Math. Z. 233, 149–164 (2000)

Mathematische Zeitschrift

On a class of conformal metrics arising in the work of Seiberg and Witten

David E. Barrett

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1109, USA (e-mail: barrett@umich.edu)

Received August 18, 1998

Abstract. We examine a class of conformal metrics arising in the "$N = 2$ supersymmetric Yang-Mills theory" of Seiberg and Witten. We provide several alternate characterizations of this class of metrics and proceed to examine issues of existence and boundary behavior and to parameterize the collection of Seiberg-Witten metrics with isolated non-essential singularities on a fixed compact Riemann surface. In consequence of these results, the Riemann sphere $\widehat{\mathbb{C}}$ does not admit a Seiberg-Witten metric, but for all $\epsilon > 0$ there is a conformal metric on $\widehat{\mathbb{C}}$ of regularity $C^{2-\epsilon}$ which is Seiberg-Witten off of a finite set.

1. Results

Theorem 1. *Let ω be the area form of a conformal metric on a Riemann surface X. Then the following conditions are equivalent:*

(1) ω is locally of the form $\eta_1 \wedge \overline{\eta}_2 - \eta_2 \wedge \overline{\eta}_1$, where η_1 and η_2 are holomorphic (1,0)-forms.

(2) For all $p \in X$ there exists a holomorphic coordinate z on a neighborhood of p together with a harmonic function h so that $\omega = \frac{i}{2} h\, dz \wedge d\overline{z}$.

(3) On the domain of definition of an arbitrary holomorphic coordinate z, ω takes the form $\frac{i}{2} e^{2u}\, dz \wedge d\overline{z}$ with u either a harmonic function or a smooth superharmonic function satisfying

$$i\partial\overline{\partial} \log |i\partial\overline{\partial} u| + 4i\partial\overline{\partial} u = 2\pi\nu$$

in the sense of distributions, ν a locally finite sum of delta masses.

Supported in part by the National Science Foundation.

(4) There is a rank 2 vector bundle $E \to X$ equipped with a holomorphically flat Hermitian (1,1)-form-valued Lorentz metric g and a time-like holomorphic section $\eta : X \to E$ satisfying $\omega = g(\eta, \overline{\eta})$.

(5) Either ω comes from a flat metric, or else there are

(a) a non-negative (1,1)-form ρ on X inducing a conformal metric of curvature -1 with isolated conical singularities having total angles of the form $2n\pi, n \geq 2$

and

(b) a non-negative (1,1)-form ξ on X inducing a flat conformal metric with isolated conical singularities

satisfying $\omega = \xi^{3/2}\rho^{-1/2}$ on $\{z \in X : \rho(z) \neq 0\}$.

(6) There exist

(a) a rank 1 affine bundle $A \to X$ associated to a line bundle which is the tensor product of the square of the holomorphic tangent bundle of X with a flat Hermitian line bundle.

and

(b) a Levi-flat real hypersurface $S \subset A$ with circular fibers over X so that the induced radius function $r(z)$ satisfies $\omega = 1/r(z)$.

Remarks on the conditions of Theorem 1.

(3) Here the absolute value of a 2-form is regarded as a positive 2-form. Also, the operator $i\partial\overline{\partial}\log$ mapping 2-forms to 2-forms is defined in local coordinates by

$$i\partial\overline{\partial}\log\left(\psi(z)\,dz\wedge d\overline{z}\right) = i\partial\overline{\partial}\log\psi,$$

This operator maps log-integrable non-negative real 2-forms to distribution-theoretic real 2-forms.
Working near the origin of a local coordinate z, the second alternative is equivalent to saying that there is a non-negative integer k so that

$$\log|\Delta u| + 4u - 2k\log|z|$$

is harmonic in a neighborhood of the origin.
Clearing denominators we find that u must solve the degenerate elliptic equation

$$u_{z\overline{z}}\left(u_{zz\overline{z}\overline{z}} + 4u_{z\overline{z}}^2\right) = u_{zz\overline{z}}u_{z\overline{z}\overline{z}},$$

but the latter equation admits solutions that do not satisfy condition (3).

(5) To explain the terminology here we introduce the following.

Definition 2. Let z be a local coordinate vanishing at $p \in X$ and let $e^{u(z)}|dz|$ be a conformal metric on a deleted neighborhood of p. Then the *order* of the metric at p is defined to be

$$\chi_p \stackrel{\text{def}}{=} \liminf_{z \to 0} \frac{u(z)}{\log |z|}.$$

This quantity is easily seen to be independent of the choice of coordinate.

If the metric $e^{u(z)}|dz|$ above has constant curvature and $\chi_p > -1$ then the metric is said to have a *conical singularity with total angle* $2\pi(\chi_p + 1)$. (Compare [HuTr, 2.1] and see Lemmata 11, 12 and 13 below.)
The flat metric may be viewed as being obtained from the other two metrics by "interpolation of norms": $\xi = \omega^{2/3}\rho^{1/3}$. Equivalently, we may say that ω is obtained from ρ and ξ by extrapolation.

