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K-THEORETIC INVARIANTS FOR FLOER HOMOLOGY

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Abstract

This paper defines two K-theoretic invariants, Wh_1 and Wh_2, for individual and one-parameter families of Floer chain complexes. The chain complexes are generated by intersection points of two Lagrangian submanifolds of a symplectic manifold, and the boundary maps are determined by holomorphic curves connecting pairs of intersection points. The paper proves that Wh_1 and Wh_2 do not depend on the choice of almost complex structures and are invariant under Hamiltonian deformations. The proof of this invariance uses properties of holomorphic curves, parametric gluing theorems, and a stabilization process.

1 Introduction

1.1 Overview. This paper adapts the Whitehead torsion from h-cobordisms and the Wh_2 obstruction from pseudo-isotopies to invariants of the Floer theory of Lagrangian intersections. The first, defined for Floer chain complexes, provides an obstruction, when Floer homology cannot, to deforming a Lagrangian away from its image under a Hamiltonian isotopy. The second, defined for a one-parameter family of Floer chain complexes, provides an obstruction to deforming a Hamiltonian isotopy of one Lagrangian to another 'away' from a third Lagrangian.

When Floer proved that Floer homology is invariant to changes in either the almost complex structure or the Lagrangian, he showed that the Floer chain complex only changes in ways which imitate the handle-slides, births and deaths of critical points in a one-parameter family of Morse functions. Because his proofs were quite analytical and in some places incomplete, symplectic geometers have abandoned his approach in favor of an elegant homotopy of homotopies method. This paper uses Floer's original method, however, as its explicit treatment of handle-slides, births and deaths are necessary when considering Whitehead torsion and Wh_2. The paper bypasses Floer's hard analysis with a geometric technique known as stabilization.

1.2 History. A *symplectic manifold* (P, ω) is a $2n$-dimensional smooth manifold P equipped with a nondegenerate closed 2-form ω, known as a *symplectic form.* A *Lagrangian submanifold* $L \subset P$ is an n-dimensional submanifold on which ω vanishes. Lagrangians are the largest submanifolds on which the symplectic form can vanish. Let

$$H : P \times \mathbf{R} \to P\,, \quad H : (x,t) \mapsto H_t(x)$$

be a time-dependent smooth Hamiltonian function. Let X_{H_t} be the time-dependent *Hamiltonian vector field* of H_t defined by $\omega(X_{H_t}(x), .) {=} {-}dH_t(x)$. Then H_t defines a family of exact symplectomorphisms, or Hamiltonian diffeomorphisms, $\phi_t : P \to P$ by $\phi_0(x) = x$ and

$$\tfrac{d\phi_t}{dt} = X_{H_t} \circ \phi_t\,.$$

This family is often called a *Hamiltonian isotopy.*

A fundamental question asks for a lower bound to the number of intersection points of L and $\phi_1(L)$. This question is closely related to the problem of finding a lower bound to the number of fixed points of a Hamiltonian isotopy $\phi_\lambda : P \to P$. To see the connection, embed $P = \{p,p\} \subset (P{\times}P, \omega{\oplus}{-}\omega)$ as the Lagrangian diagonal and embed $\phi_1(P) = \{p, \phi_1(p)\} \subset (P \times P, \omega \oplus -\omega)$ as a Hamiltonian deformation of the diagonal. The fixed point problem then becomes a Lagrangian intersection one.

Arnold conjectured that if $L \pitchfork \phi_1(L)$, then the sum of the Betti numbers of L bounds from below the number of elements in $\{L \cap \phi_1(L)\}$ [A]. In the 1960s, he made a similar conjecture for the non-degenerate fixed point problem. These statements have been proved at a number of different levels in a variety of different manners. Eliashberg first proved the fixed point conjecture for Riemann surfaces [E]. Conley and Zehnder proved this result for the torus of arbitrary even dimension [ConZ]. Chaperon proved the Lagrangian intersection conjecture when L is any torus embedded as the zero-section inside T^*L [Ch]. Laudenbach and Sikorav extended Chaperon's result to any $L \subset T^*L$ embedded as the zero-section [LS]. Floer ([F1,2,3,4]) proved the sum of the $\mathbf{Z}_2$-Betti numbers of L to be a bound by constructing what is now known as Floer homology theory. Oh extended Floer's result to the monotone case [O2]. The weakly monotone fixed point version was proved by Hofer and Salamon [HoS] and Ono [On]. Recently, the general case of the fixed point conjecture was proved by Fukaya and Ono [FuO] and Liu and Tian [LiT].

The methods Floer uses are the basis of this paper. Floer chooses an *almost complex structure* J: an element of $\mathrm{End}(TP)$ which satisfies $J^2 = -\mathrm{id}$. Using $\{L \cap \phi_1(L)\}$ as generators and 'J-holomorphic disks' to construct a boundary map, Floer defines a Floer chain complex, $CF(L, \phi_1(L), J; \mathbf{Z}_2)$, with $\mathbf{Z}_2$-coefficients. He shows that its homology, $HF_*(L, \phi_1(L), J; \mathbf{Z}_2)$, is independent of both J and ϕ_1. Floer finally equates $HF_*(L, \phi_1(L), J; \mathbf{Z}_2)$ with the Morse homology, $H_*(L; \mathbf{Z}_2)$, thereby proving a $\mathbf{Z}_2$-version of the Arnold conjecture.

1.3 Main results. There are more invariants of a Morse chain complex than just its homology. In algebraic and differential topology, certain K-theoretic invariants were developed as (sometimes complete) obstructions to certain phenomena. One invariant, Whitehead torsion, arose in the field of simple-homotopies and h-cobordisms. Another, Wh_2, appeared in the field of pseudo-isotopies.

Roughly, a *simple-homotopy equivalence* between two spaces (CW complexes) is a finite sequence of elementary collapses and expansions. An elementary expansion of a space (CW complex) is the addition to the space of two cells of consecutive dimensions, where one forms part of the boundary of the other. A collapse is the reverse process. An algebraic Whitehead torsion for a homotopy equivalence provides a complete answer to whether or not the map is a simple-homotopy equivalence, [Co], [M1].

An *h-cobordism* is a triple (W, M_-, M_+) of manifolds where M_- and M_+ make up the disjoint boundary of W, $\pi_1(M_\pm) = \pi_1(W)$ and $H_*(W, M_-) = 0$. A geometric Whitehead torsion for the pair (W, M_-) provides a complete answer to whether or not W is the trivial cobordism, $M_- \times \mathbf{R}$, [Hu]. The two versions of Whitehead torsion are related in [M1].

An isotopy of M, $f_t : (M, \partial M) \to (M, \partial M)$, can be thought of as a diffeomorphism $f : (M, \partial M) \times [0,1] \to (M, \partial M) \times [0,1]$, which is level preserving and the identity near $M \times \{0\}$. A *pseudo-isotopy* of M is similar except that it only requires the level preserving property to hold near $\partial(M \times [0,1])$. The second Whitehead element of a pseudo-isotopy serves as an obstruction to deforming the map to an isotopy within the space of pseudo-isotopies, [HW].

To define the analogous K-theoretic invariants in Floer theory, the definition of the Floer complex is extended to a complex $CF(L, \phi_1(L), J)$ with coefficients in the $\mathbf{Z}_2\pi_1(L)$ group ring (see section 2.4). Henceforth, assume $\pi_1(L) \neq 0$; otherwise all invariants defined vanish. Let $L' = \phi_1(L)$.

Defining Whitbread torsion requires $H_*(CF(L, L', J)) = 0$, defining the second Whitehead invariant further requires $HF_*(L, L', J; \mathbf{Z}_2) = 0$. Since $HF_*(L, L', J; \mathbf{Z}_2) \cong H_*(L; \mathbf{Z}_2)$, [F4], which never vanishes for compact L, Floer theory for non-compact manifolds needs to be developed. One set of manifolds for which the theory works are those which have finite-geometry at infinity (see section 2.3), a concept introduced by Gromov.

Whitehead torsion, known as Wh_1, is constructed for the acyclic chain complex $CF(L, L', J)$ in the following manner. Let d be the boundary map and $\delta_n : CF_n(L, L', J) \to CF_{n+1}(L, L', J)$ be a chain contraction such that $d\delta + \delta d = \text{ id}$. Essentially, Whitehead torsion is the matrix $d + \delta$ modulo certain matrix relations (see section 2.5). $CF(L, L', J)$ may change if either L' changes under a Hamiltonian deformation or J changes; however, this paper shows

Theorem 1.1. *Let (P, ω) be a symplectic manifold and $L \subset P$ be a Lagrangian submanifold. Let J and J' be two regular compatible almost complex structures. Let ψ_λ be a generic Hamiltonian isotopy with compact support. Let L' be the image of L under any Hamiltonian diffeomorphism (with possibly non-compact support) such that L and L' intersect transversely. Assume that J and J' can be connected by a family of compatible almost complex structures J_λ such that $(P, \omega, J_\lambda, L, L')$ has finite-geometry at infinity. For any map $u : (D^2, \partial D^2) \to (P, L)$, assume that $\int_{D^2} u^*\omega = 0$ and that the Maslov index (see section 2.2) of $u(\partial D^2)$ is even. Assume $CF(L, L', J)$is acyclic. Then*

$$Wh_1\big(CF(L, L', J)\big) = Wh_1\big(CF(L, \psi_1(L'), J')\big)\,.$$

When Floer homology vanishes, a non-zero Wh_1 for a pair of Lagrangians provides an obstruction to deforming one away from the other.

Fukaya suggests a similar invariance result in [Fu]. He defines the torsion for a Floer homology group with coefficients in a certain Novikov ring. Eliashberg and Gromov prove Theorem 1.1 in the special case when P is the cotangent bundle of L [EG]. Their methods use generating functions instead of Floer homology. Hutchings and Lee [HutL] have developed an alternative invariant to Wh_1, known as Reidemeister torsion, in the finite-dimensional setting which instead of requiring a Morse function uses the more general closed one-form. Lee is currently working on extending their results to the Floer homology of periodic orbits [Le]. The author and Lee are also working on adapting Reidemeister torsion to the theory of Lagrangian intersections. Because Reidemeister torsion does not require a well-defined symplectic action (see section 2.2), the hypothesis that L' is a Hamiltonian deformation of L can be dropped in this setting.

Whitehead torsion is often labeled Wh_1 because of the existence of higher Whitehead groups. In [HW], a two-parameter Morse theory is used to construct Wh_2 as an obstruction in pseudo-isotopy theory. This paper takes a similar approach constructing an invariant of $CF(L, \phi_\Lambda(L'), J_\Lambda)$, a one-parameter family of Floer chain complexes with $\mathbf{Z}_2\pi_1(L)$-coefficients. Here Λ is the unit interval and L' is some non-compact Hamiltonian deformation of a non-compact L. Under conditions similar to Theorem 1.1, Wh_2 can be defined for a family of acyclic chain complexes. Using terminology from pseudo-isotopy theory, define the Steinberg word of $CF(L, \phi_\Lambda(L'), J_\Lambda)$ to be a list of so-called 'handle-slides' of Lagrangian intersections. To define this word, the choice of J_Λ, though non-generic, can always be made. The associated Wh_2 element is this word modulo certain algebraic relations (see section 2.6). Suppose ϕ_Λ and ϕ'_Λ are two compactly supported Hamiltonian isotopies with $\phi_0 = \phi'_0 = \mathrm{id}$, $\phi_1 = \phi'_1$ and suppose $L \cap L' = L \cap \phi_1(L') = L \cap \phi'_1(L') = \emptyset$. Suppose J_Λ and J'_Λ are appropriately chosen one-parameter families of almost complex structures.

Theorem 1.2. *For any map $u : (D^2, \partial D^2) \to (P, L)$, assume that $\int_{D^2} u^*\omega = 0$ and that the Maslov index of $u(\partial D^2)$ is even. If there exists a family of Hamiltonian isotopies connecting ϕ_Λ to ϕ'_Λ fixing $\phi_0 = \phi'_0$ and $\phi_1 = \phi'_1$, then*

$$Wh_2\big(CF(L, \phi_\Lambda(L'), J_\Lambda)\big) = Wh_2\big(CF(L, \phi'_\Lambda(L'), J'_\Lambda)\big)\,.$$

One might ask whether a Hamiltonian isotopy taking L' to L'' can avoid a third Lagrangian L. A non-zero Wh_2 answers the question in the negative.

1.4 Applications and extensions. Examples 5.2 and 5.3 construct symplectic manifolds, Lagrangians and Hamiltonian diffeomorphisms which have well-defined non-trivial Wh_1 and Wh_2 invariants.

Consider a manifold M such that either $Wh_1(\pi_1(M))$ or $Wh_2(\pi_1(M))$ is non-zero, and $\pi_2(M) = 0$. Sections 2.5 and 2.6 define $Wh_i(G)$ for $i = 1, 2$ and G a group. Theorem 5.1 provides examples of when $Wh_i(G) \neq 0$. Let $(V, \omega) = (\mathbf{R}^1_t \times S^1_\theta \times T^*M, dt \wedge d\theta + \omega_{std})$ where ω_{std} is the standard symplectic structure on the cotangent bundle. Let $S_c(t, \theta, x) = (t, \theta + c, x)$ be the non-trivial Hamiltonian rotation. Let $\mathcal{G}$ denote the set of Hamiltonian diffeomorphisms of V of the form $f \circ S_c$ where f has compact support and $f \circ S_c$ has no fixed points. Then a corollary of Theorem 1.2 is $\pi_0(\mathcal{G}) \neq 0$. Example 5.4 proves this claim.

It is unknown whether or not the invariants can be modified so that the converse statements to Theorems 1.1 and 1.2 hold. Although the topological converses are true (with coefficients in $\mathbf{Z}\pi_1$ instead of in $\mathbf{Z}_2\pi_1$ and with an additional obstruction measurement in the case of Wh_2), there are some complications in translating the topological methods into symplectic ones.

For example, finding a Whitney disk is an important step towards proving that a zero Whitehead torsion of an h-cobordism implies the cobordism to be trivial. The existence of an analogous 'Hamiltonian disk' is unknown in the symplectic case, however. On a positive note, extending the theory to $\mathbf{Z}\pi_1$ coefficients should be feasible. See Remark 5.5.

Milnor defines Whitehead torsion for a more general class of chain complexes [M1]. He considers chain complexes whose homology is freely generated with a preferred set of bases. Unfortunately, while the Floer chain complexes are equipped with a preferred set of generators, the Floer homology is not. Otherwise Theorem 1.1 would apply to a more general setting. That is, Lemma 2.13 still holds for Milnor's torsion; thus, the techniques of section 3.4 would prove this potential generalization of Theorem 1.1.

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2 Preliminaries

2.1 Floer theory. This subsection briefly introduces the Floer homology theory developed for Lagrangian intersections [F1,2,3], [O2]. Let (P,ω) be a symplectic manifold and $L, L' \subset P$ be two transversely intersecting Lagrangian submanifolds. For now assume that P is closed; this condition will later be replaced with the more general finite-geometry at infinity condition. Let $\mathcal{J}_\omega$ denote the space of smooth t-dependent families of ω-*compatible almost complex structures*:

$$J : [0,1]_t \times P \to \mathrm{End}(TP)\,, \quad J(t)^2 = -Id\,, \quad \omega\big(\cdot, J(t)\,\cdot\big) \text{ is a metric}\,.$$

Henceforth, any almost complex structures J will be assumed to be time-dependent and ω-compatible unless otherwise stated. The time-dependence is a technical condition used in a certain surjectivity result (proved as Theorem 5 in [F3] and corrected as Proposition 3.2 in [O2]). The reader will not notice its necessity in this survey of Floer theory, however, and thus can choose to ignore it.

Let $\Theta = \mathbf{R} \times [0,1] \subset \mathbf{C}$, $k > 2/p$ and x_1, x_2 be transverse intersection points of L and L'. Define the space of strips connecting x_1 and x_2 to be

$$\mathcal{P}_k^p(x_1,x_2) = \big\{u \in L^p_{k;\mathrm{loc}}(\Theta,P) \;\big|\; (1)\ u(\mathbf{R},0) \subset L\,,\ u(\mathbf{R},1) \subset L' \\ (2)\ \exists\, \rho_i > 0\,,\ \text{and } \xi_i \in L^p_k(\Theta, T_{x_i}P) \text{ such that} \\ u(\tau,t) = \exp_{x_i} \xi_i\big((-1)^i\tau,t\big)\,,\ \text{for } \tau > \rho_i \text{ for } i=1,2\big\}\,.$$

For $u \in \mathcal{P}_k^p(x,y)$, let

$$L^q_l(u) = \big\{\xi \in L^q_{l;\mathrm{loc}}(\Theta, u^*TP) \;\big|\; \|\xi\|_{l,q} < \infty\big\}$$

$$W^q_l(u) = \big\{\xi \in L^q_l(u) \;\big|\; \xi(\tau,0) \in T_{u(\tau,0)}L \text{ and } \xi(\tau,1) \in T_{u(\tau,1)}L'\big\}\,.$$

Define the map

$$\bar\partial_J : \mathcal{P}_k^p(x,y) \to L^p_{k-1}(u)\,, \quad \bar\partial_J(u) = \frac{\partial u(\tau,t)}{\partial\tau} + J(u(\tau,t),t)\frac{\partial u(\tau,t)}{\partial t}\,. \tag{1}$$

A *holomorphic curve* (also called pseudo-holomorphic or J-holomorphic) $u \in \mathcal{P}_k^p$ is a zero of the above map, $\bar\partial_J(u) = 0$. Define the *energy*, $\mathcal{E}$, of a holomorphic curve u to be

$$\mathcal{E}(u) = \frac{1}{2}\int_0^1 \int_{-\infty}^{\infty} \big|u_\tau(\tau,t)\big|^2 + \big|u_t(\tau,t)\big|^2 d\tau\, dt\,. \tag{2}$$

Let x,y be transverse intersections of L and L'. Let $C = C(x,y)$ be some positive constant. Let

$$\mathcal{M}(x,y,J) = \big\{u \in \mathcal{P}_k^p(x,y) \;\big|\; \bar\partial_J(u) = 0 \text{ and } \mathcal{E}(u) < C\big\} \tag{3}$$

be the moduli space of such curves with finite energy. Let $f_c : \Theta \to \Theta$ be the translation $f_c(\tau,t) = (\tau + c, t)$. Note that $u \in \mathcal{M}(x,y,J) \iff u \circ f_c \in \mathcal{M}(x,y,J)$. Thus the translation induces an $\mathbf{R}$-action on the moduli space. Let $\widehat{\mathcal{M}}(x,y,J) = \mathcal{M}(x,y,J)/\mathbf{R}$ be the set of so-called *'rigid' holomorphic curves.*

Theorem 2.1 ([F3]). *Suppose x and y are transverse intersections of L and L', two Lagrangian submanifolds in a closed symplectic manifold (P,ω). Assume $\omega_{|\pi_2(P,L)} = \omega_{|\pi_2(P,L')} = 0$. Then $\bar\partial_J$ is a smooth section of a Banach space bundle $\mathcal{L}^p_{k-1}(x,y)$ over $\mathcal{P}^p_k$ whose fiber over u is L^p_{k-1}. The linearization of $\bar\partial_J$ at u*

$$E_u \equiv D\bar\partial_J(u) : T_u\mathcal{P}_k^p(x,y) = W^p_k(u) \to L^p_{k-1}(u)$$

is a Fredholm operator. There is a Baire set $\mathcal{J}_{reg} \subset \mathcal{J}_\omega$ for which $\mathcal{M}(x,y,J)$ is a manifold.

Floer also proves a higher parameter version of Theorem 2.1. Let Λ^m be an m-dimensional compact submanifold of $\mathbf{R}^m$. Assume $\{L_\lambda\}_{\lambda\in\Lambda^m}$ is a smooth family of Lagrangian submanifolds. Suppose $x = x(\lambda)$, $y = y(\lambda) \in L_\lambda \cap L$ represents two families of transverse intersections which exist for all $\lambda \in \Lambda^m$ and which vary smoothly with λ. Define

$$\mathcal{P}^p_k(x,y;\Lambda^m) = \big\{(u,\lambda) \;\big|\; u \in \mathcal{P}^p_k(x(\lambda),y(\lambda)) \text{ and } \lambda \in \Lambda^m\big\}\,.$$

Assume $J_{\Lambda^m} = \{J_\lambda\}_{\lambda\in\Lambda^m}$ is a family of almost complex structures varying smoothly with λ. Define

$$\bar\partial : \mathcal{P}^p_k(x,y;\Lambda^n) \to \mathcal{L}^p_{k-1}(x,y)\,, \quad \bar\partial(u,\lambda) = \bar\partial_{J_\lambda}(u)\,.$$

Define the moduli space

$$\mathcal{M}_{\Lambda^m}(x,y,J_{\Lambda^m}) = \big\{(u,\lambda)\bigm| u\in\mathcal{M}(x(\lambda),y(\lambda),J_\lambda) \text{ and } \lambda\in\Lambda^m\big\}$$
$$\widehat{\mathcal{M}}_{\Lambda^m}(x,y,J_{\Lambda^m}) = \mathcal{M}_{\Lambda^m}(x,y,J_{\Lambda^m})/\mathbf{R}\,.$$

Assume that the bound on energy $C(\lambda) = C(x(\lambda),y(\lambda))$ varies continuously with λ. Since Λ^m is compact, this continuity implies a uniform bound on energy.

Theorem 2.2 ([F3]). *Suppose x and y are two families of transverse intersections of L and L_λ, $\lambda\in\Lambda^m$. Assume $\omega|_{\pi_2(P,L)} = \omega|_{\pi_2(P,L_\lambda)} = 0$. Then $\bar\partial$ is a smooth section of a Banach space bundle $\mathcal{L}^p_{k-1}(x,y)$ over $\mathcal{P}^p_k(x,y;\Lambda^m)$ whose fiber over (u,λ) is L^p_{k-1}. The linearization of $\bar\partial$ at (u,λ)*

$$E_{(u,\lambda)} \equiv D\bar\partial(u,\lambda) : T_{(u,\lambda)}\mathcal{P}^p_k(x,y;\Lambda^m) = W^p_k(u)\times T_\lambda\Lambda^m \to L^p_{k-1}(u)$$

is a Fredholm operator. There is a Baire set $\mathcal{J}^{\Lambda^m}_{reg}\subset\mathcal{J}^{\Lambda^m}_\omega$ for which the set $\mathcal{M}_{\Lambda^m}(x,y,J_{\Lambda^m})$ is a manifold.

Any $J_{\Lambda^m}\in\mathcal{J}^{\Lambda^m}_{reg}$ will be called a *regular* or *generic* family of almost complex structures.

2.2 Action and index. Floer and others try to capitalize as much as possible on the similarity of Floer theory to Morse theory. In Morse theory, a fundamental object is the Morse function. In Floer theory, the analogous object, when defined, is the symplectic action. Let γ_0 be a smooth path from L to L'. Denote the path space by

$$\Omega(L,L') = \big\{\gamma\in C^\infty([0,1],P)\bigm|\gamma(0)\in L\,,\ \gamma(1)\in L'\big\}$$

and let $\Omega(L,L';\gamma_0)\subset\Omega(L,L')$ be the component containing γ_0. For $\gamma\in\Omega(L,L';\gamma_0)$ let $S(\gamma)\subset P$ be the image of any path of paths from γ to γ_0. Define the *action* $\mathcal{A}$ on $\Omega(L,L';\gamma_0)$ by

$$\mathcal{A}(\gamma) = \int_{S(\gamma)}\omega\,. \tag{4}$$

The action is not well defined, *a priori*, because $\mathcal{A}(\gamma)$ depends on the choice of $S(\gamma)$. The following lemma demonstrates when an action exists and illustrates the action's main use.

Lemma 2.3 ([F1], [O1]). *The action is well defined if L' is a Hamiltonian deformation of L and $\omega|_{\pi_2(P,L)} = 0$. If the action exists, then for any intersections $x,y\in L\cap L'$, and any $u\in\mathcal{M}(x,y,J)$*

$$\mathcal{E}(u) = \mathcal{A}(y) - \mathcal{A}(x)\,.$$

In particular, if $\mathcal{A}(y) \leq \mathcal{A}(x)$ then $\mathcal{M}(x,y,J)$ has no non-constant curves.

It is easy to show that critical points of $\mathcal{A}$ are exactly intersection points of L and L'. Choosing $J \in \mathcal{J}_\omega$ induces a metric on P, which, by integration induces one on $\Omega(L, L'; \gamma_0)$. This metric is not complete; thus, the L^2-gradient, a crucial ingredient in Morse theory, is not an element of $T\Omega$. Nevertheless, it can be readily shown that the equation for 'L^2-gradient flows,' an ODE in the infinite-dimensional path space, can be re-expressed as the Cauchy–Riemann PDE in P, $\bar{\partial}_J(u) = 0$. By the above theorem, then, $\widehat{\mathcal{M}}(x,y,J)$ is analogous to the space of rigid Morse-gradient flows.

Lemma 2.3 shows that as in Morse theory, there are no gradient flows from an intersection point to itself or to another point with a higher action. Lemma 2.3 further removes the choice of energy bound, $C(x,y)$, made in the definition of $\mathcal{M}(x,y,J)$.

In addition to the Morse function, f, with its gradient flows, there is the Morse index assigned to each critical point of f. Recall that the index of a critical point is the number of negative eigenvalues of the Hessian at that point. Furthermore, the dimension of the set of gradient trajectories between two critical points equals the difference in indices. The Morse index provides a grading for the critical points when viewed as chains in Morse homology theory. The symplectic analogy to this index is the Maslov index.

Let x and y be two transverse intersections of L and L'. Consider any $u \in \mathcal{P}_k^p(x,y)$, and temporarily reparameterize Θ so that $u : [0,1]^2 \to P$. Since $[0,1]^2$ is contractible, there exists a trivialization $F : u^*TP \to [0,1]^2 \times \mathbf{C}^n$ which is constant on $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$, and such that at the points x and y the tangent spaces of L' are i times the tangent spaces of L. Let $\mathcal{L}ag(n)$ be the space of Lagrangian planes in $(\mathbf{R}^{2n}, dx \wedge dy)$. Construct a loop $\mathcal{L}_u : \partial([0,1]^2) \to \mathcal{L}ag(n)$ by $\mathcal{L}_u(\tau, 0) = F(T_{u(\tau,0)}L)$, $\mathcal{L}_u(1,t) = e^{i\pi t/2}F(T_yL)$, $\mathcal{L}_u(\tau,1) = F(T_{u(\tau,1)}L')$, and $\mathcal{L}_u(0,t) = e^{-i\pi t/2}F(T_xL')$.

Define the relative *Maslov index* $m_u(x,y)$ to be $\mu(\mathcal{L}_u)$ where μ is the usual Maslov index for loops in $\mathcal{L}ag(n)$ (see p. 48-49 of [McS1] for a review of μ).

Floer proves the following dimension argument which clarifies the analogy between the Maslov index and Morse index. Although he gives an unparameterized version, the general parameterized version is an obvious extension.

