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On the Betti numbers of semialgebraic sets defined by few quadratic inequalities

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1 Introduction

In this paper we study the topology of a semialgebraic set defined in $\mathbb{R}^n$ by few quadratic inequalities. Let us denote by $\langle \cdot, \cdot \rangle$ the standard inner product in $\mathbb{R}^n$. Let Q be a symmetric $n \times n$ matrix, $b \in \mathbb{R}^n$ be a vector and $a \in \mathbb{R}$ be a number. The function $q : \mathbb{R}^n \longrightarrow \mathbb{R},\ q(x) = \langle Qx, x \rangle + \langle b, x \rangle - a$ is called a *quadratic polynomial*. Let us consider a semialgebraic set defined by quadratic inequalities:

$$X = \Big\{ x \in \mathbb{R}^n : q_i(x) < 0 : i = 1, \dots, s; \quad q_i(x) \leq 0 : i = s+1, \dots, k \Big\},$$

where q_i are quadratic polynomials. We are interested in the topology of the set X when the number k of inequalities is small whereas the dimension n is big. As a measure of the "topological complexity" of the set X we consider the sum of its Betti numbers

$$\operatorname{rank} H^*(X; \mathscr{F}) = \operatorname{rank} H_*(X; \mathscr{F}),$$

where H^* are singular cohomology and H_* are singular homology with the coefficients in a field $\mathscr{F}$. Our results and constructions are independent of the field, and therefore we often omit $\mathscr{F}$ in the notation.

In this paper we prove the following main result.

(1.1) Theorem. *Let us fix $k \in \mathbb{N}$. Then there exists a polynomial $P_k(n) :\ n \in \mathbb{N}$ of degree $O(k)$ such that for any $n \in \mathbb{N}$, for any k quadratic polynomials q_i : $\mathbb{R}^n \longrightarrow \mathbb{R} :\ i = 1, \dots, k$ and for any $1 \leq s \leq k$ the sum of the Betti numbers of the set*

$$X = \Big\{ x \in \mathbb{R}^n : q_i(x) < 0 : i = 1, \dots, s; \quad q_i(x) \leq 0 : i = s+1, \dots, k \Big\}$$

does not exceed $P_k(n)$.

The general estimate (see [6–8, 11]), which is applicable for an arbitrary semialgebraic set, would give us $(2k)^{O(n)}$ as an upper bound for the sum of the Betti numbers of X. Thus this estimate is better than the estimate from Theorem 1.1 if k is large. If, however, k is small (fixed) and n is large, then the estimate of Theorem 1.1 is better. As is known, any real or complex algebraic variety can be represented as an intersection of quadrics. Therefore, Theorem 1.1 provides, in principle, a tool for estimating the sum of the Betti numbers of an arbitrary real or complex algebraic variety. Of course, generally the number of obtained quadratic equations will be very large, so the estimate provided by Theorem 1.1 will be too weak. However, some algebraic varieties can be represented as the intersection of few quadrics, and, therefore, Theorem 1.1 would provide for them a better estimate (see Sect. 6 for an example). A possible application of Theorem 1.1 to lower bounds of bilinear complexity of a semialgebraic decision problem is discussed in Sect. 6.1.

We use a similar technique to that of [6] with some necessary modifications. As in [6], we prove first a version of Theorem 1.1 for a smooth variety by choosing an appropriate Morse function on it. Let $Q = (Q_1, \dots, Q_k)$ be a k-tuple of symmetric $n \times n$ matrices and $a = (a_1, \dots, a_k)$ be a k-tuple of real numbers interpreted as a vector from $\mathbb{R}^k$. First, we establish our upper bound for a compact complete intersection of k real quadrics

$$X_0(Q, a) = \Big\{x \in \mathbb{R}^n : \langle x, Q_i x\rangle = a_i,\ i = 1, \dots, k\Big\}, \tag{1.2}$$

where $a = (a_1, \dots, a_k)$ is "in general position" in $\mathbb{R}^k$ (Corollary 3.3). To prove this, as in [6] we choose an appropriate Morse function $f(x)$ on $X_0(Q, a)$ and estimate the number of critical points of f as the number of solutions to a system of polynomial equations. However, in our case we get a very special system, so we are able to show that the number of solutions is bounded by a polynomial in the dimension n rather than by $(2k)^{O(n)}$ (Lemma 3.2). Next, as in [6] we extend our bound to the general (non-smooth) case. However, we can no longer use the reduction of [6], for it would destroy the "quadratic nature" of our sets. Instead, by using a combinatorial construction we prove Theorem 1.1 for the sets

$$X(Q, a) = \Big\{x \in \mathbb{R}^n : \langle x, Q_i x\rangle \leq a_i,\ i = 1, \dots, k\Big\}, \tag{1.3}$$

where $X(Q, a)$ is compact and a is "in general position" (Lemma 4.3).

Then, using the standard approximation technique, we prove Theorem 1.1 in whole generality (Sect. 5).

2 Preliminaries. The space of quadratic forms

In this section we recall some facts about quadratic forms "in general position".

Let us consider the linear space W_n of all $n \times n$ real symmetric matrices. We can identify W_n with the space $\mathbb{R}^{\frac{n(n+1)}{2}}$. We will be interested in the subset of singular matrices in W_n.