(6) The condition (a) means that the affine bundle can be described by transition functions of the form

$$(z, w) \mapsto \left(\varphi(z), e^{i\theta} \left(\varphi'(z)\right)^2 w + \upsilon(z)\right).$$

The radius function $r(z)$ is to be computed with respect to the natural $(-1,-1)$-differential-valued metric given locally by $d((z, w_1), (z, w_2)) = |w_1 - w_2|$. (See [Leh, IV.1.4] for definitions.) The reciprocal of $r(z)$ is thus a $(1,1)$-differential or $(1,1)$-form.
The condition that S be Levi-flat means that the Levi-form of S vanishes identically, or alternatively that S is foliated by Riemann surfaces.

The condition 1 appears in the "$N = 2$ supersymmetric Yang-Mills theory" of Seiberg and Witten [SW] as the correct local form of the metric giving the kinetic terms on the moduli space of vacua. We will call a metric satisfying the equivalent conditions of Theorem 1 a (one-dimensional) *Seiberg-Witten metric*. (See [SW, 3.1] for a generalization to Kähler metrics of higher dimension.)

Definition 3. A conformal metric $e^{u(z)}\,|dz|$ on the unit disk Δ will be said to be *bounded below* if $\inf_\Delta u > -\infty$.

Theorem 4. *If $e^{u(z)}\,|dz|$ is a Seiberg-Witten metric bounded below on Δ then the function u has non-tangential boundary limits almost everywhere on $\mathbb{T} = b\Delta$.*

The function on $\mathbb{T}$ arising in Theorem 4 will be called the *boundary function* of the metric.

Theorem 5. *Let $\phi \in L^\infty(\mathbb{T})$ be real-valued. Then the set of Seiberg-Witten metrics $e^{u(z)}\,|dz|$ bounded below on Δ with boundary function ϕ is parameterized by*

$$B_{ne}\left(H_0^\infty(\Delta)\right)/S^1 \times M_{sing}^+(\mathbb{T}),$$

where $H_0^\infty(\Delta)$ is the Banach space of bounded holomorphic functions on Δ vanishing at the origin, $B\left(H_0^\infty(\Delta)\right)$ is the closed unit ball of $H_0^\infty(\Delta)$, $B_{ne}\left(H_0^\infty(\Delta)\right)$ is the set of non-extreme points of $B\left(H_0^\infty(\Delta)\right)$, the quotient $B_{ne}\left(H_0^\infty(\Delta)\right)/S^1$ is taken with respect to multiplication by unimodular scalars, and $M_{sing}^+(\mathbb{T})$ is the space of non-negative singular Borel measures on $\mathbb{T}$.

We return to the examination of Seiberg-Witten metrics on general Riemann surfaces.

Theorem 6. *All complete Seiberg-Witten metrics are flat.*

Theorem 7. *The surface X admits a Seiberg-Witten metric if and only if*

X is non-compact

or

X is compact and $\operatorname{genus}(X) = 1$.

Moreover, X admits a non-flat Seiberg-Witten metric if and only if X is non-compact and is not covered by $\mathbb{C}$.

Theorem 8. *Let $X = \widehat{X} \setminus \{z_1, \ldots, z_P\}$ where $\widehat{X}$ is a compact Riemann surface and $z_1, \ldots, z_P$ are distinct points of $\widehat{X}$ with $P \geq 1$. (If $\operatorname{genus}\widehat{X} = 0$ then we require $P \geq 3$.)*

Then any Seiberg-Witten metric on X with $\chi_{z_1}, \ldots, \chi_{z_P} > -\infty$ must satisfy

$$2(\operatorname{genus}\widehat{X} - 1) \leq \sum_{j=1}^{P} \chi_{z_j} \leq 3(\operatorname{genus}\widehat{X} - 1) + \frac{P}{2},$$

and there are no other restrictions on the χ_{z_j}.

The flat metrics correspond precisely to the case

$$\sum_{j=1}^{P} \chi_{z_j} = 2(\operatorname{genus}\widehat{X} - 1),$$

and in this case the χ_{z_j} determine the metric up to a multiplicative constant.