Theorem 2.4 ([F2]). *Assume the notation and hypotheses from Theo-*

rem 2.2. Note that m might be 0. Assume J_{Λ^m} is regular. The component of $\mathcal{M}_{\Lambda^m}(x,y,J_{\Lambda^m})$ containing (u,λ), $O \subset \mathcal{M}_{\Lambda^m}(x,y,J_{\Lambda^m})$, has manifold dimension

$$\dim O = \operatorname{Ind} E_{u,\lambda} = m_u\big(x(\lambda), y(\lambda)\big) + m = m_u(x,y) + m \tag{5}$$

where Ind is the Fredholm index.

The last equality uses the fact that the index is integer-valued and varies continuously with λ.

Let $\mathcal{M}^n_{\Lambda^m}(x,y,J_{\Lambda^m}) \subset \mathcal{M}_{\Lambda^m}(x,y,J_{\Lambda^m})$ denote the union of $(n+m)$-dimensional components. So if $(u,\lambda_1,\dots,\lambda_m) \in \mathcal{M}^n_{\Lambda^m}(x,y)$, then $m_u(x,y) = n$ for generic u. Similarly, denote the $(n+m-1)$-dimensional components of rigid curves by $\widehat{\mathcal{M}}^n_{\Lambda^m}(x,y,J_{\Lambda^m})$.

The relative Maslov index is not as 'strong' as the relative Morse index because while the former depends on u, the latter does not depend on the connecting gradient flow. Thus, the Maslov index at first does not seem to provide a grading for the Lagrangian intersection points.

Define the homomorphism

$$F : \pi_2(P,L) \to \mathbf{Z}, \quad F(v) = \mu(v(\partial D^2)) \tag{6}$$

where v is thought of as the map $v : (D^2, \partial D^2) \to (P,L)$. Define the *minimal Maslov number* $\Sigma(L)$ as the positive generator for the subgroup $F(\pi_2(P,L)) \subset \mathbf{Z}$.

Lemma 2.5 [O2, Lemma 5.2]. *Suppose L' is a Hamiltonian deformation of L. Suppose x, y are transverse intersections of L and L'. Let $u, v \in \mathcal{M}(x,y,J)$. Then there exists an integer n such that $m_u(x,y) - m_v(x,y) = n\Sigma(L)$.*

Note that in this case $\Sigma(L') = \Sigma(L)$. A corollary of Lemma 2.5 is that a $\mathbf{Z}_{\Sigma(L)}$-valued grading for the transversal intersection points exists. Pick $x \in L \cap L'$ which, as a constant path, lies in $\Omega(L,L';\gamma_0)$, and define its grading to be $0 \in \mathbf{Z}_{\Sigma(L)}$. Thus if $L \pitchfork L'$ and L' is a Hamiltonian deformation of L, then this choice determines a grading $\mu : \{L \cap L'\} \cap \Omega(L,L';\gamma) \to \mathbf{Z}_{\Sigma(L)}$.

2.3 Finite geometry at infinity. In section 4.1, Wh_2 is defined assuming $L \cap \phi_1(L) = \emptyset$ which requires $HF_*(L,\phi_1(L),J;\mathbf{Z}_2) = 0$. But

$$HF_*(L,\phi_1(L),J;\mathbf{Z}_2) = H_*(L)$$

which never vanishes for compact L; thus, the acyclic requirement motivates a generalization of the previous theorems to a class of non-compact manifolds developed by Gromov in [G]. A brief treatment of this can be found in Chapter 5 of [AuL].

Definition 2.6. Suppose (P,ω) is a symplectic manifold with a compatible almost complex structure J. Let g be the associated metric, that is, $g(v,w) = \omega(v,Jw)$. Let $L, L' \subset P$ be Lagrangian submanifolds. (P,ω,J,L,L') has *finite-geometry at infinity* if g is complete and there exist r_1, C_1, C_2 positive constants and a compact set $K \subset P$ such that

1. For all $p \in P$, $\exp_p : B(0,r_1) \to B(p,r_1)$ is a diffeomorphism.
2. For all $\gamma : S^1 \to B(p,r) \subset P$ with $r < r_1$, γ extends to $\bar\gamma : D^2 \to B(p,r)$ such that $\mathrm{Area}_g(\bar\gamma) \le C_1 \,\mathrm{length}_g(\gamma)$.
3. If $x \in L \setminus (L \cap K)$ and $x' \in L' \setminus (L' \cap K)$ then $\mathrm{dist}_{(P,g)}(x,x') > r_1$. If $x, x' \in L \setminus (L \cap K)$, and $\mathrm{dist}_{(P,g)}(x,x') < r_1$, then $\mathrm{dist}_{(L,g)}(x,x') < C_2 \,\mathrm{dist}_{(P,g)}(x,x')$. Here, $\mathrm{dist}_{(L,g)}$ refers to the Riemannian distance associated with the induced metric on L. A similar statement holds for L'.
4. For all $x \in L \cup L'$, $L \cap B(x,r_1)$ and $L' \cap B(x,r_1)$ are contractible.

A family of such 5-tuples, $(P,\omega,J_\Lambda,L_\Lambda,L'_\Lambda)$, has *uniform finite-geometry at infinity* if the constants $r_{1_\lambda}, C_{1_\lambda}, C_{2_\lambda}$ and the choice of compact sets K_λ do not depend on $\lambda \in \Lambda$.

For a given symplectic manifold (P,ω) and a pair of Lagrangians (L,L') (resp. family of pair of Lagrangians (L_Λ, L'_Λ)), denote by $\mathcal{FG}(L,L')$ (resp. $\mathcal{FG}(L_\Lambda, L'_\Lambda)$) the set of ω-compatible almost complex structures J (resp. J_Λ) for which (P,ω,J,L,L') has finite-geometry (resp. $(P,\omega,J_\Lambda,L_\Lambda,L'_\Lambda)$ has uniform finite-geometry).

Note that compact symplectic manifolds have finite-geometry at infinity. The main property enjoyed by a symplectic manifold with finite-geometry at infinity is that a holomorphic curve $u \in \mathcal{M}(x,y,J)$ is contained in a compact set.

Theorem 2.7. *Consider a family $(P,\omega,J_\Lambda,L,L'_\Lambda)$, which has uniform finite-geometry at infinity. Assume that the energy of all holomorphic strips is uniformly bounded. That is, for any $u \in \mathcal{M}_\lambda(x_\lambda,y_\lambda,J_\lambda)$, $\mathcal{E}(u) < E$ for some constant E. Then there exists a compact $K' \subset P$ such that the image of any such u lies in K'.*

Note that the theorem does not require the Lagrangian submanifolds to intersect transversely. The proof relies on a monotonicity property of holomorphic strips proved by Propositions 4.3.1(ii) and 4.7.2(ii) of [AuL, Chapter 5].

Lemma 2.8. *Suppose (P,ω,J,L,L') has finite-geometry with constants r_1, C_1, C_2 and compact set K from Definition 2.6. Choose C_3 such that for* *all $p, p' \in K$, $dist_{(P,g)}(p, p') < C_3$. Let $S \subset \Theta$ be a simply-connected connected domain in Θ. Let $f : S \to P$ be the restriction of some holomorphic strip $u : \Theta \to P$. If $(f(S), f(\partial S)) \subset (B(x, r), \partial B(x, r))$ for some $r < r_1$ and if $x \in f(S)$ then Area$(f(S)) \geq C_4 r^2$ where $C_4 = 1/4C_1(C_2 + C_3 + 1)$.*

Now for the proof of Theorem 2.7.

Proof. Let $K_0 = K$. Suppose P is not compact; otherwise, there is nothing to prove. Define inductively

$$K_i = \left\{x \in P \;\middle|\; \mathrm{dist}_{(P,g)}(x, K_{i-1}) \leq 1\right\}.$$

Note that $K_{i-1} \subsetneq K_i$ because P is non-compact, and that K_i is compact because the metric is complete. Consider a holomorphic curve $u : \Theta \to P$ with Lagrangian boundary conditions which passes through $K_i \backslash K_{i-1}$ going from K_{i-1} to $K_{i+1} \backslash K_i$. Let $\Theta' \subset \Theta$ be some (simply-connected) subdomain where this occurs. Without loss of generality, assume $r_1 \leq 1/2$. Then there exists $x \in u(\Theta')$ such that $B(x, r_1) \subset K_i \backslash K_{i-1}$. Let $S = u^{-1}(B(x, r_1)) \cap \Theta'$. Then by Lemma 2.8, Area$\,(u(S)) \geq C_4 {r_1}^2$. Note that C_4 does not depend on λ. Any strip must start (and end) at some points in $L \cap L'_\lambda \subset K_0$. If $N \geq E/C_4 {r_1}^2$, then the image of the strip lies in K_{N+1}. This last statement uses the fact that for holomorphic curves, energy and area are equal (Lemma 2.5 of [O1]). □

Corollary 2.9. *Replace the condition that L, L' and P are compact manifolds and $J \in \mathcal{J}_\omega$ with $J \in \mathcal{FG}(L, L')$. In the parameterized setting, assume that $J_\Lambda \in \mathcal{FG}(L, L'_\Lambda)$. If the other hypotheses of the theorems of sections 2.1 and 2.2 remain, then the theorems' conclusions still hold.*

Proof. The only non-local aspect of the proofs of the theorems in sections 2.1 and 2.2 is the application of Gromov's compactness theorem (see [F3, Theorem 1] or [O2, Proposition 3.7] for a specific statement). Roughly speaking, Gromov's compactness theorem states the following:

Consider a sequence of u_n of J_n-holomorphic curves in a compact symplectic manifold P with uniformly bounded energy and boundaries L and L_n. Suppose $J_n \to J$ and $L_n \to L'$ in the C^∞-topology. Then, up to certain reparameterizations in the domain, there exists a subsequence of the curves that converges uniformly to a so-called *cusp-curve* which is a union of curves in $\mathcal{M}(x_0, x_1, J) \times \mathcal{M}(x_1, x_2, J) \times \ldots \times \mathcal{M}(x_{k-1}, x_k, J)$.

The only use of the compactness of P is the assumption that the curves u_n lie in a compact set. But this assumption can be replaced by Theorem 2.7. □

The following definition encapsulates all the hypotheses needed for the analysis of the main theorems to hold. It essentially summarizes the relevant concepts from sections 2.2 and 2.3.

DEFINITION 2.10. (L, L') are called *admissible* if L' is an exact deformation of L, $\omega|_{\pi_2(P,L)} = 0$, $\Sigma(L) \in 2\mathbf{Z}$, and $\mathcal{FG}(L, L') \neq \emptyset$.

2.4 Floer homology with group ring coefficients. Henceforth assume the admissibility condition. Suppose $\pi_1(L) \neq 0$. For a regular $J \in \mathcal{FG}(L, L')$, define $CF(L, L', J)$ to be the graded module freely generated by the elements of $\{L \cap L'\} \cap \Omega(L, L'; \gamma_0)$ over the group ring $\mathbf{Z}_2\pi_1(L)$.

REMARK 2.11. Alternatively, $CF(L, L', J)$ could be the direct sum of complexes generated by all intersection points, $\{L \cap L'\}$. This counts more intersections but makes the notation a bit more complicated. Since the statements and proofs of theorems do not change when multiple components are considered, for the sake of brevity the discussion is restricted to the set of intersections in one component and will henceforth label this set $\{L \cap L'\}$.

Assign the $\mathbf{Z}_{\Sigma(L)}$-cyclic grading, μ, to $CF(L, L', J)$. Fix a base point $p \in L$. For any loop $\gamma : \mathbf{S}^1 \to L$ such that $\gamma(0) = \gamma(1) = p$, denote its homotopy class by $[\gamma] \in \pi_1(L; p) \cong \pi_1(L)$. For each $z \in \{L \cap L'\}$, pick a base path $\gamma_z : [0,1] \to L$, $\gamma_z(0) = p$, $\gamma_z(1) = z$. For each $u \in \widehat{\mathcal{M}}(x, y, J)$, define $\gamma_u : [0,1] \to L$ by $\gamma_u(s) = u\big(\frac{1}{s} - \frac{1}{1-s}, 0\big)$. That is, $\gamma_u(0) = y, \gamma_u(1) = x$ and γ_u follows the image of $u(*, 0)$. To be consistent with the conventions of others, such as [HW], write the composition of paths from left to right; thus, $\alpha = [\gamma_y \gamma_u \gamma_x^{-1}]$ is an element of $\pi_1(L)$.

Define a boundary operator

$$d_n : CF_n(L, L', J) \to CF_{n-1}(L, L', J)$$
$$d(y) = \sum_{\{x \in L \cap L' | |\mu(x) = \mu(y) - 1\}} \Big(\sum_{u \in \widehat{\mathcal{M}}^1(x,y,J)} [\gamma_y \gamma_u \gamma_x^{-1}] \Big) x \,.$$

Note that by Gromov compactness, the interior summand sums over a compact 0-dimensional manifold and hence is finite.

Suppose $h : C \to C'$ is a homomorphism between two free modules with given sets of generators (for example, $C' = C$ with the canonical basis of intersection points, and $h = d$). If $h(y) = \sum_i \alpha_i x_i$, then let $\langle h(y), x_k \rangle = \langle x_k, h(y) \rangle = \alpha_k$.

Theorem 2.12. $d^2 = 0$.

Proof. This is a brief review of the original proof of $d^2 = 0$ for Floer homology with $\mathbf{Z}_2$-coefficients, with an extension to a proof for $\mathbf{Z}_2\pi_1(L)$-

coefficients. Suppose x, y, y' and z are intersection points such that, for simplicity, $dz = y + y'$. Let u and u' be the unique rigid curves connecting z to y and y' such that $m_u(y,z) = m_u(y',z) = 1$. Suppose $\{v_1, \dots, v_k\}$ (resp. $\{v'_1, \dots, v'_l\}$) are the unique set of curves connecting y (resp. y') to x of Maslov index 1. When working with $\mathbf{Z}_2$ coefficients, it suffices to show that $l \equiv k \bmod 2$ since this implies $\langle d^2 z, x\rangle = \langle dy + dy', x\rangle = 0 \in \mathbf{Z}_2$.

Note that $\Sigma(L) \in 2\mathbf{Z}$, $x \neq z$, and Gromov compactness implies that any non-closed one-dimensional component of $\widehat{\mathcal{M}}^2(x,z,J)$ must converge to a cusp-curve in $\widehat{\mathcal{M}}^1(x,p,J) \times \widehat{\mathcal{M}}^1(p,z,J)$. Floer constructs a one-to-one correspondence between such a set of limit points and cusp-curves [F1]. Since the number of boundary points of an open bounded one-manifold is even, this correspondence proves that $l \equiv k \bmod 2$.

Now to prove $\langle d^2 z, x\rangle = 0 \in \mathbf{Z}_2\pi_1(L)$. To simplify notation, assume $l = k = 1$ and write $v = v_1$, $v' = v'_1$. Denote by $\{\tilde{u}_s\}$ the component of $\widehat{\mathcal{M}}^2(x,z,J)$ with $u \cup v$ and $u' \cup v'$ as endpoints. For $i = 0, 1$, let $\alpha = [\gamma_z\gamma_u\gamma_y^{-1}]$, $\alpha' = [\gamma_z\gamma_{u'}\gamma_{y'}^{-1}]$, $\beta = [\gamma_y\gamma_v\gamma_x^{-1}]$ and $\beta' = [\gamma_{y'}\gamma_{v'}\gamma_x^{-1}]$. Then

$$\langle d^2 z, x\rangle = \langle \alpha dy + \alpha' dy', x\rangle = \alpha\beta + \alpha'\beta' = 0 \in \mathbf{Z}_2\pi_1(L)$$

because

$$\alpha\beta = [\gamma_z\gamma_u\gamma_v\gamma_x^{-1}] = [\gamma_z\gamma_{\tilde{u}_s}\gamma_x^{-1}] = [\gamma_z\gamma_{u'}\gamma_{v'}\gamma_x^{-1}] = \alpha'\beta' \,. \qquad \square$$

Thus, $CF(L, L', J)$ is a $\mathbf{Z}_2\pi_1(L)$-chain complex and its homology can be defined:

$$HF_n\big(L, L', J; \mathbf{Z}_2\pi_1(L)\big) = \frac{\ker(d_n)}{\operatorname{Im}(d_{n+1})}\,. \tag{7}$$

2.5 Algebra of Wh_1. This subsection briefly reviews the Whitehead torsion which arises in simple-homotopy theory. Chapter 3 of [Co] offers a good introduction with more details.

For any ring R with unit, let $GL(R)$ be the direct limit of $GL(n,R)$. Let $G \subset R$ be a subgroup of the units of R. Consider matrices A and B

$$A = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & r & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & \bigcirc & \\ & & 1 & & & & \\ & & & g & & & \\ & & & & 1 & & \\ & \bigcirc & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \tag{8}$$

Here $r \in R$ and $g \in G$. Let $E(R) \subset GL(R)$ denote the subgroup generated by the matrices of type A, also known as elementary matrices. Matrices of type A shall be written as $E_{ij}(r)$. Let $E_G \subset GL(R)$ be the subgroup generated by $E(R)$ and matrices of type B. Define the quotient map, called the torsion map, by

$$\tau : GL(R) \to K_G(R) = \frac{GL(R)}{E_G}\,.$$

The reader can easily check that $K_G(R)$ is abelian. Also, if G contains the unit -1, then τ is invariant under elementary row operations.

For a general group H, denote by $\mathbf{Z}_2(H)$ the $\mathbf{Z}_2$-group ring of H. Define the first Whitehead group of the given group H to be

$$Wh_1(H) = K_H(\mathbf{Z}_2(H))\,.$$

Henceforth, consider the group G to be a subgroup of the units of the ring R with $-1 \in G$. (For example, $(G,R) = (H,\mathbf{Z}_2(H))$ satisfies this condition since $-1 = 1 \in H$.) Define an (R,G)-module to be a free R-module M, along with a preferred set $\mathcal{B}$ of bases such that for all $b,b' \in \mathcal{B}$, $\tau\langle b/b'\rangle = 0$. Here $\langle b/b'\rangle$ is the change of basis matrix from b to b'. If $f : M_1 \to M_2$ is a module homomorphism, write $\langle f\rangle_{b_1,b_2}$ to denote the matrix representing f with respect to bases b_1 and b_2 of M_1 and M_2, respectively. When the choice of basis does not matter, the subscripts will sometimes be omitted. Define an (R,G)-complex to be a free chain complex over R, $(C = \{C_i\}, d)$ where each C_i is an (R,G)-module with preferred sets of bases $\mathcal{B}_i$. Denote by $\mathcal{B} = \{b_1 \cup b_2 \cup \dots \mid b_i \in \mathcal{B}_i\}$ the preferred set of bases for C. The grading of (C,d) can be either $\mathbf{Z}$ or $\mathbf{Z}_{2k}$. Suppose (C,d) is an acyclic (R,G)-complex. A chain-contraction δ for (C,d) is a set of maps

$$\delta_n : C_n \to C_{n+1}\,, \quad \delta_{n-1}d_n + d_{n+1}\delta_n = Id : C_n \to C_n\,.$$

Pick any δ and write

$$C_{odd} = C_1 \oplus C_3 \oplus \cdots\,, \quad C_{even} = C_0 \oplus C_2 \oplus \cdots$$

Denote the preferred bases of C_{odd} by $\mathcal{B}_{odd} = \{b_1 \cup b_3 \cup \cdots \mid b_{2i+1} \in \mathcal{B}_{2i+1}\}$ and do the same for C_{even}. The torsion of the chain complex with a preferred set of bases $\mathcal{B}$ is

$$Wh_1(C,d,\mathcal{B}) = \tau\big(\langle (d+\delta)|_{C_{odd}}\rangle\big) \tag{9}$$

where the matrix is with respect to any pair of preferred bases. In section 3.1, the chain complex will come equipped with a canonical set of preferred bases; thus, the dependence of Wh_1 on $\mathcal{B}$ will henceforth notationally be omitted. In [Co], Cohen proves that the torsion is well defined;

i.e. $(d+\delta)_{odd}$ is invertible and τ is independent of the choice of bases in $\mathcal{B}$ and δ.

Let (C,d) be any (R,G)-module. Fix a preferred basis. Suppose $x,y \in C_i$ for some i are two basis elements. Let $\delta^a_b = 1$ if $a=b$ and 0 otherwise. Let $f : (C,d) \to (C, d' = f \circ d \circ f^{-1})$ be a chain map defined on the basis elements by $f(p) = p + \delta^p_x g y$ for some fixed $g \in G$ and extended as a homomorphism to all chains. Let (T, d_T) denote a trivial acyclic (R,G)-complex

$$(T,d_T) : 0 \to T_k \to T_{k-1} \to 0$$

where $T_k = T_{k-1} \cong R$ and $\mathcal{B}_k = \mathcal{B}_{k-1} \cong G$. Let (C,d) be any (R,G)-complex. The following lemma will be useful later.

Lemma 2.13. $Wh_1(C,d') = Wh_1(C,d) = Wh_1(C \oplus T, d \oplus d_T)$.

Proof. For the first equality, note that if δ is a chain contraction for (C,d), then $\delta' = f \circ \delta \circ f^{-1}$ is a chain contraction for (C,d'). Thus

$$\begin{aligned} Wh_1(C,d') &= \tau\big(\langle (d'+\delta')_{C_{odd}}\rangle\big) = \tau\big(\langle (f \circ (d+\delta) \circ f^{-1})_{C_{odd}}\rangle\big) \\ &= \tau\big(\langle f|_{C_{even}}\rangle\big)\tau\big(\langle (d+\delta)|_{C_{odd}}\rangle\big)\tau\big(\langle f^{-1}|_{C_{odd}}\rangle\big) \\ &= 1 \cdot Wh_1(C,d) \cdot 1\,. \end{aligned}$$

This last equality holds regardless of the parity of the index of x.

To prove the lemma's second equality, pick $b_{k-1} \cup b_k \in \mathcal{B}_T = G \times G$. Define $\delta_T : T_{k-1} \to T_k$ by $\delta_T(b_{k-1}) = b_k$. Then $\delta'' = \delta \oplus \delta_T$ is a chain contraction for $C'' = C \oplus T$. Choose $b_o \in \mathcal{B}_{odd}$ and $b_e \in \mathcal{B}_{even}$. If k is odd, let $b''_o = b_k \cup b_o$ and $b''_e = b_{k-1} \cup b_e$. If k is even, pair up the bases the other way; thus, $b''_o \in \mathcal{B}''_{odd}$ and $b''_e \in \mathcal{B}''_{even}$.

Regardless of the parity of k, $\langle (d_T + \delta_T)_{odd}\rangle = (1)$. Thus, the matrix $\langle (d''+\delta'')_{odd}\rangle_{b''_o,b''_e}$ differs from $\langle (d+\delta)_{odd}\rangle_{b_o,b_e}$ in that it has an extra column inserted in the jth spot and an extra row in the ith spot with all zeroes except for a 1 in the (i,j) entry. As elements of $GL(R)$, these two matrices differ by only an elementary operation and so have the same torsion. □

2.6 Algebra of Wh_2.

This subsection briefly reviews the second Whitehead group which arises in pseudo-isotopy theory. See [HW] for an introduction to pseudo-isotopies and more algebraic details.

Let R be any ring and define the *Steinberg group*, $St(R)$, to be the free group generated by the symbols $h_{ij}(r)$ where $0 \le i,j < \infty$ and $i \ne j$, modulo the relations

(St1): $h_{ij}(\alpha)h_{ij}(\beta) = h_{ij}(\alpha+\beta)$

(St2): $h_{ij}(\alpha)h_{kl}(\beta)h_{ij}(\alpha)^{-1}h_{kl}(\beta)^{-1} = 1$ for $i \ne l$ and $j \ne k$

$$\text{(St3):} \qquad h_{ij}(\alpha)h_{jk}(\beta)h_{ij}(\alpha)^{-1}h_{jk}(\beta)^{-1} = h_{ik}(\alpha\beta) \text{ for } i,j,k \text{ distinct}\,.$$

Denote by Γ the indexing set of the generators. As written above, Γ is a subset of the non-negative integers, although it need not always be. Denote the projection homomorphism $\pi : St(R) \to E(R)$ defined on the generators by $\pi(h_{ij}(\alpha)) = E_{ij}(\alpha)$. It is easy to check that π preserves the relations, and thus is well defined. Henceforth, assume $R = \mathbf{Z}_2[G]$ for some group G. Define the following subgroups of $St(R)$: let $W(G)$ be generated by words of the form $w_{ij}(g) = h_{ij}(g)h_{ji}(g^{-1})h_{ij}(g)$; let $U(R)$ be the pre-image under π of all lower-triangular matrices in $E(R)$; let $U(G) \subset U(R)$ be generated by words of the form $w_{ij}(g)w_{ij}(1)$ and generators $h_{kl}(\alpha)$ with $k < l$; let $K_2(R) = \ker\ \pi$. Define the second Whitehead group of the group G to be

$$Wh_2(G) = \frac{K_2(\mathbf{Z}_2[G])}{W(G) \cap K_2(\mathbf{Z}_2[G])} = \frac{U(R)}{U(G)}\,. \tag{10}$$

The equivalence of these two definitions is proved on p. 105–107 of [HW].

Now to construct an element of $Wh_2(G)$ for certain one-parameter families of $\mathbf{Z}$ or $\mathbf{Z}_{2k}$-graded acyclic chain complexes. Let $C_i = B_i \oplus Z_i$ where B_i and Z_i are free R-modules generated by $\{b_i^n\}$ and $\{z_i^n\}$, respectively. Here $i \in \mathbf{Z}_{2k}$ or $\mathbf{Z}$. Let $St_i(R)$ be the Steinberg group generated by $h_{pq}(r)$ where p and q are distinct generators of C_i. Note that the index set Γ is now made of generators. Γ can be well-ordered by ordering the generators $\cup_i\{b_i^0, z_i^0, b_i^1, \dots\}$. Let $\pi_i : St_i(R) \to E_i(R)$ be the projection where $E_i(R)$ is generated by elementary matrices $E_{pq}(r)$. Since Γ is well-ordered, 'row p' and 'column q' make sense. Let $C = \oplus C_i$, $St_C(R) = \oplus St_i(R)$ and $E_C(R) = \oplus E_i(R)$. Define $W_C(G), W_C(R)$, et cetera to be appropriate subspaces of $St_C(R)$. Define C_{odd} and C_{even} as in section 2.5. Define St_{even} et cetera in the obvious way.