(2.1) Proposition (see, for example, Corollary of Lemma 2 from [1]). *For any* $r = 0, \dots, n$ *the set*

$$W_n^r = \{A \in W_n : \operatorname{rank} A = n - r\}$$

is a smooth analytic variety in W_n *of codimension* $\frac{r(r+1)}{2}$. □

Proposition 2.1 is also known as the "corank formula". Thus $W_n^n = \{0\}$ is the origin in W_n, whereas W_n^0 is a dense subset in W_n.

(2.2) Definition. Let us fix a k-tuple $Q = (Q_1, \dots, Q_k)$ of symmetric $n \times n$ matrices. Let F be a symmetric $n \times n$ matrix. If the map

$$\Psi_F : \mathbb{R}^k \longrightarrow W_n, \quad \Psi_F(z_1, \dots, z_k) = F - \sum_{i=1}^{k} z_i \cdot Q_i$$

intersects every variety $W_n^r \subset W_n,\ r = 0, \dots, n$ transversally, we say that F is *generic with respect to* Q. Sometimes we say that F is *generic*, if the choice of $Q = (Q_1, \dots, Q_k)$ is obvious from the context.

Proposition 2.1 immediately implies

(2.3) Corollary.

1) For given k*-tuple* $Q = (Q_1, \dots, Q_k)$ *of* $n \times n$ *symmetric matrices the set of generic matrices* F *contains an open and dense subset of* W_n.

2) If F *is generic with respect to* Q, *then*

$$\operatorname{rank}\Big(F - \sum_{i=1}^{k} z_i Q_i\Big) \geq n - \left[\frac{\sqrt{8k+1} - 1}{2}\right]$$

for all $z_1, \dots, z_k \in \mathbb{R}$. *Here* $[\cdot]$ *denotes the integer part.*

Proof.

1) Let us consider a linear map $\phi : \mathbb{R}^k \oplus W_n \longrightarrow W_n$:

$$\phi(z, F) = F - \sum_{i=1}^{k} z_i \cdot Q_i : \quad z \in \mathbb{R}^k,\ F \in W_n.$$

The map ϕ is surjective and therefore intersects every variety W_n^r transversally. For $r = 0, \dots, n$ let $S_r = \phi^{-1}(W_n^r) \subset \mathbb{R}^k \oplus W_n$. Let us denote by p the natural projection $p : \mathbb{R}^k \oplus W_n \longrightarrow W_n$ onto the second summand. It follows then (see, for example, Lemma 4.6 of [4]) that F is generic with respect to Q whenever F is a regular value of every restriction $p_r : S_r \longrightarrow W_n$ of p on S_r. Since the set of singular values of p_r is a semialgebraic set, by Sard's Theorem (see, for example, Sect. 1 of Chapter II of [4]) we get the desired result.

2) If F is generic, then the affine variety

$$F - \sum_{i=1}^{k} z_i Q_i : \ z_1, \dots, z_k \in \mathbb{R}$$

in W_n does not intersect W_n^r for $k < \dfrac{r(r+1)}{2}$. □

3 Generic intersections of quadrics

In this section we prove a version of Theorem 1.1 for a compact complete intersection (1.2) of real quadrics.

(3.1) Definition. Let us fix a k-tuple $Q = (Q_1, \dots, Q_k)$ of $n \times n$ symmetric matrices. We say that $a \in \mathbb{R}^k$ is *generic with respect to* Q, if a is a regular value of the quadratic map $\mathbf{q} : \mathbb{R}^n \longrightarrow \mathbb{R}^k : \ \mathbf{q}(x) = \big(\langle x, Q_1 x\rangle, \dots, \langle x, Q_k x\rangle\big)$. Sometimes we just say that a is *generic*, if the choice of $Q = (Q_1, \dots, Q_k)$ is obvious from the context.

By Sard's Theorem it follows that for any $Q = (Q_1, \dots, Q_k)$ the set of generic a is open and dense in $\mathbb{R}^k$. If a is generic, then the variety

$$X_0(Q, a) = \Big\{x \in \mathbb{R}^n : \ \langle x, Q_i x\rangle = a_i, \ i = 1, \dots, k\Big\},$$

is either empty or a smooth $(n-k)$-dimensional manifold such that for every $x \in X_0(Q, a)$ the vectors $Q_1 x, \dots, Q_k x$ are linearly independent.

Let F be a positive definite matrix. Then there exists a vector $x_0 \in \mathbb{R}^n$ such that the function $f(x) = \langle x - x_0, F(x - x_0)\rangle$ is a Morse function on $X_0(Q, a)$ (see, for example, Sect. 6, Part I of [5] for the standard squared distance $l(x) = \langle x - x_0, x - x_0\rangle$. Our function $f(x)$ reduces to $l(x)$ by a coordinate transformation.) We show that if F is generic with respect to Q, then the number of critical points of f on $X_0(Q, a)$ is bounded by a polynomial in n.

(3.2) Lemma. *Let us fix $k \in \mathbb{N}$. Then there exists a polynomial $p_k(n) : \ n \in \mathbb{N}$ of degree $O(k)$ such that for any k-tuple $Q = (Q_1, \dots, Q_k)$, for any generic $a = (a_1, \dots, a_k) \in \mathbb{R}^k$, and for any Morse function $f(x) = \langle x - x_0, F(x - x_0)\rangle$, where F is a positive definite matrix which is generic with respect to Q, the number of critical points of $f(x)$ on the variety*

$$X_0(Q, a) = \Big\{x \in \mathbb{R}^n : \ \langle x, Q_i x\rangle = a_i, \ i = 1, \dots, k\Big\}$$

does not exceed $p_k(n)$.