For

$$2(\operatorname{genus}\widehat{X}-1)<\sum_{j=1}^{P}\chi_{z_j}\leq 3(\operatorname{genus}\widehat{X}-1)+\frac{P}{2}, \tag{1.1}$$

the associated metrics correspond up to a multiplicative constant to a choice of the following:

- *a set of "branch points" $w_1,\ldots,w_D\in X$ with*

$$0\leq D\leq 6(\operatorname{genus}\widehat{X}-1)+P-2\sum_{k=1}^{P}\chi_{z_j} \tag{1.2}$$

  (multiple listings allowed);

- *non-negative real numbers $s_{z_1},\ldots,s_{z_P}$ satisfying*

$$\sum_{j=1}^{P}s_{z_j}=6(\operatorname{genus}\widehat{X}-1)+P-D-2\sum_{k=1}^{P}\chi_{z_j}. \tag{1.3}$$

Theorem 9. *For all $N\geq 7$ and any collection $z_1,\ldots,z_N$ of distinct points in $\widehat{\mathbb{C}}$ there is a conformal metric of regularity class $C^{2-12/N}$ on $\widehat{\mathbb{C}}$ which is Seiberg-Witten on $\widehat{\mathbb{C}}\setminus\{z_1,\ldots,z_N\}$.*

2. Lemmata

Lemma 10. *If $\eta_1,\eta_2,\widetilde{\eta}_1$, and $\widetilde{\eta}_2$ are holomorphic (1,0)-forms on a connected open set $U\subset X$ satisfying*

$$\eta_1\wedge\overline{\eta}_2-\eta_2\wedge\overline{\eta}_1=\widetilde{\eta}_1\wedge\overline{\widetilde{\eta}}_2-\widetilde{\eta}_2\wedge\overline{\widetilde{\eta}}_1\not\equiv 0$$

on U then there are a matrix $M\in SL(2,\mathbb{R})$ and a unimodular constant $e^{i\theta}$ satisfying

$$\begin{pmatrix}\widetilde{\eta}_1\\ \widetilde{\eta}_2\end{pmatrix}=e^{i\theta}M\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}.$$

If η_1/η_2 is non-constant then the matrix $e^{i\theta}M$ is uniquely determined. If η_1/η_2 is constant then we may take $\theta=0$, and with this additional constraint M is uniquely determined.

Proof of Lemma 10. The hypothesis implies in particular that $\eta_2 \not\equiv 0$; thus we may write

$$\begin{aligned}\eta_1 &= f\eta_2\\ \widetilde{\eta}_1 &= g_1\eta_2\\ \widetilde{\eta}_2 &= g_2\eta_2\end{aligned}$$

with f, g_1, g_2 meromorphic. Our basic equation now reads

$$f - \overline{f} = g_1\overline{g}_2 - g_2\overline{g}_1 \not\equiv 0. \tag{2.1}$$

If g_2 is nonconstant then applying $\partial\overline{\partial}$ to (2.1) and rearranging we have

$$dg_1/dg_2 = \overline{dg_1/dg_2}.$$

Thus $dg_1 = \lambda\, dg_2$ for some real constant λ, yielding $g_1 = \lambda g_2 + C$. Substituting back into (2.1) we find that

$$f - \overline{f} = C\overline{g}_2 - \overline{C}g_2 \not\equiv 0$$

so that

$$C \neq 0$$

and

$$\overline{C}g_2 = -f + \mu$$

for some real constant μ. Consequently we must take

$$\theta = \arg C, \quad M = \begin{pmatrix} -\dfrac{\lambda}{|C|} & \dfrac{\lambda\mu}{|C|} + |C| \\ -\dfrac{1}{|C|} & \dfrac{\mu}{|C|} \end{pmatrix}.$$

The case where g_1 is nonconstant is similar.

If g_1 and g_2 are both constant then so are $\operatorname{Re} f$ and f. In this case we can take

$$\theta = 0, \quad M = \begin{pmatrix} \dfrac{g_1 - \overline{g}_1}{f - \overline{f}} & \dfrac{\overline{g}_1 f - g_1\overline{f}}{f - \overline{f}} \\ \dfrac{g_2 - \overline{g}_2}{f - \overline{f}} & \dfrac{\overline{g}_2 f - g_2\overline{f}}{f - \overline{f}} \end{pmatrix},$$

and the $\mathbb{R}$-linear independence of η_1 and η_2 implies that this is the only choice satisfying $\theta = 0$. □

Lemma 11. *Let ρ be the area form of a conformal metric of curvature -1 defined on a deleted neighborhood of $p \in X$. Then there exist a uniquely-determined non-negative $s_p \in \mathbb{R}$ and a local coordinate z vanishing at p so that*

$$\rho = \begin{cases} \dfrac{2is_p^2|z|^{2(s_p-1)}\,dz \wedge d\overline{z}}{(1-|z|^{2s_p})^2} & \text{if } s_p > 0, \\ \dfrac{i\,dz \wedge d\overline{z}}{2|z|^2(\log|z|)^2} & \text{if } s_p = 0. \end{cases}$$

Moreover, the form ρ is integrable near p, and the distribution-theoretic Gauss-Bonnet form

$$\kappa\rho = -i\partial\overline{\partial}\log\rho$$

is equal to

$$-2\pi(s_p - 1)\delta_p - \rho.$$

If $s_p \in \{1, 2, 3, \dots\}$ then the metric is obtained by pulling back the Poincaré metric via a holomorphic map from a neighborhood of p into the unit disk Δ. The map is determined up to composition with an automorphism of Δ, and the multiplicity of the map at p is equal to s_p.