Let $\mathcal{C}(C)$ be the set of pairs $(d,\delta) = \cup_i\{(d_i,\delta_i)\}$ where d is a boundary map for C and δ is a chain contraction map for d satisfying $\delta^2 = 0$. Let $\chi = (\chi_0, \chi_1, \dots) \in St_C(R)$ be any element. Define an action of $St_C(R)$ on $\mathcal{C}(C)$ as follows:

$$\chi : \mathcal{C}(C) \to \mathcal{C}(C)$$

$$\chi \cdot (d,\delta) = (\chi \cdot d, \chi \cdot \delta) = \cup_i\big\{(\pi(\chi_i)d_i\pi_{i-1}(\chi_{i-1})^{-1}, \pi(\chi_i)\delta_i\pi_{i-1}(\chi_{i-1})^{-1})\big\}\,.$$

Define the *standard pair* $(D,\Delta) \in \mathcal{C}(C)$ for C by the equations

$$D(b_i^n) = z_{i-1}^n, \quad \Delta(z_i^n) = b_{i+1}^n, \quad D(z_i^n) = \Delta(b_i^n) = 0.$$

A pair (d,δ) is called *elementary* if for any canonical generator p of C_i, $d_i(p) = 0$ or $d_i(p) = q$ where q is a generator of C_{i-1}. Define

$$\Upsilon = \big\{\chi \in St_C(R) \bigm| \chi \cdot (D,\Delta) \text{ is elementary}\big\}\,.$$

Hatcher and Wagoner prove ([HW, Lemma III.3.1]) that for each $\chi \in \Upsilon$ there exists a $w \in W_C(G)$ such that

$$w\chi \cdot D_{even} + w\chi \cdot \Delta_{even} \in \pi(U(R)) \subset E(R)\,.$$

Note that

$$w\chi \cdot D_{even} + w\chi \cdot \Delta_{even} = \pi\Big(\prod_{0\le i} w_{2i}\chi_{2i} \prod_{0\le i} \chi_{2i+1}^{-1} w_{2i+1}^{-1}\Big)\,.$$

Define a map

$$\Sigma : \Upsilon \to Wh_2(G)$$
$$\Sigma(\chi) = \prod_{0\le i} w_{2i}\chi_{2i} \prod_{0\le i} \chi_{2i+1}^{-1} w_{2i+1}^{-1} \mod U(G)\,.$$

Hatcher and Wagoner prove ([HW, p. 124–125]) that $\Sigma(\chi)$ does not depend on the choice of $w \in W_C(G)$ made in [HW, Lemma III.3.1]. Although their proofs are for $R = \mathbf{Z}[G]$, the proofs for $R = \mathbf{Z}_2[G]$ when not identical are simpler.

3 The Wh_1 Theorem

3.1 The statement. Suppose $CF(L, L', J)$ is acyclic. Choose as a preferred set of bases the set consisting of a single canonical basis, $\{L \cap L'\}$. Define the *Whitehead torsion* of $CF(L, L', J)$ (with basis $\{L \cap L'\}$) to be

$$Wh_1\big(CF(L, L', J)\big) \in Wh_1(\pi_1(L)) = \frac{GL(\mathbf{Z}_2\pi_1(L))}{E_{\pi_1(L)}}\,. \qquad (11)$$

Define the *stabilization* of (P, ω) to be the symplectic manifold $(\tilde{P}, \tilde{\omega}) = (P \times \mathbf{R}^2, \omega + dx \wedge dy)$. The $\mathbf{R}^2$-component is called the *fiber* of the stabilization. The (x, y) coordinates are called the stabilization's *fiber coordinates.*

The *stabilized Lagrangians* are defined to be $\tilde{L} = L \times \{(x, 0) \mid x \in \mathbf{R}\}$ and $\tilde{L}' = L' \times \{(x, \pm 2x) \mid x \in \mathbf{R}\}$. L' can be stabilized in either a '*positive*' or '*negative*' manner. Note that no additional points of intersection arise from stabilizing the Lagrangians. If L' is the time-one map of a Hamiltonian deformation of L with Hamiltonian function $H_t : P \to \mathbf{R}$, then so too is $\tilde{L}'$ of $\tilde{L}$ with Hamiltonian $\tilde{H}_t(p, x, y) = H_t(p) \pm x^2$.

Theorem 3.1. *Suppose (L, L') are admissible and ψ_1 is the time-one map of a compactly supported Hamiltonian isotopy connected to the identity. Suppose $J', J'' \in \mathcal{FG}(L, L') = \mathcal{FG}(L, \psi_\lambda(L'))$ are regular and are connected by a one-parameter family $j_\Lambda \subset \mathcal{FG}(L, L')$. Suppose $CF(L, L', J')$ is acyclic so that its Whitehead torsion can be defined. Then*

$$Wh_1\big(CF(L, L', J')\big) = Wh_1\big(CF(L, \psi_1(L'), J'')\big)\,.$$

$Wh_1(CF(L,L',J'))$ does not depend on the choice of base paths needed to define Floer chain complexes with $\mathbf{Z}_2\pi_1(L)$-coefficients.

If $\tilde{L}$ and $\tilde{L}'$ are the stabilizations of L and L' respectively, then

$$Wh_1\big(CF(L,L',J')\big) = Wh_1\big(CF(\tilde{L},\tilde{L}',J' \oplus J_{std})\big)\,.$$

REMARK 3.2. An example of two such almost complex structures is when $J' \in \mathcal{FG}(L,L')$ is regular and J'' is another almost complex structure which agrees with J' outside some suitably large set but is generically perturbed within the set. The two are obviously connected by some $j_\Lambda \subset \mathcal{FG}(L,L')$. Moreover, Theorem 2.7 implies all holomorphic curves lie in some compact set; hence, the regularity requirement for J'' is vacuous outside the set.

The proof of Theorem 3.1 constructs a one-parameter family of pairs of Lagrangians and a one-parameter family of almost complex structures connecting (L,L',J') to $(L,\psi_1(L'),J'')$. The changes that occur in the resulting one-parameter family of Floer chain complexes are then examined and proved to not affect the chain complexes' Whitehead torsion.

There are two phenomena that generically occur at discrete times in a one-parameter family which can alter the chain complex. At isolated instances, a non-generic holomorphic curve u with $m_u(x,y)=0$ can generically appear which shall be named a *handle-slide* after the analogous Morse phenomenon. The handle-slide is said to *represent* $[\gamma_y\gamma_u\gamma_x^{-1}] \in \pi_1(L)$. Since $\mathcal{A}(y) > \mathcal{A}(x)$, u represents a 'negative L^2-gradient' flow from y 'down' to x. For this reason, u will sometimes be called a handle-slide *from y to x*.

The second phenomenon is the appearance or disappearance of pairs of intersection points. These shall be called *births* and *deaths*. Let $Q : \mathbf{R}^{n-1} \to \mathbf{R}$ be a non-degenerate quadratic function. Consider the family of functions $f_\lambda(q_1,q_2,\dots,q_n) = q_1^3 + \lambda q_1 + Q(q_2,\dots,q_n)$ where $\lambda \in [-1,1]$. Let $L = \mathbf{R}^n \times \mathbf{0}$ be the Lagrangian zero-section in $(\mathbf{R}^{2n}, dq \wedge dp)$ and let $L_\lambda = \{(q_1,\dots,q_n,p_1,\dots,p_n) \mid p_i = \partial f_\lambda/\partial q_i\}$. Then L and L_λ have two transverse intersection points when $\lambda < 0$ which 'die' as a degenerate intersection at $\lambda = 0$. The general death of two intersection points is locally modeled by this example. Define an *independent birth* or *death* to be one when there are no holomorphic curves connecting the degenerate intersection point with any other intersection point.

3.2 Births, deaths and stabilization. Before proving in Theorem 3.12 that these are the only singularities to consider, a discussion of stabilization and its application to births and deaths is needed. This subsection provides an explicit construction of how to adjust the fiber coordinate of a birth or death. The subsection also takes a first step in applying stabilization to render all births and deaths independent.

To model the birth phenomenon by a one-parameter family of functions, choose a Darboux neighborhood N of the location of the birth and a symplectic chart κ

$$\kappa : (N, \omega|_N) \to \Big(\mathbf{I}^{2n}, \sum_i dq_i \wedge dp_i\Big) \text{ where } \mathbf{I} = [-3, 3]\,.$$

Assume N is small enough such that $L \cap N$ and $L_\lambda \cap N$ are both diffeomorphic to $\mathbf{R}^n$. Choose κ such that

$$\kappa(N \cap L) = \mathbf{I}^n \times \mathbf{0}\,, \tag{12}$$

$$\kappa(N \cap L_\lambda) = \Big\{ \Big(q_1, \dots, q_n, \tfrac{\partial f_\lambda}{\partial q_1}, \dots, \tfrac{\partial f_\lambda}{\partial q_n} \Big) \;\Big|\; (q_1, \dots, q_n) \in \mathbf{I}^n \Big\} \tag{13}$$

where $f_\lambda : \mathbf{I}^n \to \mathbf{R}$ is smooth and $\lambda \in \Lambda = [0, 1]$.

Let $\sigma_1 : \mathbf{R} \to \mathbf{R}$ be an even smooth bump function supported on $[-1, 1]$ and non-vanishing on $(-1, 1)$. Assume σ_1 has a unique maximum of 1 at 0, and that σ_1' has a unique minimum of -2 at $\frac{1}{2}$. Assume that for $z \in (0, \frac{1}{2})$, $\sigma_1'(z) = \sigma_1'(1-z)$. Let $\sigma_2(z) = \sigma_1\big(z + \frac{1}{2}\big)$. Thus, σ_2' is locally even around 0. There exists some non-degenerate quadratic function $Q : \mathbf{R}^{n-1} \to \mathbf{R}$ and symplectomorphism κ so that

$$f_\lambda(q_1, q_2, \dots, q_n) = \epsilon_1 q_1 + \lambda \sigma_2(q_1) + Q(q_2, \dots, q_n) \tag{14}$$

where $1 > \epsilon_1 > 0$ can be made arbitrarily small. Note that $\lambda \max\{\sigma_2'\} + \epsilon_1 < 3$ implies that $\kappa(N \cap L_\lambda)$ is a graph in $\mathbf{I}^{2n}$. Points in $L \cap L_\lambda \cap N$ correspond to critical points of f_λ. For $\lambda \in \big[0, \frac{1}{2}\epsilon_1\big)$, $\frac{\partial f_\lambda}{\partial q_1} > 0$ implies there are no such critical points, at $\lambda = \frac{1}{2}\epsilon_1$ there is a degenerate critical point at $(0, 0, \dots, 0)$, and for $\lambda \in \big(\frac{1}{2}\epsilon_1, 1\big]$, there are two non-degenerate critical points with positive and negative q_1-coordinates.

To simplify notation, assume that $\dim P = 2n = 2$; the stabilization process for manifolds of higher dimensions is similar. The few differences between the two-dimensional case and the general case will be addressed when they arise.

In the neighborhood $\tilde{N} = N \times \mathbf{R}^2$ of the birth, letting $\tilde{\kappa} = \kappa \times \mathrm{id}_{\mathbf{R}^2}$, the stabilized Lagrangians are modeled by

$$\tilde{\kappa}(\tilde{N} \cap \tilde{L}) = \big\{ (q, 0, x, 0) \;\big|\; (q, x) \in \mathbf{I} \times \mathbf{R} \big\} \tag{15}$$

$$\tilde{\kappa}(\tilde{N} \cap \tilde{L}_\lambda) = \Big\{ \Big(q, \tfrac{\partial F_\lambda}{\partial q}, x, \tfrac{\partial F_\lambda}{\partial x} \Big) \;\Big|\; (q, x) \in \mathbf{I} \times \mathbf{R} \Big\} \quad \text{where} \tag{16}$$

$$F_\lambda(q, x) = f_\lambda(q) - x^2\,. \tag{17}$$

Theorem 3.3. *Consider the one-parameter family of functions $F_\lambda : \mathbf{I} \times \mathbf{R} \to \mathbf{R}$ in equation (17) which has a unique birth at $(\lambda, q, x) = \big(\frac{1}{2}\epsilon_1, 0, 0\big)$.*

Given any constant $C \in \mathbf{R}$, there exists a one parameter family of functions $G_\lambda : \mathbf{I} \times \mathbf{R} \to \mathbf{R}$ such that

1. *$G_0 = F_0$ and $G_1 = F_1$.*
2. *$G_\lambda(q,x) = F_0(q,x) = F_1(q,x)$ for q in a neighborhood of $\partial\mathbf{I}$.*
3. *$|\partial G_\lambda/\partial q| < 3$.*
4. *No deaths of critical points occur and a unique birth occurs arbitrarily close to $(\lambda, q, x) = \left(\frac{1}{2}\epsilon_1, 0, C\right)$.*
5. *At the moment of birth, supp$(G_\lambda - F_0) \subset (-1,1) \times (C-1, C+1)$.*

Remark 3.4. An identical result holds for the positive stabilization of L_λ, when $F_\lambda(q,x) = f_\lambda(q) + x^2$ replaces equation (17).

The proof appears in the Appendix. For an overview of the proof, first consider the linear function $\mathcal{F}(q,x) = q$. The idea is to perturb $\mathcal{F}$ with a "growing bump function" to generate two critical points. That is, let

$$\mathcal{F}_\lambda(q,x) = \mathcal{F}(q,x) + \lambda\sigma_2(q)\sigma_1(x)$$

for $\lambda \in [0,1]$. A pair of critical points are born at $\lambda = 1/2$ and $(q,x) = (0,0)$.

Next replicate this perturbation for the non-linear function $\mathcal{N}(q,x) = \mathcal{F}(q,x) - x^2$. Choose some small $\epsilon > 0$ and let

$$\begin{aligned}\mathcal{N}_\lambda(q,x) &= \mathcal{N}(q,x) + \lambda\epsilon\sigma_2\left(\tfrac{q}{\epsilon}\right)\sigma_1\left(\tfrac{x}{\epsilon}\right)\\ &= \epsilon\mathcal{F}_\lambda\left(\tfrac{q}{\epsilon}, \tfrac{x}{\epsilon}\right) - \epsilon^2\left(\tfrac{x}{\epsilon}^2\right).\end{aligned}$$

Writing $\mathcal{N}_\lambda$ in this second form indicates that $\mathcal{N}_\lambda$ is locally like $\mathcal{F}_\lambda$ near $(0,0)$. Thus $\mathcal{N}_\lambda$ has similar dynamics for its critical points.

This is the basic idea to the construction behind Theorem 3.3. The theorem also adjusts the location of the critical points. See Figure 1. Throughout the proof, care must be taken to ensure that $|\partial\mathcal{N}_\lambda/\partial q| < 3$ so that the graph of the derivative lies in the appropriate neighborhood.

Remark 3.5. Recall that a critical point near $(0,C)$ corresponds to a Lagrangian intersection near $\tilde{\kappa}^{-1}(0,0,C,0) \in \tilde{P}$ which lies away from $P \times \{(0,0)\} \subset \tilde{P}$. Part of the goal of Theorem 3.3 is to 'slide' the intersection point back to $P \times \{(0,0)\}$ after the birth (STEP 4 of the proof of Theorem 3.3). This sliding is equivalent to sliding the critical point from $(0,C)$ to the $x=0$ n-plane.

As Figure 1 may indicate, pairs of critical points of relative index 1 may slide up or down in the fiber direction (x) without globally affecting the gradient flows; however, individual critical points do not have this flexibility. If $\mathcal{N}_\lambda$ slid a single critical point from $(q,x) = (0,0)$ to $(0,C)$ (or in the other direction, as done in STEP 4) for some large C, a quick check of the gradient vector field would reveal that $\frac{\partial}{\partial q}\mathcal{N}_\lambda$ would have to be large. Alternatively,

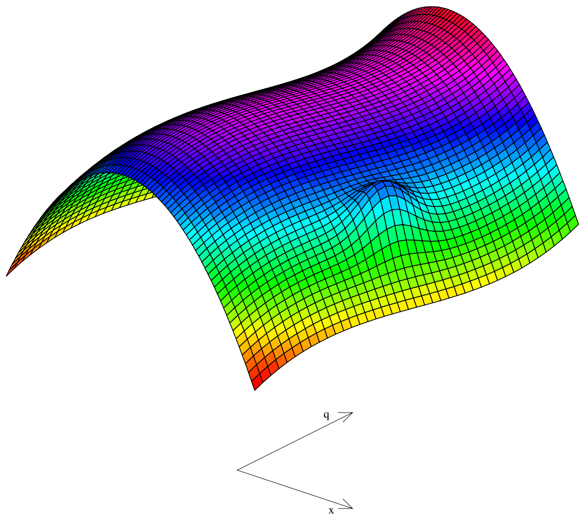

Figure 1: Two newly-born critical points near $(q, x) = (0, C)$.

the obstruction to sliding a single critical point without affecting the global structure of the function can be seen by considering the topology of nearby regular level sets.

By replacing F_λ with G_λ, a direct corollary of properties (1)–(4) of Theorem 3.3 is the following:

Corollary 3.6. *For the fixed Lagrangian $\tilde{L}$ locally given by the equation* (15)*, any constant $C \in \mathbf{R}$ and the moving Lagrangian $\tilde{L}_\lambda$ locally given by equations* (16) *and* (17)*, there exists a Hamiltonian deformation of $\tilde{L}_0$, say $\bar{L}_\Lambda$, such that*

1. $\bar{L}_0 = \tilde{L}_0$, $\bar{L}_1 = \tilde{L}_1$.
2. $\bar{L}_\lambda \cap (\tilde{P} \setminus \tilde{N}) = \tilde{L}_\lambda \cap (\tilde{P} \setminus \tilde{N})$ *for* $\lambda \in [0, 1]$.
3. *No pairs of points in $\tilde{L} \cap \bar{L}_\lambda$ become degenerate and die, and only one pair of such points is born. The birth occurs arbitrarily close to* $\tilde{\kappa}^{-1}(0, 0, C, 0) \in N \times \mathbf{R}^2 = \tilde{N}$.

When $F_\lambda(q, x) = f_\lambda(q) + x^2$ replaces equation (17)*, a similar result holds.*

The following theorem demonstrates how holomorphic curves connecting intersection points with non-zero fiber coordinates to intersection points with zero fiber coordinates often cannot exist. A future application of this theorem will make births and deaths independent by adjusting their fiber coordinates. Let J_{std} be the standard complex structure on $\mathbf{R}^2$.

Theorem 3.7. *Consider $\tilde{P}, \tilde{L}, \tilde{N}, \tilde{\kappa}$ and $\mathbf{I}$ from the discussion preceding Theorem 3.3 but this time stabilized twice.*

Let $\bar{L}$ be any Lagrangian such that

$$\bar{L} \cap (\tilde{P} \setminus \tilde{N}) = \big(L' \times \{(x_1, -2x_1, x_2, 2x_2) \mid (x_1, x_2) \in \mathbf{R}^2\}\big) \cap (\tilde{P} \setminus \tilde{N})$$

for some n-dimensional Lagrangian $L' \subset P$. Let

$$F(q_1, \dots, q_n, x_1, x_2) = f(q_1, \dots, q_n) - x_1^2 + x_2^2$$

for some smooth $f : \mathbf{I}^n \to \mathbf{R}$. Assume $\bar{L}$ is locally modeled in $\tilde{N}$ by some $G : \mathbf{I}^n \times \mathbf{R}^2 \to \mathbf{R}$ which satisfies

$$\mathit{supp}(G - F) \subset \big\{(q_1, \dots, q_n, x_1, x_2) \bigm| x_j \in O_j\big\} \tag{18}$$

where $O_j \subset \mathbf{R} \setminus \{0\}$ is some bounded interval.

Choose any $z \in \bar{L} \cap \tilde{L} \cap (P \times (\mathbf{R}^4 \setminus \{\mathbf{0}\}))$ and any $z' \in \bar{L} \cap \tilde{L} \cap (P \times \{\mathbf{0}\})$. Then

$$\widehat{\mathcal{M}}(z, z', J \oplus J_{std} \oplus J_{std}) = \widehat{\mathcal{M}}(z', z, J \oplus J_{std} \oplus J_{std}) = \emptyset$$

for any $J \in \mathcal{J}_\omega$.

Proof. Let $\mathrm{pr}_{f_j} : \tilde{P} \to \mathbf{R}^2 = \mathbf{C}$ denote the projection onto the (x_j, y_j)-coordinates for $j = 1, 2$. Note that $\mathrm{pr}_{f_j}(z') = (0,0)$ while $\mathrm{pr}_{f_j}(z) = (C_j, 0)$ for some non-zero $C_j \in O_j$.

For any $J \in \mathcal{J}_\omega$, pr_{f_j} is $(J \oplus J_{std} \oplus J_{std}, J_{std})$-holomorphic. Let $\mathcal{Q}_j = \mathrm{pr}_{f_j}(\tilde{L})$ and $\mathcal{Q}'_j = \mathrm{pr}_{f_j}(\bar{L})$. Then $\mathcal{Q}_j \subset \mathbf{C}$ is the horizontal x_j-axis while $\mathcal{Q}'_j \subset \mathbf{C}$ is the union of the line $y_j = (-1)^j 2x_j$ and some set contained in $O_j \times [-k, k] \subset \mathbf{C}$ for some positive constant k.

Consider the case $j = 1$. Let A_i, $i = 1, \dots, 4$, denote the four quadrants of $\mathbf{C}$ partitioned by the x-axis and line $y_1 = -2x_1$ where $A_1 = \{(x_1, y_1) \mid -2x_1 < y_1 < 0\}$ and A_2, A_3 and A_4 are the other quadrants in counter-clockwise order.

If $u \in \widehat{\mathcal{M}}(z', z, J \oplus J_{std} \oplus J_{std})$, then $u_1 = \mathrm{pr}_{f_1} \circ u : \Theta \to \mathbf{C}$ maps the unique complex orientation of Θ to the unique complex orientation of $\mathbf{C}$. Recall the boundary conditions

$$\text{(a)} \quad u_1(\mathbf{R}_\tau \times 0) \subset \mathcal{Q}_1\,, \qquad \text{(b)} \quad u_1(\mathbf{R}_\tau \times 1) \subset \mathcal{Q}'_1\,,$$
$$\text{(c)} \quad \lim_{\tau \to -\infty} u_1(\tau, *) = (0,0)\,, \quad \text{(d)} \quad \lim_{\tau \to \infty} u_1(\tau, *) = (C_1, 0)\,.$$

Conditions (a), (b) and (c) along with the orientation requirement force u_1 to map (at least partially) into the interior of A_2 or A_4. In either case, the open-mapping theorem for non-constant maps implies that u_1 maps onto the unbounded $A_2 \setminus \mathcal{Q}'$ or $A_4 \setminus \mathcal{Q}'$. Thus,

$$\infty = \mathcal{E}(u_1) \leq \mathcal{E}(u) < \infty .$$

And so $\widehat{\mathcal{M}}(z', z, J \oplus J_{std} \oplus J_{std})$ is empty.

To show $\widehat{\mathcal{M}}(z, z', J \oplus J_{std} \oplus J_{std})$ is empty, repeat the logic of the previous two paragraphs with $j = 2$. □

Remark 3.8. Theorem 3.7 can be generalized to an open set (in the C^∞ topology) of $J \oplus J_{std} \oplus J_{std} \in \mathcal{J}_{\omega + dx_1 \wedge dy_1 + dx_2 \wedge dy_2}$. The theorem also holds for an open set (in the C^∞ topology) of $\bar{L}$ in the space of compact Hamiltonian deformations of $\bar{L}$. The proofs are similar so only the first generalization will be justified.

Suppose there is no such open set of structures about $J \oplus J_{std} \oplus J_{std}$, then construct a sequence $\tilde{J}_n$ which converges to $J \oplus J_{std} \oplus J_{std}$ for which there are $\tilde{J}_n$-holomorphic curves, u_n, between z' and z. Without loss of generality, pass to a subsequence and assume they all lie in $\widehat{\mathcal{M}}(z', z, \tilde{J}_n)$. By Gromov's compactness theorem, the curves converge to a $J \oplus J_{std} \oplus J_{std}$-cusp-curve, $v_1 \cup v_2 \cup \ldots \cup v_k \in \widehat{\mathcal{M}}(z', z_1, J \oplus J_{std} \oplus J_{std}) \times \widehat{\mathcal{M}}(z_1, z_2, J \oplus J_{std} \oplus J_{std}) \times \ldots \times \widehat{\mathcal{M}}(z_{k-1}, z, J \oplus J_{std} \oplus J_{std})$. For some i, $z_{i-1} \in P \times \{\mathbf{0}\}$ and $z_i \in P \times (\mathbf{R}^4 \setminus \{\mathbf{0}\})$. By the above theorem, no such v_i can exist.

The next theorem and lemma assume a single stabilization; however, the below discussion has an obvious extension for multiple stabilizations.

The action functional in equation (4) extends from $\mathcal{A} : \Omega(L, L'; \gamma_0) \to \mathbf{R}$ to

$$\tilde{\mathcal{A}} : \Omega(\tilde{L}, \tilde{L}'; \gamma_0) \to \mathbf{R}\,, \quad \tilde{\mathcal{A}}(\gamma) = \int_{S(\gamma)} \omega + dx \wedge dy\,. \tag{19}$$

As before, $S(\gamma)$ is a homotopy from γ to γ_0. Note that $\Omega(L, L'; \gamma_0) \subset \Omega(\tilde{L}, \tilde{L}'; \gamma_0)$ and $\tilde{\mathcal{A}}\big|_{\Omega(L,L';\gamma_0)} = \mathcal{A}$.

Theorem 3.9. *If (L, L') are admissible then so are $(\tilde{L}, \tilde{L}')$. The map $J \to J \oplus J_{std}$ embeds $\mathcal{FG}(L, L')$ into $\mathcal{FG}(\tilde{L}, \tilde{L}')$. The curve $\tilde{u} \subset \tilde{P}$ is $(J \oplus J_{std})$-holomorphic if and only if $\tilde{u} = (u, 0)$ and $u \subset P$ is J-holomorphic. If $u \in \widehat{\mathcal{M}}(x, y, J)$ then $m_u(x, y) = m_{\tilde{u}}(x, y)$.*

Proof. The first two statements are obvious. To prove the third statement, consider the projection $\mathrm{pr}_f : \tilde{P} \to \mathbf{R}^2$ onto the fiber component. The projection is holomorphic since $\tilde{J}$ splits. Thus, if u is any holomorphic strip between two critical points in $\tilde{P}$ satisfying the Lagrangian boundary conditions, then $\mathrm{pr}_f \circ u$ is a J_{std}-holomorphic curve with boundaries on the x-axis and $y = \pm 2x$ line. By the maximum-modulus principle, $\mathrm{pr}_f \circ u = 0$.

Let $\mathcal{L}(t) \in \mathcal{L}ag(n)$ denote the loop in $\mathcal{L}ag(n+1)$ used to define $m_{\tilde{u}}(x,y)$. Since $\tilde{P} = P \times \mathbf{R}^2$ and $\tilde{u} = (u, 0)$, the loop splits $\mathcal{L}_u(t) = (\mathcal{L}'(t), \mathcal{L}''(t)) \in \mathcal{L}ag(n) \oplus \mathcal{L}ag(1)$ where $\mathcal{L}''(t)$ is the constant loop. By the additive property of the Maslov index for loops (e.g. see [McS1, p. 48–49]),

$$m_{\tilde{u}}(x,y) = \mu(\mathcal{L}_u) = \mu(\mathcal{L}') + \mu(\mathcal{L}'') = m_u(x,y) + 0\,. \qquad \square$$

Because the symplectic structure on the fiber coordinates is standard, Stokes theorem leads to a trivial observation which will be useful later.