Proof. We introduce some notation. For $z = (z_1, \dots, z_k)$ we let

$$\Phi(z) = F - \sum_{i=1}^{k} z_i Q_i .$$

For subsets $I, J \subset \{1, \dots, n\}$ we denote by $\Phi(z; I, J)$ the submatrix of $\Phi(z)$ with row indices in $I \subset \{1, \dots, n\}$ and column indices in $J \subset \{1, \dots, n\}$. Let $y = Fx_0$. We denote by $\widehat{\Phi(z)}$ the $n \times (n+1)$ matrix $(\Phi(z), y)$.

Without loss of generality we may assume that $X_0(Q, a)$ is not empty. A point $x \in \mathbb{R}^n$ is a critical point of the function f, if and only if there exist Lagrange multipliers $z = (z_1, \dots, z_k) \in \mathbb{R}^k$ such that

$$\Phi(z)x = y, \tag{3.2.1}$$

and, additionally, we have that

$$\langle x, Q_i x\rangle = a_i \quad \text{for } i = 1, \ldots, k. \tag{3.2.2}$$

For a given critical point x the vector $z = (z_1, \ldots, z_k)$ of Lagrange multipliers is necessarily unique, because the vectors $Q_1 x, \ldots, Q_k x$ are linearly independent. We will represent the set of possible Lagrange multipliers $z = (z_1, \ldots, z_k)$ as a union of polynomially many (possibly intersecting) pieces $\mathscr{B}_\alpha \subset \mathbb{R}^k$ so that the system of equations (3.2.1)–(3.2.2) with $z \in \mathscr{B}_\alpha$ has at most polynomially many solutions x for every piece $\mathscr{B}_\alpha$.

Since F is generic with respect to $Q = (Q_1, \ldots, Q_k)$, by Corollary 2.3 we have that

$$\operatorname{rank} \Phi(z) \geq n - \left[\frac{\sqrt{8k+1}-1}{2}\right] \quad \text{for all} \quad z = (z_1, \ldots, z_k).$$

(3.2.3) Pieces $\mathscr{B}_\alpha$. Our pieces $\mathscr{B}_\alpha \subset \mathbb{R}^k$ are indexed by ordered triples $\alpha = (s, I, J)$, where s is a natural number $0 \leq s \leq \left[\frac{\sqrt{8k+1}-1}{2}\right]$ and $I, J \subset \{1, \ldots, n\}$ are subsets of equal cardinali ty $n - s$. We define

$$\mathscr{B}_{s,I,J} = \Big\{ z = (z_1, \ldots, z_k) \in \mathbb{R}^k : \ \operatorname{rank} \Phi(z) = \ \operatorname{rank} \widehat{\Phi(z)} = n - s$$
$$\text{and} \ \det \Phi(z, I, J) \neq 0 \Big\}.$$

We observe that a piece $\mathscr{B}_\alpha \subset \mathbb{R}^k$ can be defined by a system of one polynomial equation in $z_1, \ldots, z_k$ of degree at most $2n$ (that is, the sum of squares of all $(n-s+1) \times (n-s+1)$ minors of the matrix $\widehat{\Phi(z)}$ is equal to zero) and one polynomial inequality of degree at most n (that is, $\det \Phi(z, I, J) \neq 0$). The number of all possible pieces $\mathscr{B}_\alpha$ does not exceed $k \cdot n^{2k}$, that is, polynomial in n for a fixed k.

Let us choose a piece $\mathscr{B}_\alpha$ for $\alpha = (s, I, J)$. For every $z \in \mathscr{B}_\alpha$ the set x of solutions of (3.2.1) is an s-dimensional affine subspace A_z in $\mathbb{R}^n$. Let us construct rational vector-functions $u_0(z), \ldots, u_s(z) : \ \mathscr{B}_\alpha \longrightarrow \mathbb{R}^n$ such that $u_0(z), \ldots, u_s(z)$ is an affine basis of A_z for every $z \in \mathscr{B}_\alpha$. We let $\{1, \ldots, n\} \setminus J = \{j_1, \ldots, j_s\}$. For $i = 0, \ldots, s$ and $j \in \{j_1, \ldots, j_s\}$ we define the j-th coordinate of $u_i(z)$ to be either identically 1, if $j = j_i$, or identically zero, otherwise. Then the other coordinates of $u_i(z)$ are uniquely determined from a non-degenerate system (3.2.1) of linear equations. By Cramer's rule it follows that every $u_i(z)$ is a rational vector-function in $z = (z_1, \ldots, z_k)$ of the type

$$u_i(z) = \left(\frac{P_{1,i}(z)}{\det \Phi(z; I, J)}, \ldots, \frac{P_{n,i}(z)}{\det \Phi(z; I, J)} \right), \tag{3.2.4}$$

where $P_{j,i}$ are polynomials and $\deg P_{j,i} \leq n$ for $i = 0, \ldots, s$ and $j = 1, \ldots, n$. Thus for every $x \in \mathbb{R}^n$, which satisfies (3.2.1) with $z \in \mathscr{B}_\alpha$, there is a unique vector $\lambda = (\lambda_0, \ldots, \lambda_s)$ such that

(3.2.5)
$$x = \sum_{i=0}^{s} \lambda_i u_i(z)$$

and

(3.2.6)
$$\sum_{i=0}^{s} \lambda_i = 1.$$

Let us write the following system $\Sigma_\alpha :\ \alpha = (s, I, J)$ of polynomial equations and inequalities in $s + k + 1$ real variables $z_1, \dots, z_k, \lambda_0, \dots, \lambda_s$:

System Σ_α.

