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SYMPLECTIC GEOMETRY AND THE UNIQUENESS OF GRAUERT TUBES

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1 Introduction

Given a differentiable manifold M, it is always possible to find an almost complex structure J on TM with respect to which the zero-section is totally-real. In this paper we study a situation in which there is a canonical choice for a complex structure with this property, a canonical complexification of M.

Suppose that (M,g) is a compact real-analytic Riemannian manifold of dimension n. Identify M with the zero section in TM. Let $\rho : TM \to \mathbb{R}$ be the length, with respect to g, of tangent vectors. Then, for r sufficiently small, $T^rM = \{v \in TM \mid \rho(v) < r\}$ carries a unique complex structure satisfying the following two conditions:

(i) ρ^2 is strictly plurisubharmonic and the corresponding Kähler metric restricts to g on M.

(ii) ρ is a solution of the homogeneous complex Monge–Ampère equation $(dd^c\rho)^n = 0$ on $T^rM \setminus M$, where $M \subset T^rM$.

This is proven by V. Guillemin and M. Stenzel in [GS] and by L. Lempert and R. Szöke in [LeS]. The resulting Stein complex manifolds will be called Grauert tubes. Alternatively, one can think of the complex structure as the unique choice making the leaves of the Riemann foliation on $T^rM \setminus M$ into holomorphic curves.

It will be important for our purposes that the map

$$\sigma : T^rM \to T^rM\,, \quad v \mapsto -v\,,$$

is an antiholomorphic involution with respect to these special complex structures.

Another characterisation of Grauert tubes is given by the following theorem, taken from [B].

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Theorem 1. *Let X be a connected complex manifold and u a smooth, bounded, non-negative, strictly plurisubharmonic exhaustion function such that $\sqrt{u}$ solves the Monge–Ampère equation on $X \setminus \{u = 0\}$.*

Then $M = \{u = 0\}$ is a connected real-analytic submanifold of X and there exists a biholomorphism ϕ from X to a Grauert tube on $T^r M$ with $\rho = \sqrt{u} \circ \phi$, and $r = \sup_{x \in X} \sqrt{u(x)}$.

We call the r in the statement of the theorem the radius of the Grauert tube. The purpose of this paper is to demonstrate that there is a unique way of associating a Riemannian manifold to a Grauert tube. Specifically, the remainder of the paper is devoted to proving the following.

Theorem 2. *Suppose that two Grauert tubes, X_1 associated to a compact Riemannian manifold (M_1, g_1) and X_2 associated to (M_2, g_2), of equal finite radius r, are biholomorphic via a map ϕ. Then ϕ maps $M_1 \subset X_1$ to $M_2 \subset X_2$, restricting to an isometry with respect to g_1 and g_2.*

In particular, the biholomorphisms of a Grauert tube are just the differentials of the isometries of the underlying Riemannian manifold.

The last statement of the theorem is simply the functoriality of the Grauert tube construction [LeS]: every isometry ϕ of M extends uniquely by its differential $d\phi_* : T^r(M) \to T^r(M)$ to a biholomorphic map of the Grauert tube.

A weak form of Theorem 2 was shown in [B]. Several other previous partial results are due to S.-J. Kan [K1,2] and S.-J. Kan and D. Ma [KM1,2], which employed an interesting variety of different methods.

The case of Grauert tubes of infinite radius is quite different from the case treated here. In particular, biholomorphic automorphism groups for such a Grauert tube can be infinite dimensional, and can move the submanifold M off itself in the tube. Simple examples of this phenomenon are shown in [S2]. Partial uniqueness results (*i.e.*, under more restrictive hypotheses) in the case of infinite radius appear in [B] and [S2]. Although Grauert tubes can also be constructed over non-compact manifolds, Theorem 2 is no longer true in general. We will discuss this case in a forthcoming paper.

Finally, it is a pleasure for us to express our thanks to Yasha Eliashberg for several very stimulating conversations we have had with him during the course of this work, from which we have benefited significantly.