Proof of Lemma 11. Choose a coordinate $\tilde{z}$ vanishing at p so that ρ is defined for $0 < |\tilde{z}| < 1$. Then $\exp^* \rho$ comes from a metric of curvature -1 on the left half-plane L. Any such metric is the pullback of the Poincaré metric on the upper half-plane H for some holomorphic map $f : L \to H$. The metric on L determines f up to composition with an automorphism of H; since the metric on L is invariant under the translation $T : w \mapsto w + 2\pi i$, there is an automorphism Φ of H satisfying $f \circ T = \Phi \circ f$.

If Φ is hyperbolic then the quotient of H by the group generated by Φ is biholomorphic to an annulus A with inner radius $r_1 > 0$ and outer radius $r_2 < \infty$; let $\Psi : H \to A$ denote the corresponding quotient map. Then $\Psi \circ f \circ \log$ is a well-defined holomorphic map from $\Delta^* \stackrel{\text{def}}{=} \Delta \setminus \{0\}$ to A which induces an isomorphism of the corresponding fundamental groups. But Riemann's removable singularity theorem implies that $\Psi \circ f \circ \log$ extends to a holomorphic map of Δ into $A \cup \mathrm{b}A$; the maximum principle implies that the extended map in fact maps Δ into A, so that $\Psi \circ f \circ \log$ is homotopic to a constant map. The contradiction shows that Ψ cannot be hyperbolic.

If Φ is parabolic then the quotient of H by the group generated by Φ is biholomorphic to Δ^*. The corresponding map $\Psi \circ f \circ \log : \Delta^* \to \Delta^*$ again induces an isomorphism of fundamental groups and thus extends to a map $\Delta \to \Delta$ which is unbranched at the origin. We may thus replace the coordinate $\tilde{z}$ by $z \stackrel{\text{def}}{=} (\Psi \circ f \circ \log)(\tilde{z})$; with respect to the new coordinate our metric is just the Poincaré metric on Δ^*. This yields the desired formula for ρ in the case $s_p = 0$.

If Φ is elliptic then Φ may be conjugated to the map $\Delta \to \Delta, \tau \mapsto e^{2\pi\tilde{s}i}\tau$ with $0 \leq \tilde{s} < 1$. Thus we may replace f by a map $f : L \to \Delta$ satisfying $f(w + 2\pi i) = e^{2\pi\tilde{s}i} f(w)$. It follows that $g(\tilde{z}) \stackrel{\text{def}}{=} (f \circ \log)(\tilde{z})/\tilde{z}^{\tilde{s}}$ is single-valued. The boundedness of f implies that g has a removable singularity at p. Writing $g(\tilde{z}) = \tilde{z}^k h(\tilde{z})$ with $h(0) \neq 0$ and setting $s_p = \tilde{s}+k$, $z = \tilde{z}\ \sqrt[s_p]{h(\tilde{z})}$ we have $(f \circ \log)(\tilde{z}) = z^{s_p}$. Using z as the coordinate now, we find that our metric is defined by pulling back the Poincaré metric of Δ via the map $z \mapsto z^{s_p}$, yielding the desired formula for ρ when $s_p > 0$.

The remaining claims follow by inspection and direct computation along with the standard formula $i\partial\overline{\partial} \log |z| = \pi\delta_0$. □

Lemma 12. *If S is a discrete subset of a simply-connected Riemann surface X and ρ is the area form of a conformal metric of curvature -1 on $X \setminus S$ satisfying $s_p \in \{1, 2, 3, \dots\}$ for $p \in S$ then there is a non-constant holomorphic map $f : X \to \Delta$ so that*

$$\rho = \frac{2i\, df \wedge d\overline{f}}{(1 - |f|^2)^2}, \tag{2.2}$$

the pull-back of the Poincaré area form for Δ.

Proof. Lemma 11 shows that the sheaf of germs of maps f satisfying (2.2) forms a covering space for X; thus Lemma 12 follows from the monodromy theorem. □

Lemma 13. *The order χ_p of a metric satisfying condition 5 of Theorem 1 with an isolated singularity at p must be $< \infty$. If $\chi_p > -\infty$ then (with notation as in Definition 2) precisely one of the limits*

$$\lim_{z\to 0} |z|^{-\chi_p} e^{u(z)}$$

$$\lim_{z\to 0} |z|^{-\chi_p} e^{u(z)} \Big/ \sqrt{\big|\log |z|\big|}$$

exists and is finite and non-zero.