– we express the condition $z = (z_1, \dots, z_k) \in \mathscr{B}_\alpha$ by means of a polynomial equation and an inequality (see (3.2.3));

– we substitute (3.2.4) into (3.2.5) and the resulting expression for x we substitute into (3.2.2);

– finally we write the equation (3.2.6).

Since the number of variables in Σ_α does not exceed $2k$ (see (3.2.3)), the degrees of equations and inequalities do not exceed $2n + 1$, and the number of equations and inequalities is $k + 2$, by the results of [8], Sect. 3 (see also [6]) we conclude that the number of connected components of the set of solutions to the system Σ_α does not exceed $\big((k + 2)(2n + 1)\big)^{O(k)}$. The set of solutions $(z_1, \dots, z_k, \lambda_0, \dots, \lambda_s)$ of the system Σ_α is in one-to-one correspondence via (3.2.4)–(3.2.5) with the set of critical points $x \in X_0(Q, a)$ of the function f, whose vector $z = (z_1, \dots, z_k)$ of Lagrange multipliers from (3.2.1) belongs to the piece $\mathscr{B}_\alpha$. In particular, we conclude that the set of solutions to system Σ_α is finite. Therefore the number of solutions to a system Σ_α does not exceed $\big((k + 2)(2n + 1)\big)^{O(k)}$. In other words, for every fixed k the number of critical points $x \in X(Q, a)$ of f, whose vector of Lagrange multipliers belongs to a piece $\mathscr{B}_\alpha$, is bounded by a polynomial in n of degree $O(k)$. Since the total number of pieces $\mathscr{B}_\alpha$ is also bounded by a polynomial in n of degree $O(k)$ (see (3.2.3)), the proof follows. □

(3.3) Corollary. *Let us fix $k \in \mathbb{N}$. Then there exists a polynomial $p_k(n) :\ n \in \mathbb{N}$ of degree $O(k)$ such that for any $n \in \mathbb{N}$, for any k-tuple $Q = (Q_1, \dots, Q_k)$ of symmetric $n \times n$ matrices, and for any generic vector $a \in \mathbb{R}^k$ the sum of the Betti numbers of the set*

$$X_0(Q, a) = \Big\{ x \in \mathbb{R}^n :\ \langle x, Q_i x \rangle = a_i,\ i = 1, \dots, k \Big\}$$

is bounded by $p_k(n)$ provided $X_0(Q, a)$ is compact.

Proof. The sum of the Betti numbers of a compact smooth manifold does not exceed the number of critical points of a Morse function (see, for example, [5]). Now we apply Lemma 3.2. □

For further convenience we assume that $p_k(n)$ is a monotone function in n.

4 Generic quadratic inequalities

In this section we prove Theorem 1.1 for compact semialgebraic sets

$$X(Q,a)=\Big\{x\in\mathbb{R}^n:\ \langle x,Q_ix\rangle\le a_i:\ i=1,\dots,k\Big\},$$

where $a=(a_1,\dots,a_k)$ is "in general position". To do that, we consider a formally larger class of semialgebraic sets defined by quadratic equations and inequalities and use the induction on the number of inequalities. We use Corollary 3.3 as a base fo r the induction.

Below we define the main object of this section.

(4.1) Sets $W_{S,I,E}(Q,a)$. Let us choose a k-tuple $Q=(Q_1,\dots,Q_k)$ of $n\times n$ symmetric matrices and a vector $a=(a_1,\dots,a_k)\in\mathbb{R}^k$. For any partition of the set $\{1,\dots,k\}$ into pairwise disjoint (possibly empty) subsets $S,I,E:\ S\cup I\cup E=\{1,\dots,k\}$ we consider a semialgebraic set $W_{S,I,E}(Q,a)\subset\mathbb{R}^{n+s}$, where $s=\operatorname{card}S$. We interpret a point in $\mathbb{R}^{n+s}$ as a pair (x,u), where $x\in\mathbb{R}^n$ and $u=(u_i:\ i\in S)$ is an s-tuple of real numbers u_i indexed by the elements of S. We define

$$\begin{aligned}W_{S,I,E}(Q,a)=\Big\{(x,u)\in\mathbb{R}^{n+s}:\ &\langle x,Q_ix\rangle\le a_i \text{ for } i\in I;\\ &\langle x,Q_ix\rangle=a_i \text{ for } i\in E;\\ &\langle x,Q_ix\rangle+u_i^2=a_i \text{ for } i\in S\Big\}.\end{aligned}$$

For a subset $S\subset\{1,\dots,k\}$ we define a quadratic map $\mathbf{q}^S:\mathbb{R}^{n+s}\longrightarrow\mathbb{R}^k$ as follows: $\mathbf{q}^S=(q_1^S,\dots,q_k^S)$,where

$$q_i^S(x,u)=\begin{cases}\langle x,Q_ix\rangle+u_i^2 & \text{if } i\in S;\\ \langle x,Q_ix\rangle & \text{if } i\notin S.\end{cases}$$

(4.2) Definition. Let $Q=(Q_1,\dots,Q_k)$ be a k-tuple of $n\times n$ symmetric matrices. We say that a vector $a\in\mathbb{R}^k$ is *strictly generic with respect to* Q, if a is a regular value of every map $\mathbf{q}^S:\mathbb{R}^{n+s}\longrightarrow\mathbb{R}^k$ for $S\subset\{1,\dots,k\}$. Sometimes we just say that a is *strictly generic*, if the tuple Q is obvious from the context.