2 Proof of Theorem 2

Let X be a Grauert tube of finite radius r associated to a compact Riemannian manifold (M, g). Let $Aut_{\mathbb{C}}(X)$ denote the group of biholomorphisms of X, let $\pm Aut_{\mathbb{C}}(X)$ be the group of holomorphic or antiholomorphic automorphisms of X, and $Isom(M) \hookrightarrow Aut_{\mathbb{C}}(X)$ be the group of isometries of M with respect to the original metric g, acting via differentials on T^rM. We note that the index of $Aut_{\mathbb{C}}(X) \subset \pm Aut_{\mathbb{C}}(X)$ is exactly two as X has an antiholomorphic involution. It follows from Theorem 6.3 of Szőke [S2] that $Aut_{\mathbb{C}}(X)$, and hence $\pm Aut_{\mathbb{C}}(X)$, is a compact Lie group.

Now, following [B], we let $\tau = \rho^2$, the length function squared, and proceed as follows. The function $\tau : X \to [0, r^2)$ is a proper strictly plurisubharmonic exhaustion function of X. Let $\tilde{\tau}$ be the average of $h^*(\tau)$ over $h \in \pm Aut_{\mathbb{C}}(X)$ with respect to the Haar measure on $\pm Aut_{\mathbb{C}}(X)$. Then, as explained in [B], $\tilde{\tau}$ is again a smooth, strictly plurisubharmonic exhaustion of X which is now invariant under $\pm Aut_{\mathbb{C}}(X)$. Therefore choosing c to be slightly less than r^2, the set $\tilde{X} = \{z \in X | \tilde{\tau} \leq c\}$ is a strictly pseudoconvex subset of X which is invariant under $\pm Aut_{\mathbb{C}}(X)$. As $\tilde{X}$ lies in a slightly larger Stein manifold, namely X, we can apply a theorem of S.-Y. Cheng and S.-T. Yau, see [CheY], to say that $\tilde{X}$ carries a unique complete Kähler–Einstein metric of negative Ricci curvature.

Remark. There is a maximal radius R, which is not necessarily $+\infty$, for which a Grauert tube structure can be defined on T^RM. The above discussion is of course only necessary if our $r = R$. Otherwise a Kähler–Einstein metric exists on X itself and in what follows we can just assume $\tilde{X} = X$.

Let g_{KE} be the real (Riemannian) part of the Kähler–Einstein metric and ω_{KE} the imaginary part, which is a symplectic form on $\tilde{X}$. Since the involution σ is anti-holomorphic, we have that $\sigma^* g_{KE} = g_{KE}$ and $\sigma^*\omega_{KE} = -\omega_{KE}$. The zero-section $M \subset X$ is fixed by σ, and hence M is a totally geodesic submanifold of $\tilde{X}$ with the metric g_{KE}. Also, ω_{KE} must be zero when restricted to M, in other words, M is a Lagrangian submanifold of $(\tilde{X}, \omega_{KE})$.

These observations lead to the following lemma (see [B]).

Lemma 3. *$Aut^0_{\mathbb{C}}(X) \simeq Isom^0(M)$.*

Here, $Aut^0_{\mathbb{C}}(X)$ denotes the identity component of the Lie group of biholomorphisms of X, and $Isom^0(M)$ is the identity component of the isometries of M with respect to the original metric g, acting via differentials on T^rM.

Proof. It is a result of Chen, Leung and Nagano, see [Ch], that a closed, maximal-dimensional, totally-real, minimal submanifold M of a Kähler manifold $\tilde{X}$ of negative Ricci curvature is strictly stable. In our situation this implies in particular that $Aut^0_{\mathbb{C}}(X)$ must preserve M as all biholomorphisms act as isometries of g_{KE} when restricted to $\tilde{X}$. Such biholomorphisms preserving M must act as isometries of g when restricted to M, see [S2], and it has already been remarked that the differentials of isometries of g are biholomorphisms of X.

