell^{p,[z;n],q}$, the following result verifies the proper inclusions only for these spaces. We recall that by (2.3) the corresponding norm in this case can be written as

$$(2.5)\qquad \|c\|_{\ell^{p,[z;n],q}} := \left(\sum_{j=0}^{\infty}\left(\sum_{i_1=0}^{j-1}\sum_{i_2=0}^{i_1-1}\cdots\sum_{i_n=0}^{i_{n-1}-1}\|d^{ji_1\cdots i_n}c\|_{\ell^p}^{z}\right)^{q/z}\right)^{1/q} < \infty,$$

$$\|c\|_{\ell^{p,[\infty;n],q}} := \left(\sum_{j=0}^{\infty}\left(\sup_{i_n<i_{n-1}<\cdots i_1<j}\|d^{ji_1\cdots i_n}c\|_{\ell^p}\right)^{q}\right)^{1/q} < \infty.$$

Finally we remark that for the sake of simplicity we shall sometimes write

$$\left(\sum_j\left(\sum_{i_1}\sum_{i_2}\cdots\sum_{i_n}\|d^{ji_1\cdots i_n}c\|_{\ell^p}^{z}\right)^{q/z}\right)^{1/q}$$

for the expression defining $\|c\|_{\ell^{p,[z;n],q}}$ in (2.5) and use similar abbreviations for other such expressions.

PROPOSITION 2. *Let $p,q,z,r,s\in[1,\infty]$. Then*

i) $\ell^{p,[z;n],q}\subset\ell^{r,[z;n],s}$ *if $p\leqq r$ and $q\leqq s$; the inclusion is proper if $p<r$ or $q<s$;*

ii) $\ell^{p,[z;n-1],q}\subset\ell^{p,[z;n],q}$ *properly if $1\leqq p<z$; $\ell^{p,[z;n],q}\subset\ell^{p,[z;n-1],q}$ properly if $p>z$ and $\ell^{z,[z;n-1],q}=\ell^{z,q}$.*

In order to get on with our main theorems we defer the proof of Proposition 2 to the Appendix.

3. Multipliers of mixed-norm spaces

In this section we characterize the multipliers of multiple mixed-norm spaces of the type considered in Proposition 2, i.e., multipliers of the form $\left(\ell^{r,[t;n],s},\ell^{u,[w;n],v}\right)$ for arbitrary indices $r,s,t,u,v,w\in[1,\infty]$. Our result extends the mentioned theorem of Kellogg [10, Theorem 1] in a desirable fashion. Naturally, the proof goes along the standard lines, with somewhat more involved calculations. Since for the purpose of this paper it suffices to characterize the multiplers of mixed-norm spaces of the above type we do not present here a more general result which could be proved similarly.

THEOREM 1. *Let $1\leqq r,s,t,u,v,w\leqq\infty$ and let the indices p, q, z be defined by $1/p=1/u-1/r$, if $r>u$, $p=\infty$ if $r\leqq u$; $1/q=1/v-1/s$, if $s>v$, $q=\infty$ if $s\leqq v$; $1/z=1/w-1/t$, if $t>w$, $w=\infty$ if $t\leqq w$. Then*

$$\left(\ell^{r,[t;n],s},\ell^{u,[w;n],v}\right)=\ell^{p,[z;n],q}.$$

PROOF. Suppose that $1 \leqq r \leqq u \leqq \infty$, $1 \leqq s \leqq v \leqq \infty$ and $1 \leqq t \leqq w \leqq \infty$. Then by Proposition 2 clearly $\ell^{r,[t;n],s} \subset \ell^{u,[w;n],v}$. We shall verify that

$$\ell^{\infty} \subset \left(\ell^{r,[t;n],s}, \ell^{u,[w;n],v}\right) \subset \ell^{\infty}.$$

The first inclusion follows immediately observing that $\ell^{r,[t;n],s}$ is solid, by Proposition 1. To verify the second inclusion let $\lambda \in \left(\ell^{r,[t;n],s}, \ell^{u,[w;n],v}\right)$ and define T_λ by $T_\lambda c = \lambda c$ for $c \in \ell^{r,[t;n],s}$. Then by the Closed Graph Theorem T_λ is a bounded linear operator from $\ell^{r,[t;n],s}$ into $\ell^{u,[w;n],v}$ and it is trivial to see that $\|T_\lambda\| = \|\lambda\|_{\ell^\infty}$, so that $\lambda \in \ell^\infty$. Observing that for $p, q = \infty$, $\ell^\infty = \ell^{p,[\infty;n],q}$, the proof is complete in this case.

We suppose now that $r > u$, $v > s$ and $t > w$ and that p, q, z are as defined above. We first verify that $\ell^{p,[z;n],q} \subset \left(\ell^{r,[t;n],s}, \ell^{u,[w;n],v}\right)$. So let $\lambda \in \ell^{p,[z;n],q}$ and $c \in \ell^{r,[t;n],s}$. Then from (2.5), applying Hölder's inequality first to the innermost sum, with respect to the index $r/u > 1$, then to the next outer sum, with respect to the index $t/w > 1$, and finally to the outermost sum, with respect to the index $s/v > 1$ we obtain

$$\text{(3.1)}\quad \|\lambda c\|_{\ell^{u,[w;n],v}} = \left(\sum_{j=0}^{\infty}\left(\sum_{i_1=0}^{j-1}\cdots\sum_{i_n=0}^{i_{n-1}-1}\left(\sum_{k\in d_{ji_1\cdots i_n}}|\lambda_k c_k|^u\right)^{w/u}\right)^{v/w}\right)^{1/v}$$

$$\leqq \left(\sum_{j=0}^{\infty}\left(\sum_{i_1=0}^{j-1}\cdots\sum_{i_n=0}^{i_{n-1}-1}\left(\sum_{k\in d_{ji_1\cdots i_n}}|\lambda_k|^p\right)^{w/p}\left(\sum_{k\in d_{ji_1\cdots i_n}}|c_k|^r\right)^{w/r}\right)^{v/w}\right)^{1/v}$$

$$\leqq \Bigg[\sum_{j=0}^{\infty}\left(\sum_{i_1=0}^{j-1}\cdots\sum_{i_n=0}^{i_{n-1}-1}\left(\sum_{k\in d_{ji_1\cdots i_n}}|\lambda_k|^p\right)^{z/p}\right)^{v/z} \cdot\left(\sum_{i_1=0}^{j-1}\cdots\sum_{i_n=0}^{i_{n-1}-1}\left(\sum_{k\in d_{ji_1\cdots i_n}}|c_k|^r\right)^{t/r}\right)^{v/t}\Bigg]^{1/v}$$

$$\leqq \left[\sum_{j=0}^{\infty}\left(\sum_{i_1=0}^{j-1}\cdots\sum_{i_n=0}^{i_{n-1}-1}\left(\sum_{k\in d_{ji_1\cdots i_n}}|\lambda_k|^p\right)^{z/p}\right)^{q/z}\right)^{1/q} \cdot\left(\sum_{j=0}^{\infty}\left(\sum_{i_1=0}^{j-1}\cdots\sum_{i_n=0}^{i_{n-1}-1}\left(\sum_{k\in d_{ji_1\cdots i_n}}|c_k|^r\right)^{t/r}\right)^{s/t}\right]^{1/s} = \|\lambda\|_{\ell^{p,[z;n],q}}\|c\|_{\ell^{r,[t;n],s}}.$$

Hence, $\lambda c \in \ell^{u,[w;n],v}$. Consequently $\ell^{p,[z;n],q} \subset \left(\ell^{r,[t;n],s}, \ell^{u,[w;n],v}\right)$.