Proof. In the non-flat case we have $\xi = \frac{i}{2} e^{2v}\, dz \wedge d\overline{z}$ with v harmonic. Let

$$t_p = \frac{1}{2\pi} \int_{|z|=\epsilon} *dv.$$

Then $e^{v(z)} = |z|^{t_p} \left|e^{g(z)}\right|$, g holomorphic in a deleted neighborhood of p.

If g has a non-removable singularity at p then ξ and (in view of Lemma 11) ω have order $-\infty$ at p.

If g has a removable singularity at p then using Lemma 11 we have

$$\chi_p = \frac{1}{2}(3t_p - s_p + 1)$$

with

$$\lim_{z\to 0} |z|^{-\chi_p} e^{u(z)} = \frac{e^{\frac{3}{2}g(0)}}{\sqrt{2s_p}}$$

if $s_p > 0$ and

$$\lim_{z\to 0} |z|^{-\chi_p} e^{u(z)} \Big/ \sqrt{\big|\log|z|\big|} = e^{\frac{3}{2}g(0)}$$

if $s_p = 0$.

The argument for the flat case is similar. □

3. Proofs of the theorems

Proof of Theorem 1. For possible future convenience, some redundancy is incorporated into the following chain of implications.

(1)⇒(2): Locally we may write $\eta_2 = dz, \eta_1 = f\,dz,\ f$ holomorphic, yielding (2) with $h = 4\,\mathrm{Im}\, f$.

(2)⇒(1): Locally we may choose f holomorphic with $\mathrm{Im}\, f = \frac{h}{4}$. Then take $\eta_2 = dz, \eta_1 = f\,dz$.

(1)⇒(4): From Lemma 10 we find that that the germs of η_1, η_2 satisfying $\omega = \eta_1 \wedge \overline{\eta}_2 - \eta_2 \wedge \overline{\eta}_1$ form a flat $S^1 \cdot SL(2,\mathbb{R})$ bundle over X. (If η_2/η_1 is constant then the structure group is just $SL(2,\mathbb{R})$.) Let F be the corresponding rank 2 vector bundle. Continuation of a germ η_1, η_2 yields a holomorphic section $\eta = \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}$ of $F \otimes T^*X$ so that $\omega = g(\eta, \overline{\eta})$, where g is the flat Hermitian (1,1)-form-valued Lorentz metric on $E \overset{\text{def}}{=} F \otimes T^*X$ defined by

$$g\left(\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}, \begin{pmatrix}\overline{\zeta}_1\\ \overline{\zeta}_2\end{pmatrix}\right) = \eta_1 \wedge \overline{\zeta}_2 - \eta_2 \wedge \overline{\zeta}_1. \tag{3.1}$$

(4)⇒(1): This follows from the observation that the metric g is locally equivalent to the metric on $T^*X \oplus T^*X$ defined by (3.1).

(4)⇒(6): The proof of the equivalence of (4) and (1) shows in particular that we may assume that $E = F \otimes T^*X$, where F is a flat bundle with monodromy in $S^1 \cdot SL(2,\mathbb{R})$, and that g is given by (3.1).

We construct the affine bundle A by removing the graphs of multiples of η from E and projectivizing. In local coordinates on $U \subset X$ we may accomplish this by the map

$$\begin{aligned} &\left(E\,\big|_U\right) \setminus (\text{graphs of multiples of } \eta) \to TU \otimes TU \\ (3.2)\quad &\left(z, \begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}\right) \mapsto \left(z, \zeta_2 \otimes \eta_2(z)^{-1} \otimes (\zeta_1 \otimes \eta_2(z) - \zeta_2 \otimes \eta_1(z))^{-1}\right). \end{aligned}$$

Continuing along a loop with associated monodromy matrix $e^{i\theta}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$, $\alpha\delta - \beta\gamma = 1$, we find that

$$\tau \overset{\text{def}}{=} \zeta_2 \otimes \eta_2(z)^{-1} \otimes (\zeta_1 \otimes \eta_2(z) - \zeta_2 \otimes \eta_1(z))^{-1}$$

transforms to

$$e^{-i2\theta}\left(\tau + \frac{\gamma}{\eta_2(\gamma\eta_1 + \delta\eta_2)}\right).$$

This shows that the line bundle associated to A is indeed $TX \otimes TX$ tensored with a flat Hermitian line bundle.

We let S be the image of the light cone of g under the quotient map

$$E \setminus (\text{graphs of multiples of } \eta) \to A.$$

Since the light cone is given by $\zeta_1/\zeta_2 \in \mathbb{R} \cup \{\infty\}$ and the map (3.2) is linear fractional in ζ_1/ζ_2, the fibers of S are indeed circles. The sets $\zeta_1/\zeta_2 \equiv \lambda \in \mathbb{R} \cup \{\infty\}$ project to Riemann surfaces foliating S, so S is indeed Levi-flat.