LEMMA 3.10. *Consider $\tilde{P}, \tilde{L}, \tilde{N}, \tilde{\kappa}$ and $\mathbf{I}$ from the discussion preceding Theorem 3.3. Let $\tilde{L}'$ be any stabilized Lagrangian locally given by $F : \mathbf{I}^n \times \mathbf{R} \to \mathbf{R}$. Let $\Pi : \mathbf{I}^{2n} \times \mathbf{R}^2 \to \mathbf{I}^n \times \mathbf{R}$ be the projection onto the $(q_1, \dots, q_n, x)$ coordinates. Suppose $w, z \in \tilde{L} \cap \tilde{L}' \cap \tilde{N}$. Then*

$$\tilde{\mathcal{A}}(w) - \tilde{\mathcal{A}}(z) = F\big(\Pi(\tilde{\kappa}(w))\big) - F\big(\Pi(\tilde{\kappa}(z))\big)\,.$$

That is, the relative action values of the intersection points equal the corresponding relative critical values of the function.

The proofs of the main theorems will set standard the almost complex structures near a degenerate intersection point. In this situation, the following result holds. To simplify notation, let $\mathcal{M}(x,y) = \mathcal{M}(x,y,J)$.

LEMMA 3.11. *Suppose the pair of intersection points (x,y) of Lagrangians L and L_λ are born at $\lambda = 0$ with $\mathcal{A}(y) > \mathcal{A}(x)$. Assume there exists an $\epsilon > 0$ and a Darboux neighborhood $N \subset P$ of the degenerate point and symplectomorphism κ such that for all $\lambda \in (-\epsilon, \epsilon)$, $\kappa : (N, J_\lambda, \omega) \to (\mathbf{I}^{2n}, J_{std}, dq \wedge dp)$. In this lemma, let $\mathbf{I}$ be some possibly small interval which contains $(-\delta, \delta)$, $\delta > 0$.*

Then there exist $0 < \epsilon' < \epsilon$ such that for $\lambda \in (0, \epsilon')$, $\widehat{\mathcal{M}}_\lambda(x,y) = \widehat{\mathcal{M}}^1_\lambda(x,y)$ contains a unique element. In particular, $\mu(y) = \mu(x) + 1$.

Proof. Let $x_\lambda = x$ and $y_\lambda = y$ denote the newly born pair of points which exist when $\lambda > 0$. Assume ϵ' is small enough so that $x_\lambda, y_\lambda \in N$ for $\lambda \in (0, \epsilon')$. The proof of Theorem 2.7 shows that for any $u_\lambda \in \widehat{\mathcal{M}}_\lambda(x,y)$ not contained in N, $\mathcal{E}(u_\lambda) \geq f(\delta) > 0$ for some strictly increasing function $f : \mathbf{R}^+ \to \mathbf{R}^+$. But $\mathcal{E}(u_\lambda) = \mathcal{A}(y_\lambda) - \mathcal{A}(x_\lambda) \to 0$ as $\lambda \to 0^+$. Thus, for small λ, $u_\lambda \subset N$.

Decompose the curve into its complex projections $u_\lambda = u_{\lambda 1} + \ldots + u_{\lambda n}$. Let $\mathrm{pr}_k : \mathbf{C}^n \to \mathbf{C}$ be the holomorphic projection onto the k-th coordinate. After possibly shrinking N, and thus restricting λ to a smaller interval, assume that equations (12) and (13) hold, where $f_\lambda(q_1, \dots, q_n) = q_1^3 + \lambda q_1 + Q(q_2, \dots, q_n)$ for some non-degenerate quadratic Q. For $k > 1$, the projections $\mathrm{pr}_k(\kappa(N \cap L))$ and $\mathrm{pr}_k(\kappa(N \cap L_\lambda))$ are lines which intersect at one point. Thus, by the maximum-modulus theorem, $u_\lambda = u_{\lambda_1}$. Since such a solution always exists in this local model of $\mathbf{C}$, $\widehat{\mathcal{M}}_\lambda(x,y) = \{u_\lambda\}$.

Let $\mathcal{L} \in \mathcal{L}ag(n)$ denote the loop in $\mathcal{L}ag(n)$ used to define $m_u(x,y)$. Note that the loop splits $\mathcal{L}_u = (\mathcal{L}', \mathcal{L}'') \subset \mathcal{L}ag(1) \oplus \mathcal{L}ag(n-1)$ where $\mathcal{L}' = \{e^{i\pi t}\}\mathbf{R} \subset \mathbf{C}$ and $\mathcal{L}''$ is the constant loop. By the additive and normalization properties of the Maslov index for loops

$$m_{u_\lambda}(x,y) = \mu(\mathcal{L}_{u_\lambda}) = \mu(\mathcal{L}') + \mu(\mathcal{L}'') = 1 + 0\,. \tag{20}$$

It is easy to check that this curve represents a regular value of $D\bar{\partial}_J$, since the cokernel of $D\bar{\partial}_{J_{std}} = \bar{\partial}_{J_{std}}$ is empty when the intersections are transverse ([F1]). Thus $\widehat{\mathcal{M}}^1_\lambda(x,y) = \{u_\lambda\}$. □

3.3 Perturbing the one-parameter family. Consider the two almost complex structures $J', J'' \in \mathcal{FG}(L, L')$ given in Theorem 3.1 which are connected by j_Λ. Let $\mathcal{J}_{\mathbf{\Lambda}}$ be the space of one-parameter families of almost complex structures, J_Λ, such that $J_0 = J'$, $J_1 = J''$, and $J_\lambda = j_\lambda$ outside some compact set $K \subset P$ (which can depend on the family J_Λ). Note that $J_\lambda \in \mathcal{FG}(L, L')$.

Let $\mathbf{\Phi}_\Lambda$ be the space of one-parameter families of compactly supported Hamiltonian deformations, ϕ_Λ, such that $\phi_0 = \mathrm{id}$ and $\phi_1 = \psi_1$.

Suppose P and its Lagrangian submanifolds were stabilized. The Hamiltonian isotopy ϕ_Λ then would extend to $\tilde{\phi}_\Lambda$:

$$\tilde{\phi}_\lambda : \tilde{P} = P \times \mathbf{R}^2 \to \tilde{P}\,, \quad \tilde{\phi}_\lambda = \phi_\lambda \times \mathrm{id}_{\mathbf{R}^2}\,. \tag{21}$$

Define $\tilde{\mathbf{\Phi}}_\Lambda$ to be the set of compactly supported Hamiltonian isotopies connecting $\mathrm{id} : \tilde{P} \to \tilde{P}$ to $\tilde{\psi}_1$.

Note that $J \in \mathcal{FG}(L, L')$ extends to $\tilde{J} = J \oplus J_{std} \in \mathcal{FG}(\tilde{L}, \tilde{L}')$. Define $\tilde{\mathcal{J}}_{\mathbf{\Lambda}}$ to be the space of one-parameter families of almost complex structures, $\tilde{J}_\lambda$, such that $\tilde{J}_0 = J' \oplus J_{std}$, $\tilde{J}_1 = J'' \oplus J_{std}$ and $\tilde{J}_\lambda = j_\lambda \oplus J_{std}$ outside of a compact set in $\tilde{P}$.

Let $p_i : X_1 \times \ldots \times X_m \to X_i$ be the projection map onto the i-th component.

Theorem 3.12. *There exists a non-empty set $A \subset \mathbf{\Phi}_\Lambda \times \mathcal{J}_{\mathbf{\Lambda}}$ such that $(\phi_\Lambda, J_\Lambda) \in A$ satisfies the following properties:*

- (i) *L intersects $\phi_\lambda(L')$ transversely for all but a discrete set of λ. Non-transverse intersections are isolated in λ and have only one degenerate direction with a quadratic tangency. Furthermore, there are no* simultaneous death/births *(i.e. no quadratic tangencies of the form $f_\lambda(x) = x^2 - \lambda^2$).*
- (ii) *After stabilizing twice, all births and deaths are independent.*

(iii) *At a degenerate intersection at time λ_0, there exists a Darboux neighborhood $U \subset P$ of the point and an open interval $O \subset \Lambda$ about λ_0 such that for all $\lambda \in O$, $(U, J_\lambda|_U) = (\mathbf{C}^n, J_{std})$.*

(iv) *If $\Lambda' \subset \Lambda$ is a subinterval and $x_{\Lambda'}$ and $y_{\Lambda'}$ are two families of transverse intersections denoted by x and y, then for each $J_{\Lambda'} \in p_2(A)$, $\widehat{\mathcal{M}}_{\Lambda'}(x, y, J_{\Lambda'})$ is a smooth manifold. Also, for $(u, \lambda) \in \widehat{\mathcal{M}}_{\Lambda'}(x, y, J_{\Lambda'})$, $\mathrm{Ind}\, E_{u,\lambda} = m_u(x, y) + 1$.*

(v) *Handle-slides occur only at isolated λ, and do not occur when there is a degenerate intersection.*

Furthermore, for each $\phi_\Lambda \in \mathbf{\Phi}_\Lambda$ which satisfies condition (i), $\phi_\Lambda \in p_1(A)$.

Proof. (i) The first statement follows from Darboux' Theorem and the classification of one-parameter families of smooth functions [AGV], [C]. Consider an intersection x at λ_0. There exists a family of functions $f_\lambda : \mathbf{R}^n \to \mathbf{R}$ for $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, a neighborhood U of x and a symplectomorphism $\kappa : (U, \omega) \to (T^*L, dq \wedge dp)$ such that $\kappa(L_\lambda \cap U) = \left\{q_1, \dots, q_n, \frac{\partial f_\lambda}{\partial q_1}, \dots, \frac{\partial f_\lambda}{\partial q_n}\right\}$. Let $g_{k\lambda} = \partial f_\lambda / \partial q_k$. Arnold, Cerf and others have shown that any one parameter-family of smooth functions can be deformed so that its degenerate zeroes are isolated and the derivative has at most one quadratic degeneracy.

(ii) Suppose a birth occurs at $\lambda = \lambda_1$. Denote the degenerate intersection by $z \in L \cap \phi_{\lambda_1}(L')$. Let $z' \in L \cap \phi_{\lambda_1}(L') \setminus \{z\}$. To be consistent with the notation of Corollary 3.6, temporarily denote $\phi_\lambda(L')$ by L_λ. Stabilize L_λ twice, in both the positive and negative manner.

Choose $N \subset P$ a Darboux neighborhood of z and κ a symplectomorphism. Let $\tilde{N} = N \times \mathbf{R}^2_{x_1,y_1} \times \mathbf{R}^2_{x_2,y_2}$ and $\tilde{\kappa} = \kappa \times \mathrm{id}_{\mathbf{R}^4}$. Assume equations (15), (16) and (17) hold with $F_\lambda(q, x_1, x_2) = f_\lambda(q) - x_1^2 + x_2^2$. Here $\tilde{L}$ and $\tilde{L}_\lambda$ are the stabilized Lagrangians. Apply Corollary 3.6 twice to replace $\tilde{L}_\lambda$ by $\bar{L}_\lambda$ so that the birth now occurs at $\tilde{\kappa}^{-1}(\mathbf{0}, C, 0, C, 0)$ for some $C \gg 1$. Here $\mathbf{0} \in N$.

Without loss of generality, assume the birth still occurs at $\lambda = \lambda_1$ and denote the degenerate intersection again by z. Let $\bar{L}_\lambda$ be locally modeled by $G_\lambda(q, x_1, x_2)$. Then property 5 of Theorem 3.3 and $C - 1 > 0$ implies $\bar{L}_{\lambda_1}$ (that is, the support of $G_{\lambda_1} - F_0$) satisfies equation (18) when setting $O_j = (C - 1, C + 1)$ for $j = 1, 2$.

Thus, by Theorem 3.7, no $J \oplus J_{std} \oplus J_{std}$-holomorphic curves exist between z' and z (for any almost complex structure J on P). The result now follows from Remark 3.8.

(iii) Since $\mathcal{J}_{\mathbf{\Lambda}}$ is contractible and hence path-connected, there exists a path from J' to J'' which, near designated parameter values, is standard in a certain Darboux neighborhood.

(iv) If either x_λ or y_λ lies outside of a neighborhood in which J_λ is fixed, (iv) essentially restates Theorems 2.2 and 2.4. Only the argument of the surjectivity of E_u requires a generic choice of almost complex structures. But provided J_λ can be deformed on some open set of the image of u (where u is somewhere injective), then surjectivity can be achieved. The parameterized version is no different. The one exception which must be handled separately is when x_λ and y_λ are about to cancel each other. For some time λ before the death at λ_0, both intersection points lie in a neighborhood U on which J_λ cannot be deformed. But for this special case, the proof of Lemma 3.11 offers a precise description of the curves connecting the two newly-born points and so the statement still holds.

(v) Floer makes this same claim in Proposition 3.2 [F1] although his proof is incomplete. Lee [Le] proves an analogous statement in detail for the Floer theory of fixed points. To see how Lee's details can apply to the Lagrangian intersection version, apply the following "naturality" trick [S2].

Suppose $\bar\partial_J(u) = 0$. Let $v(\tau,t) = \phi_t^{-1}(u(\tau,t))$. Then v satisfies a perturbed Cauchy–Riemann equation:

$$\frac{\partial v}{\partial \tau} + (\phi_t)^* J\big(v(\tau,t),t\big)\left(\frac{\partial v}{\partial t} + X_{H_t}\right) = 0 \tag{22}$$

where H_t generates the Hamiltonian isotopy ϕ_t. Note that by adding the zero-order term, the previously moving Lagrangian boundary condition becomes fixed.

Solutions to equation (22), with $\mathbf{R}_\tau \times S^1_t$ replacing the domain Θ, represent gradient flows in the Floer theory for fixed points. Not all analysis for the fixed point version applies to the Lagrangian intersection version since the former does not have to consider a boundary condition; however, since all small neighborhoods, as well as some dense neighborhoods, of $\mathbf{R} \times S^1$ are naturally diffeomorphic to neighborhoods in Θ, much of the local analysis transfers between the two theories. In particular, Lee's method easily adapts to prove (v). $\square$

3.4 Proof of Theorem 3.1.

Proof. Pick $(\phi_\Lambda, J_\Lambda) \in A$ (defined in Theorem 3.12) connecting (L', J') to $(\psi_1(L'), J'')$. To simplify notation, let $L_\lambda = \phi_\lambda(L')$.

Let $v_1,\dots,v_k$ be a set of J_{λ_0}-holomorphic curves such that $v_i \in \widehat{\mathcal{M}}_{\lambda_0}(x_i,x_{i+1})$. Denote the union of the images by $v_1\cup\ldots\cup v_k$. This cusp-curve can also be thought of as a path in $\Omega(L,L_{\lambda_0};\gamma_0)$. When thought of as such a path, define the *neighborhood* of the cusp-curve to be

$$\begin{aligned}&U_\varepsilon(v_1,\dots,v_k;\lambda_0)\subset\Omega(L,L_\Lambda;\gamma_0)\,,\\&U_\varepsilon(v_1,\dots,v_k;\lambda_0)=\big\{(\gamma,\lambda)\in\Omega(L,L_\lambda;\gamma_0)\times\mathbf{R}\;\big|\;|\lambda-\lambda_0|<\varepsilon\quad\text{and}\\&\qquad\max_{t\in[0,1]}\operatorname{dist}\big(\gamma(t),v_j(\tau,t)\big)<\varepsilon\text{ for some }\tau\in\mathbf{R},\text{ and }1\le j\le k\big\}\,.\end{aligned}$$

Theorem 3.13 ([F1]). *Suppose there are no births or deaths throughout Λ, but that at $\lambda=0\in\Lambda$ there exists a handle-slide u from y to x. For any other intersection point w, there exists $\rho_0>0$ and a local diffeomorphism*

$$\varpi:\{u\}\times[\rho_0,\infty)\times\widehat{\mathcal{M}}^1_0(w,x)\to\widehat{\mathcal{M}}^1_\Lambda(w,y)\,.$$

Furthermore, for all $v\in\widehat{\mathcal{M}}^1_0(w,x)$, there exists $\epsilon>0$ such that ϖ is onto $\widehat{\mathcal{M}}^1_\Lambda(w,y)\cap U_\epsilon(u,v;0)$. A similar result holds for $\{u\}\times[\rho_0,\infty)\times\widehat{\mathcal{M}}^1_0(y,w)\to\widehat{\mathcal{M}}^1_\Lambda(x,w)$.

Corollary 3.14. *Suppose a handle-slide from y to x representing $\alpha\in\pi_1(L)$ occurs at $\lambda=0$. Let (C,d_-) and (C,d_+) denote the $\mathbf{Z}_2\pi_1(L)$-chain complexes immediately before and after $\lambda=0$. Define the module homomorphism $f:C\to C$ by setting $f(p)=p+\delta^p_y\alpha x$ for each basis element b. Then*

$$d_+=fd_-f^{-1}\,.$$

Proof. Let $\widehat{\mathcal{M}}^1_0(x,y)=\{u\}$. Let z,w be two intersections with $\mu(z)=\mu(w)+1\in\mathbf{Z}_{\Sigma(L)}$. The only subtlety in checking that $\langle fd_-f^{-1}z,w\rangle=\langle d_+z,w\rangle$ is when either $z=y$ or $w=x$, but not both. Consider the case $z=y$ and $w\neq x$. The other case is similar. The equation can be easily verified if $\langle d_-x,w\rangle=0$. Instead suppose $\widehat{\mathcal{M}}^1_0(w,x)=\{v\}$ and $\langle d_-y,w\rangle=0$. If $\widehat{\mathcal{M}}^1_0(w,x)$ has multiple elements and/or $\langle d_-y,w\rangle\neq0$, the arguments are similar. Let $\beta=[\gamma_x\gamma_v\gamma_w^{-1}]$. By the manifold property of $\widehat{\mathcal{M}}_\Lambda(w,y)$, $\widehat{\mathcal{M}}^1_\lambda(w,y)=\{u_\lambda\}$ for small $\lambda>0$. Furthermore, the convergence of u_λ to $u\cup v$ implies that

$$\begin{aligned}&\langle fd_-f^{-1}y,w\rangle=\big\langle fd_-(y+\alpha x),w\big\rangle=\big\langle d_-(y+\alpha x),w\big\rangle\\&\quad=\alpha\beta=[\gamma_y\gamma_u\gamma_x^{-1}\gamma_x\gamma_v\gamma_w^{-1}]=[\gamma_y\gamma_u\gamma_v\gamma_w^{-1}]=[\gamma_y\gamma_{u_\lambda}\gamma_w^{-1}]=\langle d_+y,w\rangle\,.\quad\square\end{aligned}$$

Let (T,d_T) denote a trivial acyclic chain complex

$$(T,d_T):0\to\mathbf{Z}_2\pi_1(L)\to\mathbf{Z}_2\pi_1(L)\to0\,.$$

LEMMA 3.15. *Let (C, d_-) and (C_+, d_+) denote the Floer chain complex immediately before and after the birth of intersection points x and y. Then $(C_+, d_+) = (C_- \oplus T, d_- \oplus d_T)$, where T is generated by x and y and $d_T(y) = \alpha x$ for some $\alpha \in \pi_1(L)$.*

Proof. By Theorem 3.12 (ii), the independence of the birth implies that $d_+ = d_- \oplus d_T$. By Theorem 3.12 (iii), there exists an open interval $O \subset \Lambda$ about 0 and a neighborhood $N \subset P$ of the degenerate point at $\lambda = 0$ such that for all $\lambda \in O$, $\kappa : (N, J_\lambda, \omega) \to (\mathbf{I}^{2n}, J_{std}, dq \wedge dp)$ under the symplectomorphism κ. The result now follows from Lemma 3.11. □

Lemmas 2.13, 3.15 and Corollary 3.14 show that births, deaths and handle-slides do not change $Wh_1(CF(L, L'J')) \in Wh_1(\mathbf{Z}_2\pi_1(L))$. Theorem 3.12 proves that these are the only 'local' singularities that need to be considered; however, 'globally' $CF(L, L, J')$ may undergo one other type of alteration. Suppose $\psi_1(L') = L'$, $J' = J''$ and there are neither handle-slides nor birth-deaths for $\lambda \in [0, 1]$. For each family of intersection points $x = \{x_\lambda\}$, define the loop $\beta_x : [0, 1] \to L$ by $\beta_x(\lambda) = x_\lambda$. The matrix for d_0 and for d_1 then differ by a finite combination of matrices of type B in equation (8), where $g = [\beta_x] \in \mathbf{Z}_2\pi_1(L)$. Since the definition of the $Wh_1(\pi_1(L))$ mods out such matrices,

$$Wh_1\big(CF(L, L', J')\big) = Wh_1\big(CF(L, \psi_1(L'), J'')\big) .$$

The invariance of the torsion under this last alteration also shows that a different choice of base paths γ_x for any intersection x does not affect $Wh_1(CF(L, L', J'))$.

The last statement of Theorem 3.1 follows trivially from Theorem 3.9. □

4 The Wh_2 Theorem

4.1 The statement. Consider Lagrangians L, L' and L'' such that (L, L') are admissible and L' and L'' are connected by a compactly supported Hamiltonian isotopy, ϕ_Λ. Assume that $L \cap L' = L \cap L'' = \emptyset$. Note that this implies automatically that L is not compact, since otherwise a non-vanishing Floer homology would imply that L' and L'' could not be separated from L.

Let $\mathbf{\Phi}_\Lambda$ be the set of compactly supported Hamiltonian isotopies which connect ϕ_1 with id. Perturb ϕ_Λ (fixing ϕ_0 and ϕ_1) so that it lies in $p_1(A)$ where A is defined in Theorem 3.12. (Note that $A \subset \mathbf{\Phi}_\Lambda \times \mathcal{J}_{\mathbf{\Lambda}}$ and so cannot be defined until $\mathcal{J}_{\mathbf{\Lambda}}$ is defined, and hence not until J' and J'' are chosen. But for ϕ_Λ to be in $p_1(A)$, ϕ_Λ need only satisfy condition (i) of Theorem 3.12. This condition does not require almost complex structures to be chosen.) Since any two perturbed ϕ_Λ are connected by the first perturbation composed with the inverse of the second, the proof of the main theorem shows that the Wh_2 element does not depend on the choice of the perturbation.

Choose compatible almost complex structures $J', J'' \in \mathcal{FG}(L, L') = \mathcal{FG}(L, L'')$ and a path $j_\Lambda \subset \mathcal{FG}(L, L')$ connecting them. Let $\mathcal{J}_{\boldsymbol{\Lambda}}$ be the space of one-parameter families of compatible almost complex structures as defined in section 3.3. Choose any $J_\Lambda \in \mathcal{J}_{\boldsymbol{\Lambda}}$ such that $(\phi_\Lambda, J_\Lambda) \in A$. Such a J_Λ exists by the last statement of Theorem 3.1.

Fix a base point $p \in L$ and a path γ_{x_λ} from p to $x_\lambda \in L \cap \phi_\lambda(L')$ as before. If x and y represent a family of pairs of intersection points which are born at λ_0, then choose paths so that $\gamma_{y_\lambda} = \gamma_{x_\lambda}\gamma_{u_\lambda}^{-1}$ where u_λ is the unique element in $\widehat{\mathcal{M}}_\lambda^1(x, y)$ from the proof of Lemma 3.11. Denote by (C_λ, d_λ) the chain complex with $\mathbf{Z}_2\pi_1(L)$-coefficients. Note that $L \cap L' = \emptyset$ implies that the chain complex is acyclic. Label each birth pair of intersections (x, y) as (z_{i-1}^n, b_i^n) where $i = \mu(y)$. Note that by choice of base paths, $\langle d_\lambda(b_i^n), z_{i-1}^n\rangle = 1$ for λ shortly after the birth.

Let $St_C(\mathbf{Z}_2\pi_1(L))$ be the Steinberg group whose indexing set includes all (one-parameter families of) intersection points in $L \cap L_\lambda$ that may ever exist. Let C be the chain complex generated by this set of intersection points. Note that $C_\lambda \subset C$. Let $D : C \to C$ be the canonical boundary operator from section 2.6 defined by $D(b_i^n) = z_{i-1}^n$ and $D(z_i^n) = 0$. Similarly, let Δ be the canonical contraction map. Extend the boundary operator d_λ from C_λ to C in the following way: for an intersection point $x \in C \setminus C_\lambda$ which is not yet born, define $d_\lambda(x) = D(x)$; for a pair of points $x, y \in C \setminus C_\lambda$ which died together at some earlier time $\lambda_0 < \lambda$ with $\mu(y) = \mu(x) + 1$, define $d_\lambda(y) = \langle d_{\lambda_0 - \epsilon} y, x\rangle x$ and $d_\lambda(x) = 0$ (where $\epsilon > 0$ is small enough such that $\lambda_0 - \epsilon \in O$ from Theorem 3.12 (iii)). Note that before any intersections are born, $d_\lambda = D$, whereas after they have all died, d_λ maps each generator to another or to 0.

Let d_- and d_+ denote d_λ just before and after some time λ_0 when a handle-slide exists from z to w representing $\alpha \in \pi_1(L)$. Then,

$$d_+ = h_{zw}(\alpha) \cdot d_-$$

is a restatement of Corollary 3.14 in the language of section 2.6. Suppose a handle-slide from z_i to w_i representing α_i occurs at $\lambda = \lambda_i$. Since $(\phi_\Lambda, J_\Lambda) \in A$, the handle-slides can be re-ordered so that $i < j \Rightarrow \lambda_i < \lambda_j$.

Define

$$St(\phi_\Lambda, J_\Lambda) = h_{z_k w_k}(\alpha_k) \cdots h_{z_1 w_1}(\alpha_1) \in St_C(\mathbf{Z}_2\pi_1(L))\,. \tag{23}$$

Suppose $\lambda \in (\lambda_i, \lambda_{i+1})$. Define $\delta_\lambda : C \to C$ by $\delta_\lambda = \chi_i \cdots \chi_1 \cdot \Delta$. It is easy to check that δ_λ is a contraction map for d_λ. Furthermore (d_1, δ_1) is elementary; thus, $St(\phi_\Lambda, J_\Lambda) \in \Upsilon$. Now use the methods of section 2.6 to construct $Wh_2(\phi_\Lambda, J_\Lambda)$ from $St(\phi_\Lambda, J_\Lambda)$. Hatcher and Wagoner prove ([HW, p. 129–130]) that the choices of base point, base paths, and ordering of intersection points (the choice of n made when labeling the birth pair (x, y) as (z_{i-1}^n, b_i^n)) does not alter $Wh_2(\phi_\Lambda, J_\Lambda)$.

Theorem 4.1. *Let $\phi'_\Lambda \in \mathbf{\Phi}_\Lambda$, which again after perturbation can be assumed to be in $p_1(A)$. Choose another $J'_\Lambda \in \mathcal{J}_{\mathbf{\Lambda}}$ such that $(\phi'_\Lambda, J'_\Lambda) \in A$. If ϕ_Λ can be deformed to ϕ'_Λ in the space of one-parameter families of compactly supported Hamiltonian isotopies, then*

$$Wh_2(\phi_\Lambda, J_\Lambda) = Wh_2(\phi'_\Lambda, J'_\Lambda)\,. \tag{24}$$

Furthermore, Wh_2 is invariant under stabilization.

4.2 Perturbing the two-parameter family. Define the *Cerf diagram* of $(\phi_\Lambda, J_\Lambda) \in A$ to be the collection of graphs of the symplectic action's (critical) values at $\{L \cap L_\lambda\}$, thought of as $\#\{L \cap L_\lambda\}$ continuous functions from (subsets of) Λ to $\mathbf{R}$. See Figure 2. For generic $(\phi_\Lambda, J_\Lambda) \in A$, $\mathcal{A}|_{\{L\cap L_\lambda\}} : \{L \cap L_\lambda\} \to \mathbf{R}$ is injective except for possible discrete $\lambda \in \Lambda$; thus, an obvious correspondence exists between $\{L \cap L_\lambda\}$ and individual curves of the Cerf diagram.