To verify the reverse inclusion let $\lambda \in \left(\ell^{r,[t;n],s}, \ell^{u,[w;n],v}\right)$ and define, as before, $T_\lambda c = \lambda\, c$ for $c \in \ell^{r,[t;n],s}$. By the Closed Graph Theorem T_λ is a bounded linear operator from $\ell^{r,[t;n],s}$ into $\ell^{u,[w;n],v}$, with the norm $\|T_\lambda\|$. Now, for each positive integer J define a bounded linear operator T_λ^J from $\ell^{r,[t;n],s}$ into $\ell^{u,[w;n],v}$ by the equation

$$T_\lambda^J c = s^{2^J} \lambda c = \sum_{k=0}^{2^J} \lambda_k\, c_k\, e^k .$$

Applying the inequality (3.1) to the above product of sequences $s^{2^J}\lambda$ and c we see that

$$\|s^{2^J}\lambda c\|_{\ell^{u,[w;n],v}} \leqq \|s^{2^J}\lambda\|_{\ell^{p,[z;n],q}} \|c\|_{\ell^{r,[t;n],s}} .$$

Consequently,

$$\|T_\lambda^J\| \leqq \|s^{2^J}\lambda\|_{\ell^{p,[z;n],q}} . \tag{3.2}$$

We shall show now that the reverse inequality also holds for $J \geqq J_0$ where the integer J_0 is chosen for the above sequence λ so that $\|s^{2^{J_0}}\lambda\|_{\ell^1} \neq 0$.

To prove the reverse of (3.2), for a fixed $J \geqq J_0$ we define a sequence $c = (c_k)_{k\in\mathbf{Z}}$ as follows:

$$c_k = \|s^{2^J}\lambda\|_{\ell^{p,[z;n],q}}^{-q/s} \|d^j\lambda\|_{\ell^{p,[z;n],q}}^{q/s-z/t} \|d^{ji_1\cdots i_n}\lambda\|_{\ell^p}^{z/t-p/r} |\lambda_k|^{p/r} \tag{3.3}$$

for $k \in d_j$, with $\|d^j\lambda\|_{\ell^{p,[z;n],q}} \neq 0$ and $\|d^{ji_i\cdots i_n}\lambda\|_{\ell^p} \neq 0$, and $c_k = 0$ otherwise. We will verify now that $\|c\|_{\ell^{r,[t;n],s}} = 1$ and $\|T_\lambda^J c\|_{\ell^{u,[w;n],v}} = \|s^{2^J}\lambda\|_{\ell^{p,[z;n],q}}$, which imply the reverse inequality of (3.2) and therefore the conclusion that for all $J \geqq J_0$,

$$\|T_\lambda^J\| = \|s^{2^J}\lambda\|_{\ell^{p,[z;n],q}} . \tag{3.4}$$

From (2.5) and (3.3) after performing simple calculations we see that

$$\|c\|_{\ell^{r,[t;n],s}} = \|s^{2^J}\lambda\|_{\ell^{p,[z;n],q}}^{-q/s}$$

$$\cdot \left[\sum_{j=J_0}^{J-1} \|d^j\lambda\|^{q-sz/t}{}_{\ell^{p,[z;n],q}} \left(\sum_{i_1=0}^{j-1} \cdots \sum_{i_n=0}^{i_{n-1}-1} \|d^{ji_1\cdots i_n}\lambda\|_{\ell^p}^{t(z/t-p/r)} \right.\right.$$

$$\left.\left. \cdot \left(\sum_{k\in d_{ji_1\cdots i_n}} |\lambda_k|^p \right)^{t/r} \right)^{s/t} \right]^{1/s}$$

$$= \|s^{2^J}\lambda\|_{\ell^{p,[z;n],q}}^{-q/s} \cdot \left[\sum_{j=J_0}^{J-1} \left(\sum_{i_1=0}^{j-1} \cdots \sum_{i_n=0}^{i_{n-1}-1} \left(\sum_{k\in d_{ji_1\cdots i_n}} |\lambda_k|^p \right)^{z/p} \right)^{q/z-s/t} \right.$$

$$\left. \cdot \left(\sum_{i_1=0}^{j-1} \cdots \sum_{i_n=0}^{i_{n-1}-1} \left(\sum_{k\in d_{ji_1\cdots i_n}} |\lambda_k|^p \right)^{z/p} \right)^{s/t} \right]^{1/s}$$

$$= \|s^{2^J}\lambda\|_{\ell^{p,[z;n],q}}^{-q/s} \|s^{2^J}\lambda\|_{\ell^{p,[z;n],q}}^{q/s} = 1.$$

This verifies the first claim, i.e., that $\|c\|_{\ell^{r,[t;n],s}} = 1$.

Similarly, from (2.5) and (3.3) we have

$$\|T_\lambda^J c\|_{\ell^{u,[w;n],v}}^v = \|s^{2^J}\lambda\|_{\ell^{p,[z;n],q}}^{-v\frac{q}{s}}$$

$$\cdot \sum_{j=J_0}^{J-1} \|d^j\lambda\|_{\ell^{p,[z;n],q}}^{v\left(\frac{q}{s}-\frac{z}{t}\right)} \left(\sum_{i_1=0}^{j-1} \cdots \sum_{i_n=0}^{i_{n-1}-1} \left(\sum_{k\in d_{ji_1\cdots i_n}} |\lambda_k|^p \right)^{\frac{w}{p}\left(\frac{z}{t}-\frac{p}{r}\right)} \right.$$

$$\left. \cdot \left(\sum_{k\in d_{ji_1\cdots i_n}} |\lambda_k|^p \right)^{\frac{w}{u}} \right)^{\frac{v}{w}}.$$

Observing that $w/u - w/r + (w/p)(z/t) = z/p$ and that $vq/s - vz/t + vz/w = q$, we obtain

$$\|T_\lambda^J c\|_{\ell^{u,[w;n],v}}$$

$$= \|s^{2^J}\lambda\|_{\ell^{p,[z;n],q}}^{-q/s} \cdot \left(\sum_{j=J_0}^{J-1} \|d^j\lambda\|_{\ell^{p,[z;n],q}}^{v(q/s-z/t)} \cdot \|d^j\lambda\|_{\ell^{p,[z;n],q}}^{vz/w} \right)^{1/v} = \|s^{2^J}\lambda\|_{\ell^{p,[z;n],q}}$$

which is the second equality claimed above.

Thus (3.4) has been established to hold for all $J \geqq J_0$. But clearly, by i) of Proposition 1, the right side of (3.4) converges to $\|\lambda\|_{\ell^{p,[z;n],q}}$. Furthermore, since T_λ is a bounded operator and $\|T_\lambda^J\| \leqq \|T_\lambda\|$, the last observation and (3.4) imply that $\|\lambda\|_{\ell^{p,[z;n],q}}$ is finite, i.e., that $\lambda \in \ell^{p,[z;n],q}$. Hence, $\left(\ell^{r,[t;n],s}, \ell^{u,[w;n],v}\right) \subset \ell^{p,[z;n],q}$. Consequently $\left(\ell^{r,[t;n],s}, \ell^{u,[w;n],v}\right) = \ell^{p,[z;n],q}$ in this case.

The equality $\left(\ell^{r,[t;n],s}, \ell^{u,[w;n],v}\right) = \ell^{p,[z;n],q}$ in the other remaining cases is verified in almost precisely the same way, just interpreting in the above argument the corresponding indices p, z or q as ∞ and the corresponding norms as sup-norms.

REMARK 4. Taking $n = 0$ in Theorem 1 we obtain Theorem 1 in [10]. We conjecture that a similar characterization holds for multipliers of general $(n+1)$-fold mixed-norm spaces, namely that $\left(\ell^{r,s_1,s_2,\dots,s_n}, \ell^{u,v_1,v_2,\dots,v_n}\right) = \ell^{p,q_1,q_2,\dots,q_n}$ where p and q_i for $i = 1, 2, \dots, n$ are defined similarly as in Theorem 1.

4. A generalization of the Hausdorff–Young Theorem

Our next task is to prove the following result concerning multipliers of special mixed-norm spaces which, combined with Theorem 1, yields an extension of the first inclusion of Theorem K and consequently a desired generalization of Theorem K.