A few comments will be needed on the symplectic manifolds $(\tilde{X}, \omega_{KE})$.

Remark. We observe that for the remainder of this paper the symplectic form ω_{KE} on $\tilde{X}$ or X could in fact be replaced by any other symplectic form which is invariant under $\pm Aut_{\mathbb{C}}(X)$ and, like ω_{KE}, satisfies the conclusions of the forthcoming Lemma 5. Such symplectic forms can be constructed directly from the canonical form on the cotangent bundle of M without appealing to Cheng and Yau's analysis. We thank the referee for this remark.

Let $h : [0, r) \to [0, \infty)$ be a smooth, convex and strictly increasing proper map satisfying $h(x) = x$ for x near 0.

Lemma 4. *$(\tilde{X}, \omega_{KE})$ is symplectomorphic via a symplectomorphism fixing M to $(T^r M, -dd^c f)$, where $f = h \circ \rho^2$.*

Proof. The Kähler–Einstein metric can also be written in the form $\omega_{KE} = -dd^c f'$ for some strictly plurisubharmonic exhaustion function f' on $\tilde{X}$ ([CheY]). Now the existence of a symplectomorphism follows from Theorem 1.4.A in [EG]. Since σ is anti-holomorphic, we have $\sigma^* \circ dd^c = -dd^c \circ \sigma^*$, and thus replacing f', if necessary, by $F = \frac{1}{2}(f' + \sigma^* f')$ we can assume without loss of generality that the exhaustion function f' is σ-invariant. Hence the whole construction of [EG] can be carried out in a σ-invariant fashion. In particular the resulting symplectomorphism must commute with σ and hence fix M, the fixed point set of σ.

The symplectic manifold $(T^r M, \omega = -dd^c f)$ has, in the terminology of [EG], a contracting vector field v given by the negative gradient of f with respect to the metric $G(x, y) = \omega(x, Jy)$, where J is our complex structure. Equivalently, v satisfies $v \rfloor \omega = d^c f$. We note that v vanishes only along M.

Furthermore,

$$\mathcal{L}_v \omega = d(v \rfloor \omega) = -\omega$$

and

$$\mathcal{L}_v d^c f = -v \rfloor \omega + d(v \rfloor d^c f) = -d^c f \,.$$

In particular, if (v_t) denotes the 1-parameter group of diffeomorphisms generated by v we have

$$v_t^* d^c f = e^{-t} d^c f\,. \tag{1}$$

We can now in fact observe the following.

LEMMA 5. *$(\tilde{X}, \omega_{KE})$ is symplectomorphic via a symplectomorphism ψ fixing M to $(T^*M, d(pdq))$ where pdq denotes the canonical Liouville 1-form on the cotangent bundle.*

Proof. It suffices to show that $(T^rM, \omega = -dd^c f)$ is symplectomorphic to $(TM, d\mu)$ where μ is the pull-back of pdq to TM using the isomorphism given by the metric g. The conditions on a Grauert tube imply that $\mu = -d^c f$ in the neighbourhood of M where $h(x) = x$. Recall that $f = h \circ \rho^2$ where $h : [0, r) \to [0, \infty)$ is any smooth, convex and strictly increasing proper map satisfying $h(x) = x$ for x near 0.

Now,

$$d^c f = h'(\rho^2) d^c \rho^2$$

and

$$\omega = -dd^c f = -h''(\rho^2) d(\rho^2) \wedge d^c \rho^2 - h'(\rho^2) dd^c \rho^2\,.$$

We choose h to grow sufficiently fast that $h'' \gg h'$ for x close to r. Then if v is the corresponding contracting vector field we see that $d(\rho^2)(v)$ must be of the order of h'/h'' near the boundary of T^rM. Hence by choosing a suitable h we are able to assume that the vector field $-v$ is complete.

We can define an expanding vector field w on TM with respect to $d\mu$ by $w \rfloor d\mu = \mu$. Let (w_t) denote the corresponding 1-parameter group.