Sard's Theorem implies that for any given Q the set of strictly generic a is an open and dense subset in $\mathbb{R}^k$.

(4.3) Lemma. *Let us fix $k\in\mathbb{N}$. Then there exists a polynomial $p_{k,1}(n):\ n\in\mathbb{N}$ of degree $O(k)$ such that for any k-tuple $Q=(Q_1,\dots,Q_k)$, for any partition $S\cup I\cup E=\{1,\dots,k\}$, and for any $a\in\mathbb{R}^k$, which is strictly generic with respect to Q, the sum of the Betti numbers of the set $W_{S,I,E}(Q,a)$ does not exceed $p_{k,1}(n)$ provided this set is compact.*

Proof. Let $p_{k,1}(n)=p_k(n+k)$, where $p_k(n)$ is a polynomial from Corollary 3.3. We proceed by induction on the cardinality of I (the set of inequalities). If card

$I = 0$ then the estimate follows by Corollary 3.3. Suppose that the set I is not empty. Let us pick up a $j \in I$. Let $S' = S \cup \{j\}$, $I' = I \setminus \{j\}$ and $E' = E \cup \{j\}$. Starting from $W_{S,I,E}(Q,a) \subset \mathbb{R}^{n+s}$ we get either $W_{S',I',E}(Q,a) \subset \mathbb{R}^{n+s+1}$, if we replace the inequality $\langle x, Q_j x\rangle \leq a_j$ by the equation $\langle x, Q_j x\rangle + u_j^2 = a_j$, or $W_{S,I',E'}(Q,a) \subset \mathbb{R}^{n+s}$, if we replace this inequality by the equation $\langle x, Q_j x\rangle = a_j$. If $W_{S,I,E}(Q,a)$ is compact, then both $W_{S',I',E}(Q,a)$ and $W_{S,I',E'}$ are compact. Our aim is to estimate rank $H^*\ W_{S,I,E}(Q,a)$ in terms of rank $H^*\ W_{S',I',E}(Q,a)$ and rank $H^*\ W_{S,I',E'}(Q,a)$. Note, that the number of inequalities participating in the definition of either of the sets $W_{S',I',E}(Q,a)$ and $W_{S,I',E'}(Q,a)$ is smaller than that of $W_{S,I,E}(Q,a)$.

The set $W_{S',I',E}(Q,a)$ can be dissected into two pieces

$$W^+_{S',I',E}(Q,a) = \Big\{(x,u) \in W_{S',I',E}(Q,a) :\ u_j \geq 0\Big\} \text{ and}$$

$$W^-_{S',I',E}(Q,a) = \Big\{(x,u) \in W_{S',I',E}(Q,a) :\ u_j \leq 0\Big\}$$

with the intersection

$$W^0_{S',I',E}(Q,a) = \Big\{(x,u) \in W_{S',I',E}(Q,a) :\ u_j = 0\Big\}.$$

Let us consider the natural projection $\mathbb{R}^{n+s+1} \longrightarrow \mathbb{R}^{n+s}$ that maps a point $(x, u_i :\ i \in S')$ to the point $(x, u_i :\ i \in S)$ (we erase u_j). This projection homeomorphically maps each set $W^+_{S',I',E}(Q,a)$ and $W^-_{S',I',E}(Q,a)$ onto $W_{S,I,E}(Q,a)$ and the set $W^0_{S',I',E}(Q,a)$ onto $W_{S,I',E'}(Q,a)$. Applying the induction hypothesis to $W_{S',I',E}(Q,a)$ and $W_{S,I',E'}(Q,a)$ we get

$$\begin{aligned} \text{rank } H^*\ W_{S',I',E}(Q,a),\quad \text{rank } H^*\ W^0_{S',I',E}(Q,a) &= \text{rank } H^*\ W_{S,I',E'}(Q,a) \\ &\leq p_{k,1}(n). \end{aligned}$$

Therefore, applying the Mayer–Vietoris exact sequence (see, for example, Sect. 6 of Chapter 4 of [9]) we get that

$$\text{rank } H^*\ W^+_{S',I',E}(Q,a) + \text{rank } H^*\ W^-_{S',I',E}(Q,a) \leq 2p_{k,1}(n),$$

and hence

$$\begin{aligned} \text{rank } H^*\ W^+_{S',I',E}(Q,a) = \text{rank } H^*\ W^-_{S',I',E}(Q,a) &= \text{rank } H^*\ W_{S,I,E}(Q,a) \\ &\leq p_{k,1}(n). \end{aligned}$$

□

(4.4) Corollary. *Let us fix $k \in \mathbb{N}$. Then there exists a polynomial $p_{k,1}(n) : n \in \mathbb{N}$ of degree $O(k)$ such that for any k-tuple $Q = (Q_1, \ldots, Q_k)$ of $n \times n$ symmetric matrices and for any strictly generic $a \in \mathbb{R}^k$ the sum of the Betti numbers of the set*

$$X(Q,a) = \Big\{x \in \mathbb{R}^n :\ \langle x, Q_i x\rangle \leq a_i,\ i = 1, \ldots, k\Big\}$$

does not exceed $p_{k,1}(n)$ provided $X(Q,a)$ is compact.