THEOREM 2. *Let $1 < p \leqq 2$. Then for each $n = 0, 1, 2, \dots$*

$$\widehat{L^p} \subset \left(\ell^{\infty,[2;n],2}, \ell^{p',[1;n],1}\right).$$

COROLLARY 1. *Let $1 < p \leqq 2$. Then $\widehat{L^p} \subset \ell^{p',[2;n],2}$ for each $n = 0, 1, 2, \dots$*

PROOF. Corollary 1 is a direct consequence of Theorems 1 and 2.

REMARK 4. The statement of Theorem 2 does not hold for $p = 1$. Suppose on the contrary that $\widehat{L^1} \subset \left(\ell^{\infty,[2;n],2}, \ell^{\infty,[1;n],1}\right)$. Then by Theorem 1, $\widehat{L^1} \subset \ell^{\infty,[2;n],2}$. If $n = 0$ this yields a contradiction in view of the fact that $\widehat{L^1} \not\subset \ell^{\infty,2}$ by Remark 1. Let $n \geqq 1$. Then from the assumption and Theorem 1 it follows that $\widehat{L^1} \subset \left(\ell^{2,[\infty;n],\infty}, \ell^2\right)$ and consequently with a restriction to sequences whose terms are zero on $\mathbf{Z}^-$,

$$\ell^{2,[\infty;n],\infty} \subset (\widehat{L^1}, \ell^2) \subset (\widehat{H^1}, \ell^2) = (\widehat{H^1}, \widehat{H^2}).$$

Now by [8, Proposition 1] and the remark stated there, $(\widehat{H^1}, \widehat{H^2}) = \ell^{2,\infty}$. Thus the above assumption leads to the inclusion $\ell^{2,[\infty;n],\infty} \subset \ell^{2,\infty}$ which is clearly false for $n \geqq 1$ since the spaces $\ell^{2,[\infty;n],\infty}$ increase properly with n by Proposition 2.

From what we have just shown we extract

REMARK 5. If $n \geqq 1$ then $\ell^{2,[\infty;n],\infty} \not\subset (\widehat{H^1}, \widehat{H^2})$.

The last remark shows that the corresponding generalization of Theorem H, i.e., the inclusion $\ell^{q,[\infty;n],\infty} \subset (\widehat{H^p}, \widehat{H^2})$ where $q = 2p/(2-p)$, does not hold for $p = 1$ if $n \geqq 1$. This points to difficulties in the proof of Theorem 2, since Kellogg's approach involving the use of the operators of strong type on Hardy's spaces and the result of Hedlund [8, Theorem 2] for operators of strong type, cannot be applied here. Instead, we shall prove Theorem 2 and consequently a corresponding generalization of Theorem H for $1 < p < 2$ by employing an interpolation-type argument for operators of weak type, extending the latter notion to operators on multiple mixed-norm spaces, using parts of the proof of the Marcinkiewicz's theorem [6, 13.8.1] and the original Hausdorff–Young Theorem, i.e., the inclusion $\widehat{L^p} \subset \ell^{p'}$ for $1 < p \leqq 2$.

PROOF OF THEOREM 2. The claim in the case $p = 2$ is trivially true since, from Theorem 1, it follows that $\widehat{L^2} = \ell^2 = \left(\ell^{\infty,[2;n],2}, \ell^{2,[1;n],1}\right)$. So suppose that $1 < p < 2$. Let $\widehat{g} \in \ell^{\infty,[2;n],2}$ and for this fixed g let T be a linear operator defined on L^p by the equation $Tf = \widehat{g} \cdot \widehat{f}$. We shall prove that for some constants A and α, determined only by p and g,

$$\|Tf\|_{\ell^{p'},[1;n],1} \leqq A\|f\|^{\alpha}_{L^p} \quad \text{for every} \quad f \in L^p, \tag{4.1}$$

that is $\widehat{g} \cdot \widehat{f} \in \ell^{p',[1;n],1}$ for all $f \in L^p$, which clearly verifies the above claim.

To prove (4.1) we shall use interpolation between properly selected pairs of indices (p_0, q_0) and $(2, q_1)$ with $1 < p_0 < p < 2$. We begin by choosing the index p_0 in the following way:

$$\begin{cases} 1 < p_0 < \frac{p+1}{2} & \text{if } p^2 > 2, \\ \dfrac{p(4-p-p^2)}{2(1+p-p^2)} < p_0 < \dfrac{p+1}{2} & \text{if } p^2 \leqq 2. \end{cases} \tag{4.2}$$

From the assumption that $p > 1$ and (4.2) it is trivial to see that

$$1 < p_0 < p. \tag{4.3}$$

Let $q_0 := p_0'$ and let the index q_1 be defined by the equation

$$q_1 := p' \frac{\left(1 + \frac{1}{p}\right) \frac{p_0'}{p'} \left(\frac{p}{p_0} - 1\right)}{\left(1 - \frac{p}{2}\right)\left(\frac{p_0'}{p'} - 1 - \frac{1}{p}\right) + \frac{p_0'}{p'}\left(\frac{p}{p_0} - 1\right)}. \tag{4.4}$$

Next we observe that the right side of the inequalities (4.2) implies that

$$1 + \frac{1}{p} < \frac{p_0'}{p'}. \tag{4.5}$$

Using this in (4.4) we see immediately that

$$\frac{q_1}{p'} < 1 + \frac{1}{p}. \tag{4.6}$$

Finally, we shall establish that the above definitions also imply the inequality

$$\frac{q_1}{p'} > 1. \tag{4.7}$$

Using (4.4), a simple calculation shows that (4.7) is equivalent to

$$\frac{p_0'}{p'}\left(\frac{p}{2} - \frac{1}{p} - \frac{1}{p_0'}\right) > -\left(1 + \frac{1}{p}\right)\left(1 - \frac{p}{2}\right),$$

i.e.,

$$\frac{p_0'}{p'}(p^2 - 2) > p^2 + p - 4. \tag{4.8}$$

Thus, it suffices to show that (4.2) implies (4.8). Suppose first that $p^2 > 2$. Then it is easy to see that

$$1 + \frac{1}{p} > \frac{p^2 + p - 4}{p^2 - 2}$$

so in this case (4.8) follows immediately from (4.5), and therefore from (4.2). Next, suppose that $p^2 \leqq 2$. Then (4.8) is trivial if $p^2 = 2$; and if $p^2 < 2$, (4.8) is equivalent to

$$\frac{p_0'}{p'} < \frac{4 - p - p^2}{2 - p^2},$$

which in turn is equivalent to

$$p_0 > \frac{p(4 - p - p^2)}{2(1 + p - p^2)},$$

and this is clearly implied by (4.2). Hence (4.8) holds under either case of (4.2), and this completes the verification of (4.7).

We now proceed with the main part of this proof referring to some of the concepts and notation used in [6, 13.7 and 13.8].

Let ν denote the ordinary counting measure on $\mathbf{Z}$ and μ the Lebesgue measure on the circle group, i.e., normalized by 2π. For the $(n+1)$-tuple of integers $(j, i_1, \ldots, i_n)$ and $f \in L^1$ let $d^{ji_1 \cdots i_n} f$ denote the function whose

Fourier coefficients coincide with $\widehat{f}(k)$ for $k \in d_{ji_1\cdots i_n}$ and are zero otherwise. Let $D^{\nu}_{Td^{ji_1\cdots i_n}f}$ denote the corresponding distribution function (in the sense of [6, 13.7.2]) of its T image with respect to the counting measure ν, i.e.,

$$D^{\nu}_{Td^{ji_1\cdots i_n}f}(b) := \nu\left\{ k \in d_{ji_1\cdots i_n} \,:\, \left|\widehat{f}(k)\widehat{g}(k)\right| > b\right\}.$$

For $f \in L^1$, let D^{μ}_f denote the the distribution function of f with respect to the Lebesgue measure μ, i.e.,

$$D^{\mu}_f(b) := \mu\left\{ x \in [0, 2\pi) \,:\, \left|f(x)\right| > b\right\}.$$

Using the result cited in [6, 13.7.3] we can write

$$\sum_{k \in d_{ji_1\cdots i_n}} \left|\widehat{f}(k)\widehat{g}(k)\right|^{p'} = \int_0^\infty p' b^{p'-1} D^{\nu}_{Td_{ji_1\cdots i_n}f}(b)\, db,$$

$$\frac{1}{2\pi}\int_0^{2\pi} |f|^p\, d\mu = \int_0^\infty p b^{p-1} D^{\mu}_f(b)\, db.$$

Furthermore, just as in the proof of Marcinkiewicz' theorem [6, 13.8.1], we consider truncations of f. Namely for $a \in (0, \infty)$ we define

$$f_{1,a}(x) := f(x), \quad \text{if } \left|f(x)\right| \leqq a \quad \text{and} \quad f_{1,a}(x) := a, \quad \text{otherwise};$$
$$f_{2,a} := f - f_{1,a}.$$

Clearly, $|f_{1,a}| = \min(|f|, a)$ and $f = f_{1,a} + f_{2,a}$.