To find the radius function for S we note that since $\zeta_1/\zeta_2 = \tau^{-1} \otimes \eta_2^{-2} + \eta_1/\eta_2$ it follows that

$$(z,\tau) \in S \text{ if and only if } \tau^{-1} \otimes \eta_2^{-2} + \eta_1/\eta_2 = \overline{\tau^{-1} \otimes \eta_2^{-2} + \eta_1/\eta_2}.$$

This last condition may be rewritten as

$$\left|\tau + \frac{1}{\eta_1 \otimes \eta_2 - \frac{\overline{\eta}_1}{\overline{\eta}_2}\eta_2^2}\right| = \frac{1}{\eta_1 \wedge \overline{\eta}_2 - \eta_2 \wedge \overline{\eta}_1}$$

so that the radius is indeed $r(z) = 1/\left(\eta_1 \wedge \overline{\eta}_2 - \eta_2 \wedge \overline{\eta}_1\right) = 1/\omega$.

(6)⇒(3): In local coordinates S has a center function $c(z)$ and radius $e^{-2u(z)}$. The Levi-flatness condition [Ber, Prop. 2.3] reads

$$(3.3)\qquad 2u_{z\overline{z}} = -|c_{\overline{z}}|^2 e^{4u}, \left(c_{\overline{z}} e^{4u}\right)_z = 0.$$

(Here subscripts denote differentiation.)

If the holomorphic function $\overline{c}_z e^{4u}$ vanishes identically then of course u is harmonic. Otherwise we can write

$$2u_{z\overline{z}} = -\left(c_{\overline{z}} e^{4u}\right)\left(\overline{c}_z e^{4u}\right) e^{-4u}$$

and it follows easily that u is superharmonic and that $i\partial\overline{\partial}\log|i\partial\overline{\partial}u| + 4i\partial\overline{\partial}u$ is $2\pi\nu$, where ν is the sum of delta masses corresponding to the zeros of $\overline{c}_z e^{-4u}$ (counted with multiplicity).

(3)⇒(5): If u is harmonic then ω comes from a flat metric.

If u is superharmonic and $i\partial\overline{\partial}\log|i\partial\overline{\partial}u| + 4i\partial\overline{\partial}u = 2\pi\nu$, ν a locally finite sum of delta masses, then we may set $\rho = -4i\partial\overline{\partial}u$. We have

$$\begin{aligned} -i\partial\overline{\partial}\log\rho &= -i\partial\overline{\partial}\log|i\partial\overline{\partial}u| \\ &= 4i\partial\overline{\partial}u - 2\pi\nu \\ &= -\rho - 2\pi\nu, \end{aligned}$$

which implies that the conformal metric induced by ρ on $X \setminus \operatorname{supp}\nu$ has curvature -1. Lemma 11 shows that this metric has conical singularities at points of $\operatorname{supp}\nu$ and that the total angles at these singularities are of the form $2n\pi, n \geq 2$. We now set

$$\xi = \omega^{2/3}\rho^{1/3}.$$

We then have

$$\begin{aligned} -i\partial\overline{\partial}\log\xi &= -\frac{4}{3}i\partial\overline{\partial}u + \frac{4}{3}i\partial\overline{\partial}u - \frac{2}{3}\pi\nu \\ &= -\frac{2}{3}\pi\nu \end{aligned}$$

which shows that the metric induced by ξ is flat off of $\operatorname{supp}\nu$. Moreover, for $p \in \operatorname{supp}\nu$ we find that $\log\xi - \frac{2}{3}\nu(\{p\})\log|z|$ extends harmonically across p, showing that the flat metric has a conical singularity at p.

(5)⇒(1): If ω comes from a flat metric then we may choose a local coordinate z so that $\omega = \frac{i}{2}\,dz \wedge d\overline{z}$. Then we may set

$$\begin{aligned} \eta_1 &= \frac{i}{2}\,dz, \\ \eta_2 &= \frac{1}{2}\,dz. \end{aligned}$$

If the metric is not flat then by the end of Lemma 11 with H replacing Δ we may represent ρ locally as

$$\frac{2\,df \wedge d\overline{f}}{i(f-\overline{f})^2}$$

for some f mapping holomorphically into H. Since the orders of ξ at the singularities are of the form $2n\pi/3$, ξ admits a local representation of the form

$$\frac{i}{2}|\xi(z)|^{2/3}\,dz\wedge d\bar{z}, \tag{3.4}$$

ξ holomorphic. Then we may set

$$\eta_1 = \frac{1}{2}\sqrt{\frac{\xi(z)}{df/dz}}f(z)\,dz,$$
$$\eta_2 = \frac{1}{2}\sqrt{\frac{\xi(z)}{df/dz}}\,dz.$$

(1)⇒(5): Lemma 10 and the positivity of $\eta_1\wedge\overline{\eta}_2-\eta_2\wedge\overline{\eta}_1$ imply that η_1/η_2 maps into the upper half-plane H and is well-defined up to composition with automorphisms of H.