Proof. Follows by Lemma 4.3 since the set $W_{S,I,E}(Q,a)$ for $S = E = \emptyset$ and $I = \{1, \ldots, k\}$ coincides with $X(Q,a)$. □

5 Proof of Theorem 1.1

First, using the standard approximation technique we prove Theorem 1.1 for the sets $X(Q, a)$ defined by (1.3).

(5.1) Lemma. *Let us fix $k \in \mathbb{N}$. Then there exists a polynomial $p_{k,2}(n) : n \in \mathbb{N}$ of degree $O(k)$ such that for any $(k+1)$-tuple $Q = (Q_0, Q_1, \ldots, Q_k)$ of $n \times n$ symmetric matrices, where Q_0 is a positive definite matrix, and for any $a = (a_0, \ldots, a_k)$ the sum of the Betti numbers of the set*

$$X(Q, a) = \left\{ x \in \mathbb{R}^n : \langle x, Q_i x \rangle \leq a_i, \ i = 0, \ldots, k \right\}$$

does not exceed $p_{k,2}(n)$.

Proof. Let $p_{k,2}(n) = p_{k+1,1}(n)$, where $p_{k,1}$ is the polynomial from Corollary 4.4. We define

$$\mathscr{E}(Q, a) = \Big\{ \epsilon = (\epsilon_0, \ldots, \epsilon_k) : \ a + \epsilon \text{ is strictly generic with respect to } Q \\ \text{and } \ 0 < \epsilon_i < 1 \text{ for } i = 0, \ldots, k \Big\}$$

(see Definition 4.2). The set $\mathscr{E}(Q, a)$ is open and dense in the unit cube $0 \leq \epsilon_i \leq 1$. Therefore

$$X(Q, a) = \bigcap_{\epsilon \in \mathscr{E}(Q,a)} X(Q, a + \epsilon).$$

Since Q_0 is positive definite, the sets $X(Q, a)$ and $X(Q, a+\epsilon)$ are compact and we have that

$$H^* \, X(Q, a) = \varinjlim H^* \, X(Q, a + \epsilon),$$

where the direct limit is taken with respect to natural inclusions $X(Q, a+\epsilon) \subset X(Q, a+\delta)$ whenever $\epsilon_j \leq \delta_j$ for $j = 0, \ldots, k$ (see, for example, Sect. 6 of Chapter 6 of [9]). Now we apply Corollary 4.4 to the sets $X(Q, a+\epsilon) : \ \epsilon \in \mathscr{E}(Q, a)$. □

(5.2) Corollary. *Let us fix $k \in \mathbb{N}$. Then for any k-tuple $Q = (Q_1, \ldots, Q_k)$ of $n \times n$ symmetric matrices and for any $a = (a_1, \ldots, a_k)$ the sum of the Betti numbers of the set*

$$X(Q, a) = \left\{ x \in \mathbb{R}^n : \langle x, Q_i x \rangle \leq a_i, \ i = 1, \ldots, k \right\}$$

does not exceed $p_{k,2}(n)$, where $p_{k,2}$ is the polynomial from Lemma 5.1.

Proof. For $R > 0$ let us define

$$X_R(Q, a) = \left\{ x \in X(Q, a) : \ x_1^2 + \ldots + x_n^2 \leq R \right\}.$$

Then

$$X(Q, a) = \bigcup_{R > 0} X_R(Q, a),$$

and therefore

$$H_* \ X(Q, a) = \varinjlim H_* \ X_R(Q, a).$$

On the other hand, $X_R(Q, a)$ can be considered as the set of solutions of a system of quadratic inequalities with at least one positive definite matrix. By Lemma 5.1

$$\operatorname{rank} H_* \ X_R(Q, a) \leq p_{k,2}(n)$$

and the proof follows. □

(5.3) Corollary. *Let us fix $k \in \mathbb{N}$. Then for any k symmetric matrices $Q_1, \ldots, Q_k$, for any $a_1, \ldots, a_k$ and for any $s \leq k$ the sum of the Betti numbers of the set*

$$X = \left\{ x \in \mathbb{R}^n : \ \langle x, Q_i x\rangle < a_i, \ i = 1, \ldots, s; \quad \langle x, Q_i x\rangle \leq a_i : i = s+1, \ldots, k \right\}$$

does not exceed $p_{k,2}(n)$, where $p_{k,2}$ is the polynomial from Corollary 5.2.

Proof. Let $\mathscr{E} = \left\{ \epsilon = (\epsilon_1, \ldots, \epsilon_s) : 0 < \epsilon_i \leq 1 : i = 1, \ldots, s \right\}$ and

$$X_\epsilon = \left\{ x \in \mathbb{R}^n : \ \langle x, Q_i x\rangle \leq a_i - \epsilon_i, \ i = 1, \ldots, s; \ \langle x, Q_i x\rangle \leq a_i : i = s+1, \ldots, k \right\}$$

Then

$$X = \bigcup_{\epsilon \in \mathscr{E}} X_\epsilon.$$

Now we observe that

$$H_*(X) = \varinjlim H_*(X_\epsilon),$$

where the limit is taken with respect to natural inclusions $X_\epsilon \subset X_\delta$ whenever $\epsilon_i \geq \delta_i$ and apply Corollary 5.2 to the sets X_ϵ. □