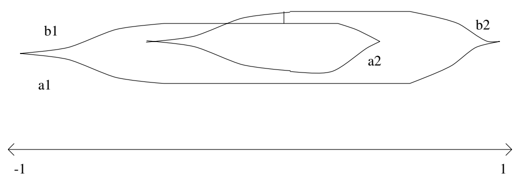

Figure 2: An example of a marked Cerf diagram. After the birth of the pairs (a_1, b_1) and (a_2, b_2), there is a handle-slide from b_2 to b_1 at approximately $\lambda = 0$. By Lemma 3.11, $\mu(b_1) = \mu(a_1) + 1 = \mu(b_2)$; thus, such a handle-slide could exist.

A *marked Cerf diagram* is a Cerf diagram with handle-slides between intersection points represented by vertical line segments connecting the corresponding critical values. See Figure 2. Because $\mathcal{A}(x) > \mathcal{A}(y) \Rightarrow \mathcal{M}(x, y) = \emptyset$, the marked Cerf diagram also determines the 'direction' of the handle-slide.

Let $\mathbf{\Phi}_{\Lambda^2}$ be the space of two-parameter families of compactly supported Hamiltonian symplectomorphisms, ϕ_{Λ^2}, such that $\phi_{(0,s)} = \mathrm{id}$, $\phi_{(1,s)}{=}\phi_1{=}\phi'_1$, $\phi_{(\lambda,0)} = \phi_\lambda$ and $\phi_{(\lambda,1)} = \phi'_\lambda$. Let $\mathcal{J}_{\mathbf{\Lambda^2}}$ be the space of two-parameter families of almost complex structures, J_{Λ^2}, such that $J_{(\lambda,0)} = J_\lambda$, $J_{(\lambda,1)} = J'_\lambda$, and $J_{(0,s)}$ and $J_{(1,s)}$ are any two paths in $\mathcal{FG}(L, L') = \mathcal{FG}(L, L'')$ which connect J_0 to J'_0 and J_1 to J'_1, respectively. Such paths exist because $\mathcal{J}_{\mathbf{\Lambda}}$ is contractible. Note that $J_{\lambda,0} = J_{\lambda,1} = j_\lambda$ outside a compact set since $J_\Lambda, J'_\Lambda \subset \mathcal{J}_{\mathbf{\Lambda}}$.

Let $\tilde{A} \subset \tilde{\mathbf{\Phi}}_\Lambda \times \tilde{\mathcal{J}}_{\mathbf{\Lambda}}$ be the 'stabilized' analogy to $A \subset \mathbf{\Phi}_\Lambda \times \mathcal{J}_{\mathbf{\Lambda}}$. Note that $(\phi_\Lambda, J_\Lambda) \in A$ implies $(\tilde{\phi}_\Lambda, \tilde{J}_\Lambda) \in \tilde{A}$ where $\tilde{\phi}_\Lambda$ is the family of $\tilde{\phi}_\lambda$ defined in equation (21) and $\tilde{J}_\Lambda = J_\Lambda \oplus J_{std}$. By the third statement of Theorem 3.9, the set of holomorphic curves does not change after stabilization; thus, each property in Theorem 3.12 by which $\tilde{A}$ is defined is easily verified. Define $\tilde{\mathbf{\Phi}}_{\Lambda^2} \times \tilde{\mathcal{J}}_{\mathbf{\Lambda^2}}$ in a manner analogous to the definition of $\mathbf{\Phi}_{\Lambda^2} \times \mathcal{J}_{\mathbf{\Lambda^2}}$.

In section 3.3, the types of singularities encountered when considering a one-parameter family (L_Λ, J_Λ) were reduced to births, deaths and handle-slides. There is an analogous list of 'unavoidable' phenomena in a generic $(\phi_{\Lambda^2}, J_{\Lambda^2}) \in \mathbf{\Phi}_{\Lambda^2} \times \mathcal{J}_{\mathbf{\Lambda^2}}$. The birth or death phenomenon is a codimension one singularity since it results from an intersection of two Lagrangians with a one-dimensional tangency. Hence the set of (λ, s) where a birth or death occurs for a given pair of critical points is one-dimensional. Similarly, just as the handle-slide occurred at isolated moments in a one-parameter family, the set of $(\lambda, s) \in \Lambda^2$ where it occurs between a given pair of critical points is one-dimensional.

Each of the singularities in a generic $(\phi_{\Lambda^2}, J_{\Lambda^2})$ can be represented as a bifurcation in a family, parameterized by s, of marked Cerf diagrams. To help illustrate the singularities appearing in Figure 3, some of the corresponding bifurcations accompany their descriptions.

1. *Point 'a'*: There are two types of singularities which result from the intersection of two families of handle-slides. The point 'a' closer to the bottom-left corner involves two handle-slides whose union as curves in P does not make a handle-slide; thus, no other handle-slides are affected by this singularity. A cusp-curve of two handle-slides exists at the other point 'a', however, and hence represents a boundary point of the one-dimensional moduli space associated with the third handle-slide. See Figure 4.
2. *Points 'b' and 'c'*: At point 'b' the order of two families of births is exchanged. At point 'c' the order of a family of deaths and a family

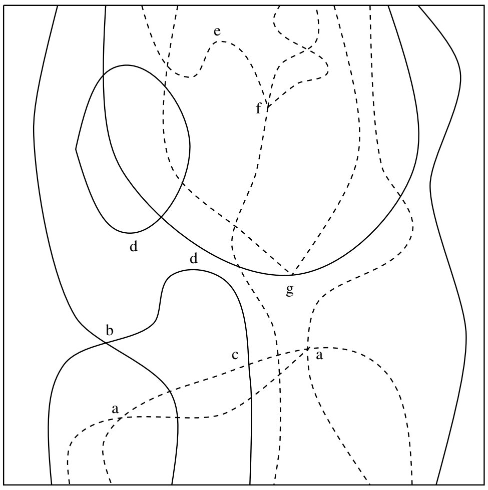

Figure 3: Handle-slides, births and deaths in $(\phi_{\Lambda^2}, J_{\Lambda^2})$. The λ-axis and s-axis correspond to the horizontal and vertical axes, respectively. Dashed lines represent handle-slides and solid lines, births or deaths.

of handle-slides is exchanged. See Figure 5.

3. *Points 'd' and 'e'*: The points marked 'd' represent the death of a birth and death, and the birth of a birth and death. A death of two handle-slides lies at point 'e'. These points exist when the manifolds are tangent to the $s = k$ lines for constants k.
4. *Point 'f'*: At an isolated (λ, s), a holomorphic curve with $m_u(x, y) = -1$ can appear. This curve shall be called a *super handle-slide*. As illustrated in Figures 3 and 6, the super handle-slide paired with index 1 curves can be a boundary point for several moduli spaces of handle-slides.
5. *Point 'g'*: Point 'g' represents what [HW] and others call a '*dovetail*' or 'swallowtail.' Suppose L and $L_{(\lambda,s)}$ are one dimensional Lagrangians in $T^*\mathbf{R}$ and $L = \mathbf{R} \times 0$ while $L_{(\lambda,s)} = \{x, x^3 - sx + \lambda\}$, then a dovetail singularity occurs at $\lambda = s = 0$. This can also be thought of as a cubic tangency, just as a birth or death is a quadratic tangency. Several families of handle-slides might appear with a dovetail. The dovetail also marks the birth of a birth and a death. See Figure 7.

Note that since $\phi_{0,s}(L') = L'$ and $\phi_{1,s}(L') = L''$, there are no intersection points at those (λ, s) values and hence no births, deaths or handle-slides.

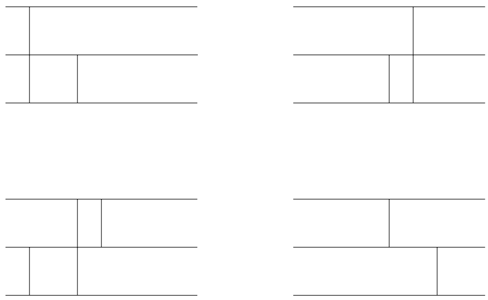

Figure 4: The order of two families of handle-slides is exchanged. In the bottom example, the resulting cusp-curve represents the boundary point of the moduli space of a third handle-slide.

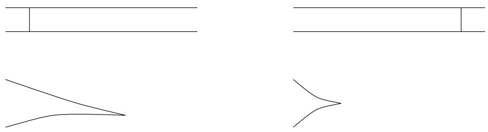

Figure 5: The order of a family of deaths and a family of handle-slides is exchanged.

As the Cerf diagram for point 'g' indicates, a dovetail involves one intersection point of index i and two of index $i-1$. Label them a, b and b' respectively. (The upside-down dovetail is similar.) The dependence of these points on (λ, s), when they exist, will usually be suppressed in the notation. Suppose the dovetail occurs at $(\lambda, s) = (0, 0)$ where a and b' do not exist for $s < 0$. Let $d_{(\lambda,s)}$ denote the boundary map of the chain complex at (λ, s). Without loss of generality, assume that for small positive s, the birth of (b', a) occurs at $-s < -\lambda_{BD}(s) < 0$ and the death of (b, a) at $\lambda_{BD}(s)$.

Definition 4.2. A dovetail is *nice* if there exists $\epsilon > 0$ and a fourth intersection point, e, such that

1. $\mu(a) = \mu(e)$.
2. For all $(\lambda, s) \in [-\epsilon, \epsilon] \times [-\epsilon, \epsilon]$, for all $x \in X = \{a, b, b', e\}$ and for all $y \in \{L \cap L'\} \setminus X$
$$\widehat{\mathcal{M}}_{\lambda,s}(x, y) = \widehat{\mathcal{M}}_{\lambda,s}(y, x) = \emptyset\,.$$
(Note that this condition is vacuous for $x \in \{a, b'\}$ and $s < 0$.)
3. There exist some $\epsilon' < \lambda_{BD}(\epsilon)$ arbitrarily close to $\lambda_{BD}(\epsilon)$ such that

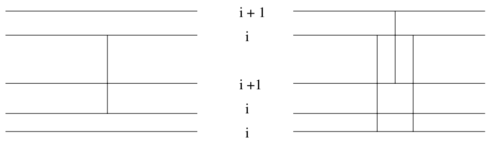

Figure 6: This marked Cerf diagram for a super handle-slide lists the indices of the intersection points next to the graphs of their action values. Assume the super handle-slide occurs between the points with the second and third highest action values.

Figure 7: There is an 'opposite' dovetail whose Cerf diagrams are turned upside-down.

$d_{(-\epsilon',\epsilon)}e = b$, $d_{(-\epsilon',\epsilon)}a = b'$, $d_{(\epsilon',\epsilon)}e = b'$ and $d_{(\epsilon',\epsilon)}a = b$.

4. For all $(\lambda, s) \in [-\epsilon, 0) \times [0, \epsilon]$ $\mathcal{A}(e) \geq \mathcal{A}(a) \geq \mathcal{A}(b') \geq \mathcal{A}(b)$ and for all $(\lambda, s) \in (0, \epsilon] \times [0, \epsilon]$ $\mathcal{A}(e) \geq \mathcal{A}(a) \geq \mathcal{A}(b) \geq \mathcal{A}(b')$ with equalities only holding for degenerate intersection points.

Theorem 4.3. *After four stabilizations, there exists a non-empty set $B \subset \mathbf{\Phi}_{\Lambda^2} \times \mathcal{J}_{\mathbf{\Lambda^2}}$ such that any $(\phi_{\Lambda^2}, J_{\Lambda^2}) \in B$ satisfies the following properties:*

(i) *L intersects $\phi_{(\lambda,s)}(L')$ transversely for all but a codimension one set of (λ, s). The set of parameter values for which an intersection has only one degenerate direction with a quadratic tangency forms a transversely intersecting collection of one dimensional curves in Λ^2. The only other degenerate intersections are the cubic degeneracies associated with dovetails. These are isolated from all other degeneracies except, of course, from the birth and death pairs which define them.*

(ii) *All dovetails are nice.*

(iii) *All births and deaths away from a neighborhood of the dovetails are independent.*

(iv) *Consider a one-parameter family of quadratically degenerate intersections occurring at $\Gamma \subset \Lambda^2$. There exists a family of Darboux neighborhoods $U_\lambda \subset P$ of the points and an open set $O \subset \Lambda^2$ of Γ such that for all $(\lambda, s) \in O$, $(U_\lambda, J_{(\lambda,s)}\big|_U) = (\mathbf{C}^n, J_{std})$.*

(v) *If $\Lambda' \subset \Lambda^2$ is an open subset and $x_{\Lambda'}$ and $y_{\Lambda'}$ are two families of transverse intersections denoted by x and y, then for each $J_{\Lambda'} \in p_2(B)$, $\widehat{\mathcal{M}}_{\Lambda'}(x,y)$ is a smooth manifold. Also, for $(u,\lambda,s) \in \widehat{\mathcal{M}}_{\Lambda'}(x,y)$, $\mathrm{Ind}\, E_{u,\lambda,s} = m_u(x,y) + 2$.*

(vi) *Away from dovetail occurrences, the set of parameter values (λ, s) for which there are handle-slides is, in general, a transversely intersecting collection of one dimensional curves. The only exception to this is the crossing of two handle-slides with a common critical point, which can create a third family of handle-slides. See point 'a'. These curves also transversely intersect the curves associated with births or deaths. The collection of curves associated with handle-slides, births and deaths intersects the $s = k$ lines transversely except for possible tangencies at isolated points.*

(vii) *Super handle-slides occur only at isolated (λ, s).*

Proof. The proofs of statements (i), (iii), (iv), (v) and (vii) are very similar to their counterpart statements (i), (ii), (iii), (iv) and (v), respectively, in Theorem 3.12 and will not be reproved here. The Darboux charts for the fourth statement are chosen small enough so that when two degenerate points occur at the same time, their charts do not intersect (except in the cubic case, when they come together). For the manifold results in the fifth statement, again almost all holomorphic curves exit the Darboux charts of statement (iv). The exceptions can again be studied as curves entirely in $\mathbf{C}^n$.

(ii) Consider $(\phi_{\Lambda^2}, J_{\Lambda^2}) \in \mathbf{\Phi}_{\Lambda^2} \times \mathcal{J}_{\mathbf{\Lambda^2}}$ such that statement (i) holds for ϕ_{Λ^2} and a dovetail occurs at $(\lambda, s) = (0,0)$. Let $L_{\lambda,s} = \phi_{\lambda,s}(L')$.

Choose some small Darboux neighborhood N of the cubic degeneracy and chart $\kappa : N \to \mathbf{R}^{2n}$ such that $\kappa(L \cap N)$ is the zero section while, for (λ, s) near $(0,0)$, $\kappa(L_{\lambda,s} \cap N)$ is modeled by the graph of $df_{\lambda,s}$ with

$$f_{\lambda,s}(q_1,\ldots,q_n) = q_1^4 - sq_1^2 + \lambda q_1 + Q(q_2,\ldots,q_n) \tag{25}$$

for some non-degenerate quadratic Q. To simplify notation, henceforth assume $(q_1, q_2, \ldots, q_n) = q_1 = q$. The more general case is similar. Also, the symplectomorphism κ will be omitted.

Figure 8: A dovetail. The two figures graph the critical values of $f_{\lambda,\pm 1}$ from equation (25) as λ varies in $[-1,1]$. Figures 8, 9 and 10 are unmarked Cerf diagrams.

Note that by Lemma 3.10, the Cerf diagram for the relative action values of the intersection points agrees with Figure 8. The main idea is to replace the one-parameter family of Cerf diagrams in Figure 8 with the family in Figure 9. Let b denote the intersection point in the first Cerf diagram in Figure 8 ($s = -1$). In the second diagram ($s = 1$), let (b', a) denote the pair of intersections created at the birth with $\mu(b') = \mu(a) - 1$. Thus, to be consistent with the notation of the first diagram, the pair (b, a) dies at some later λ. Let (e_2, e_1) be the pair of (families of) intersection points which appear in Figure 9 but not in Figure 8. Here $\mu(e_2) = \mu(e_1) - 1$. The following paragraph sketches the purpose of (e_2, e_1). Details will follow.

The pair (e_2, e_1) can be created with the techniques from section 3.2. Pairing e_1 with b before the dovetail occurs, the techniques in section 3.2 allow the two intersection points to slide in the fiber directions. Note that e_1 and b slide 'up' one fiber and 'down' the other; thus, by Lemma 3.10, the relative action values can be assumed to be those represented by Figure 9. See Remark 3.5 for a description of sliding in the fiber direction. The main purpose of creating e_1 is to adjust the fiber coordinates of b, so that Theorem 3.7 and Remark 3.8 ensure that property 2 of Definition 4.2 holds when b undergoes a dovetail. Between the third (furthest right on top row, $s = -s_0 < 0$) and fourth (furthest left on bottom row, $s = s_0 > 0$) Cerf diagrams in Figure 9, the dovetail occurs.

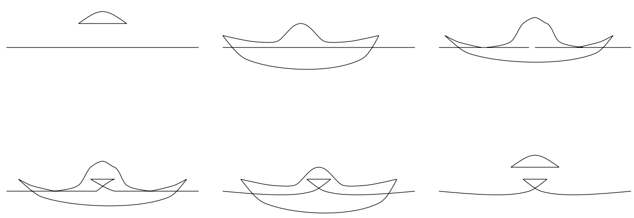

Figure 9: Making the dovetail nice.

Details of the family of functions in N will be given for the fourth diagram in Figure 9, which corresponds to a one-parameter family (in λ) for $s = s_0$. There is a birth, a simultaneous death/birth (see Theorem 3.12 (i)), a birth, a death, a simultaneous death/birth and a death which occur at $-1 < \lambda_1 < \ldots < \lambda_6 < 1$, respectively. See Figure 10. Without loss of generality, assume that $\lambda_1(s)$ and $\lambda_6(s)$ are independent of s for $s \in [-s_0, s_0]$, $-\lambda_2(s) = \lambda_5(s) = s_0$ for $s \in [-s_0, s_0]$, and $-\lambda_3(s) = \lambda_4(s) = \frac{1}{2}s$ for $s \in [0, s_0]$. Define $\lambda_3(s) = \lambda_4(s) = 0$ for $s \in [-s_0, 0)$ even though

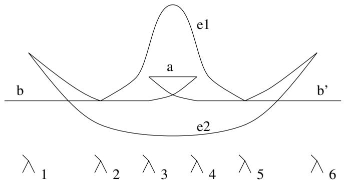

Figure 10: An enlargement of the fourth diagram in Figure 9.

they do not mark degenerate intersection points.

Stabilize P four times. Let $F_{\lambda,s}(q,x_1,\dots,x_4) = f_{\lambda,s}(q) - x_1^2 + x_2^2 - x_3^2 + x_4^2$ model the stabilized $\tilde{L}_{\lambda,s}$. Let $\tilde{N} = N \times \mathbf{R}^8$. Rather then repeat the calculus from Theorem 3.3, a verbal explanation will describe how to construct a $G_{\lambda,s}$ from $F_{\lambda,s}$, such that after replacing $\tilde{L}_{\lambda,s} \cap \tilde{N}$ with some $\bar{L}_{\lambda,s} \cap \tilde{N}$ locally modeled by $G_{\lambda,s}$, the dovetail is nice. To reduce the notation, represent points in $\tilde{L} \cap \tilde{N}$ (often in $\{\tilde{L} \cap \tilde{L}_{\lambda,s} \cap \tilde{N}\}$) by their (q,x_1,x_2,x_3,x_4)-coordinates. The explicit description will only be given for $s = s_0$; however, the deformation for more general s can be easily seen. Figure 11 illustrates the dovetail at $s = s_0$ after stabilizing but before sliding critical points in the fiber directions. Figure 12 illustrates the graphs of $G_{\lambda,s_0}(q,x_1,0,0,0)$ for some $\lambda \in (\lambda_i, \lambda_{i+1})$, $i = 1,\dots,5$.

Set $\epsilon = s_0$. Note that $(\phi_{\Lambda^2}, J_{\Lambda^2})$ will be altered for (λ, s) in some set larger than $[-\epsilon,\epsilon] \times [-\epsilon,\epsilon] = [\lambda_2,\lambda_5] \times [-s_0,s_0]$. Indeed, the set contains at least all $\lambda \in [\lambda_1,\lambda_6]$ and s ranging from diagram 1 to diagram 6 in Figure 9. Nonetheless, the properties of a nice dovetail need only be verified for $(\lambda,s) \in [-\epsilon,\epsilon] \times [-\epsilon,\epsilon]$. Choose $\delta > 0$, independent of $s \in [-s_0,s_0]$, such that

$$\delta \ll \tfrac{1}{4} \min\left\{\lambda_2 - \lambda_1, \lambda_3(s_0) - \lambda_2, \lambda_4(s_0) - \lambda_3(s_0), \lambda_5 - \lambda_4(s_0), \lambda_6 - \lambda_5\right\}.$$

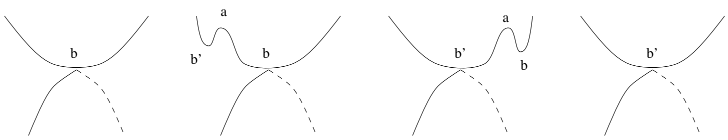

Figure 11: The graphs of $F_{\lambda,s_0}(q,x_1,0,0,0)$ for selected $\lambda \in [-1,1]$. This figure only shows the stabilization associated to x_1. Note that by Lemma 3.10, the relative action values of a, b and b' are similar to those drawn in the $s = 1$ diagram in Figure 8.

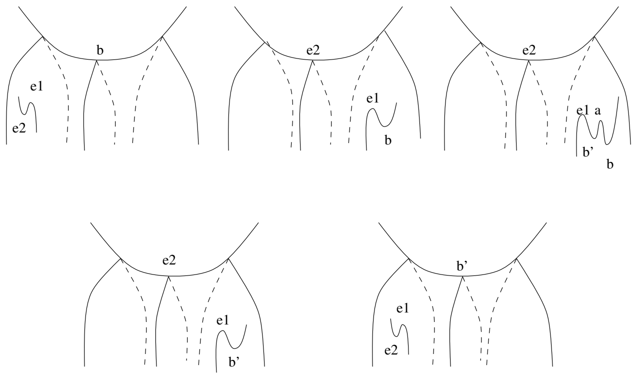

Figure 12: The graphs of $G_{\lambda,s_0}(q,x_1,0,0,0)$ for selected $\lambda \in (\lambda_1,\lambda_6)$. Each little 'squiggle' represents a pair of critical points of relative index one at some non-zero x_1 value, C. Note that the relative indices implied by the squiggles agree with those from Definition 4.2. This figure only shows one stabilization. Since a second stabilization is always required to make births and deaths independent, the relative action values of a, b, b', e_1 and e_2 can be assumed to be those drawn in Figure 10.

STEP 1: The birth of (e_2, e_1).

When $(\lambda, s) = (-1, s_0)$, assume the intersection b occurs at $(q, x_1, x_2, x_3, x_4) = (0,0,0,0,0)$. For $\lambda \in [0, \lambda_1 + \delta]$, deform F_{λ,s_0} away from b such that there is an independent birth of the pair (e_2, e_1) at $(q, x_1, x_2, x_3, x_4) = (-q_0, C, C, 0, 0)$ which slides to $(-q_0, 0, 0, 0, 0)$. Here $0 < q_0 < 1$ and $C \gg 1$. Fixing J in a neighborhood of the birth, an argument as in the proof of Lemma 3.11 implies that $\mu(e_1) = \mu(e_2) + 1$. The first diagram in Figure 12 occurs at some $\lambda \in (\lambda_1, \lambda_1 + \delta)$, after the birth of (e_2, e_1) but before the pair slides to $x_1 = x_2 = 0$.

STEP 2: Sliding (b, e_1).

For $\lambda \in [\lambda_1 + \delta, \lambda_2 - \delta]$, adjust the critical points so that e_1 is close to b instead of e_2. In other words, replicate the changes of F_{λ,s_0} from diagrams 2 to 3 in Figure 11. Assume that for $\lambda \in [\lambda_1 + \delta, \lambda_2 - \delta]$,

$$G_{\lambda,s_0}(q, x_1, x_2, x_3, x_4) = g_{\lambda,s_0}(q) - x_1^2 + x_2^2 - x_3^2 + x_4^2 \qquad (26)$$

for some $g_{\lambda,s_0} : \mathbf{I} \to \mathbf{R}$.

Choose $\tilde{N}' \subset \tilde{N} \setminus \{e_2\}$ containing e_1, b and $(\mathbf{I}')^2 \times \mathbf{R}^8$ for some small set $\mathbf{I}' \subset [-1,1]$. For $\lambda \in [\lambda_2 - \delta, \lambda_2 + \delta]$, deform the function in $\tilde{N}'$ such that e_1 and b slide in the (x_1, x_2)-coordinates and degenerate in a simultaneous death/birth at $(q, x_1, x_2, x_3, x_4) = (q_0, C, C, 0, 0)$ and $\lambda = \lambda_2$. The second diagram of Figure 12 illustrates this slide. Again, fixing J in a neighborhood of this simultaneous death/birth demonstrates that $\mu(e_1) = \mu(b) + 1$. This verifies property 1 of Definition 4.2. As will be detailed in STEP 3, Theorem 3.7 and Remark 3.8 show that no holomorphic curves connect any other intersection points to e_1 and b.

Let $B_r(\mathbf{0})$ denote the ball of radius r about $\mathbf{0} \in \mathbf{R^8}$. Fix $r \ll 1$. Since the pair (e_1, b) is born (with names unchanged), a set of base paths must be chosen. Choose γ_{e_1}, and hence γ_b, such that $\gamma_{e_1} \subset \tilde{N} \cup (P \times B_r(\mathbf{0}))$.

STEP 3: Property 2 of Definition 4.2.

For $(\lambda, s) \in [\lambda_2, \lambda_5] \times [-s_0, s_0]$ the deformation of $G_{\lambda,s}$ will remain restricted to some small neighborhood of $(x_1, x_2) = (C, C)$ in $\mathbf{R}^2$ away from $(0, 0)$. That is,

$$\operatorname{supp}(G_{\lambda,s} - G_{\lambda_2-\delta,s}) \subset \big\{(q, x_1, x_2, x_3, x_4) \bigm| q \in \mathbf{I}\,,\ x_1 \in O_1\,,\ x_2 \in O_2\big\} \tag{27}$$

where $O_j \subset \mathbf{R} \setminus \{0\}$ is some bounded interval. The discussion in STEPS 1 and 2, including equation (26), clearly generalizes to any $s \in [-s_0, s_0]$. Applying equations (26) and (27) to Theorem 3.7 and Remark 3.8 thus proves that intersection points which are born as a result of the deformation at some $(\lambda, s) \in [-s_0, s_0] \times [-s_0, s_0]$ (namely a, b, b' and e_1) cannot be connected via holomorphic curves to intersection points whose (x_1, x_2)-coordinates are zero (such as e_2). This justifies property 2 of Definition 4.2.