If η_1/η_2 is constant then $\eta_2\wedge\overline{\eta}_1-\eta_1\wedge\overline{\eta}_2$ induces a flat metric.

If η_1/η_2 is not constant we may set ρ to be the pullback

$$\frac{2\,d(\eta_1/\eta_2)\wedge d\overline{(\eta_1/\eta_2)}}{i\left((\eta_1/\eta_2)-\overline{(\eta_1/\eta_2)}\right)^2}$$

of the Poincaré metric via η_1/η_2. If we now set

$$\xi = 2^{1/3}\,|d(\eta_1/\eta_2)\otimes\eta_2\otimes\eta_2|^{2/3}$$

we find that

$$\xi^{3/2}\rho^{-1/2} = \eta_1\wedge\overline{\eta}_2-\eta_2\wedge\overline{\eta}_1=\omega.$$

□

Proof of Theorem 4. We will use condition 6 of Theorem 1 as our definition of a Seiberg-Witten metric. Since we are working over $\varDelta$, we may view S as a Levi-flat hypersurface in $\varDelta\times\mathbb{C}$ with circular fibers and radius function $r(z)=e^{-2u(z)}$ [For, 26.1 & 30.5].

Since $\varDelta$ is simply-connected, S is the disjoint union of graphs of holomorphic functions. Choosing one such function f, and transforming S via the map $(z,w)\mapsto(z,w-f(z))$, we may assume that $\varDelta\times\{0\}\subset S$.

The fibers of S are now seen to be circles passing through 0 with bounded radii. It follows that the centers are also bounded and thus that the functions whose graphs lie in S are in $H^\infty(\varDelta)$. Let f_1,f_2,f_3 be three such functions. (We can take $f_3\equiv 0$.) Since $f_1(z),f_2(z),f_3(z)$ determine $r(z)$, r has a well-defined non-negative non-tangential limit at each boundary point where f_1, f_2, f_3 have distinct non-tangential limits. (The boundedness of $r(z)$ implies that the limits cannot be distinct and collinear.) The theorem now follows from Fatou's theorem and the fact that an H^∞ function is determined by its boundary values on any set of positive measure [Rud, 15.19]. $\square$

Proof of Theorem 5. Let $e^{u(z)}\,|dz|$ be a Seiberg-Witten metric on Δ.

If $e^{u(z)}\,|dz|$ is non-flat then by condition 5 of Theorem 1, Lemma 12 and a similar globalization of (3.4) there exist a non-constant holomorphic map $f : \Delta \to \Delta$ with $f(0) = 0$ and a holomorphic function $\xi(z)$ so that

$$e^{2u(z)} = \frac{1 - |f(z)|^2}{2|f'(z)|}|\xi(z)|;$$

f and ξ are determined up to multiplication by a unimodular scalar.

Since the right-hand side has no zeros or poles, $\xi(z) = 2f'(z)e^{2h(z)}$, where h is determined up to a purely imaginary additive constant. Thus

$$u(z) = \frac{1}{2}\log(1 - |f(z)|^2) + \operatorname{Re} h.$$

Allowing $f \equiv 0$ we have included the flat metrics as well.

Assume now that u is bounded below with boundary function ϕ. Since $\log(1-|f(z)|^2)$ is negative, $\operatorname{Re} h$ is also bounded below. Thus the boundary function of $\operatorname{Re} h$ is integrable [Rud, 11.30]; but this boundary function is just ϕ minus the boundary function ψ of $\frac{1}{2}\log(1-|f(z)|^2)$. Thus ψ is integrable, which says precisely that f is non-extreme in $B\,(H_0^\infty(\Delta))$ [dLR, Thm. 12]. Moreover, there is a non-negative singular Borel measure μ on $\mathbb{T}$ so that $\operatorname{Re} h$ is the Poisson integral of $(\phi - \psi)\ d\theta + \mu$ [Rud, 11.30].

Conversely, given $f \in B_{ne}\,(H_0^\infty(\Delta))$ and $\mu \in M_{sing}^+(\mathbb{T})$ the argument above can be reversed to construct a Seiberg-Witten metric bounded below on Δ with boundary function ϕ. $\square$

Proof of Theorem 6. Suppose that the Riemann surface X admits a complete non-flat Seiberg-Witten metric. The pullback of this metric to the universal cover $\widetilde{X}$ will also be Seiberg-Witten and complete [Cha, Prop. 4.1].