Proof of Theorem 1.1. Let $P_k(n) = \frac{1}{2} p_{k+2,2}(n+1)$, where $p_{k,2}$ is the polynomial from Corollary 5.3. Suppose that $q_i(x) = \langle x, Q_i x\rangle + \langle b_i, x\rangle - a_i$, where Q_i is an $n \times n$ symmetric matrix, $b_i \in \mathbb{R}^n$ is a vector and $a_i \in \mathbb{R}$ is a number. We introduce a new real variable t and consider the semialgebraic set $\widehat{X} \subset \mathbb{R}^{n+1}$ given by the following family of $k+2$ quadratic inequalities without linear terms:

$$\begin{aligned} \widehat{X} = \Big\{ (x, t) \in \mathbb{R}^{n+1} : \ & \langle x, Q_i x\rangle + t \cdot \langle b_i, x\rangle < a_i : \ i = 1, \ldots, s; \\ & \langle x, Q_i x\rangle + t \cdot \langle b_i, x\rangle \leq a_i : \ i = s+1, \ldots, k; \\ & t^2 \leq 1, \ -t^2 \leq -1 \text{ (that is, } t^2 = 1) \Big\}. \end{aligned}$$

By Corollary 5.3 it follows that

$$\operatorname{rank} H^* \ \widehat{X} \leq 2P_k(n).$$

On the other hand, the set $\widehat{X}$ consists of the two non-intersecting components $\widehat{X}_+$ and $\widehat{X}_-$ corresponding to the cases $t = 1$ and $t = -1$ respectively. The map $(x, t) \longmapsto x$ is a homeomorphism between $\widehat{X}_+$ and X whereas the map $(x, t) \longmapsto -x$ is a homeomorphism between $\widehat{X}_-$ and X. This observation completes the proof. □

6 Corollaries, examples and possible applications

Theorem 1.1 allows us to distinguish several classes of (semi)algebraic varieties with "small" Betti numbers. For example, for any fixed k the sum of the Betti numbers of an intersection of k real quadrics in $\mathbb{R}^n$ is bounded by a polynomial in n of degree $O(k)$ (since a system of k quadratic equations $q_i(x) = a_i$: $i = 1, \dots, k$ can be written as a system of $2k$ quadratic inequalities $q_i(x) \leq a_i,\ -q_i(x) \leq -a_i$: $i = 1, \dots, k$). The same remains true for intersections of complex quadrics. Note, that we don't require that the quadrics must intersect transversally. Another corollary of Theorem 1.1 is that a fixed number of quadrics in $\mathbb{R}^n$ dissects the space into polynomially many pieces.

As is known, any system of polynomial equations can be reduced to a system of quadratic equations by substitutions of the type $z = xy$. Sometimes the number of obtained quadratic equations is small and we can apply Theorem 1.1. For example, let us consider a real algebraic variety in $\mathbb{R}^n$ defined by a polynomial of degree 4

$$X = \Big\{ x \in \mathbb{R}^n :\ q_1(x) \cdot q_2(x) + q_3(x) \cdot q_4(x) = 1 \Big\},$$

where q_1, q_2, q_3, q_4 are quadratic polynomials. Theorem 1.1 implies that the sum of the Betti numbers of this variety is bounded by a polynomial in n. Indeed, let us consider a variety $Z \subset \mathbb{R}^{n+4}$

$$Z = \Big\{ (x, y_1, y_2, y_3, y_4) :\ q_i(x) - y_i = 0 :\ i = 1, 2, 3, 4 \text{ and } y_1 y_2 + y_3 y_4 = 1 \Big\}.$$

The variety Z is given by 5 quadratic equations (and, therefore, by 10 quadratic inequalities), and hence we can use Theorem 1.1 to estimate the sum of the Betti numbers of Z. The natural projection $(x, y_1, y_2, y_3, y_4) \longmapsto x$ maps Z homeomorphically onto X.

This construction suggests that the number of quadratic inequalities needed to describe a semialgebraic set may be the "right" measure of its complexity. This idea can be formalized as follows.

(6.1) Relation to the computational complexity. Theorem 1.1 can be used to establish some lower complexity bounds in algebraic computations. Here we only briefly sketch a possible approach in maximal generality. Let us consider a computational model over the real numbers (see, for example, [2]). Suppose there is a machine that can perform arithmetic operations (addition, subtraction, multiplication and division) and comparison over the real numbers. It is assumed that every operation has cost 1. The machine has a finite number of states S and a memory realized by a finite number of boxes, indexed by $1, \dots, N$, each of which contains at most one real number. At every step the machine either performs an arithmetic operation with the numbers from some two boxes and stores the result in some other box, or changes its state in accordance with the sign (positive, negative or zero) of the number contained in some box. The indices of the used boxes and the performed operation are determined by the current state of the machine.

Let us fix some semialgebraic set $X \subset \mathbb{R}^n$. We are interested in the computational complexity of the following fundamental

X-membership problem. Given $x \in \mathbb{R}^n$ decide whether $x \in X$.