STEP 4: The birth of (b', a).

Thus far, all holomorphic curves and intersection points have zero x_3 and x_4 coordinates. In particular

$$G_{\lambda_3-\delta,s_0}(q, x_1, x_2, x_3, x_4) = g_{\lambda_3-\delta,s_0}(q, x_1, x_2) - x_3^2 + x_4^2 \tag{28}$$

for some $g_{\lambda_3-\delta,s_0} : \mathbf{I} \times \mathbf{R}^2 \to \mathbf{R}$.

Choose a neighborhood $\tilde{N}'' \subset \tilde{N}' \backslash \{b, e_1\}$ which contains $(\mathbf{I}')^2 \times (\mathbf{I}'')^4 \times \mathbf{R}^4$ for some small interval $\mathbf{I}''$. See Figure 13. For $\lambda \in [\lambda_3 - \delta, \lambda_3 + \delta]$, deform the function in $\tilde{N}''$ such that at $\lambda = \lambda_3$, the pair (b', a) is born near $(q, x_1, x_2, x_3, x_4) = (q_0, C, C, C, C)$ with $\mu(b') = \mu(a) - 1$. Construct G_{λ,s_0} such that

$$\operatorname{supp}(G_{\lambda_3,s_0} - G_{\lambda_3-\delta,s_0}) \subset \big\{(q, x_1, x_2, x_3, x_4) \bigm| q \in \mathbf{I}\,,\ x_3 \in O_3\,,\ x_4 \in O_4\big\} \tag{29}$$

where $O_j \subset \mathbf{R} \setminus \{0\}$ is some bounded interval. Applying equations (28) and (29) to Theorem 3.7 and Remark 3.8 ensures that the birth of (b', a) is independent of b and e_1.

Note that by convention, $d_{(\lambda_3+\delta,s_0)} a = b'$. Moreover, property 2 of

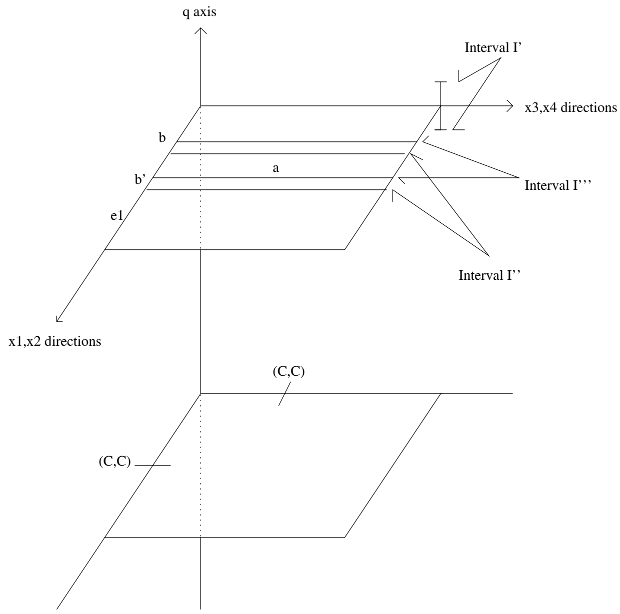

Figure 13: The birth and death of (e_2, e_1) occur in the lower plane ($q = -q_0$) at $(x_1, x_2, x_3, x_4) = (C, C, 0, 0)$. These events occur at $\lambda = \lambda_1$ and $\lambda = \lambda_6$. In the upper plane ($q = q_0$), the points marked e_1, b and b' indicate where those intersections lie when $\lambda \in (\lambda_3 + 2\delta, \lambda_4 - 2\delta)$. This takes place after b' is born and slides to $x_3 = x_4 = 0$ but before b slides to $x_3 = x_4 = C$ to die. The point marked a approximates the locations of the birth of (b', a) and the death of (b, a) which occur at $\lambda = \lambda_3$ and $\lambda = \lambda_4$, respectively. All other intersection points lie on the q-axis.

Definition 4.2 and the independence of the birth of (b', a) imply that no handle-slides exist starting or ending at e_1 or b for $\lambda \in [\lambda_2, \lambda_3 + \delta]$. Thus, $d_{(\lambda_3+\delta,s_0)}e_1 = gb$ for some $g \in \pi_1(L)$. By setting $0 < \mathcal{A}(e_1) - \mathcal{A}(b) \ll 1$ for $\lambda \in [\lambda_2, \lambda_3 + \delta]$, the proof of Theorem 2.7 implies that any holomorphic curve $u \in \widehat{\mathcal{M}}_{\lambda,s_0}(b, e_1)$ remains in $\tilde{N}$. Since $\tilde{L} \cap \tilde{N}$ is contractible and $d_{(\lambda_2,s_0)}e_1 = b$, $[\gamma_{e_1}\gamma_u\gamma_b^{-1}] = 1 \in \pi_1(L)$; thus, $d_{(\lambda_3+\delta,s_0)}e_1 = b$. Setting $-\epsilon'$ from the Definition 4.2 equal to $\lambda_3 + \delta$, half of property 3 of Definition 4.2 now holds.

Choose γ_a, and hence $\gamma_{b'}$, such that $\gamma_a \subset \tilde{N} \cup (P \times B_r(\mathbf{0}))$ and $\gamma_a \cap (P \times B_r(\mathbf{0})) = \gamma_{e_1} \cap (P \times B_r(\mathbf{0}))$. For $\lambda \in [\lambda_3 + \delta, \lambda_3 + 2\delta]$, slide (b', a) back to the $x_3 = x_4 = 0$ plane. Diagram 3 of Figure 12 shows the function for some $\lambda \in [\lambda_3 + 2\delta, \lambda_4 - 2\delta]$.

STEP 5: The death of (b, a).

Choose a neighborhood $\tilde{N}''' \subset \tilde{N}' \setminus \{b', e_1\}$ which contains a, b and $(\mathbf{I}')^2 \times (\mathbf{I}''')^4 \times \mathbf{R}^4$ for some small interval $\mathbf{I}'''$. See Figure 13. For $\lambda \in [\lambda_4 - 2\delta, \lambda_4 + \delta]$, deform the function in $\tilde{N}'''$ such that for $\lambda \in [\lambda_4 - 2\delta, \lambda_4 - \delta]$, the pair slides in the (x_3, x_4)-coordinates to a small neighborhood of $(q, x_1, x_2, x_3, x_4) = (q_0, C, C, C, C)$, and at $\lambda = \lambda_4$, (b, a) dies near (q_0, C, C, C, C). For $\lambda \in [\lambda_4 - \delta, \lambda]$, apply Remark 3.8 to ensure that no curves connect e_1 or b' to a or b. Note that for $\lambda \in (\lambda_3 + \delta, \lambda_4 - \delta)$ there may be holomorphic curves starting and ending at $\{a, b, b', e_1\}$.

Assume for $\lambda \in [\lambda_4 - \delta, \lambda_4]$, $0 < \mathcal{A}(a) - \mathcal{A}(b) \ll 1$. Then, by Lemma 3.11 and the argument of Theorem 2.7, there is a unique curve in $\widehat{\mathcal{M}}^1_{\lambda_4 - \delta, s_0}(b, a)$ that stays in $\tilde{N}$. Since $\tilde{N} \cap \tilde{L}$ is contractible, it then follows from the manner in which a slid to and from the $x_3 = x_4 = 0$ plane, as well as the assumption that $\gamma_z \cap (P \times B_r(\mathbf{0}))$ is the same for all $z \in \{a, b, b', e_1\}$, that $d_{(\lambda_4 - \delta, s_0)} a = b$. A similar argument verifies $d_{(\lambda_4 - \delta, s_0)} e_1 = b'$. Since $\epsilon' = -(\lambda_3 + \delta) = \lambda_4 - \delta$, the second half of property 3 of Definition 4.2 now holds.

STEP 6: STEPS 4 and 5 for $s < s_0$.

Unlike STEPS 1-3, STEPS 4 and 5 will not work for all $s < s_0$. Since $\lim_{s \to 0^+} \lambda_4(s) - \lambda_3(s) = 0$, the pair (b', a) has a decreasing interval of time to slide from (q_0, C, C, C, C) to $(q_0, C, C, 0, 0)$. Instead, for $s < s_0$, the birth will occur near $(q, C, C, C(s), C(s))$ where $C(s_0) = C$ and $C(0) = 0$. A similar adjustment applies to STEP 5.

Note that for small (positive) s, equation (29) and its analogue in STEP 5 may fail; thus, the birth and death may no longer be independent. For example, at $(\lambda_3(s), s)$, some holomorphic curves may connect $x \in \{b, e_1\}$ to the degenerate intersection $a = b'$. Nevertheless, none of the properties of a nice dovetail are compromised.

STEP 7: The rest.

Repeating the part of the discussion for $\lambda \in [0, \lambda_2 + \delta]$ in reverse completes the description of G_{λ, s_0} for $\lambda \in [\lambda_5 - \delta, 1]$. See diagrams 4 and 5 of Figure 12.

As mentioned in the caption of Figure 12, with both negative (x_1 and x_3) and positive (x_2 and x_4) stabilization, the critical levels of $G_{\lambda, s}$ can be adjusted so that property 4 of Definition 4.2 holds. For example, if necessary, the birth of (b', a) can occur at $(q_0, C_1, C_2, C_3, C_4)$ for $C_i \neq C_j$ instead of at (q_0, C, C, C, C).

(vi) There are several claims in the sixth statement. To show that the parameter values of handle-slides intersect those of other handle-slides, births and deaths transversely, repeat the arguments from Theorem 3.12(v). To ensure the transversality of the $s = k$ lines with parameter values associated to births, deaths and handle-slides, first note that there are no points of type 'd' or 'e' (as labeled in Figure 3) near $\partial\Lambda^2$. It suffices then to reparameterize Λ^2 if necessary by a diffeomorphism of the square Λ^2 to itself which is the identity near the boundary. □

4.3 Proof of Theorem 4.1.

Proof. STEP 1: Two handle-slides.

Suppose two families of handle-slides cross at $(\lambda, s) = (0, 0)$. That is, if they lie in $\widehat{\mathcal{M}}^0_{(\lambda_\eta(s),s)}(x, w)$ and $\widehat{\mathcal{M}}^0_{(\lambda_\nu(s),s)}(z, y)$ and represent the elements η and ν in $\pi_1(P)$, then $\lambda_\nu > \lambda_\eta$ for $s < 0$ while $\lambda_\nu < \lambda_\eta$ for $s > 0$. There are several cases to consider: (i) w, x, y, z are distinct, (ii) $w = y$ and/or $x = z$, and (iii) either $x = y$ or $w = z$ but not both.

The goal is to show that $Wh_2(\phi_{\Lambda,s_-}, J_{\Lambda,s_-}) = Wh_2(\phi_{\Lambda,s_+}, J_{\Lambda,s_+})$ for some small $s_- < 0 < s_+$. It suffices to show that the Steinberg word is unchanged. In cases (i) and (ii), the second Steinberg relation

$$h_{wx}(\eta)h_{yz}(\nu)h^{-1}_{wx}(\eta)h^{-1}_{yz}(\nu) = 1\,,$$

indicates that the order of the handle-slides does not matter. There are no cusp-curves in these two cases. Hence, by Gromov compactness and the manifold property of holomorphic curves, $\widehat{\mathcal{M}}^0_{\Lambda,s_-}(p, q) = \widehat{\mathcal{M}}^0_{\Lambda,s_+}(p, q)$. That is, no handle-slides can appear or disappear. Thus, Wh_2 does not change.

In case (iii), the second Steinberg relation does not apply. The presence of a cusp-curve and its subsequent gluing theorem below shows how the order is relevant. Consider the $x = y$ case; the other case is similar.

Define a neighborhood of a cusp-curve as a path in $\Omega(L, L_{\Lambda^2}; \gamma_0)$ in a manner analogous to the one-parameter case:

$$\begin{aligned} &U_\varepsilon(v_1, \ldots, v_k; \lambda_0, s_0) \\ &\quad = \bigl\{(\gamma, \lambda, s) \in \Omega(L, L_{\lambda,s}; \gamma_0) \times \mathbf{R}^2 \bigm| |\lambda - \lambda_0| + |s - s_0| < \varepsilon \,\text{and} \\ &\qquad \max_{t\in[0,1]} \operatorname{dist}(\gamma(t), v_j(\tau, t)) < \varepsilon \,\text{for some}\, \tau \in \mathbf{R},\ \text{and}\ 1 \le j \le k\bigr\}\,. \end{aligned}$$

Theorem 4.4. *Suppose $w, x = y$, and z are as above. Then there exist positive constants ρ_0 and C and a local diffeomorphism*

$$\varpi : \widehat{\mathcal{M}}^0_{(0,0)}(z, x) \times [\rho_0, \infty) \times \widehat{\mathcal{M}}^0_{(0,0)}(x, w) \to \widehat{\mathcal{M}}^0_{\Lambda^2}(z, w).$$

Furthermore, for all $(u,u') \in \widehat{\mathcal{M}}^0_{(0,0)}(z,x) \times \widehat{\mathcal{M}}^0_{(0,0)}(x,w)$*, there exists* $\varepsilon > 0$ *such that* ϖ *is onto* $\widehat{\mathcal{M}}^0_{\Lambda^2}(z,w) \cap U_\varepsilon(u,u';0,0)$.

This is similar to Theorem 3.13. The difference is that the parameter space is one dimension larger and, this time, the curves u and u' to be glued are handle-slides. The proof is nearly identical to the proof for Theorem 4.5, presented in the Appendix, and thus will not be given.

In case (iii), there is one cusp-curve, $(u,u') \in \widehat{\mathcal{M}}^0_{(0,0)}(z,x) \times \widehat{\mathcal{M}}^0_{(0,0)}(x,w)$. So if $\widehat{\mathcal{M}}^0_{(\lambda,s)}(z,w) = \emptyset$ for $s < 0$ then there exists unique $v_{\lambda,s} \in \widehat{\mathcal{M}}^0_{(\lambda,s)}(z,w)$ for $s > 0$ and $\lambda = \lambda(s)$. Furthermore, because these curves converge to the cusp-curve, $\gamma_w \gamma_{v_{(\lambda,s)}} \gamma_z^{-1}$ converges to $\gamma_w \gamma_{u'} \gamma_x^{-1} \gamma_x \gamma_u \gamma_z^{-1}$ as $(\lambda,s) \to (0,0)$; thus, $[\gamma_w \gamma_{v_{(\lambda,s)}} \gamma_z^{-1}] = \eta\nu$. Assume $|s_-| + |s_+| < \varepsilon$ where $\varepsilon = \varepsilon(u,u') > 0$ is the same ε from Theorem 4.4. Then $St(\phi_{\Lambda,s_-}, J_{\Lambda,s_-}) = St(\phi_{\Lambda,s_+}, J_{\Lambda,s_+})$ by the third Steinberg relation:

$$h_{wx}(\eta) h_{xz}(\nu) h^{-1}_{wx}(\eta) h^{-1}_{xz}(\nu) = h_{wz}(\eta\nu)\,.$$

STEP 2: A super handle-slide.

Suppose a super handle-slide occurs at $(\lambda,s) = (0,0)$ then the following gluing theorem holds:

Theorem 4.5. *Suppose that for an isolated* $(0{,}0) \in \Lambda^2$, $\widehat{\mathcal{M}}^{-1}_{(0,0)}(y{,}z) = \{v\}$. *Let* w *and* x *be any other critical points such that* $\mu(x) = \mu(z) = \mu(w) - 1$. *Then there exist positive constants* ρ_0 *and* C *and a local diffeomorphism*

$$\varpi : \widehat{\mathcal{M}}^1_{(0,0)}(x,y) \times [\rho_0,\infty) \times \{v\} \to \widehat{\mathcal{M}}^0_{\Lambda^2}(x,z)\,.$$

Furthermore, for all $u \in \widehat{\mathcal{M}}^1_{(0,0)}(x,y)$*, there exists* $\varepsilon > 0$ *such that* ϖ *is onto* $\widehat{\mathcal{M}}^0_{\Lambda^2}(x,z) \cap U_\varepsilon(u,v;0,0)$*. A similar diffeomorphism exists between* $\widehat{\mathcal{M}}^1_{(0,0)}(z,w) \times [\rho_0,\infty) \times \{v\}$ *and* $\widehat{\mathcal{M}}^0_{\Lambda^2}(y,w)$.

Proof. Define a family of charts $\exp_y : [0,1] \times T_yP \to P$ by $\exp_y(0,T_yL) \subset L$ and $\exp_y(1,T_yL') \subset L'$. Recall the domain of the curves is $\Theta = \mathbf{R}_\tau \times [0,1]_t$. Pick $\xi^u \in C^\infty(\Theta{,}T_yP)$ such that for large enough τ, $u(\tau{,}t) = \exp_y(t{,}\xi^u(\tau{,}t))$. Let $\beta : \mathbf{R} \to \mathbf{R}$ be a smooth function such that

$$\beta(\tau) = 0 \ \text{ for } \ \tau \le 0 \quad \beta(\tau) = 1 \ \text{ for } \ \tau \ge 1 \quad \text{and} \ |\beta'| < 2\,. \tag{30}$$

Define the *pregluing* of u and v, $\varpi_1 : \widehat{\mathcal{M}}^1_{(0,0)}(x,y) \times [\rho_0,\infty) \times \{v\} \to \mathcal{P}(x,z)$ by

$$\begin{aligned} \chi = (u,\rho,v) \mapsto \varpi_1(\chi) &= w_\chi(\tau,t) \\ &= \begin{cases} u(\tau+\rho,t) & \text{if } \tau \le -1 \\ \exp_y\big(t, \beta(-\tau)\xi^u(\tau+\rho,t) + \beta(\tau)\xi^v(\tau-\rho,t)\big) & \text{if } -1 \le \tau \le 1 \\ v(\tau-\rho,t) & \text{if } \tau \ge 1\,. \end{cases} \end{aligned}$$

This is the first step in the gluing process. Although the result is not a holomorphic curve from x to z, by the finiteness of $\widehat{\mathcal{M}}^1_{(0,0)}(x,y)$, there exists $\rho_0, k > 0$ and $\epsilon : \mathbf{R}_+ \to \mathbf{R}_+$ with $\lim_{\rho\to\infty} \epsilon(\rho) = 0$ such that for all $\rho > \rho_0$ and for all $u \in \widehat{\mathcal{M}}^1_{(0,0)}(x,y)$, $\|\bar{\partial}_J w_{(u,\rho,v)}\|_p < k\epsilon(\rho)$. A basic contraction mapping theorem for Fredholm maps, stated in [F1], [Su] and proved in [Sc2], adds a correction term to the preglued curve $(w_\chi, 0, 0) \in \mathcal{P}(x,z;\Lambda^2)$ making it holomorphic. Moreover, the contraction map provides a bijection, ϖ_2, between the preglued terms with small $\bar{\partial}$, and $\bar{\partial}^{-1}(0) \subset \mathcal{P}(x,z;\Lambda^2)$. Recall that the Maslov index is based on the homotopy of a loop in $\mathcal{L}ag(n)$; thus, for ρ large enough, $\operatorname{Ind} E_{w_\chi} = \operatorname{Ind} E_u + \operatorname{Ind} E_v$. Hence, letting $\varpi = \varpi_2 \circ \varpi_1$ gives the required diffeomorphism into $\widehat{\mathcal{M}}^0_{\Lambda^2}(x,z)$. The last step which shows that the curves in the image of ϖ converge to $u \cup v$ uniformly is proved on [F1, p. 532–533]. □

The main difficulty in this and similar gluing situations is to prove that for ρ large enough, $E_{(\varpi_1(u,\rho,v),0,0)}$ has a right inverse bounded uniformly in ρ. Such a bound is necessary to apply the contraction mapping theorem. Several papers provide such a bound for the periodic orbit version of Floer homology. Floer proves this bound ([F1]) for the unparameterized version of Lagrangian intersections; however, he spends only a few lines on how to extend this to the parameterized version. In proving Theorem 4.6, the Appendix expands on those few lines and corrects some of the errors he makes in the unparameterized version.

To simplify notation, denote $(w_{(u,\rho,v)}, 0, 0) = (w_\chi, 0, 0)$ by $\vec{\chi}$ and let $\rho(\vec{\chi}) = \rho$, $u(\vec{\chi}) = u$, et cetera.

Theorem 4.6. *There exist positive constants C, ρ_0 such that for all $\vec{\chi}$ which satisfy $\rho(\vec{\chi}) > \rho_0$, $E_{\vec{\chi}}$ is invertible with right inverse $G_{\vec{\chi}}$. Furthermore,*

$$\| G_{\vec{\chi}}(\eta) \|_{p,k} \le C \| \eta \|_{p,k-1} \, .$$

Now consider what effects a super handle-slide $v \in \widehat{\mathcal{M}}^{-1}_{(0,0)}(y,z)$ has on the Steinberg word. Let $\alpha = [\gamma_z \gamma_v \gamma_y^{-1}]$. Assume $\widehat{\mathcal{M}}^0_{(\lambda,s)}(x,z) = \emptyset$ for $s < 0$ where x is some third intersection not involved with the super handle-slide. Choose some small $s_+ > 0$, $s_- < 0$ satisfying $|s_+| + |s_-| < \varepsilon$ where ε is from Theorem 4.5. Since $L \cap L_{(0,0)}$ is finite, $s_\pm$ can be chosen independent of x. Pick any $u \in \widehat{\mathcal{M}}^1_{(0,0)}(x,y)$ and let $\beta = [\gamma_y \gamma_u \gamma_x^{-1}]$. As in the gluing theorem for two handle-slides, the cusp-curve $u \cup v$ is the limit of a sequence $v_{(\lambda,s)} \in \widehat{\mathcal{M}}^0_{(\lambda,s)}(x,z)$ with $s > 0$ and $\lambda = \lambda(s)$, since $\widehat{\mathcal{M}}^0_{(\lambda,s)}(x,z) = \emptyset$ for $s < 0$. Since the loops $\gamma_z \gamma_{v_{(\lambda,s)}} \gamma_x^{-1}$ converge to $\gamma_z \gamma_v \gamma_u \gamma_x^{-1}$, as (λ, s)

approaches $(0,0)$, $[\gamma_z \gamma_{v_{(\lambda,s)}} \gamma_x^{-1}] = \alpha\beta$. Since this equality holds for each $u \in \widehat{\mathcal{M}}^1_{(0,0)}(x,y)$, $St_{\mu(x)}(\phi_{\Lambda,s_+}, J_{\Lambda,s_+})$ differs from $St_{\mu(x)}(\phi_{\Lambda,s_-}, J_{\Lambda,s_-})$ by the addition of

$$\prod_{\{x|\mu(x)=\mu(z),x\neq z\}} h_{zx}\big(\alpha\langle d_{(0,0)}y, x\rangle\big)\,.$$

Note that the order of the slides does not matter by the second Steinberg relation. A similar argument shows that $St_{\mu(y)}(\phi_{\Lambda,s_+}, J_{\Lambda,s_+})$ differs from $St_{\mu(y)}(\phi_{\Lambda,s_-}, J_{\Lambda,s_-})$ by the addition of

$$\prod_{\{w|\mu(w)=\mu(y),w\neq y\}} h_{wy}\big(\langle d_{(0,0)}w, z\rangle\alpha\big)\,.$$

Hatcher and Wagoner in [HW] call this change in the Steinberg word the "exchange relation" based on an example they provide in [HW, p. 142–143]. They prove in [HW, p. 156–159], that such a change does not affect the Wh_2 element.

STEP 3: A dovetail.

Suppose a nice dovetail occurs at $(\lambda,s) = (0,0)$ involving $X = \{a,b,b',e\}$. Let ϵ and ϵ' be from Definition 4.2. Suppose no other singularity occurs for $(\lambda,s) \in [-\epsilon,\epsilon] \times [-\epsilon,\epsilon]$. The upside-down story is similar.

Because of property 2 of Definition 4.2, there are no handle-slides connecting points in X with points in $\{L \cap L_{\lambda,s}\} \setminus X$ for $(\lambda,s) \in [-\epsilon,\epsilon] \times [-\epsilon,\epsilon]$. Because of properties 1 and 4 of Definition 4.2, the only handle-slides which might be affected by the dovetail (that is, which might appear or disappear with the dovetail) are of the form $h_{b'b}(\alpha_1), h_{e_1a}(\alpha_2), h_{bb'}(\alpha_3)$. Using the first Steinberg relation, let $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{Z}_2\pi_1(L)$ represent the summed handle-slides between the three possible pairs of points. By the second Steinberg relation the handle-slides can be assumed to occur in the above order. The goal is to show that $Wh_2(\phi_{\Lambda,-\epsilon}, J_{\Lambda,-\epsilon}) = Wh_2(\phi_{\Lambda,\epsilon}, J_{\Lambda,\epsilon})$.

At $s = \epsilon$, assume without loss of generality that the handle-slides occur while $\lambda \in [-\epsilon',\epsilon']$. Let $d_\pm = d_{(\pm\epsilon',\epsilon)}$.

For $i = 1,2,3$, let $f_i : C \to C$ represent the change in chain complexes induced by the handle-slides:

$$f_1(p) = p + \delta_p^{b'}\alpha_1 b\,, \quad f_2(p) = p + \delta_p^{e}\alpha_2 a\,, \quad f_3(p) = p + \delta_p^{b}\alpha_3 b'\,.$$

Here p are the intersection points which generate the chain complex. Since $f_i = f_i^{-1}$, $d_+ = f_3 f_2 f_1 d_- f_1 f_2 f_3$. Property 3 of Definition 4.2 and an easy computation then show

$$\begin{aligned} d_+e &= f_3 f_2 f_1 d_- f_1 f_2 f_3 e \quad \Longrightarrow \\ b' &= b + \alpha_3 b' + \alpha_2 b' + \alpha_2\alpha_1 b + \alpha_2\alpha_1\alpha_3 b' \end{aligned}$$

$$\begin{aligned} d_+a = f_3f_2f_1d_-f_1f_2f_3a \quad &\Longrightarrow \\ b = b' + \alpha_1 b + \alpha_1\alpha_3 b' \,. \end{aligned}$$

This implies $\alpha_1 = \alpha_2 = \alpha_3 = 1$, which is a specific case of

$$h_{b'b}(\nu),\ h_{e_1a}(-\langle d_-e, b\rangle\nu^{-1}),\ h_{bb'}(-\nu^{-1}).$$

Hatcher and Wagoner prove that this particular algebraic change does not affect Wh_2 when they treat an even more general dovetail singularity ([HW, p. 153]).

STEP 4: Other singularities.

Most of the remaining singularities have no effect on the Steinberg word. Switching the order of two births (or two deaths), or a birth (or death) and a handle-slide does not change any algebraic invariants because the births and deaths are independent. If an $h_{xw}(\nu)$ is born at $(0,0)$, then obviously at $s > 0$ there are two of them, which by the first Steinberg relation in $\mathbf{Z}_2$-coefficients does not alter the Steinberg word. The death of two handle-slides (point 'e') is the same. A birth or death of a birth-death pair (point 'd') does not affect the Steinberg word because of Theorem 4.3 (iii). There is something to prove when a birth at $s = 0$ of a death-birth pair occurs. (A death-birth is where for a given $s < 0$, the death proceeds the birth in λ. As s approaches 0, the death and birth converge and cancel each other out for $s > 0$.) The death of a death-birth pair is similar. Although $St(\phi_{\Lambda,s_-}, J_{\Lambda,s_-}) \neq St(\phi_{\Lambda,s_+}, J_{\Lambda,s_+})$, the Wh_2 elements are the same. The algebraic proof is no different than Hatcher and Wagoner's because there are no analytical gluing theorems to apply; thus, it will be omitted.