In a neighbourhood U of M we have $w = -v$.

We now define the map ψ of T^rM to TM as follows. Given $x \in TM$, choose t sufficiently large that $v_t(x) \in U$. Then set $\psi(x) = w_t \circ v_t(x)$. This is clearly well-defined (that is, independent of t), is a symplectomorphism (which is surjective since $-v$ is complete), and is the identity near M.

We can now prove Theorem 2. Suppose that we have a biholomorphism ϕ between two Grauert tubes $X_1 = T^{r_1}M_1$ and $X_2 = T^{r_2}M_2$. We will denote $\phi(M_1)$ also by M_1, and construct $\tilde{X}_2$ as above. There is an antiholomorphic involution σ_1 of X_2 which preserves M_1, namely the push-forward of the involution on X_1. As σ_1 also preserves $\tilde{X}_2$, we see that M_1, alongside M_2, must be a Lagrangian submanifold of $(\tilde{X}_2, \omega_{KE})$. Combining with the results of [S1] and [B], to prove Theorem 2 it will suffice for us to show that as submanifolds of X_2 we have $M_1 = M_2$.

Let λ and v denote the pull-backs via ψ from Lemma 5 of the primitive 1-form pdq and the corresponding contracting vector field respectively to $\tilde{X}_2$. Also, let $\omega = d\lambda = \omega_{KE}$.

There are two cases to consider.

Case 1. $M_1 \subset \tilde{X}_2$ is not an exact Lagrangian submanifold. That is, the form $\lambda|_{M_1}$, which is closed because M_1 is Lagrangian, is not exact.

In this case, there is a closed loop $\gamma_1 \subset M_1$ along which λ has a non-zero integral. Project γ_1 onto M_2 by the bundle projection of $X_2 = T^{r_2}M_2$ and connect each point of γ_1 with its projection by a segment in the corresponding fiber to construct (the image of) a cylinder C in $\tilde{X}_2$ whose boundary components are γ_1 and the projection, say γ_2, of γ_1 in M_2. As $\lambda|_{M_2} = 0$, we have

$$\int_C \omega = \int_{\gamma_1} \lambda \neq 0$$

by Stokes' theorem.

We recall now the antiholomorphic involution σ of a Grauert tube about the underlying Riemannian manifold. In our case we have antiholomorphic involutions σ_1 about M_1 and σ_2 about M_2.

In fact there is a sequence of (Lagrangian) submanifolds of $\tilde{X}_2$ about which there are antiholomorphic involutions. We define (unfortunately) $N_1 = M_2$, $N_2 = M_1$, N_3 to be the reflection of N_1 in N_2, and in general N_k to be the reflection of N_{k-2} in N_{k-1}. Each N_k can be written in the form $\sigma_1^{\epsilon}(\sigma_2\sigma_1)^{n(k)}M_2$ for k odd and $\sigma_2^{\epsilon}(\sigma_1\sigma_2)^{n(k)}M_1$ for k even. In these formulas, ϵ is either 0 or 1 depending on $k(\mathrm{mod}4)$ and similarly $n(k)$ is some exponent depending on k.

As already noted above, it follows from Theorem 6.3 of Szőke [S2] that $Aut_{\mathbb{C}}(X)$ is a compact Lie group for any Grauert tube X of finite radius. Combining this with the fact that the identity component of $Aut_{\mathbb{C}}(X_2)$ must preserve M_2 we deduce that for some large (odd) value of k, say l, $N_l = M_2$.

Let $C_1 = C$, C_2 be the reflection of C_1 in N_2, and in general C_k be the reflection of C_{k-1} in N_k. These cylinders can be joined together end-to-end for $1 \leq k \leq l-1$ to obtain one long cylinder $\tilde{C}$ with both boundaries in M_2. Now, each of the reflections σ is an orientation reversing map of $\tilde{C}$ wherever it is defined, but also satisfies $\sigma^*\omega = -\omega$. Therefore, we have that

$$\int_{\tilde{C}} \omega = (l-1)\int_C \omega \neq 0\,.$$

However, this is a contradiction to Stokes' Theorem.