It can be seen that the computational complexity of the X-membership problem is directly related to the "complexity" of the representation of X in terms of quadratic inequalities. Suppose that the X-membership problem can be solved in k steps on some machine with the number of states S. Let us choose some computation τ (that is, a sequence of states of the machine) which leads to the answer "yes" in the X-membership problem and let $X_\tau \subset X$ be a set of x for which this computation is realized. Then X_τ is homeomorphic to a semialgebraic set Y_τ in $\mathbb{R}^{n+m} : m \leq k$ given by at most k quadratic inequalities. Indeed, every arithmetic operation $x * y \longmapsto z$, where $* = +, -, \cdot,$: can be recorded by an at most quadratic equation with one new variable whereas branching fixes signs of some variables. So $x \in X_\tau$ iff x can be appended (in a unique way) by new variables $(z_1, \dots, z_m)$ to a vector from some semialgebraic set $Y_\tau \subset \mathbb{R}^{n+m}$ which is a set of solutions of at most k quadratic inequalities. The natural projection $Y_\tau \longrightarrow X_\tau : (x, z_1, \dots, z_m) \longmapsto x$ is a homeomorphism. So X can be represented as a disjoint union of at most S^k subsets X_τ each of which is homeomorphic to an at most $(n+k)$-dimensional semialgebraic set Y_τ given by at most k quadratic inequalities. It is seen that an upper bound on the number $\beta_0(X_\tau)$ of connected components of X_τ and a lower bound on the number $\beta_0(X)$ of connected components of X produce some lower bound on the complexity of the machine that solves the X-membership problem.

This approach was used, for example, in [12] to obtain lower bounds for algebraic decision trees. It was based on the general estimates of [6, 7, 11] for the Betti numbers of semialgebraic sets. The estimate of Theorem 1.1 appears to be too weak in this general situation.

However, the same construction can be applied to the "bilinear complexity" model (see, for example, [10]). In the bilinear model our machine operates with the vectors from $\mathbb{R}^n$ instead of the real numbers. Instead of addition (subtraction) we are allowed to perform linear operations with real vectors: addition $(x, y) \longmapsto x+y$ and an application of a linear operator $x \longmapsto Ax$ and instead of multiplication we are allowed to compute the value of a bilinear form $B(x, y)$ for any given pair of vectors x, y. All the operations have unit cost. Similarly, every operation can be recorded by an at most quadratic equation except that the new variable can be a vector from $\mathbb{R}^n$. In this case we gain by using Theorem 1.1 instead of the general bounds [6, 7, 11] since our bound depends better on the dimension. Indeed, we deduce that the number of connected components of X should not exceed S^k times the maximal number of connected components of X_τ. In other words,

$$\beta_0(X) \leq S^k P_k(kn),$$

where $P_k(n)$ is a polynomial of degree $O(k)$ from Theorem 1.1.

In other words, the number of states S and the number of steps k can not be both small if the number of connected components of X is large.

For particular types of programs (decision trees, straight line programs) one can obtain sharper estimates. For example, if we can solve the X-membership problem in k steps on a machine which allows branching ("yes" or "no") on the last step only we must have $S = 1$ and hence

$$\text{rank } H_*(X) \leq P_k(kn).$$

The details of this construction will be described elsewhere.

7 Remarks

(7.1) How optimal is the bound of Theorem 1.1? One can show that the type of the upper bound from Theorem 1.1 can not be improved. More precisely, for every fixed k we present an intersection $X(n, k)$ of $2k$ real quadrics in $\mathbb{R}^{2n}$, whose sum of the Betti numbers grows at least as fast as n^{ck} for some $c > 0$ depending on k only. Let us fix $k \in \mathbb{N}$. Let $X(n, k) \subset \mathbb{C}^n$ be a transversal intersection of k complex affine quadrics. The topological space $X(k, n)$ is uniquely defined, and explicit recursions for rank $H_p\big(X(n, k); \mathbb{Z}\big)$ are known (see [3]). In particular, it follows that

$$\text{rank } H_{n-k}\ X(n, k) \geq n^{ck} \text{ for some } c = c(k) > 0 \text{ and all even } n.$$

On the other hand, identifying $\mathbb{C}^n \approx \mathbb{R}^{2n}$ we may consider $X(n, k)$ as a (non-transversal) intersection of $2k$ real quadrics in $\mathbb{R}^{2n}$.

(7.2) The case of $\mathscr{F} = \mathbb{Z}_2$. Corollary 3.3, which establishes an upper bound for the sum of the Betti numbers of a generic intersection of real quadrics, plays the crucial role in our proof of Theorem 1.1. This corollary is proven by constructing an appropriate Morse function. For (co)homology with the coefficients in $\mathbb{Z}_2$ there is an alternative proof based on the Smith Theory.

Let $X(n, k)$ be a transversal intersection of k complex quadrics in $\mathbb{C}^n$ (see (7.1)). Using [3] one can estimate $H_*\big(X(n, k); \mathbb{Z}_2\big)$. Let $Z(n, k)$ be the set of real points in $X(n, k)$. In other words, $Z(n, k)$ is the set of fixed points of the complex conjugation on $\mathbb{C}^n$. Then

$$\text{rank } H_*\big(Z(n, k); \mathbb{Z}_2\big) \leq \text{rank } H_*\big(X(n, k); \mathbb{Z}_2\big)$$

(see [11]). On the other hand, $Z(n, k)$ may be viewed as an intersection of k real quadrics in $\mathbb{R}^n$. This approach, in principle, can give a sharper estimate for the sum of the Betti numbers, but since it is applicable only for a specific field of coefficients, we do not discuss this topic here.

Interesting results on the topology of the set $X(Q, a)$ defined by (1.2) were obtained in [1] by different methods. In particular, a spectral sequence was constructed in [1], which converges to $H^*(X(Q, a); \mathscr{F})$.

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