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UNIFIED DEFINITION OF DIVERGENCE, CURL, AND GRADIENT

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Abstract

In this note the differential expressions of divergence, curl, and gradient are derived based on one common model. Each of them involves the limiting value of a differential quantity per unit volume. By taking advantage of some differential relations of the unit vectors weighted by the metric coefficients, the full expressions of these three quantities in vector analysis can be readily derived.

In this note we will derive the differential expressions of divergence, curl, and gradient in an orthogonal system based on their fundamental definitions as described below. All these definitions are found on one common model. Before we discuss them the concept of differential length, differential area, and differential volume will be reviewed, and the relations between the derivatives of unit vectors, weighted by the corresponding metric coefficients, will be derived first.

In an orthogonal system the differential length of a segment in space can be written in the form

$$d\bar{l}=h_1dx_1\mathbf{x}_1+h_2dx_2\mathbf{x}_2+h_3dx_3\mathbf{x}_3 \tag{1}$$

where x_1, x_2, x_3 denote the coordinate variables in that system and h_1, h_2, h_3 the corresponding metric coefficients. $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ are mutually perpendicular to each other in an orthogonal system, and we assume right-hand system for these unit vectors in that order, i.e., $\mathbf{x}_1\times\mathbf{x}_2=\mathbf{x}_3$, etc.

The differential areas are then given by

$$\left.\begin{aligned}d\bar{A}_1&=(h_2dx_2\mathbf{x}_2)\times(h_3dx_3\mathbf{x}_3)=h_2h_3dx_2dx_3\mathbf{x}_1\\ d\bar{A}_2&=(h_3dx_3\mathbf{x}_3)\times(h_1dx_1\mathbf{x}_1)=h_3h_1dx_3dx_1\mathbf{x}_2\\ d\bar{A}_3&=(h_1dx_1\mathbf{x}_1)\times(h_2dx_2\mathbf{x}_2)=h_1h_2dx_1dx_2\mathbf{x}_3\end{aligned}\right\} \tag{2}$$

and the differential volume is expressed by

$$dv=h_1h_2h_3dx_1dx_2dx_3 \tag{3}$$

From Eq. (1) we find

$$\frac{\partial \bar{l}}{\partial x_1}=h_1\mathbf{x}_1, \qquad \frac{\partial \bar{l}}{\partial x_2}=h_2\mathbf{x}_2, \qquad \frac{\partial \bar{l}}{\partial x_3}=h_3\mathbf{x}_3 \tag{4}$$

hence,

$$\left.\begin{aligned}\frac{\partial(h_1\mathbf{x}_1)}{\partial x_2}&=\frac{\partial(h_2\mathbf{x}_2)}{\partial x_1}\\ \frac{\partial(h_2\mathbf{x}_2)}{\partial x_3}&=\frac{\partial(h_3\mathbf{x}_3)}{\partial x_2}\\ \frac{\partial(h_3\mathbf{x}_3)}{\partial x_1}&=\frac{\partial(h_1\mathbf{x}_1)}{\partial x_3}\end{aligned}\right\} \tag{5}$$

The coordinate variables and their corresponding metric coefficients for three commonly used systems are tabulated below:

System	x_1	x_2	x_3	h_1	h_2	h_3
Cartesian	x	y	z	1	1	1
Cylindrical	ρ	ϕ	z	1	ρ	1
Spherical	r	θ	ϕ	1	r	$r\sin\theta$

It is not difficult to show that Eq. (5) is indeed true for these systems. Of course, they are valid for any orthogonal system.

I. Divergence of a Vector Function

The divergence of a vector function, denoted by $\nabla\cdot F$, is defined by

$$\nabla\cdot\bar{F}=\lim_{\Delta V\to 0}\frac{\sum_i \bar{F}\cdot\Delta\bar{A}_i}{\Delta V}, \tag{1.1}$$

where $\Delta\bar{A}_i$ denotes a typical differential area of the differential volume ΔV, and the direction of $\Delta\bar{A}_i$ is pointed outward. If we let ΔV to be the differential volume in an orthogonal system bounded by the six coordinate surfaces as shown in Fig. 1, then the terms in the numerator of Eq. (1.1) are:

$$\begin{aligned}\sum_i \bar{F}\cdot\Delta\bar{A}=&[(h_2h_3F_1)_{x_1+\Delta x_1}-(h_2h_3F_1)_{x_1}]\Delta x_2\Delta x_3\\ &+[(h_1h_3F_2)_{x_2+\Delta x_2}-(h_1h_3F_2)_{x_2}]\Delta x_1\Delta x_3\\ &+[(h_1h_2F_3)_{x_3+\Delta x_3}-(h_1h_2F_3)_{x_3}]\Delta x_1\Delta x_2\end{aligned}$$