Case 2. $M_1 \subset \tilde{X}_2$ is an exact Lagrangian submanifold, that is, $\lambda|_{M_1}$ is exact.

First we set-up the Lagrangian submanifolds N_k exactly as in Case 1. Again assume that $N_l = M_2$.

We now study the isotopy L_t for $t \geq 0$ of M_1 given by $L_t = v_t(M_1)$. Recall that (v_t) is the 1-parameter group generated by the contracting vector field v. By equation 1, this is an exact Hamiltonian isotopy, see for instance [C].

Suppose that M_1 does not map into M_2. Then we can find a point $p \in M_1 \backslash M_2$.

Let $Z = \mathbb{R} + i(0,1) \subset \mathbb{C}$. It is proven by H. Hofer in [H] that for any fixed t there exists a holomorphic map $u : Z \to X_2$ with a continuous extension to the boundary such that $u(i) = p$, $u(\mathbb{R}) \subset L_t$, $u(\mathbb{R} + i) \subset M_1$ and $\int_Z u^*\omega < \infty$.

As $\tilde{X}_2$ is exhausted by pseudoconvex domains, all holomorphic strips with boundary on $M_1 \cup L_t$, for any t, must lie in a fixed compact region. This observation is sufficient to be able to apply Hofer's results which are stated for Lagrangian submanifolds of a compact symplectic manifold. Also we will let m be an upper bound for the norm of λ with respect to the Kähler metric on this compact set, which, without loss of generality, includes all of the N_k for $1 \leq k \leq l$.

In the situation when M_1 and L_t do not intersect transversally, the behaviour of such maps $u(x+iy)$ as $x \to \pm\infty$ may be difficult to describe precisely. However the following is true (see [H] again).

For any $\delta > 0$ there exist numbers $R_1, R_2 \in \mathbb{R}$ with $-R_1 > \delta^{-1}$, $R_2 > \delta^{-1}$ and

$$\left| \int_Z u^*\omega - \int_{[R_1,R_2]+i[0,1]} u^*\omega \right| < \delta \,.$$

Furthermore, for suitable R_1, R_2 and $j = 1, 2$, the length of $t \mapsto u(R_j + it)$ is less than δ. To see this, note that

$$\int_Z \big(|u_x|^2 + |u_y|^2\big) ds dt = 2 \int_Z u^*\omega < \infty \,.$$

Now,

$$\begin{aligned} \left| \int_Z u^*\omega \right| &< \delta + \left| \int_{[R_1,R_2]+i[0,1]} u^*\omega \right| \\ &< (1+2m)\delta + \left| \int_{[R_1,R_2]} u^*\lambda \right| + \left| \int_{[R_1,R_2]+i} u^*\lambda \right| \end{aligned}$$

$$= (1+2m)\delta + e^{-t}\left|\int_{[R_1,R_2]} u^*(v_t^{-1})^*\lambda\right| + \left|\int_{[R_1,R_2]+i} u^*\lambda\right|.$$

But the integral of λ along any path in M_1 is uniformly bounded (as λ is exact it depends only on the endpoints). Hence, $|\int_Z u^*\omega| < C$, where C is a uniform constant independent of t.

Now let u_t be the holomorphic map corresponding to the Lagrangian L_t, and for any holomorphic map f from an open subset $E \subset \mathbb{C}$ into X_2 write

$$\partial f = \left\{ q \in X_2 \mid q = \lim_{n\to\infty} f(z_n), z_n \in \mathrm{E}, z_n \to \partial E \cup \{\infty\} \right\}.$$

LEMMA 6. *There exists a sequence $t_j \to \infty$ such that the maps u_{t_j} converge uniformly on compact sets to a non-constant holomorphic map $u_\infty : Z \to X_2$ with $\partial u_\infty \subset M_1 \cup M_2$.*