To prove (4.1) we first observe that from the above and (2.5),

$$(4.9) \qquad \|Tf\|_{\ell^{p'},[1;n],1} = \sum_j \sum_{i_1} \cdots \sum_{i_n} \left(\int_0^\infty p' b^{p'-1} D^{\nu}_{Td_{ji_1\cdots i_n}f}(b)\, db\right)^{1/p'}.$$

Choosing $a = a(b) \in (0, \infty)$ to be the value $a(b)$ of a monotonically decreasing function of $b \in (0, \infty)$, that will be suitably selected later, we may write $f = f_{1,a} + f_{2,a}$. Noticing as in [6, (13.8.4)] that

$$D^{\nu}_{Td_{ji_1\cdots i_n}f}(2b) \leqq D^{\nu}_{Td_{ji_1\cdots i_n}f_{1,a}}(b) + D^{\nu}_{Td_{ji_1\cdots i_n}f_{2,a}}(b),$$

from (4.9) it follows that

$$(4.10) \quad \|Tf\|_{\ell^{p'},[1;n],1} = 2\sum_j \sum_{i_1} \cdots \sum_{i_n} \left(\int_0^\infty p' b^{p'-1} D^{\nu}_{Td_{ji_1\cdots i_n}f_{1,a}}(b)\, db\right)^{1/p'}$$

$$+2\sum_j\sum_{i_1}\cdots\sum_{i_n}\left(\int_0^\infty p'b^{p'-1}D^\nu_{Td_{ji_1\cdots i_n}f_{2,a}}(b)\,db\right)^{1/p'} := 2(I_1+I_2).$$

We now estimate I_1 and I_2 using the fact that for $q_0 = p_0'$ and q_1 given by (4.4), the operator T is of weak type (p_0, q_0) and $(2, q_1)$ uniformly on each $(n+1)$-tuple block $d_{ji_1\cdots i_n}$. Namely, we have

$$b^{q_1}\, D^\nu_{Td_{ji_1\cdots i_n}f_{1,a}}(b) \leqq \left\| d^{ji_1\cdots i_n}\widehat{g}\right\|^{q_1}_{\ell^\infty} \sum_{k\in d_{ji_1\cdots i_n}} \left|\widehat{f_{1,a}}(k)\right|^{q_1},$$

$$b^{p_0'}\, D^\nu_{Td_{ji_1\cdots i_n}f_{2,a}}(b) \leqq \left\| d^{ji_1\cdots i_n}\widehat{g}\right\|^{p_0'}_{\ell^\infty} \sum_{k\in d_{ji_1\cdots i_n}} \left|\widehat{f_{2,a}}(k)\right|^{p_0'}.$$

In view of the fact that by (4.7), $q_1/p' \geqq 1$ so that $q_1 \ge p' > 2$ the above inequalities yield

$$\text{(4.11)}\qquad \begin{cases} D^\nu_{Td_{ji_1\cdots i_n}f_{1,a}}(b) \leqq b^{-q_1}\left\| d^{ji_1\cdots i_n}\widehat{g}\right\|^{q_1}_{\ell^\infty}\left\| d^{ji_1\cdots i_n}\widehat{f_{1,a}}\right\|^{q_1}_{\ell^2},\\ D^\nu_{Td_{ji_1\cdots i_n}f_{2,a}}(b) \leqq b^{-p_0'}\left\| d^{ji_1\cdots i_n}\widehat{g}\right\|^{p_0'}_{\ell^\infty}\left\| d^{ji_1\cdots i_n}\widehat{f_{2,a}}\right\|^{p_0'}_{\ell^{p_0'}}. \end{cases}$$

To estimate I_1 we use (4.11) to obtain

$$I_1 \leqq \sum_j\sum_{i_1}\cdots\sum_{i_n}\left(\int_0^\infty p'b^{p'-q_1-1}\left\| d^{ji_1\cdots i_n}\widehat{g}\right\|^{q_1}_{\ell^\infty}\left\| d^{ji_1\cdots i_n}\widehat{f_{1,a}}\right\|^{q_1}_{\ell^2}\,db\right)^{1/p'}$$

$$\leqq \sum_j\sum_{i_1}\cdots\sum_{i_n}\int_0^\infty p'^{1/p'}b^{1-q_1/p'-1/p'}\left\| d^{ji_1\cdots i_n}\widehat{g}\right\|^{\frac{q_1}{p'}}_{\ell^\infty}\left\| d^{ji_1\cdots i_n}\widehat{f_{1,a}}\right\|^{\frac{q_1}{p'}}_{\ell^2}\,db$$

$$\leqq \int_0^\infty p'^{\frac{1}{p'}}b^{1+\frac{1}{p}-\frac{q_1}{p'}-1}\left(\sum_j\sum_{i_1}\cdots\sum_{i_n}\left\| d^{ji_1\cdots i_n}\widehat{g}\right\|^{\frac{q_1}{p'}}_{\ell^\infty}\left\| d^{ji_1\cdots i_n}\widehat{f_{1,a}}\right\|^{\frac{q_1}{p'}}_{\ell^2}\right)db.$$

Using the fact that by (4.7) the exponent is greater than or equal to 1, i.e., $q_1/p' \geqq 1$, and applying Hölder's inequality to the above sum it follows that

$$\sum_j\sum_{i_1}\cdots\sum_{i_n}\left\| d^{ji_1\cdots i_n}\widehat{g}\right\|^{\frac{q_1}{p'}}_{\ell^\infty}\left\| d^{ji_1\cdots i_n}\widehat{f_{1,a}}\right\|^{\frac{q_1}{p'}}_{\ell^2}$$

$$\leqq \left(\sum_j\sum_{i_1}\cdots\sum_{i_n}\left\| d^{ji_1\cdots i_n}\widehat{g}\right\|^2_{\ell^\infty}\right)^{\frac{q_1}{2p'}}\cdot\left(\sum_j\sum_{i_1}\cdots\sum_{i_n}\left\| d^{ji_1\cdots i_n}\widehat{f_{1,a}}\right\|^2_{\ell^2}\right)^{\frac{q_1}{2p'}}$$

$$= \left\|\widehat{g}\right\|^{\frac{q_1}{p'}}_{\ell^{\infty,[2;n],2}}\left\|\widehat{f_{1,a}}\right\|^{\frac{q_1}{p'}}_{\ell^2}.$$

Consequently,

$$I_1 \leqq \|\widehat{g}\|^{\frac{q_1}{p'}}_{\ell^\infty,[2;n],2} \int_0^\infty p'^{\frac{1}{p'}} b^{1+\frac{1}{p}-\frac{q_1}{p'}-1} \|\widehat{f_{1,a}}\|^{\frac{q_1}{p'}}_{\ell^2}$$

$$= \|\widehat{g}\|^{\frac{q_1}{p'}}_{\ell^\infty,[2;n],2} p'^{\frac{1}{p'}} \int_0^\infty b^{1+\frac{1}{p}-\frac{q_1}{p'}-1} \left(\int_0^{a(b)} 2c\, D_f^\mu(c)\, dc \right)^{\frac{q_1}{2p'}} db.$$

Assuming $a(\cdot)$ to be monotonically decreasing we observe that in this case the characteristic function χ satisfies the equation $\chi_{[0,a(b)]}(x) = \chi_{[0,b(c)]}(x)$ where $b(\cdot)$ denotes the inverse of $a(\cdot)$.