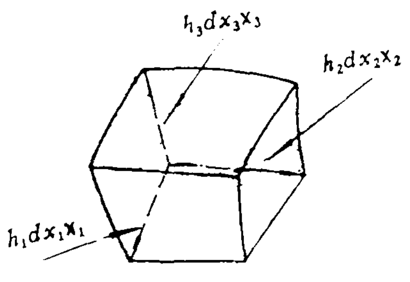

Fig. 1 A differential volume in an orthogonal system

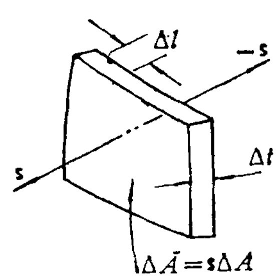

Fig. 2 Model used in the alternative definition of the components of $\nabla\times\bar{F}$ and ∇f

Since $\Delta V = h_1 h_2 h_3 \Delta x_1 \Delta x_2 \Delta x_3$, we find

$$\nabla \cdot F = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial (h_2 h_3 F_1)}{\partial x_1} + \frac{\partial (h_1 h_3 F_2)}{\partial x_2} + \frac{\partial (h_1 h_2 F_3)}{\partial x_3}\right]$$

or

$$\nabla \cdot F = \frac{1}{\Omega} \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{\Omega F_i}{h_i}\right) \tag{1.2}$$

where $\Omega = h_1 h_2 h_3$.

II. Curl of a Vector Function

The curl of a vector function, denoted by $\nabla \times \bar{F}$, is defined by

$$\nabla \times \bar{F} = \lim_{\Delta V \to 0} \frac{\sum_i \Delta \bar{A}_i \times F}{\Delta V} \tag{2.1}$$

where $\Delta \bar{A}_i$ and ΔV have the same meaning as the ones used in Eq. (1.1) for the divergence. We now use the same model for ΔV as shown in Fig. 1. Then the six terms contained in the numerator of Eq. (2.1) are given by

$$\begin{aligned}\sum_i \Delta \bar{A}_i \times \bar{F} = &(h_2 h_3 F_2 \mathbf{x}_3 - h_2 h_3 F_3 \mathbf{x}_2)_{x_1}^{x_1+\Delta x_1} \Delta x_2 \Delta x_3 \\ &+ (h_1 h_3 F_3 \mathbf{x}_1 - h_1 h_3 F_1 \mathbf{x}_3)_{x_2}^{x_2+\Delta x_2} \Delta x_1 \Delta x_3 \\ &+ (h_1 h_2 F_1 \mathbf{x}_2 - h_1 h_2 F_2 \mathbf{x}_1)_{x_3}^{x_3+\Delta x_3} \Delta x_1 \Delta x_2\end{aligned}$$

where the following notation is used:

$$(h_2 h_3 F_2 \mathbf{x}_3)_{x_1}^{x_1+\Delta x_1} = (h_2 h_3 F_2 \mathbf{x}_3)_{x_1+\Delta x_1} - (h_2 h_3 F_2 \mathbf{x}_3)_{x_1}$$

and similarly for the remaining terms. Thus we obtain

$$\begin{aligned}\nabla \times \bar{F} = \frac{1}{h_1 h_2 h_3}\Big[&\frac{\partial}{\partial x_1}(h_2 h_3 F_2 \mathbf{x}_3 - h_2 h_3 F_3 \mathbf{x}_2) \\ &+ \frac{\partial}{\partial x_2}(h_1 h_3 F_3 \mathbf{x}_1 - h_1 h_3 F_1 \mathbf{x}_3) \\ &+ \frac{\partial}{\partial x_3}(h_1 h_2 F_1 \mathbf{x}_2 - h_1 h_2 F_2 \mathbf{x}_1)\Big]\end{aligned} \tag{2.2}$$