Proof. For each t, the holomorphic map u_t can be Schwarz reflected in M_1 using σ_1. After a reparameterization of the resulting maps such that they are now defined on the open disk Δ we have a sequence of holomorphic maps $\tilde{u}_t : \Delta \to X_2$ satisfying $|\int_\Delta \tilde{u}_t^*\omega| < 2C$, $\tilde{u}_t(0) = p$ and $\partial\tilde{u}_t \subset L_t \cup \sigma_1(L_t)$. Since the $\tilde{u}_t$ all have their image in a bounded open set in a Stein manifold, Montel's theorem implies that after taking a subsequence t_j the $\tilde{u}_{t_j}$ converge uniformly on compact sets to another holomorphic map $\tilde{u}_\infty : \Delta \to X_2$ with $\tilde{u}_\infty(0) = p$. As the $\tilde{u}_t$ all have area uniformly bounded by $2C$, a result of F. Labourie, see [L], implies that $\partial\tilde{u}_\infty$ is contained in the Hausdorff limit of the sets $\partial\tilde{u}_{t_j}$. This Hausdorff limit is contained in $M_2 \cup \sigma_1(M_2)$ which is disjoint from p. Hence the map $\tilde{u}_\infty$ is non-constant. This is enough to establish the lemma, letting u_∞ be a suitable reparameterization of $\tilde{u}_\infty$.

Now let $K = \int_Z u_\infty^*\omega$. As u_∞ is holomorphic and non-constant, $K > 0$.

Now choose δ to be much less than K and find R_1 and R_2 as before. As the length of $u_\infty(R_j + i(0,1))$ is less than δ, for $j = 1, 2$ we can continuously extend u_∞ to $R_j + i[0,1]$.

We claim that the map u_∞ also extends continuously to (R_1, R_2), and maps (R_1, R_2) into $M_2 = N_1$, and therefore has a holomorphic reflection by means of σ_2 across (R_1, R_2). We first prepare the target manifold X_2.

X_2 is a Stein manifold, and so may be embedded properly in $\mathbb{C}^N$, for some N. Let $f_1, \dots, f_N$ be the component functions of this embedding, and set $f_j^*(z) = \overline{f_j(\sigma_2(z))}$, $j = 1, \dots, N$. Consider the embedding $F : X_2 \to \mathbb{C}^{2N}$ with components $F_{2j-1} = \frac{1}{2}(f_j + f_j^*)$, $F_{2j} = \frac{1}{2i}(f_j - f_j^*)$, $j = 1, \dots, N$. Note that this embedding has the property that $F(\sigma_2(z)) = \overline{F(z)}$, for all $z \in X_2$, and so $F(M_2) = F(X_2) \cap \mathbb{R}^{2N}$. To prove the map $u_\infty : (R_1, R_2) + i(0,1) \to X_2$ extends by reflection, it suffices to show each of

$F_k \circ u_\infty$ extends by reflection to $(R_1, R_2)+i(-1,1)$, and to do this it suffices, by classical Schwarz reflection, to show that $v_k = \Im(F_k \circ u_\infty)$ is continuous up to the boundary arc (R_1, R_2) and equals 0 there.

Now each of the harmonic functions $v_{t_j,k} = \Im(F_k \circ u_{t_j})$ is continuous up to the boundary of $[R_1, R_2] + i[0,1]$, and hence can be written as a Poisson integral of its boundary values there. Note that the functions $v_{t_j,k}$ are uniformly bounded on $[R_1, R_2] + i[0,1]$. Furthermore, their boundary values converge to the function $w_{\infty,k}$ which is identically 0 along $[R_1, R_2]$ (since $\max_{L_{t_j}} |\Im(F_k)|$ converges to 0 as $t_j \to \infty$) and which equals $\Im(F_k \circ u_\infty)$ on the rest of the boundary, by the interior convergence of the sequence u_{t_j}. Hence, by the bounded convergence theorem, we have that v_k on $(R_1, R_2) + i(0,1)$ is the Poisson integral of $w_{\infty,k}$. By standard properties of Poisson integrals, v_k is continuous up to the boundary along the arc (R_1, R_2) in $[R_1, R_2] + i[0,1]$, completing the proof of the claim.