This finishes the proof of Theorem 4.1. □

5 Some Examples and Concluding Remarks

The following calculations provide some examples of non-trivial Whitehead groups.

Theorem 5.1 ([Co], [W]). *$Wh_1(\mathbf{Z}_5) \neq 0$. In fact, if G is any abelian group that contains an element of order $q \neq 1,2,3,4,6$ then $Wh_1(G) \neq 0$. If H is any group, then $Wh_2(H \times \mathbf{Z}) = Wh_2(H) \oplus Wh_1(H) \oplus (?)$.*

The third summand is unknown in the second result. Wagoner proves this second result for his $R = \mathbf{Z}(H)$ version of Wh_2, but the proof does not change with the current definition. He first shows that $K_2(R[t,t^{-1}]) \cong K_2(R) \oplus K_1(R) \oplus (?)$ where $R[t,t^{-1}]$ is the Laurent polynomial ring with coefficients in an arbitrary ring R. Note that $\mathbf{Z}_2(H \times \mathbf{Z}) = \mathbf{Z}_2(H)[t, t^{-1}]$ under the identification

$$\sum n_i \cdot (g_i, z_i) \leftrightarrow \sum n_i g_i t^{z_i}.$$

Wagoner then proves the theorem by showing that the equivalence on the K_2 level factors through to one on the Wh_2 level. The proof of this factoring through is the same for both definitions of Wh_2.

EXAMPLE 5.2. *A Lagrangian h-cobordism.* Consider any closed manifold, M, of arbitrary dimension, such that $\pi_2(M) = 0$ and $\pi_1(M)$ is a finitely-generated abelian group with elements of finite order $\neq 1, 2, 3, 4, 6$. For example, construct a 3-skeleton with the appropriate homotopy groups, embed it in some $\mathbf{R}^N$ and fill it out (a strong homotopy equivalence). After gluing a copy of it along its boundary, it becomes closed with the same lower homotopy groups.

By the above theorem, $Wh_1(\pi_1(M)) \neq 0$. Suppose $\dim M \geq 5$. By a realization theorem [Co], for any $\tau_0 \neq 0 \in Wh_1(\pi_1(M))$, there exists an h-cobordism, (W, M, M_1) such that

$$H_*\big(C(W, M; \mathbf{Z}_2\pi_1(M))\big) = 0 \text{ and } Wh_1\big(C(W, M; \mathbf{Z}_2\pi_1(M))\big) = \tau_0 .$$

Here $C(Y, X)$ denotes the relative simplicial complex of (Y, X). Suppose $K \subset W$ is a large enough compact set such that $W = K \cup M \times (-\infty, -1]_r \cup M_1 \times [1, \infty)_r$. For any Morse function f on W such that $f|_{(W \setminus K)} = r$, $CM(W, f; \mathbf{Z}_2\pi_1(W)) = C(W, M; \mathbf{Z}_2\pi_1(M))$. Here $CM(X, f)$ is the Morse chain complex of X defined by f.

Embed $W = \{q, 0\} \subset (T^*W, dq \wedge dp)$ as the Lagrangian zero-section. Define the Hamiltonian $H : T^*W \to \mathbf{R}$ by $H(q, p) = f(q)$. If ϕ_t is the associated Hamiltonian and $W_t = \phi_t(W)$ then W_1 is the graph of df. When Floer proves that Floer homology is the same as Morse homology in [F4], he equates the two chain complexes by constructing a specific J and a bijection from the gradient flows of f between two critical points to boundaries of the J-holomorphic curves $u(\tau, 0) \subset W$ between the corresponding intersection points. He does not require any compactness of P. He does state that f must be C^2-small; however, in his proof he only requires that $|\nabla\nabla f|$ be small. This can be accomplished by scaling f. This provides the corresponding equivalence between Morse chain complexes with $\mathbf{Z}_2\pi_1(W)$ coefficients and $CF(W, W_1, J)$. Note that $\pi_2(T^*W, W) = 0$ and $\pi_1(W) = \pi_1(M)$; thus, (W, W_1) are admissible. Since $Wh_1(CF(W, W_1, J; \mathbf{Z}_2\pi_1(W))) = \tau_0 \neq 0$, by the above theorem, W_1 can never be separated from W by a compact Hamiltonian isotopy. For an h-cobordism, $\mathbf{Z}_2$-Morse homology is zero, so $\mathbf{Z}_2$-Floer homology does not detect this.

As a final remark, assume that W embeds as a Lagrangian submanifold in some arbitrary symplectic manifold. Suppose that the Lagrangian neighborhood theorem provides W with a non-thinning-out neighborhood symplectomorphic to T^*W, despite the non-compactness of W. Then the above example can be applied to this case. Again construct f small enough so that the graph of its derivative (W_1) remains in the Lagrangian neighborhood symplectomorphic to T^*W. Then the Whitehead torsion obstructs compact deformations of W_1 off of W within the full symplectic manifold, P.

EXAMPLE 5.3. *A realization of non-trivial Wh_2 elements.* Construct M as in Example 5.2 but this time assume at least one copy of $\mathbf{Z}$ in $\pi_1(M)$. Let $W = M \times \mathbf{R}$. By the above theorem, $Wh_2(\pi_1(W)) \neq 0$. Any non-trivial element $Y \in Wh_2(\pi_1(W))$ can be realized by a function $f : W \to \mathbf{R}$ which, outside of a compact set, is the projection onto the $\mathbf{R}$-component of W, [HW]. Scale f to get the bound on the second derivative needed in [F4]. Let W_1 denote the graph of df and W_0 denote the graph of the derivative of the projection, $\mathrm{pr}_t : W \to \mathbf{R}$. W_0 and W_1 are deformations of W under the Hamiltonian isotopies generated by the Hamiltonian vector fields $X_{H_{\mathrm{pr}_t}}$ and X_{H_f}, respectively. Although these isotopies are not compact, they agree outside a compact set; thus, composing one with the inverse of the other provides a compactly supported Hamiltonian isotopy ϕ_Λ taking (say) W_0 to W_1. Note that with an appropriate perturbation of ϕ_Λ and choice of J_Λ, $Wh_2(\phi_\Lambda, J_\Lambda) = Y \neq 0$. Thus by the above theorem, ϕ_Λ can never be deformed into some ϕ'_Λ taking W_0 to W_1 such that for all λ, $\phi'_\lambda(W_0) \cap W = \emptyset$. Again, this obstruction applies to a more general symplectic manifold P in which an embedded W is Lagrangian, provided there exists a non-thin cotangent neighborhood of W.

EXAMPLE 5.4. *π_0 of the space of fixed-point-free compact Hamiltonian deformations of a shift.* Consider M as in Example 5.2. Let $(V, \omega) = (\mathbf{R}^1_t \times S^1_\theta \times T^*M, dt \wedge d\theta + \omega_0)$ where ω_0 is the standard symplectic form on the cotangent bundle. By Theorem 5.1, $Wh_2(\pi_1(V)) \neq 0$. Let $S_c : V \to V$ denote the non-trivial Hamiltonian shift by $c \in S^1$ in the S^1_θ-direction. Let $\mathcal{G}$ denote the set of Hamiltonian diffeomorphisms of V of the form $f \circ S_c$ where f has compact support and $f \circ S_c$ has no fixed points. Then a corollary of Theorem 4.1 is that $\pi_0(\mathcal{G}) \neq 0$.

Embed $V \hookrightarrow (V \times V, \omega \oplus -\omega)$ as the Lagrangian diagonal $\mathcal{D}$. Let $Gr_{S_c} \subset (V \times V, \omega \oplus -\omega)$ be the Lagrangian graph of S_c. Although $\mathcal{D}$ is not compact in either the x-direction ($x \in T^*M$) or the (t, θ)-direction, there exists a neighborhood $N \supset \mathcal{D}$ symplectomorphic to $T^*\mathcal{D}$ which contains Gr_{S_c}. This follows because $Gr_{S_c} = \{t, \theta, x, t, \theta + c, x\}$ and hence N need only remain 'thick' in the $T^*(\mathbf{R}^1_t, S^1_\theta)$-coordinates of $T^*\mathcal{D}$. Such a requirement holds trivially because of the translation invariance of $\mathcal{D}$ in the (t,θ)-direction. Note that Gr_{S_c} is graphical in $T^*\mathcal{D}$; that is, it is the graph of the derivative of some $g : \mathcal{D} \to \mathbf{R}$ standard outside a compact set.

As in Example 5.3, choose some $h : \mathcal{D} \to \mathbf{R}$ which agrees with g outside a compact set, has appropriate derivative bounds, has no critical points and represents a non-trivial element in $Wh_2(\pi_1(\mathcal{D}))$. Denote by $W \subset N$ the Lagrangian graph of dh. This can be done since $h(t,\theta,v) = h(t,\theta)$ outside a compact set. By Theorem 4.1, the obvious deformation of W to Gr_{S_c} cannot be deformed to avoid $\mathcal{D}$. Thus W corresponds to an element in $\mathcal{G}$ which lies in a path-connected component that does not contain S_c.

Remark 5.5. *Coherent orientation and* **Z***-coefficients.* Floer and Hofer in [FH], use 'coherent orientation' to introduce **Z**-coefficients for the periodic orbits version of Floer homology. This homology theory addresses Arnold's conjecture on fixed points. Fukaya, Oh, Ohta and Ono [FuOOO] have recently developed coherent orientation for the Lagrangian intersection version. There are additional hypotheses which are needed in this situation but are unnecessary in [FH]. Essentially, if $i : L \to P$ represents the inclusion map and $w_2(TL) \in H^2(L;\mathbf{Z}_2)$ is the second Stiefel–Whitney class of L, then one must assume that $w_2(TL) = i^*a$ for some $a \in H^2(P;\mathbf{Z}_2)$. With coherent orientation, Floer homology can be easily defined with $\mathbf{Z}\pi_1$-coefficients. Assuming the determinant bundles (a section of which orients the moduli space of holomorphic curves) can incorporate the parameter space, then **Z**-versions of Theorems 3.13, 4.4 and 4.5 hold. The rest of the proofs for **Z**-versions of Theorems 3.1 and 4.1 then readily follow.

Remark 5.6. *Degenerate gluing instead of stabilization.* There is an alternative proof to Theorem 3.1 (but not Theorem 4.1) in [Su] which does not use stabilization. It relies on gluing theorems and a version of Theorem 2.1 for degenerate intersections. Essentially, if the birth (or death) of (x,y) is not independent, Floer proves a gluing theorem which establishes a bijection between $\widehat{\mathcal{M}}^1_-(p,q)$ and $\widehat{\mathcal{M}}^1_+(p,y) \times \widehat{\mathcal{M}}^1_+(x,q)$ [F1]. Unfortunately, as mentioned in section 1.1, Floer's complicated analysis is in some places incorrect or incomplete. A partial completion of this degenerate intersection theory can be found in [Su].

Appendix A. Proof of Theorem 3.3

Proof. STEP 1: Constructing a birth at $(q,x)=(0,C)$ in the linear case.

Let $F(q,x)=F_0(q,x)=\epsilon_1 q-x^2$ from equation (17) and rewrite it as the sum of a quadratic and a linear function

$$\begin{aligned} F(q,x) &= -C^2+\epsilon_1 q-2C(x-C)-(x-C)^2 \\ &= l(q,x)-(x-C)^2 . \end{aligned} \tag{31}$$

STEP 1 constructs a bump function, growing with λ, which when added to $l(q,x)$ results in the birth of two critical points for $l(q,x)$ near $(q,x)=(0,C)$.

Let $C'=\frac{1}{2}(\epsilon_1^2+(2C)^2)^{1/2}$. Let $A\in O(2,\mathbf{R})$ be a linear change of coordinates from (q,x) to (r,w) defined by

$$A=\frac{1}{2C'}\begin{pmatrix} 2C & \epsilon_1 \\ \epsilon_1 & -2C \end{pmatrix}. \tag{32}$$

The r and w directions are easily verified to be the level set and gradient directions of l, respectively. Let $(r_0,w_0)=A(0,C)$. If $\dim P=2n>2$, the change of coordinates is done so that w,r_1 are again q_1,x transformed by A, while $r_i=q_i$ for $2\le i\le n$. Note that the linear term in equation (31) becomes

$$l(r,w)=-C^2+2C'w\,. \tag{33}$$

Let $M=\big(C'+\frac{1}{2}\epsilon_1\big)/\epsilon_1$. Let $j:\big[0,\frac{1}{2}\big]\to[0,1]$ be a smooth strictly increasing function such that

$$j(0)=0\,,\quad j\left(\frac{1}{2}\epsilon_1\right)=\frac{C'}{C'+\frac{1}{2}\epsilon_1}\ \text{and}\ j\left(\frac{1}{2}\right)=1\,. \tag{34}$$

Perturb the function from equation (33) to be

$$l_\lambda(r,w)=-C^2+2C'w+j(\lambda)\epsilon_1\sigma_2\big(M(w-w_0)\big)\sigma_1(r-r_0)\,. \tag{35}$$

For the $2n>2$ case, replace the last term $\sigma_1(r-r_0)$ by $\sigma_1(((r_1-r_0)^2+(r_2-0)^2+\ldots+(r_n-0)^2)^{1/2})$.

LEMMA A.1. *For $\lambda\in\big[0,\frac{1}{2}\epsilon_1\big)$, l_λ has no critical points. At $\lambda=\frac{1}{2}\epsilon_1$, l_λ has a degenerate critical point at (r_0,w_0). For some λ_0, $\frac{1}{2}>\lambda_0>\frac{1}{2}\epsilon_1$, $\lambda\in\big(\frac{1}{2}\epsilon_1,\lambda_0\big]$ implies l_λ has exactly two nondegenerate critical points.*

Proof.

$$\tfrac{\partial l_\lambda}{\partial r}=j(\lambda)\epsilon_1\sigma_2\big(M(w-w_0)\big)\sigma_1'(r-r_0)$$

which, by the definition of σ_1 and σ_2 vanishes only when

$$|r-r_0|\ge 1\,,\quad r=r_0\,,\quad w-w_0\ge\tfrac{1}{2M}\ \text{or}\ w-w_0\le-\tfrac{3}{2M}\,. \tag{36}$$

See Figure 14. Whereas,

$$\tfrac{\partial l_\lambda}{\partial w}=2C'+j(\lambda)\epsilon_1 M\sigma_2'\big(M(w-w_0)\big)\sigma_1(r-r_0)\,.$$

For $\lambda \in [0,1]$ define

$$S(\lambda) = \left(\frac{\partial l_\lambda}{\partial w}\right)^{-1}(0)$$
$$= \left\{ (r,w) \;\Big|\; -\sigma_2'(M(w-w_0))\sigma_1(r-r_0) = \frac{2C'}{j(\lambda)\big(C'+\frac{1}{2}\epsilon_1\big)} \right\}. \quad (37)$$

The intersection of the set in equation (36) with the set in equation (37) represents the critical points of l_λ.

For $\lambda \in \big[0, \frac{1}{2}\epsilon_1\big)$, $j(\lambda) < C'/\big(C'+\frac{1}{2}\epsilon_1\big)$ (from equation (34)) implies that $2C'/j(\lambda)\big(C'+\frac{1}{2}\epsilon_1\big) > 2$, but $|\sigma_2'(M(w-w_0))\sigma_1(r-r_0)| \le 2$ by the definition of σ_1 and σ_2. Thus, $S(\lambda) = \emptyset$ when $\lambda \in \big[0, \frac{1}{2}\epsilon_1\big)$. This proves the first statement of the lemma.

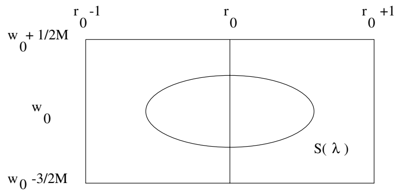

Figure 14: The zero sets in equations (36) and (37).

Next consider $S(\lambda)$ for $\lambda \ge \frac{1}{2}\epsilon_1$. Note that $2C'/j(\lambda)\big(C'+\frac{1}{2}\epsilon_1\big)$ monotonically decreases in λ, with $\lim_{\lambda\to(\frac{1}{2}\epsilon_1)^+} 2C'/j(\lambda)\big(C'+\frac{1}{2}\epsilon_1\big) = 2$. Furthermore, the graph of $-\sigma_1(r-r_0)\sigma_2'(M(w-w_0))$ has an isolated global maximum of 2 at (r_0,w_0). Thus as λ increases, $S(\lambda)$ represents the decreasing level sets of the graphs. The first non-trivial level set is $S\big(\frac{1}{2}\epsilon_1\big) = \{(r_0,w_0)\}$. This proves the second statement of the lemma.

Recall that σ_1 is globally even while σ_2' is locally even. Thus for some $\lambda_0 > \frac{1}{2}\epsilon_1$, if $\lambda \in \big(\frac{1}{2}\epsilon_1, \lambda_0\big]$ then $S(\lambda)$ is a small embedded closed curve, symmetric in $w-w_0$ and $r-r_0$, and contained in the set

$$\big\{(r,w) \;\big|\; |r-r_0| < 1\,,\ \tfrac{1}{2M} > w-w_0 > \tfrac{-3}{2M}\big\}.$$

See Figure 14. Thus the intersection of the sets in equations (36) and (37) consists of exactly two points. □

STEP 2: A bound on the partial derivatives.

On the linear level, l has been deformed to l_λ by the addition of a bump function whose support is contained in $\{(r,w) \mid |r-r_0| < 1\,,\ |w-w_0| < 3/2M\} \subset \mathbf{I} \times \mathbf{R}$. To ensure that the support of the perturbation lies in

$\tilde{\kappa}(\tilde{N}) = \{(q,p,x,y) \mid |q| < 3\,,\ |p| < 3\}$, $\partial l_\lambda/\partial q$ must be bounded by 3. Rewrite equation (35) in the original coordinates

$$\begin{aligned} l_\lambda(q,x) &= -C^2 + \epsilon_1 q - 2C(x-C) \\ &\quad + j(\lambda)\epsilon_1\sigma_2\left(M\frac{\epsilon_1 q - 2C(x-C)}{2C'}\right)\sigma_1\left(\frac{2Cq+\epsilon_1(x-C)}{2C'}\right). \end{aligned} \tag{38}$$

Let $k = 2 \geq \max(|\sigma_1|, |\sigma_1'|, |\sigma_2|, |\sigma_2'|)$. Then

$$\begin{aligned} \left|\frac{\partial l_\lambda}{\partial q}\right| &\leq \epsilon_1 + j(\lambda)\epsilon_1\left(M\tfrac{\epsilon_1}{2C'}k^2 + \tfrac{2C}{2C'}k^2\right) \\ &\leq \epsilon_1 + k^2\left(\epsilon_1\tfrac{3C}{2C'} + \epsilon_1\tfrac{2C}{2C'}\right). \end{aligned} \tag{39}$$

Since $C' > C$, it suffices to choose $\epsilon_1 < 3/\big(1+\big(\frac{3}{2}+\frac{2}{2}\big)k^2\big)$. Since $\sigma_2(M(w-w_0))$ is not a function of $q_2,\dots,q_n$, this choice works for the case $2n > 2$ as well, i.e. $|\partial l_\lambda/\partial q_i|$ is appropriately bounded. Note that the choice of ϵ_1 determines C' and M.

STEP 3: Perturbing the non-linear case.

To emphasize the homogeneity, rewrite the bump function from equation (38) as

$$\sigma(\lambda,q,x-C) = j(\lambda)\epsilon_1\sigma_2\left(M\frac{\epsilon_1 q - 2C(x-C)}{2C'}\right)\sigma_1\left(\frac{2Cq+\epsilon_1(x-C)}{2C'}\right) \tag{40}$$

and the linear part of l_λ as

$$l_h(q,x-C) = -C^2 + \epsilon_1 q - 2C(x-C)\,. \tag{41}$$

Lemma A.2. *Since $l_h(q,x-C)+\sigma(\lambda,q,x-C)$ has two newly born critical points near $(q,x) = (0,C)$, there exists $\epsilon_2 > 0$ such that $l_h(q,x-C)+\epsilon_2\sigma\big(\lambda,\frac{q}{\epsilon_2},\frac{x-C}{\epsilon_2}\big)-(x-C)^2$ also has exactly two critical points near $(0,C)$.*

Proof. Note that $l_h(q,x-C) = \epsilon_2 l_h\big(\frac{q}{\epsilon_2},\frac{x-C}{\epsilon_2}\big)$ while $(x-C)^2 = \epsilon_2\big(\epsilon_2(\frac{x-C}{\epsilon_2})^2\big)$. Thus, scaling down by a factor of ϵ_2, the added $(x-C)^2$ term effectively is made C^2-small while the linear part is unchanged. And so, after adding an appropriately scaled down bump function to create critical points, the C^2-small quadratic term does not affect the Morse structure of the function.

To prove the lemma more rigorously, it suffices to take partial derivatives after changing coordinates to $(\bar{q},\bar{x}) = (q/\epsilon_2, x-C/\epsilon_2)$.

$$\begin{aligned} &\tfrac{\partial}{\partial q}\big(l_h(q,x-C)+\sigma(\lambda,q,x-C)-(x-C)^2\big) \\ &\quad = \tfrac{\partial}{\partial\bar{q}}\big(l_h(\bar{q},\bar{x})+\sigma(\lambda,\bar{q},\bar{x})\big) \end{aligned} \tag{42}$$

$$\tfrac{\partial}{\partial x}\big(l_h(q,x-C)+\sigma(\lambda,q,x-C)-(x-C)^2\big) \\ = \tfrac{\partial}{\partial \bar{x}}\big(l_h(\bar{q},\bar{x})+\sigma(\lambda,\bar{q},\bar{x})\big)-2\epsilon_2\bar{x}\,. \tag{43}$$

Thus the size of the change of the zero set is controlled by ϵ_2 and can be made small enough so that Lemma A.1 holds for the non-linear function. (Perhaps λ_0 must decrease a bit, but the non-empty interval $\big(\frac{1}{2}\epsilon_1,\lambda_0\big]$ in Lemma A.1 still exists for small enough ϵ_2.) □

Using Lemma A.2, define for $\lambda\in[0,\lambda_0]$

$$\begin{aligned}G_\lambda(q,x)&=F_0(q,x)+\epsilon_2\sigma\left(\lambda,\frac{q}{\epsilon_2},\frac{x-C}{\epsilon_2}\right)\\ &=-C^2+\epsilon_1 q-2C(x-C)-(x-C)^2+\epsilon_2\sigma\left(\lambda,\frac{q}{\epsilon_2},\frac{x-C}{\epsilon_2}\right)\\ &=-C^2+\epsilon_1 q-2C(x-C)-(x-C)^2\\ &\quad+\epsilon_2 j(\lambda)\epsilon_1\sigma_2\left(M\frac{\epsilon_1\frac{q}{\epsilon_2}-2C\frac{(x-C)}{\epsilon_2}}{2C'}\right)\sigma_1\left(\frac{2C\frac{q}{\epsilon_2}+\epsilon_1\frac{(x-C)}{\epsilon_2}}{2C'}\right).\end{aligned} \tag{44}$$

Note that the bound in equation (39) still applies to G_λ because of equation (42). So $G_\lambda=F_0$, in a neighborhood of $\partial(\mathbf{I}\times\mathbf{R})$, the birth of the two critical points of G occurs within ϵ_2 of $(0,C)$ and $|\partial G_\lambda/\partial q_i|<3$.

Now to prove property 5 of the theorem. The definition of σ in equation (40) implies that $\mathrm{supp}(\sigma)\subset(-3,3)\times(-K,K)$ where $K=K(M,\epsilon_1,C)<\infty$. Choose $\epsilon_2\ll K^{-1}$. Then for $\lambda\in[0,\lambda_0]$ (which includes the moment of birth), equation (44) implies

$$G_\lambda(q,x)-F_0(q,x)=\epsilon_2\sigma\left(\lambda,\frac{q}{\epsilon_2},\frac{x-C}{\epsilon_2}\right)$$

which has support in $(-3\epsilon_2,3\epsilon_2)\times(-K\epsilon_2+C,K\epsilon_2+C)\subset(-1,1)\times(C-1,C+1)$.

STEPS 4 and 5 will construct G_λ such that $G_1=F_1$ without disrupting the above properties of G_λ.

STEP 4: Sliding the pair of critical points back to $x=0$.

Let $b\in\mathbf{I}\times\mathbf{R}$ denote the midpoint between the two critical points of G_{λ_0}. So $b\approx(0,C)$. Let $\gamma:[\lambda_0,3/4]\to\mathbf{I}\times\mathbf{R}$ denote the line-segment starting at b and ending at $(0,0)$. Write $\gamma(\lambda)=(\gamma_q(\lambda),\gamma_x(\lambda))$.

For $\lambda\in\big[\lambda_0,\frac{3}{4}\big]$, define $C(\lambda)=\gamma_x(\lambda)$, $C'(\lambda)=\frac{1}{2}(2C(\lambda)^2+\epsilon_1)^{1/2}$ and $M(\lambda)=\big(C'(\lambda)+\frac{1}{2}\epsilon_1\big)/\epsilon_1$. Replace C,C' and M in equation (44) by $C(\lambda), C'(\lambda)$ and $M(\lambda)$ to define G_λ for $\lambda \in [\lambda_0, 3/4]$.

$$G_\lambda(q,x) = -C(\lambda)^2 + \epsilon_1 q - 2C(\lambda)\big(x - C(\lambda)\big) - \big(x - C(\lambda)\big)^2$$
$$+ \epsilon_2 j(\lambda_0)\epsilon_1\sigma_2\Bigg(M(\lambda)\frac{\epsilon_1\frac{q}{\epsilon_2} - 2C(\lambda)\frac{(x-C(\lambda))}{\epsilon_2}}{2C'(\lambda)}\Bigg)\sigma_1\Bigg(\frac{2C(\lambda)\frac{q}{\epsilon_2} + \epsilon_1\frac{(x-C(\lambda))}{\epsilon_2}}{2C'(\lambda)}\Bigg). \tag{45}$$

Repeat bounding of the partial in STEP 2 and the scaling down in STEP 3 as λ increases from λ_0 to $3/4$. No additional births or deaths occur because the dynamics of the zero sets (equations (36), (37) where the coordinates r and w now vary with λ according to equation (32)) are unchanged. The constants ϵ_1, ϵ_2 might need to be made smaller, but the partial derivatives vary continuously with λ in the compact set $[\lambda_0, 3/4]$; thus, there exist $\epsilon_i > 0$ which keep the partials appropriately bounded.

To see intuitively how no other births or deaths occur in STEP 4, note that $\nabla(\epsilon_1 q + x^2)(\gamma(\lambda))$ always approximates the line connecting the two critical points (see Figure 1).

STEP 5: Making $G_1 = F_1$.