Noticing that the index $1 + \frac{1}{p} - \frac{q_1}{p'}$ is positive by (4.6), due to our choice of q_1, comparing the double integral appearing above with the similar expression in [6, 13.8.1] and applying an analogous argument as in the proof of Marcinkiewicz' theorem [6, (13.8.5) through (13.8.8)], we conclude that

$$I_1 \leqq A_1 \|\widehat{g}\|^{\frac{q_1}{p'}}_{\ell^\infty,[2;n],2} \left(\int_0^\infty c D_f^\mu(c)\, b^{\left(1+\frac{1}{p}-\frac{q_1}{p'}\right)\frac{2p'}{q_1}}\, dc \right)^{\frac{q_1}{2p'}}.$$

We now define $b(a)$ to be of the form $b(a) = Ka^\rho$. In order that the above estimate for I_1, and a similar estimate for I_2, should yield (4.1) we see that ρ must be chosen so that

$$1 + \rho \left(1 + \frac{1}{p} - \frac{q_1}{p'} \right) \frac{2p'}{q_1} = p - 1,$$

i.e., ρ should be given by

$$\rho := \frac{p/2 - 1}{1 + 1/p - q_1/p'} \frac{q_1}{p'}. \tag{4.12}$$

Clearly, ρ defined by (4.12) is negative, due to (4.6) and the assumption that $p < 2$. Hence, the function $b(\cdot)$ is indeed monotonically decreasing as it was assumed in the above argument. Furthermore, we obtain the desired inequality for I_1 of the form

$$I_1 \leqq A_1 K^{\left(1+\frac{1}{p}-\frac{q_1}{p'}\right)} \|\widehat{g}\|^{\frac{q_1}{p'}}_{\ell^\infty,[2;n],2} \|\widehat{f}\|^{\frac{q_1 p}{2p'}}_{L^p}. \tag{4.13}$$

We now proceed to estimate I_2 for the above choice of $a = a(b)$ and other parameters. From (4.11) we clearly have

$$I_2 \leqq \sum_j \sum_{i_1} \cdots \sum_{i_n} \left(\int_0^\infty p' b^{p'-p'_0-1} \| d^{ji_1\cdots i_n} \widehat{g} \|^{p'_0}_{\ell^\infty} \| d^{ji_1\cdots i_n} \widehat{f_{2,a}} \|^{p'_0}_{\ell^{p'_0}}\, db \right)^{\frac{1}{p'}}$$

$$\leqq \int_0^\infty p'^{\frac{1}{p'}} b^{1-\frac{p_0'}{p'}-\frac{1}{p'}} \sum_j \sum_{i_1} \cdots \sum_{i_n} \left\| d^{ji_1\cdots i_n}\widehat{g}\right\|_{\ell^\infty}^{\frac{p_0'}{p'}} \left\| d^{ji_1\cdots i_n}\widehat{f_{2,a}}\right\|_{\ell^{p_0'}}^{\frac{p_0'}{p'}} db.$$

Consequently, applying Hölder's inequality to the $(n+1)$-tuple sum and indices p' and p we obtain

$$I_2 \leqq \int_0^\infty p'^{\frac{1}{p'}} b^{1-\frac{p_0'}{p'}-\frac{1}{p'}} \left(\sum_j \sum_{i_1} \cdots \sum_{i_n} \left\| d^{ji_1\cdots i_n}\widehat{g}\right\|_{\ell^\infty}^{\frac{p_0' p}{p'}} \right)^{\frac{1}{p}} \left\| \widehat{f_{2,a}}\right\|_{\ell^{p_0'}}^{\frac{p_0'}{p'}} db.$$

Observing from (4.5) that $p_0'/p' > 2/p$, the last inequality implies that

$$I_2 \leqq p'^{\frac{1}{p'}} \left\|\widehat{g}\right\|_{\ell^\infty,[2;n],2}^{\frac{p_0'}{p'}} \int_0^\infty b^{1-\frac{p_0'}{p'}-\frac{1}{p'}} \left\| \widehat{f_{2,a}}\right\|_{\ell^{p_0'}}^{\frac{p_0'}{p'}} db.$$

From Hausdorff–Young Theorem it then follows that

$$I_2 \leqq p'^{\frac{1}{p'}} \left\|\widehat{g}\right\|_{\ell^\infty,[2;n],2}^{\frac{p_0'}{p'}} \int_0^\infty b^{1+\frac{1}{p}-\frac{p_0'}{p'}-1} \left\| \widehat{f_{2,a}}\right\|_{\ell^{p_0'}}^{\frac{p_0'}{p'}} db$$

$$= p'^{\frac{1}{p'}} \left\|\widehat{g}\right\|_{\ell^\infty,[2;n],2}^{\frac{p_0'}{p'}} \int_0^\infty b^{1+\frac{1}{p}-\frac{p_0'}{p'}-1} \left(\int_{a(b)}^\infty p_0 c^{p_0-1} D_f^\mu(c) \right)^{\frac{p_0'}{p'}} db.$$

Since $a(b)$ is monotonically decreasing the characteristic function satisfies the equality $\chi_{[a(b),\infty)}(x) = \chi_{[b(c),\infty)}(x)$. Hence, recalling (4.5), the corresponding argument used in [6, (13.8.5)–(13.8.8)] can also be applied to the above double integral, yielding the estimate

$$I_2 \leqq A_2 \left\|\widehat{g}\right\|_{\ell^\infty,[2;n],2}^{\frac{p_0'}{p'}} \left(\int_0^\infty c^{p_0-1} D_f^\mu(c) b(c)^{\left(1+\frac{1}{p}-\frac{p_0'}{p'}\right)\frac{p_0 p'}{p_0'}} dc \right)^{\frac{p_0 p'}{p_0'}}.$$

But from the definition of q_1 given by (4.4) and the definition of ρ given by (4.12) a simple calculation shows that ρ also satisfies the equation

$$\rho = \frac{p/p_0 - 1}{1 + 1/p - p_0'/p'} \frac{p_0'}{p'}.$$

Hence

$$p_0 - 1 + \rho \left(1 + \frac{1}{p} - \frac{p_0'}{p'} \right) \frac{p_0 p'}{p_0'} = p - 1,$$

and therefore

$$I_2 \leqq A_2 K^{\left(1+\frac{1}{p}-\frac{p_0'}{p'}\right)} \left\|\widehat{g}\right\|^{\frac{p_0'}{p'}}_{\ell^{\infty,[2;n],2}} \left\|\widehat{f}\right\|^{\frac{p_0' p}{p_0 p'}}_{L^p}. \tag{4.14}$$

Taking $K = \left\|\widehat{f}\right\|^{p\beta}_{L^p}$ for a properly chosen β, and combining the estimates (4.13) and (4.14) with (4.10) we obtain (4.1). A simple calculation shows that by letting β satisfy $\beta(p_0' - q_1) = p_0'/p_0 - q_1/2$ the parameter α appearing in (4.1) is given by $\alpha = p\beta\left(1 + \frac{1}{p} - \frac{p_0'}{p'}\right) + \frac{p_0' p}{p_0 p'}$. This completes the proof of Theorem 2.

We are now ready to state and prove our main results, i.e. the claimed generalizations of Theorem K and Theorem H.

THEOREM 3. *Let $1 < p \leqq 2$. Then for each $n = 0, 1, 2, \ldots$, $\widehat{L^p} \subset \ell^{p',[2;n]}$ and $\ell^{p,[2;n]} \subset \widehat{L^{p'}}$. Consequently $\widehat{L^p} \subset \bigcap_{n=0}^{\infty} \ell^{p',[2;n]}$ and $\bigcup_{n=0}^{\infty} \ell^{p,[2;n]} \subset \widehat{L^{p'}}$.*

PROOF. The inclusion $\widehat{L^p} \subset \ell^{p',[2;n]}$ for each $n = 0, 1, 2, \ldots$, is the statement of Corollary 1 of Theorem 2. Consequently $\widehat{L^p} \subset \bigcap_{n=0}^{\infty} \ell^{p',[2;n]}$.