We can now apply the reflections successively across the N_k exactly as in Case 1 to the mapping u_∞. Again as in Case 1 we stop reflecting after $(l-1)$-steps, so the resulting mapping, which we still call u_∞ is defined and holomorphic on some rectangle $\mathcal{R} = [R_1, R_2]+i[0, l-1]$, and $u_\infty([R_1, R_2]) \subset M_2$ and $u_\infty([R_1, R_2] + i(l-1)) \subset M_2 = N_1 = N_l$.

But

$$(l-1)(K-\delta) < \int_{\mathcal{R}} u_\infty^* \omega = \int_{\partial\mathcal{R}} u_\infty^* \lambda \leq 2m\delta(l-1)$$

giving a contradiction as required for δ sufficiently small.

References

[B] D. Burns, On the uniqueness and characterization of Grauert tubes, in "Complex Analysis and Geometry (Trento, 1993)", Lecture Notes in Pure and Appl. Math. 173, Dekker, New York (1996), 119–133,

[C] M. Chaperon, Questions de géometrie symplectique, in "Séminaire Bourbaki, Astérique 105/106 (1983), 231–249.

[Ch] B.Y. Chen, Geometry of Submanifolds and its Applications, Tokyo, Science University of Tokyo, 1981.

[CheY] S.-Y. Cheng, S.-T. Yau, On the existence of a complete Kähler–Einstein metric on non-compact complex manifolds and the regularity of Fefferman's equation, Comm. Pure Appl. Math. 33 (1980), 507–544.

[EG] Y. Eliashberg, M. Gromov, Convex Symplectic Manifolds, in "Several Complex Variables and Complex Geometry, Part 2 (Santa Cruz, CA, 1989), Proc. Sympos. Pure Math. 52:2, American Math. Soc., Providence, RI (1991) 135–162.

[GS] V. Guillemin, M. Stenzel, Grauert tubes and the homogeneous Monge–Ampère equation, J. Diff. Geom. 34 (1991), 561–570.

[H] H. Hofer, Lusternik–Schnirelman theory for Lagrangian intersections, Ann. Inst. Henri Poincaré 5 (1988), 465–499.

[K1] S.-J. Kan, On the characterization of Grauert tubes covered by the ball, Math. Ann. 309 (1997), 71–80.

[K2] S.-J. Kan, On the rigidity of non-positively curved Grauert tubes, Math. Z. 229 (1998), 349–363.

[KM1] S.-J. Kan, D. Ma, On the rigidity of Grauert tubes over Riemannian manifolds of constant curvature, to appear.

[KM2] S.-J. Kan, D. Ma, On the rigidity of Grauert tubes over locally symmetric spaces, to appear.

[L] F. Labourie, Appendix to Exemples de courbes pseudo-holomorphic en geometrie Riemannienne, in 'Holomorphic Curves in Symplectic Geometry' (M. Audin, J. Lafontaine, eds.), Birkhäuser (1994), 233–249.

[LeS] L. Lempert, R. Szöke, Global solutions of the homogeneous complex Monge–Ampère equation and complex structures on the tangent bundle of Riemannian manifolds, Math. Ann. 290 (1991), 689–712.

[M] M.-P. Muller, Gromov's Schwarz lemma as an estimate of the gradient for holomorphic curves, in 'Holomorphic Curves in Symplectic Geometry' (M. Audin, J. Lafontaine, eds.), Birkhäuser (1994), 217–231.

[S1] R. Szőke, Complex structures on the tangent bundles of Riemannian manifolds, Math. Ann. 291 (1991), 409–428.

[S2] R. Szőke, Automorphisms of certain Stein manifolds, Math. Z. 219 (1995), 357–385.

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