Note that
$$G_{3/4}(q,x) = \epsilon_1 q - x^2 + \epsilon_2 j(\lambda_0)\epsilon_1\sigma_2\left(\tfrac{q}{\epsilon_2}\right)\sigma_1\left(\tfrac{x}{\epsilon_2}\right) \tag{46}$$

For $i = 1,2,3$, define $j_i : [3/4, 1] \to \mathbf{R}$ to be smooth strictly monotonic functions such that
$$j_1\left(\tfrac{3}{4}\right) = \epsilon_2 j(\lambda_0)\epsilon_1\,,\ j_2\left(\tfrac{3}{4}\right) = j_3\left(\tfrac{3}{4}\right) = \tfrac{1}{\epsilon_2}\,,\ j_1(1) = j_2(1) = 1\,,\ j_3(1) = 0\,.$$
And for $\lambda \in [3/4, 1]$, define
$$G_\lambda(q,x) = \epsilon_1 q - x^2 + j_1(\lambda)\sigma_2(j_2(\lambda)q)\sigma_1(j_3(\lambda)x)\,.$$
It it easy to verify that $G_1 = F_1$ and G_λ satisfies all the required properties, as in STEP 4. □

Appendix B. Proof of Theorem 4.6

Proof. Construct a map $\varpi_\chi : \ker E_u \oplus \ker E_v \to T_{w_\chi}\mathcal{P}(x,z)$ by
$$\varpi_\chi(\xi,\eta) = \begin{cases} \beta(-\tau-1)\xi(\tau+\rho,t) & \text{if } \tau \le -1\,, \\ 0 & \text{if } -1 \le \tau \le 1\,, \\ \beta(\tau-1)\eta(\tau-\rho,t) & \text{if } \tau \ge 1\,. \end{cases}$$
Here β is defined in equation (30). Note that as $\rho(\vec{\chi}) \to \infty$, $\varpi_\chi(\ker E_u \oplus \ker E_v)$ converges to $\ker E_{w_\chi}$. Define $A(w_\chi) \subset T_{\vec{\chi}}\mathcal{P}(x,z;\Lambda^2)$ to be
$$A(w_\chi) = \big\{(\xi,\mu_1,\mu_2)\ \big|\ \langle\xi,\eta\rangle_{L^2} = 0\ \forall\,\eta \in \varpi_\chi(\ker E_u \oplus \ker E_v)\big\}\,.$$

The main inequality to show is that for all $(\xi,\mu_1,\mu_2)\in A(w_\chi)$,

$$\|\,\xi\,\|_{p,k}+|\mu_1|+|\mu_2|\le C\big\|\vec{\chi}(\xi+\mu_1X_1+\mu_2X_2)\big\|_{p,k-1} \tag{47}$$

for some constant C. Here $X_1,X_2\in C^\infty(\Theta,w_\chi^*(TP))$ are the vector fields

$$X_1=\tfrac{\partial\phi_{\lambda,s}}{\partial\lambda}\big|_{\lambda,s=0}\,,\quad X_2=\tfrac{\partial\phi_{\lambda,s}}{\partial s}\big|_{\lambda,s=0}\,. \tag{48}$$

Note that $\mathrm{Dom}\,E_{\vec{\chi}}=\mathrm{Dom}\,E_{w_\chi}\oplus\mathbf{R}^2$, $\mathrm{Range}\,E_{\vec{\chi}}=\mathrm{Range}\,E_{w_\chi}$, and

$$\begin{aligned}&\dim\ker E_{w_\chi}-\dim\operatorname{coker}E_{w_\chi}\\&\quad=\dim\ker E_u-\dim\operatorname{coker}E_u+\dim\ker E_v-\dim\operatorname{coker}E_v\\&\quad=\dim\ker E_u+\dim\ker E_v-2\\&\quad\ge\dim\ker E_{w_\chi}-2\,.\end{aligned}$$

The last inequality follows because equation (47) implies that

$$\|\,\xi\,\|_{p,k}\le C\big\|E_{\vec{\chi}}(\xi,0,0)\big\|_{p,k-1}=C\|\,E_{w_\chi}(\xi)\,\|_{p,k-1}$$

for any $(\xi,0,0)\in A(w_\chi)$. Further note that equation (47) implies $\ker E_{\vec{\chi}}\subset\mathrm{Im}\,\varpi_\chi$ which converges to $\ker E_{w_\chi}$; thus, $\dim\ker E_{w_\chi}=\dim\ker E_{\vec{\chi}}$. $E_{\vec{\chi}}$ is therefore surjective because $\dim\mathrm{Dom}\,E_{\vec{\chi}} = \dim\mathrm{Dom}\,E_{w_\chi}+2$ and $\dim\operatorname{coker}E_{w_\chi}\le 2$.

Let $G_{\vec{\chi}}$ denote the right inverse of $E_{\vec{\chi}}$. Suppose $G_{\vec{\chi}}(\eta)=\xi+\mu_1X_1+\mu_2X_2$. Then

$$\begin{aligned}\|\,G_{\vec{\chi}}(\eta)\,\|_{p,k}&=\|\,\xi\,\|_{p,k-1}+|\mu_1|+|\mu_2|\le C\big\|E_{\vec{\chi}}(\xi+\mu_1X_1+\mu_2X_2)\big\|_{p,k-1}\\&=C\|\,E_{\vec{\chi}}G_{\vec{\chi}}\eta\,\|_{p,k-1}=C\|\,\eta\,\|_{p,k-1}\,.\end{aligned}$$

This shows that Theorem 4.6 follows from equation (47). Now to prove equation (47) by contradiction.

Consider a family $\chi_n=(u_n,\rho_n,v)$ and $(\xi_n,\mu_n,\nu_n)\in T_{\vec{\chi}_n}\mathcal{P}(x,z;\Lambda^2)$ such that $\rho_n\to\infty$ and

$$\|\,\xi_n\,\|_{p,k}+|\mu_n|+|\nu_n|=1\,,\quad \big\|E_{\vec{\chi}_n}(\xi_n+\mu_nX_1+\nu_nX_2)\big\|_{p,k-1}\to 0\,. \tag{49}$$

By the finiteness of $\widehat{\mathcal{M}}^1_{(0,0)}(x,y)$, pass to a subsequence and assume that $u_n=u$. Lemma B.1 ensures that although $\dim\operatorname{coker}\ E_v=2$, $\operatorname{coker}\ E_{(v,0,0)}=\emptyset$.

Lemma B.1. *Recall that at $(\lambda,s)=(0,0)$, there exists a super handle-slide, v, connecting two intersection points of L and $L'_{(0,0)}=L'$. The two-parameter family $L'_{(\lambda,s)}$ can be generically perturbed in a neighborhood of $(0,0)\in\Lambda^2$ and a neighborhood $U\subset P$ of the image of the curve, fixing $L'_{(0,0)}$ (so that the super handle-slide, v, still appears isolated at $(0,0)$), so that $E_{(v,0,0)}$ is surjective.*

Proof. Given any $w \in \mathcal{P}(x_{\lambda,s}, z_{\lambda,s})$, note that $\phi^{-1}_{\lambda t,st}(w(\tau,t))$ has boundary values in L and $L'_{0,0}$. Redefine the parameterized Cauchy–Riemann equation so that it has fixed boundary conditions. Let

$g(w,\lambda,s) = \bar\partial\big(\phi_{\lambda t,st}(w),\lambda,s\big) = \frac{\partial}{\partial\tau}\big(\phi_{\lambda t,st}(w)\big) + J_{\lambda,s}\big(\phi_{\lambda t,st}(w),t\big)\frac{\partial}{\partial t}\big(\phi_{\lambda t,st}(w)\big)$.

Since $\phi_{0\cdot t,0\cdot t}(w) = w$, note that $Dg_{(w,0,0)}(\xi + 0X_1 + 0X_2) = E_w(\xi)$. On the other hand,

$$\begin{aligned}
&Dg_{(w,0,0)}(0 + \mu X_1 + \nu X_2)\\
&\quad= \nabla_\tau(t\mu X_1 + t\nu X_2) + J_{0,0}(w,t)\nabla_t(t\mu X_1 + t\nu X_2)\\
&\qquad+ (\nabla_{(t\mu X_1 + t\nu X_2)}J_{0,0})(w,t)\left(\tfrac{\partial w}{\partial t}\right) + \left(\tfrac{\partial}{\partial\lambda}J_{0,\lambda}(w,t)\right)\big|_{\lambda=0}\left(\tfrac{\partial w}{\partial t}\right)\\
&\qquad\quad+ \left(\tfrac{\partial}{\partial s}J_{s,0}(w,t)\right)\big|_{s=0}\left(\tfrac{\partial w}{\partial t}\right)\\
&\quad= t\big(\nabla_\tau(\mu X_1 + \nu X_2) + J_{0,0}(w,t)\nabla_t(\mu X_1 + \nu X_2)\\
&\qquad+ (\nabla_{(\mu X_1 + \nu X_2)}J_{0,0})(w,t)\left(\tfrac{\partial w}{\partial t}\right)\big) + Y + J_{0,0}(w,t)(\mu X_1 + \nu X_2)\\
&\quad= tE_u(\mu X_1 + \nu X_2) + Y + J_{0,0}(w,t)(\mu X_1 + \nu X_2)
\end{aligned}$$

where $Y = \left(\frac{\partial}{\partial\lambda}J_{0,\lambda}(w,t)\right)\big|_{\lambda=0}\left(\frac{\partial w}{\partial t}\right) + \left(\frac{\partial}{\partial s}J_{s,0}(w,t)\right)\big|_{s=0}\left(\frac{\partial w}{\partial t}\right)$ is a vector field determined only by w and the family of almost complex structures, J_{Λ^2}.

If $w = v$ is the super handle-slide, then choose X_1 and X_2 such that $-Y + JX_1$ and $-Y + JX_2$ span the two-dimensional $\operatorname{coker} E_v$. (If this were a proof of Theorem 4.4, choose each X_i such that $-Y + JX_i$ span the two one-dimensional cokernels of the two handle-slides.) This choice is consistent with a generic choice of $\phi_{\lambda,s}$.

To see that the surjectivity of $Dg_{(u,0,0)}$ implies that of $E_{(v,0,0)} = D\bar\partial_{(v,0,0)}$, note that $h_{\lambda,s}(w) = \phi^{-1}_{\lambda t,st}(w)$ is a local diffeomorphism. Because $\bar\partial(w,\lambda,s) = g(h_{\lambda,s}(w),\lambda,s)$, the chain rule says

$$D\bar\partial_{(v,0,0)} = Dg_{(v,0,0)} \cdot \begin{pmatrix} Dh_{\lambda,s_v} & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Dh_{λ,s_v} is invertible as the linearization of a diffeomorphism; thus, the larger matrix is invertible as well. □

Let $\beta_n(\tau) = \beta(\tau - \rho'_n)$ for some $\rho'_n \in \left[\frac{1}{2}\rho_n, \rho_n\right]$. Let $\eta_n(\tau,t) = \beta_n(\tau)\xi_n(\tau,t)$ and $Y_i^n = \beta_n(\tau)X_i$ for $i = 1,2$. Then

$$\begin{aligned}
&\big\|E_{\vec\chi}(\eta_n + \mu_n Y_1^n + \nu_n Y_2^n)\big\|_{p,k-1}\\
&\quad\le \big\|1 \cdot E_{\vec\chi}(\xi_n + \mu_n X_1 + \nu_n X_2)\big\|_{p,k-1} + \big\|\beta'_n(\xi_n + \mu_n X_1 + \nu_n X_2)\big\|_{p,k-1}\\
&\quad\le \varepsilon(n) + \big\|2 \cdot (\xi_n + \mu_n X_1 + \nu_n X_2)\big|_{[\rho'_n,\rho'_{n+1}]\times[0,1]}\big\|_{p,k-1}.
\end{aligned}$$

The first term converges to zero from equation (49), while the second term converges to zero for some appropriate choice of ρ'_n. Define $\hat{\eta}(\tau,t) = \eta(\tau-\rho_n,t)$ and $\hat{Y}_i^n(\tau,t) = Y_i^n(\tau-\rho_n,t)$ for $i=1,2$. These vector fields can be considered as sections of v^*TP. Furthermore,

$$\begin{aligned}
&\left\|E_{\vec{\chi}}(\eta_n+\mu_n Y_1^n+\nu_n Y_2^n)\right\|_{p,k-1}\to 0 \Longrightarrow\\
&\left\|E_{(v,0,0)}(\hat{\eta}_n+\mu_n\hat{Y}_1^n+\nu_n\hat{Y}_2^n)\right\|_{p,k-1}\to 0 \Longrightarrow\\
&\left\|\mu_n E_{(v,0,0)}(\hat{Y}_1^n)\right\|_{p,k-1},\left\|\nu_n E_{(v,0,0)}(\hat{Y}_2^n)\right\|_{p,k-1}\to 0 \quad \Longrightarrow \mu_n,\,\nu_n\to 0\,. \qquad (50)
\end{aligned}$$

To get from the second convergence to the third one, note that the proof of the surjectivity of $E_{(v,0,0)}$ ensures that $E_{(v,0,0)}(\hat{Y}_i)$ lies outside the range of E_v. To get from the third one to the last, note that the reparameterization keeps the support of $\hat{Y}_i^n$ in a fixed region.

Now that $\mu_n,\nu_n\to 0$, the remainder of the proof is nearly identical to Floer's proof for the unparameterized gluing theorem. To be complete, however, it is presented here with a few extra details.

The idea is to split ξ_n into three parts concentrated over $u(\Theta)$, y, and $v(\Theta)$ and show that the (p,k)-norm on each vanishes. That fact, coupled with equation (50) will contradict equation (49). Pick a small enough neighborhood $U\subset P$ of y such that for any $w:\Theta\to U$ with Lagrangian boundary conditions, there exists a section ζ of T_yP satisfying $w(\tau,t) = \exp_y(t,\zeta(\tau,t))$. Recall that the image of w_{χ_n} converges to that of $u\cup v$. Furthermore, due to the increasing shifts towards y during the pregluing, there exists a sequence of shrinking neighborhoods $U_n\subset U$ and sequence of numbers $r_n\to\infty$, $3<r_n<\rho_n$, such that $w_{\chi_n}([-r_n,r_n]\times[0,1])\subset U_n$.

Define vector fields ξ_{0n}

$$D_2\exp_y\big(t,\xi_{0n}(\tau,t)\big) = \xi_n(\tau,t)$$

for $-r_n<\tau<r_n$. Note that

$$\|\,\xi_n\,\|_{p,k}\le 1 \implies \|\xi_{0n}|_{[-r_n,r_n]\times[0,1]}\|_{p,k}\le c \implies \|\xi_{0n}|_{[-3,3]\times[0,1]}\|_{p,k}\le c\,.$$

By Rellich's theorem, passing to a subsequence, $\xi_{0n}|_{[-3,3]\times[0,1]}$ converges to some ξ_∞ in the $L^p_{k-1}([-3,3]\times[0,1])$-norm.

For the sequence $r'_n\to\infty$, $3<r'_n<r_n$, define $\hat{\beta}_n(\tau) = 1-\beta(\tau-r'_n)-\beta(-\tau+r'_n)$. Note that equations (49) and (50) imply

$$\|\,E_{\vec{\chi}}(\xi_n)\,\|_{p,k-1}\to 0.$$

Choose the sequence r'_n appropriately to ensure that $\|\,\hat{\beta}_n\xi_{0n}\,\|_{p,k}$ is bounded uniformly and $\|\,E_{0n}(\hat{\beta}_n\xi_{0n})\,\|_{p,k}\to 0$. Since the $\hat{\beta}_n\xi_{0n}$ are now defined on all of Θ, and their $L^p_k(\Theta,T_yP)$-norms are uniformly bounded, by Alaoglu's theorem, there exists some $\xi'_\infty \in L^p_k(\Theta, T_yP)$ which is the weak* limit of ξ_{0n}. Since $(L^p_k)^{**} = L^p_k$, ξ'_∞ is in fact a weak limit. As $w_{\chi_n}([-r_n, r_n] \times [0,1])$ converges to y, the (coefficients of the) operators $E_{0,n}$ converge smoothly to (the constant ones of) $\bar\partial_0$, the standard Cauchy Riemann operator. And so, for any test function $f \in (L^p_k(\Theta, T_yP))^* = L^q_k(\Theta, T_yP)$,

$$\int \left| E_{0n}(\hat\beta_n \xi_{0n}) f \right| \le \left\| E_{0n}(\hat\beta_n \xi_{0n}) \right\|_p \cdot \| f \|_q \to 0$$

by Holder's inequality. But at the same time,

$$\begin{aligned} \int E_{0n}(\hat\beta_n \xi_{0n}) f &= \int \hat\beta_n \xi_{0n} E^*_{0n}(f) = \int \hat\beta_n \xi_{0n} (E^*_{0n} - \bar\partial^*_0)(f) + \int \hat\beta_n \xi_{0n} \bar\partial^*_0(f) \\ &\to \int \xi'_\infty \bar\partial^*_0(f) \text{ by the } (L^\infty)\text{-convergence of operators} \\ &= \int \bar\partial_0(\xi'_\infty) f \,. \end{aligned}$$

The standard Cauchy–Riemann equation with these boundary conditions has no non-zero solutions. To see this invertibility, note that by ellipticity, the kernel is finite-dimensional; however, any non-constant solution provides an infinite-dimensional space of solutions by translation. And so,

$$\bar\partial_0(\xi'_\infty) = 0 \text{ weakly} \quad \Longrightarrow \quad \bar\partial_0(\xi'_\infty) = 0 \text{ strongly} \quad \Longrightarrow \quad \xi'_\infty = 0 \,.$$

The inclusion map $i : L^p_k \hookrightarrow L^p_{k-1}$ induces an embedding of $i^* : (L^p_{k-1})^* \hookrightarrow (L^p_k)^*$. Restricting the sequence to the subset $[-3,3] \times [0,1]$, $\hat\beta_n \, \xi_{0n}|_{[-3,3]\times[0,1]} = \xi_{0n}|_{[-3,3]\times[0,1]}$ converges weakly to $\xi'_\infty|_{[-3,3]\times[0,1]} = 0$. 'Weak' here means tested against functions in $(L^p_{k-1})^*$. Because a strong limit is also a weak one, and weak limits are unique, $\xi_{0n}|_{[-3,3]\times[0,1]}$ converges to 0 in the $L^p_{k-1}([-3,3] \times [0,1], T_yP)$-norm. Similarly, since these sections sit above a fixed neighborhood U, $\xi_n|_{[-3,3]\times[0,1]}$ converges to 0 in the $L^p_{k-1}([-3,3] \times [0,1], TP)$-norm.

Consider the portion of ξ_n supported in the middle. Let $\tilde\beta_1(\tau) = \beta(-\tau+2)$, $\tilde\beta_3(\tau) = \beta(\tau-2)$, and $\tilde\beta_2(\tau) = 1 - \tilde\beta_1(\tau) - \tilde\beta_3(\tau)$. Let $\xi^i_n(\tau,t) = \tilde\beta_i(\tau)\xi_n(\tau,t)$; thus, $\xi_n = \sum \xi^i_n$. Then

$$\begin{aligned} \| \xi^2_n \|_{p,k} = \| \tilde\beta_2 \xi_n \|_{p,k} &\le k_1 \| \tilde\beta_2 \xi_{0n} \|_{p,k} \\ &\le k_2 \| \bar\partial_0(\tilde\beta_2 \xi_{0n}) \|_{p,k-1} + k_2 \| \tilde\beta_2 \xi_{0n} \|_{p,k-1} \\ &\le k_2 \| \bar\partial_0(\xi_{0n}) \|_{p,k-1} + k_2 \| \tilde\beta'_2 \xi_{0n} \|_{p,k-1} + k_2 \| \tilde\beta_2 \xi_{0n} \|_{p,k-1} \\ &\le k_3 \| E_{0n}(\xi_{0n}) \|_{p,k-1} + 10 k_2 \| \; \xi_{0n}|_{[-3,3]\times[0,1]} \; \|_{p,k-1} \to 0 \,. \end{aligned}$$

The third step is from the ellipticity of $\bar\partial_0$ while the last step is from the (L^∞)-convergence of operators and properties of $\tilde\beta_i$ and $\tilde\beta'_i$.

Next consider the right and left portions of ξ_n. Define $\hat{\xi}_n^1 \in W_k^p(u)$ and $\hat{\xi}_n^3 \in W_k^p(v)$ by $\hat{\xi}_n^1(\tau,t) = \xi_n^1(\tau - \rho_n, t)$ and $\hat{\xi}_n^3(\tau,t) = \xi_n^3(\tau + \rho_n, t)$. Since $(\xi_n, \mu_1, \nu_1) \in A(w_{\chi n})$, $\hat{\xi}_n^1(\tau,t)$ and $\hat{\xi}_n^3(\tau,t)$ lie away from the kernels of E_u and E_v, respectively. Since E_u and E_v are uniformly invertible away from their kernels,

$$\begin{aligned}\| \xi_n^1 \|_{p,k} = \| \hat{\xi}_n^1 \|_{p,k} &\le c\| E_u(\hat{\xi}_n^1) \|_{p,k-1} = c\| E_{w_{\chi n}}(\xi_n^1) \|_{p,k-1} \\ &= c\| \tilde{\beta}_1 E_{w_{\chi n}}(\xi_n) \|_{p,k-1} + c\| \tilde{\beta}_1' \xi_n \|_{p,k-1} \\ &= c\| E_{w_{\chi n}}(\xi_n) \|_{p,k-1} + 2c\| \ \xi_n|_{[-3,3]\times[0,1]} \ \|_{p,k-1} \to 0\,.\end{aligned}$$

Similarly, $\| \xi_n^3 \|_{p,k} \to 0$.

The contradiction needed to prove the uniform bound on the right inverses now arises:

$$1 = \| \xi_n + \mu_n X_1 + \nu_n X_2 \|_{p,k} \le \| \xi_n^1 \|_{p,k} + \| \xi_n^2 \|_{p,k} + \| \xi_n^3 \|_{p,k} + |\mu_n| + |\nu_n| \to 0\,.$$

□

References

[A] V.I. Arnold, First steps in symplectic geometry, Russian Mathematical Surveys 41 (1986), 1–21.

[AGV] V.I. Arnold, S.M. Guseĭn-Zade, A.N. Varchenko, Singularities of Differentiable Maps. Vol. I. The Classification of Critical Points, Caustics and Wave Fronts, translated from Russian by I. Porteous and M. Reynolds, Monographs in Mathematics 82, Birkhäuser, 1985.

[AuL] M. Audin, J. Lafontaine, Eds., Holomorphic Curves in Symplectic Geometry, Birkhäuser, Basel, Boston, Berlin, 1994.

[C] J. Cerf, La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie, Inst. Hautes Etudes Sci. Publ. Math. 39, (1970), 5–173.

[Ch] M. Chaperon, Quelques questions de géométrie symplectique, Seminaire Bourbaki, Astérisque 105-6 (1983), 231–250.

[Co] M. Cohen, A Course in Simple-Homotopy Theory, Springer-Verlag, New York, 1973.

[ConZ] C. Conley, E. Zehnder, The Birkhoff–Lewis fixed point theorem and a conjecture of V.I. Arnold, Invent. Math. 73 (1983), 33–49.

[E] Y. Eliashberg, Estimates on the number of fixed points of area preserving transformations, Syktyvkar University, preprint (1979).

[EG] Y. Eliashberg, M. Gromov, Lagrangian intersection theory: finite dimensional approach, Amer. Math. Soc. Transl. 186 (1996), 27–118.

[F1] A. Floer, Morse theory for Lagrangian intersections, J. Diff. Geom. 28 (1988), 513–547.

[F2] A. Floer, A relative Morse index for the symplectic action, Comm. Pure Appl. Math. 41 (1988), 393–407.

[F3] A. Floer, Unregularized gradient flow of the symplectic action, Comm. Pure Appl. Math. 41 (1988), 775–813.

[F4] A. Floer, Witten's complex and infinite dimensional Morse theory, J. Diff. Geom. 30 (1989), 207–221.

[FH] A. Floer, H. Hofer, Coherent orientation for periodic orbit problems in symplectic geometry, Math. Z. 212 (1993), 13–38.

[Fu] K. Fukaya, The symplectic S-cobordism conjecture: a summary, Geometry and Physics, Aarhus (1995), 209–219.

[FuO] K. Fukaya, K. Ono, Arnold conjecture and Gromov–Witten invariants, Topology 38 (1999), 933–1048.

[FuOOO] K. Fukaya, Y.-G. Oh, H. Ohta, K. Ono, Lagrangian intersection Floer theory: anomaly and obstruction, preprint.

[G] M. Gromov, Pseudo-holomorphic curves in symplectic manifolds, Invent. Math. 82 (1985), 307–347.

[HW] A. Hatcher, J. Wagoner, Pseudo-Isotopies of Compact Manifolds, Asterisque 6 (1973).

[HoS] H. Hofer, D. Salamon, Floer homology and Novikov rings, Floer Memorial Volume, Birkhäuser (1995), 483–524.

[Hu] J. Hudson, Piecewise Linear Topology, W.A. Benjamin, Inc., 1968.

[HutL] M. Hutchings, Y.-J. Lee, Circle-valued Morse theory and Reidemeister torsion, and Seiberg–Witten invariants of 3-manifolds, Topology 38:4 (1999), 861–888.

[LS] F. Laudenbach, J.P. Sikorav, Persistence d'intersection avec la section nulle au cours d'une isotopie Hamiltonienne dans un fibré cotangent, Invent. Math. 82 (1985), 349–357.

[Le] Y.-J. Lee, in preparation.

[LiT] G. Liu, G. Tian, Floer homology and Arnold conjecture, J. Diff. Geo. 49 (1998) 1–74.

[M1] J. Milnor, Whitehead torsion, Bull. Amer. Math. Soc. 72 (1966), 358–426.

[McS1] D. McDuff, D. Salamon, Introduction to Symplectic Topology, Oxford Mathematical Monographs, Oxford Science Publications (1995).

[McS2] D. McDuff, D. Salamon, J-holomorphic Curves and Quantum Cohomology, University Lecture Series 6, AMS, Providence, RI (1994).

[O1] Y.-G. Oh, Removal of boundary singularities of Pseudo-holomorphic curves with Lagrangian boundary conditions, Comm. Pure Appl. Math. 45 (1992), 121–139.

[O2] Y.-G. Oh, Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks I, Comm. Pure Appl. Math. 46 (1993), 949–994.

[On] K. Ono, The Arnold conjecture for weakly monotone symplectic manifolds, Invent. Math. 119 (2995), 519–537.

[S1] D. Salamon, Lectures on Floer Homology, IAS/Park City Mathematics Series 7 (1998).

[S2] D. Salamon, personal communication.

[SZ] D. Salamon, E. Zehnder, Morse theory for periodic solutions of Hamiltonian systems and the Maslov index, Comm. Pure Appl. Math. 45 (1992), 1303–1360.

[Sc1] M. Schwarz, Morse Homology, Progress in Mathematics, 111, Birkhäuser (1993).

[Sc2] M. Schwarz, Ph.D. Thesis (1995).

[Su] M. Sullivan, Ph.D. Thesis (1999).

[W] J. Wagoner, On K_2 of the Laurent polynomial ring, Amer. J. Math. 93 (1971), 123–138.

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