To prove the second inclusion we first observe that $L^1 * L^p \subset L^p$, see for example [6, 3.1.6]. Let $f \in L^1$ and $g \in L^p$. Then, from what was just observed and the already proved inclusion we have $\widehat{f} \cdot \widehat{g} \in \widehat{L^p} \subset \ell^{p',[2;n]}$, and therefore $\widehat{L^1} \subset \left(\widehat{L^p}, \ell^{p',[2;n]}\right)$. By well known properties of multipliers and applying Proposition 1, the last inclusion yields $\widehat{L^1} \subset \left(\ell^{p,[2;n]}, \widehat{L^{p'}}\right)$ and therefore $\ell^{p,[2;n]} \subset (\widehat{L^1}, \widehat{L^{p'}})$, for each $n = 0, 1, 2, \ldots$. Recalling that by [6, 16.3.4], $(\widehat{L^1}, \widehat{L^{p'}}) = \widehat{L^{p'}}$ we conclude that $\ell^{p,[2;n]} \subset \widehat{L^{p'}}$ for each $n = 0, 1, 2, \ldots$ and hence $\bigcup_{n=0}^{\infty} \ell^{p,[2;n]} \subset \widehat{L^{p'}}$.

REMARK 6. Since by Proposition 2 the spaces $\ell^{p',[2;n]}$ decrease and $\ell^{p,[2;n]}$ increase properly with n, whenever $1 < p < 2$, the above result clearly generalizes Theorem K. Furthermore, just as in the case of the Hausdorff–Young Theorem, the statement cannot be extended to $p > 2$. Namely, if $p > 2$, then by Proposition 2 we can choose a sequence $c = (c_k)_{k \in \mathbf{Z}}$ such that $c \in \ell^2$ and $c \notin \ell^{p',[2;n]}$. Now from [6, 14.3.2] it is possible to choose signs in $\sum \pm c_k\, e^{ikx}$ such that the latter series is a Fourier series of a function $g \in L^p$, for any $p < \infty$. Hence, $\widehat{L^p} \not\subset \ell^{p',[2;n]}$ for any $2 < p < \infty$. Similarly the example considered in [6, 14.4 that is 14.3.6] shows that this is also the case when $p = \infty$. As for the dual inclusion in Theorem 3, it also fails for $p > 2$ for the same reasons as argued in [6, 14.4] with the observation that by Proposition 2, $\ell^2 \subset \ell^{p,[2;n]}$ holds properly in this case.

REMARK 7. The result in the above theorem can also be deduced, somewhat briefly and admittedly with greater elegance, by using the Littlewood–Paley theory and the results of Gaudry, namely Theorem 4.1 and Lemma 3.4

in [7]. This was recently suggested to us. However, we have chosen to retain the above approach in keeping with the spirit of this paper.

THEOREM 4. *Let* $1 < p < 2$ *and let* $q = 2p/(2-p)$. *Then for each* $n = 0, 1, 2, \dots,$ $\ell^{q,[\infty;n],\infty} \subset (\widehat{H^p}, \widehat{H^2})$ *and consequently* $\bigcup_{n=1}^{\infty} \ell^{q,[\infty;n]} \subset (\widehat{H^p}, \widehat{H^2})$.

REMARK 8. Theorem 4 is also a direct consequence of Theorems 1 and 2, that is of Corollary 1 of Theorem 2. As was pointed out in Remark 5 the statement is not valid for $p = 1$.

PROOF. Suppose that $1 < p < 2$ and let $q = 2p/(2-p)$. Then from Theorem 1 we have $\ell^{p',[2;n]} = \left(\ell^{q,[\infty;n]}, \ell^2\right)$ for each $n = 0, 1, 2, \dots$. Consequently by Corollary 1 of Theorem 2 it follows that $\widehat{L^p} \subset \left(\ell^{q,[\infty;n]}, \ell^2\right)$ so that also $\widehat{H^p} \subset \left(\ell^{q,[\infty;n]}, \widehat{H^2}\right)$, which in turn implies the claimed inclusion $\ell^{q,[\infty;n]} \subset (\widehat{H^p}, \widehat{H^2})$.

In view of Theorems 3 and 4 we are now able to derive other results giving sufficient conditions for sequences to belong to multiplier spaces of the form $(\widehat{L^p}, \widehat{L^q})$ and $(\widehat{H^p}, \widehat{H^q})$. The following theorem extends Theorems 5 and 6 in [10].

THEOREM 5. *Suppose that* $1 < p \leqq 2 \leqq q < \infty$ *and let* $1/s = 1/p - 1/q$. *Then* $\ell^{s,[\infty;n]} \subset (\widehat{L^p}, \widehat{L^q})$ *and* $\ell^{s,[\infty;n]} \subset (\widehat{H^p}, \widehat{H^q})$.

PROOF. We first note that the second inclusion follows from the first, by properties of Fourier coefficients of functions in Hardy spaces.

To prove the first inclusion let $\lambda \subset \ell^{s,[\infty;n]}$ and $f \in L^p$. By Theorem 3 then $\widehat{f} \in \ell^{p',[2;n]}$. But by Theorem 1 clearly $\ell^{s,[\infty;n]} = \left(\ell^{p',[2;n]}, \ell^{q',[2;n]}\right)$ and therefore $\lambda \cdot \widehat{f} \in \ell^{q',[2;n]}$. Since by assumption we have $1 < q \leqq 2$, from the last observation and Theorem 3 we see that $\lambda \cdot \widehat{f} \in \widehat{L^q}$. This shows that $\ell^{s,[\infty;n]} \subset (\widehat{L^p}, \widehat{L^q})$.

We improve on the conclusions in Theorem 3 by showing that the linear space $E = \cap_{n=0}^{\infty} \ell^{p',[2,n]}$ strictly contains $\widehat{L^p}$ and that the linear space $F = \bigcup_{n=0}^{\infty} \ell^{p,[2;n]}$ is strictly contained in $\widehat{L^{p'}}$.

THEOREM 6. *For* $1 < p < 2$, $E \neq \widehat{L^p}$ *and* $F \neq \widehat{L^{p'}}$.

PROOF. Let p be fixed and $1 < p < 2$, so that $p' > 2$. Clearly $\widehat{L^p}$, $\|\widehat{f}\|_{\widehat{L^p}} := \|f\|_{L^p}$ and $\widehat{L^{p'}}$, $\|\widehat{f}\|_{\widehat{L^{p'}}} := \|f\|_{L^{p'}}$ are Banach spaces.

For convenience, write $E_n = \ell^{p',[2;n]}$ and $F_n = \ell^{p,[2;n]}$ for $n = 0, 1, 2, \dots$. Let $E = \cap_{n=0}^{\infty} E_n$ and $F = \bigcup_{n=0}^{\infty} F_n$. From Theorem 3 we have $\widehat{L^p} \subset E$ and

$F \subset \widehat{L^{p'}}$. We shall prove that the linear spaces E and F cannot be Banach spaces, under any norm. Consequently, by the preceding observation, $E \neq \widehat{L^p}$ and $F \neq \widehat{L^{p'}}$.

We first prove that E cannot be a normed linear space. Suppose on the contrary, that $(E, \|\,\|_E)$ is a normed space. Recall, by Propositions 1 and 2, that for each n, E_n is a Banach space, E_{n+1} is a proper subspace of E_n and that the inclusion (identity) map is clearly continuous. For each n let $i_n : E \to E_n$ denote the canonical injection or identity map. Clearly, for each n, $E \subset E_n$ and the identity map i_n from $(E, \|\,\|_E)$ into the Banach space $(E_n, \|\,\|_{E_n})$ is continuous. Observe that the $\|\,\|_E$ topology on E is a locally convex topology. Now, equip E with the projective limit topology τ, see [11 or 13]. By definition, see [13, p. 52], τ is the coarsest locally convex topology which makes each i_n continuous, and hence τ is weaker than the $\|\,\|_E$ topology. Furthemore, (E, τ) is a Fréchet space and therefore E is a Fréchet space under two comparable topologies. By a corollary of the Open Mapping Theorem, see [17, 5.2.7, p. 59] the two topologies are equivalent and therefore (E, τ) is a normed space. We shall show next that this contradicts Proposition 2.