The above expression can be expanded into the form:

$$\begin{aligned}\nabla \times F = \frac{1}{h_1 h_2 h_3}\Big[&h_3 \mathbf{x}_3 \frac{\partial}{\partial x_1}(h_2 F_2) + h_2 F_2 \frac{\partial}{\partial x_1}(h_3 \mathbf{x}_3) \\ &- h_2 \mathbf{x}_2 \frac{\partial}{\partial x_1}(h_3 F_3) - h_3 F_3 \frac{\partial}{\partial x_1}(h_2 \mathbf{x}_2) \\ &+ h_1 \mathbf{x}_1 \frac{\partial}{\partial x_2}(h_3 F_3) + h_3 F_3 \frac{\partial}{\partial x_2}(h_1 \mathbf{x}_1) \\ &- h_3 \mathbf{x}_3 \frac{\partial}{\partial x_2}(h_1 F_1) - h_1 F_1 \frac{\partial}{\partial x_2}(h_3 \mathbf{x}_3)\end{aligned}$$

$$+h_2\mathbf{x}_2\frac{\partial}{\partial x_3}(h_1F_1)+h_1F_1\frac{\partial}{\partial x_3}(h_2\mathbf{x}_2)$$

$$-h_1\mathbf{x}_1\frac{\partial}{\partial x_3}(h_2F_2)-h_2F_2\frac{\partial}{\partial x_3}(h_1\mathbf{x}_1)\Big] \tag{2.3}$$

In view of Eq. (5), the terms involving the derivatives of the unit vectors weighted by their metric coefficients cancel each other. Hence

$$\nabla\times F=\frac{1}{h_1h_2h_3}\Big\{h_1\mathbf{x}_1\Big[\frac{\partial(h_3F_3)}{\partial x_2}-\frac{\partial(h_2F_2)}{\partial x_3}\Big]$$

$$+h_2\mathbf{x}_2\Big[\frac{\partial(h_1F_1)}{\partial x_3}-\frac{\partial(h_3F_3)}{\partial x_1}\Big]$$

$$+h_3\mathbf{x}_3\Big[\frac{\partial(h_2F_2)}{\partial x_1}-\frac{\partial(h_1F_1)}{\partial x_2}\Big]\Big\} \tag{2.4}$$

Equation (2.4) can be written in the determinant form as

$$\nabla\times F=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{x}_1 & h_2\mathbf{x}_2 & h_3\mathbf{x}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix} \tag{2.5}$$

III. Gradient of a Scalar Function

The gradient of a scalar function, denoted by ∇f, is defined by

$$\nabla f=\lim_{\Delta V\to 0}\frac{\sum_i f\Delta\bar{A}_i}{\Delta V} \tag{3.1}$$

Using the same model as before we obtain

$$\nabla f=\lim_{\Delta V\to 0}\Big[\frac{(h_2h_3f\mathbf{x}_1)_{x_1}^{x_1+\Delta x_1}\Delta x_2\Delta x_3+(h_1h_3f\mathbf{x}_2)_{x_2}^{x_2+\Delta x_2}\Delta x_1\Delta x_3+(h_1h_2f\mathbf{x}_3)_{x_3}^{x_3+\Delta x_3}\Delta x_1\Delta x_2}{h_1h_2h_3\Delta x_1\Delta x_2\Delta x_3}\Big]$$

$$=\frac{1}{h_1h_2h_3}\Big[\frac{\partial}{\partial x_1}(h_2h_3f\mathbf{x}_1)+\frac{\partial}{\partial x_2}(h_1h_3f\mathbf{x}_2)+\frac{\partial}{\partial x_3}(h_1h_2f\mathbf{x}_3)\Big] \tag{3.2}$$

The above expression can be decomposed as follows:

$$\frac{\partial}{\partial x_1}(h_2h_3f\mathbf{x}_1)=h_2h_3\mathbf{x}_1\frac{\partial f}{\partial x_1}+f\frac{\partial}{\partial x_1}(h_2h_3\mathbf{x}_1)$$

and

$$\frac{\partial}{\partial x_1}(h_2h_3\mathbf{x}_1)=\frac{\partial}{\partial x_1}(h_2h_3\mathbf{x}_2\times\mathbf{x}_3)=h_2\mathbf{x}_2\times\frac{\partial(h_3\mathbf{x}_3)}{\partial x_1}-h_3\mathbf{x}_3\times\frac{\partial(h_2\mathbf{x}_2)}{\partial x_1}$$

and similarly for the other two terms in Eq. (3.2). Thus Eq. (3.2) can be written as

$$\nabla f=\frac{1}{h_1h_2h_3}\Big[h_2h_3\mathbf{x}_1\frac{\partial f}{\partial x_1}+fh_2\mathbf{x}_2\times\frac{\partial(h_3\mathbf{x}_3)}{\partial x_1}-fh_3\mathbf{x}_3\times\frac{\partial(h_2\mathbf{x}_2)}{\partial x_1}$$

$$+h_1h_2\mathbf{x}_2\frac{\partial f}{\partial x_2}+fh_3\mathbf{x}_3\times\frac{\partial(h_1\mathbf{x}_1)}{\partial x_2}-fh_1\mathbf{x}_1\times\frac{\partial(h_3\mathbf{x}_3)}{\partial x_2}$$

$$+h_2h_3\mathbf{x}_3\frac{\partial f}{\partial x_3}+fh_1\mathbf{x}_1\times\frac{\partial(h_2\mathbf{x}_2)}{\partial x_3}-fh_2\mathbf{x}_2\times\frac{\partial(h_1\mathbf{x}_1)}{\partial x_3}\Bigg] \tag{3.3}$$