Let U and U_n, for each n, denote the open unit balls in E and E_n, respectively, under their respective norms. Then U is a bounded set and a 0-neighborhood. Note that, by definition of τ, $\alpha(E \cap U_n) = E \cap \alpha U_n$, $n = 0, 1, \ldots$ and $\alpha > 0$ rational, form the 0-neighborhood base of (E, τ). Hence by properties of the neighborhood base, there exists an $n \in \mathbf{N}$ such *that $E \cap U_n \subset \lambda \cdot U$ for a suitable $\lambda > 0$. Since U is bounded there exists* $\mu > 0$ such that $U \subset \mu \cdot (E \cap U_{n+1})$. Consequently $E \cap U_{n+1} \subset E \cap U_n \subset \lambda\mu \cdot (E \cap U_{n+1})$. The last relation shows that E_n and E_{n+1} induce the same topology on E. Recalling that the space of finite sequences Φ is dense in E_n (because, by Proposition 1, each E_n has AK), by taking closures of Φ, we obtain that $E_n = E_{n+1}$. However, as shown in Proposition 2, this is clearly false. Thus we have proved that E cannot be a normed space and therefore $E \neq \widehat{L^p}$.

Next, we show that F cannot be a Banach space, under any norm. Recall that $F_0 \subset F_1 \subset F_2 \subset \cdots$ where each F_n is a Banach space. By Proposition 2 each inclusion is strict and it can be easily seen that the inclusion maps are also continuous. Equip $F = \bigcup_{n=0}^{\infty} F_n$ with the inductive limit topology $(\mathrm{ind}_{n\to\infty} F_n)$, see [11, 12 or 13]. In the notation of the Grothendieck's factorization theorem, see [12, p. 271], let $u : F \to \omega$ and $u_n : F_n \to \omega$, $n = 0, 1, 2, \ldots$, be the canonical inclusion maps, where ω is the locally convex space of all complex-valued sequences with its usual Frechet topology. If a norm can be assigned to F so that $(F, \|\,\|_F)$ becomes a Banach space then by Grothendieck's *factorization theorem [12], there exists n_0 such that F is con*tinuously injected into F_{n_0} and thus $F = F_n$ for each $n \geqq n_0$, topologically. By Proposition 2, this last assertion that $F_n = F_{n_0}$ for all $n \geqq n_0$ is clearly false and hence F cannot be a Banach space under any norm. Consequently $F \neq \widehat{L^{p'}}$.

REMARK 9. From the above proof it follows that none of E and F are Banach spaces.

This completes the presentation of our main results. The following section contains a detailed proof of Proposition 2 that was stated in Section 2 and that is so essential in the above discussion.

5. Appendix

PROOF OF PROPOSITION 2. i) The inclusion is obvious by (2.5). To see that it is proper, suppose that $p < r$ or $q < s$ and consider the sequence $c = (c_k)_{k\in\mathbf{Z}}$ whose terms are defined by

(5.1)
$$c_k = \frac{1}{j^{1/p+1/q}\log^{1/q}(j+1)}\,\frac{1}{i_1^{1/z}\cdots i_n^{1/z}}\,\frac{1}{(k-2^j-2^{i_1}-\cdots-2^{i_n}+1)^{1/p}}$$

for $k \in d_{ji_1i_2\cdots i_n}$ and $c_k = 0$ otherwise. Then, on the one hand, from (2.5) and (5.1) it can be is easily verified that

$$\|c\|^q_{\ell^{p,[z;n],q}} \geqq \text{const.}\sum_{j=1}^{\infty}\frac{1}{j^{q/p+1}\log(j+1)}j^{q/p} = \text{const.}\sum_{j=1}^{\infty}\frac{1}{j\log(j+1)} = \infty$$

so that $(c_k) \notin \ell^{p,[z;n],q}$. On the other hand, from (2.5) and (5.1) we have

$$\|c\|^q_{\ell^{r,[z;n],s}}$$
$$\leqq \sum_{j=1}^{\infty}\frac{1}{j^{s(1/p+1/q)}\log^{s/q}(j+1)}\left(\sum_{i_1=1}^{j-1}\sum_{i_2=1}^{i_1-1}\cdots\sum_{i_n=1}^{i_{n-1}-1}\frac{1}{i_1\cdots i_n}\left(\sum_{\nu=1}^{2^{i_n}}\frac{1}{\nu^{r/p}}\right)^{z/r}\right)^{s/z}$$

from which it can be deduced that $(c_k)_{k\in\mathbf{Z}} \in \ell^{r,[z;n],s}$. Namely, if $r/p > 1$ then $\sum_{\nu=1}^{2^{i_n}}\frac{1}{\nu^{r/p}}$ is bounded, and observing that $s(1/p+1/q) > 1$, from the above inequality it follows that

$$\|c\|^q_{\ell^{r,[z;n],s}} \leqq \text{const.}\sum_{j=1}^{\infty}\frac{1}{j^{s(1/p+1/q)}\log^{s/q}(j+1)}\log^{ns/z}(j+1) < \infty.$$

If $s/q > 1$, then, since

$$\left(\sum_{\nu=1}^{2^{i_n}} \frac{1}{\nu^{r/p}}\right)^{z/r} = O(1)\, i_n^{z/r}$$

and $s(1/p + 1/q - 1/r) > 1$, the above inequality yields

$$\|c\|^q_{\ell^{r,[z;n],s}} \leqq \text{const.} \sum_{j=1}^{\infty} \frac{1}{j^{s(1/p+1/q)} \log^{s/q}(j+1)} j^{s/r} < \infty.$$

Hence, in either case $(c_k)_{k\in\mathbf{Z}} \in \ell^{r,[z;n],s}$. This shows that the inclusion $\ell^{p,[z;n],q} \subset \ell^{r,[z;n],s}$ is proper whenever $p < r$ or $s < q$.

ii) Suppose that $1 \leqq p < z$. From $z/p > 1$ and (2.5) we clearly have

$$\|c\|^q_{\ell^{p,[z;n],q}} \leqq \sum_{j=0}^{\infty} \left(\sum_{i_1=0}^{j-1} \cdots \sum_{i_{n-1}=0}^{i_{n-2}-1} \left(\sum_{i_n=0}^{i_{n-1}-1} \sum_{k\in d_{ji_1\cdots i_n}} |c_k|^p \right)^{z/p} \right)^{q/z}$$

$$= \sum_{j=0}^{\infty} \left(\sum_{i_1=0}^{j-1} \cdots \sum_{i_{n-1}=0}^{i_{n-2}-1} \left(\sum_{k\in d_{ji_1\cdots i_{n-1}}} |c_k|^p \right)^{z/p} \right)^{q/z} =: \|c\|^q_{\ell^{p,[z;n-1],q}},$$

which shows the inclusion $\ell^{p,[z;n-1],q} \subset \ell^{p,[z;n],q}$.