In view of Eq. (5) the six terms involving the derivatives of the weighted unit vectors cancel each other, hence,

$$\nabla f=\frac{1}{h_1}\frac{\partial f}{\partial x_1}\mathbf{x}_1+\frac{1}{h_2}\frac{\partial f}{\partial x_2}\mathbf{x}_2+\frac{1}{h_3}\frac{\partial f}{\partial x_3}\mathbf{x}_3$$

or

$$\nabla f=\sum_{i=1}^{3}\frac{1}{h_i}\frac{\partial f}{\partial x_i}\mathbf{x}_i \tag{3.4}$$

IV. Alternative Definitions of the Components of $\nabla\times\bar{F}$ and ∇f

If we choose ΔV to be a flat volume of uniform thickness Δt and with a broad surface pointed in the s direction as shown in Fig. 2. Then by taking the scalar product of Eq. (2.1) with s, we obtain

$$\mathbf{s}\cdot\nabla\times F=\lim_{\Delta V\to 0}\frac{\mathbf{s}\cdot\sum_i\Delta\bar{A}_i\times F}{\Delta V}=\lim_{\substack{\Delta A\to 0\\ \Delta t\to 0}}\frac{\sum_i F\cdot(\mathbf{s}\times\Delta\bar{A}_i)}{\Delta A\Delta t} \tag{4.1}$$

The vector product $\mathbf{s}\times\Delta\bar{A}_i$ vanishes on the two broad surfaces because $\Delta\bar{A}_i$ there is parallel to $\mathbf{s}$. At the side surface.

$$\mathbf{s}\times\Delta\bar{A}_i=\mathbf{l}\Delta l\Delta t=\Delta\bar{l}\Delta t \tag{4.2}$$

hence

$$\mathbf{s}\cdot\nabla\times F=\lim_{\Delta A\to 0}\frac{\sum_i F\cdot\Delta\bar{l}}{\Delta A} \tag{4.3}$$

Equation (4.3) gives the definition of the component of $\nabla\times F$ in the direction of s. If we let $\mathbf{s}=\mathbf{x}_1$, then $\Delta A=h_2h_3\Delta x_2\Delta x_3$, and

$$\sum_i F\cdot\Delta\bar{l}=-(h_2F_2)_{x_3}^{x_3+\Delta x_3}\Delta x_2+(h_3F_3)_{x_2}^{x_2+\Delta x_2}\Delta x_2$$

Hence

$$\mathbf{x}_1\cdot\nabla\times F=\frac{1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3F_3)-\frac{\partial}{\partial x_3}(h_2F_2)\right] \tag{4.4}$$

which agrees with Eq. (2.4) or (2.5).

The component of ∇f can now be discussed in the same manner. By taking the scalar product of Eq. (3.1) with s we obtain

$$\mathbf{s}\cdot\nabla f=\lim_{\Delta V\to 0}\frac{\sum f\mathbf{s}\cdot\Delta\bar{A}_i}{\Delta V}$$

The scalar product $\mathbf{s}\times\Delta\bar{A}_i$ vanishes at the side surface of the flat volume illustrated in Fig. 2. At the two broad surfaces

$$\sum f\mathbf{s}\cdot\Delta\bar{A}_i=(f)_t^{t+\Delta t}\Delta A$$

and $\Delta V = \Delta A \Delta t$. Hence

$$\mathbf{s} \cdot \nabla f = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} \tag{4.5}$$

If we identify s as $\mathbf{x}_1$ then $\Delta t = h_1 \Delta x_1$, thus

$$\mathbf{x}_1 \cdot \nabla f = \frac{1}{h_1} \frac{\partial f}{\partial x_1} \tag{4.6}$$

which agrees with Eq. (3.4). By identifying s with $\mathbf{x}_2$ or $\mathbf{x}_3$ we can derive the other components of ∇f in any orthogonal system. Here we treat (4.5) as an alternative definition of a typical component of ∇f. This definition gives the directional directive of ∇f, i.e.,

$$\mathbf{s} \cdot \nabla f = \frac{\partial f}{\partial t} \tag{4.7}$$

We interpret 't' as the arc length measured in the direction of s, not merely a variable such as x_l in Eq. (4.6)