We shall prove now that this inclusion is also proper. Let $c = (c_k)_{k\in\mathbf{Z}}$ be given by

$$c_k = \frac{1}{j^{1/q} \log^{1/q+1/z} j} \frac{1}{(i_1 + i_2 + \cdots + i_n)^{(n-1)/z+1/p}} \tag{5.2}$$

for $k = 2^j + 2^{i_1} + \cdots + 2^{i_n}$, where $j \geqq n+1$ and $1 < i_n < i_{n-1} < \cdots < i_1 < j$, and $c_k = 0$ otherwise. Observing that this sequence has only one non-zero term inside the block $d_{ji_1\cdots i_n}$, from (2.5) and (5.2) it follows that

$$\|c\|^q_{\ell^{p,[z;n],q}} \leqq \frac{1}{\left(n-2+\frac{z}{p}\right)^{\frac{q}{z}}} \sum_{j=n+2}^{\infty} \frac{1}{j \log^{1+\frac{q}{z}} j}$$

$$\cdot \left(\sum_{i_1=n+1}^{j-1} \cdots \sum_{i_{n-1}=3}^{i_{n-2}-1} \frac{1}{(i_1 + i_2 + \cdots + i_{n-1})^{n-2+\frac{z}{p}}} \right)^{q/z}$$

$$\leqq \cdots \leqq \frac{1}{(n-2+\frac{z}{p})!^{\frac{q}{z}}} \sum_{j=n+2}^{\infty} \frac{1}{j \log^{1+\frac{q}{z}} j} \left(\sum_{i_1=n+1}^{j-1} \frac{1}{i_1^{\frac{z}{p}}} \right)^{q/z}.$$

Since by the assumption $z/p > 1$ and $1 + q/z > 1$, the latter series clearly converges and therefore $c = (c_k)_{k\in\mathbf{Z}} \in \ell^{p,[z;n],q}$.

On the other hand, observing that $i_1 + i_2 + \cdots + i_n < ni_1$, from (2.5) and (5.2)

$$\|c\|^q_{\ell^{p,[z;n-1],q}}$$

$$\geqq \sum_{j=n+2}^{\infty} \frac{1}{j\log^{1+\frac{q}{z}} j}\Bigg(\sum_{i_1=n+1}^{j-1} \frac{1}{(ni_1)^{n-1+\frac{z}{p}}} \sum_{i_2=n}^{i_1-1} \cdots \sum_{i_{n-1}=3}^{i_{n-2}-1} (i_{n-1}-2)^{\frac{z}{p}}\Bigg)^{q/z}$$

$$> \sum_{j=n+2}^{\infty} \frac{1}{j\log^{1+\frac{q}{z}} j}\Bigg(\sum_{i_1=n+1}^{j-1} \frac{1}{(ni_1)^{n-1+\frac{z}{p}}}$$

$$\cdot \sum_{i_2=n}^{i_1-1} \cdots \sum_{i_{n-2}=4}^{i_{n-3}-1} \frac{1}{\frac{z}{p}+1}(i_{n-2}-3)^{\frac{z}{p}+1}\Bigg)^{q/z} > \text{const.} \sum_{j=4n^2}^{\infty} \frac{1}{j\log^{1+\frac{q}{z}} j} \log^{\frac{q}{z}} j = \infty.$$

Consequently $c = (c_k)_{k\in\mathbf{Z}} \notin \ell^{p,[z;n-1],q}$, which completes the argument that $\ell^{p,[z;n],q} \subset \ell^{p,[z;n-1],q}$ holds properly whenever $p \leqq z$.

Suppose now that $p > z$. Then, since $z/p < 1$, (2.5) yields the inequality

$$\|c\|^q_{\ell^{p,[z;n-1],q}} = \sum_{j=0}^{\infty}\Bigg(\sum_{i_1=0}^{j-1} \cdots \sum_{i_{n-1}=0}^{i_{n-2}-1}\Bigg(\sum_{i_n=0}^{i_{n-1}-1} \sum_{k\in d_{ji_1\cdots i_n}} |c_k|^p\Bigg)^{z/p}\Bigg)^{q/z}$$

$$\leqq \sum_{j=0}^{\infty}\Bigg(\sum_{i_1=0}^{j-1} \cdots \sum_{i_{n-1}=0}^{i_{n-2}-1} \sum_{i_n=0}^{i_{n-1}-1}\Bigg(\sum_{k\in d_{ji_1\cdots i_n}} |c_k|^p\Bigg)^{z/p}\Bigg)^{q/z} =: \|c\|^q_{\ell^{p,[z;n],q}}.$$

Hence $\ell^{p,[z;n],q} \subset \ell^{p,[z;n-1],q}$ in this case. To see that this inclusion is also proper consider the sequence $c = (c_k)_{k\in\mathbf{Z}}$ given by

$$c_k = \frac{1}{j^{1/q}\log^{1/q+1/z} j}\,\frac{1}{(i_1+i_2+\cdots+i_n)^{n/z}} \tag{5.3}$$

for $k = 2^j + 2^{i_1} + \cdots + 2^{i_n}$, where $j \geqq n+1$ and $1 < i_n < i_{n-1} < \cdots < i_1 < j$, and $c_k = 0$ otherwise. Then, arguing similarly as in the preceding example we have

$$\|c\|^q_{\ell^{p,[z;n-1],q}} \leqq \sum_{j=n+2}^{\infty} \frac{1}{j\log^{1+\frac{q}{z}} j}$$

$$\cdot\left(\sum_{i_1=n+1}^{j-1}\cdots\sum_{i_{n-1}=3}^{i_{n-2}-1}\frac{1}{\left(n^{\frac{p}{z}}-1\right)^{\frac{z}{p}}}\frac{1}{(i_1+i_2+\cdots+i_{n-1})^{n-\frac{z}{p}}}\right)^{q/z}$$

$$\leqq\frac{1}{\left(n^{\frac{p}{z}}-1\right)^{\frac{q}{p}}}\frac{1}{(n-1-\frac{z}{p})!^{\frac{q}{z}}}\sum_{j=n+2}^{\infty}\frac{1}{j\log^{1+\frac{q}{z}}j}\left(\sum_{i_1=n+1}^{j-1}\frac{1}{i_1^{2-\frac{z}{p}}}\right)^{q/z}<\infty,$$

since $2-z/p>1$ and $1+q/z>1$. Hence, $c=(c_k)_{k\in\mathbf{Z}}\in\ell^{p,[z;n-1],q}$.
On the other hand,

$$\|c\|^q_{\ell^{p,[z;n],q}}\geqq\sum_{j=n+2}^{\infty}\frac{1}{j\log^{1+\frac{q}{z}}j}\left(\sum_{i_1=n+1}^{j-1}\frac{1}{(ni_1)^n}\sum_{i_2=n}^{i_1-1}\cdots\sum_{i_{n-1}=3}^{i_{n-2}-1}(i_{n-1}-2)\right)^{q/z}$$

$$\geqq\frac{1}{n^n(n-1)!^{\frac{q}{z}}}\sum_{j=n+2}^{\infty}\frac{1}{j\log^{1+\frac{q}{z}}j}\left(\sum_{i_1=n+1}^{j-1}\frac{1}{i_1^n}(i_1-n)^{n-1}\right)^{q/z}$$

$$\frac{1}{n^n(n-1)!^{\frac{q}{z}}}\frac{1}{2^{(n-1)\frac{q}{z}}}\sum_{j=4n^2}^{\infty}\frac{1}{j\log^{1+\frac{q}{z}}j}\log^{\frac{q}{z}}j=\infty.$$

Consequently, $c=(c_k)_{k\in\mathbf{Z}}\notin\ell^{p,[z;n],q}$, which completes the proof that $\ell^{p,[z;n],q}\subset\ell^{p,[z;n-1],q}$ holds properly.

Finally we verify the equalities of statement ii). If $z=p$ then from (2.5) we see that

$$\|c\|_{\ell^{z,[z;n-1],q}}:=\left(\sum_{j=0}^{\infty}\left(\sum_{i_1=0}^{j-1}\cdots\sum_{i_{n-1}=0}^{i_{n-2}-1}\left(\sum_{k\in d_{ji_1\cdots i_{n-1}}}|c_k|^z\right)\right)^{q/z}\right)^{1/q}=\|c\|_{\ell^{z,q}},$$

so that $\ell^{z,[z;n-1],q}=\ell^{z,q}$.

The same arguments can be applied if any of the above parameters is ∞. In particular, it can be easily seen that the first inclusion of statement ii) holds for $z=\infty$, i.e., that $\ell^{p,[\infty;n-1],q}\subset\ell^{p,[\infty;n],q}$.

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