Lemma 12 shows that the form ρ on $\widetilde{X}$ induced by the metric by way of condition 5 of Theorem 1 comes from a non-constant map from $\widetilde{X}$ to Δ. By Liouville's theorem and the uniformization theorem for Riemann surfaces [FK, IV.5.6], $\widetilde{X}$ is biholomorphic to the unit disk Δ. As in the beginning of the proof of Theorem 5, the induced complete non-flat Seiberg-Witten metric on Δ may be written in the form $e^{u(z)}\,|dz|$ with

$$u(z) = \frac{1}{2}\log(1 - |f(z)|^2) + \operatorname{Re} h,$$

f and h holomorphic. The flat metric $|e^{h(z)}|\,|dz|$ majorizes $e^{u(z)}\,|dz|$ and is thus also complete. The corresponding exponential map $\mathbb{C} \cong \mathbb{T}_0\Delta \to \Delta$ is thus entire and non-constant; but this contradicts Liouville's theorem.

Thus no complete non-flat Seiberg-Witten exists. □

Proof of Theorem 7. We use condition 5 of Theorem 1 as the definition of a Seiberg-Witten metric.

If X is non-compact and hyperbolic then X is covered by the disk Δ; the Poincaré metric on Δ induces the Poincaré metric on X. We may construct a non-flat Seiberg-Witten metric by taking ρ to be the corresponding area form and taking ξ to be the absolute value of any nowhere-vanishing quadratic differential on X [For, Thm. 30.3].

If X is covered by $\mathbb{C}$ (that is, X is biholomorphic to the plane or to the punctured plane, or else X is compact of genus 1) then X inherits a flat metric from $\mathbb{C}$. But the proof of Theorem 6 shows that X does not admit a non-flat Seiberg-Witten metric.

If X is a compact of genus $\neq 1$ then the Gauss-Bonnet theorem shows that X does not admit a flat metric, and Theorem 6 shows that X does not admit a non-flat Seiberg-Witten metric. □

Proof of Theorem 8. We begin by accounting for the flat metrics. Letting g be a non-singular reference metric on $\widehat{X}$, a flat metric $e^{\psi}g$ on X with orders $\chi_{z_1}, \dots \chi_{z_P} > -\infty$ must satisfy

$$\partial\overline{\partial}\psi = -\partial\overline{\partial}\log g - i\pi \sum_{j=1}^{P} \chi_{z_j}\delta_{z_j}. \tag{3.5}$$

Applying the (distribution-theoretic) Gauss-Bonnet theorem to $e^{\psi}g$ we have

$$4\pi(1 - \operatorname{genus}\widehat{X}) = -2i\int_{\widehat{X}} \partial\overline{\partial}\log(e^{\psi}g) = -2\pi\sum_{j=1}^{P}\chi_{z_j}$$

so that

$$\sum_{j=1}^{P}\chi_{z_j} = 2(\operatorname{genus}\widehat{X} - 1). \tag{3.6}$$

On the other hand, if (3.6) holds then the right-hand side of (3.5) has mean value zero and thus Hodge theory allows us to solve for ψ.

Turning now to the non-flat case and using the notation of condition 5 of Theorem 1, let b_p denote the order at p of the metric induced by ξ so that

$$\chi_p = \frac{1}{2}(3b_p - s_p + 1).$$

In particular, $3b_p = s_p - 1$ for $p \in X$.

Applying the Gauss-Bonnet theorem to the metrics induced ξ and ρ we have, respectively,

$$\sum_{p\in\widehat{X}} b_p = 2(\text{genus}\,\widehat{X} - 1)$$

and

$$\frac{1}{2\pi}\int_{\widehat{X}} \rho + \sum_{p\in\widehat{X}} (s_p - 1) = 2(\text{genus}\,\widehat{X} - 1).$$

Thus

$$\begin{aligned}\sum_{j=1}^{P} \chi_{z_j} &= \sum_{p\in\widehat{X}} \chi_p \\ &= \frac{1}{2}\left(\sum_{p\in\widehat{X}} 3b_p - \sum_{p\in\widehat{X}} (s_p - 1)\right) \\ &= 2(\text{genus}\,\widehat{X} - 1) + \frac{1}{4\pi}\int_{\widehat{X}} \rho \\ &> 2(\text{genus}\,\widehat{X} - 1),\end{aligned}$$

yielding the left-hand side of (1.1) Also, letting $D = \sum_{p\in X}(s_p - 1)$ we have

$$\begin{aligned}\sum_{j=1}^{P} \chi_{z_j} &= \sum_{p\in\widehat{X}} \chi_p \\ &= \frac{1}{2}\left(\sum_{p\in\widehat{X}} 3b_p - \sum_{p\in\widehat{X}} (s_p - 1)\right) \\ &= \frac{1}{2}\left(6(\text{genus}\,\widehat{X} - 1) + P - D - \sum_{j=1}^{P} s_{z_j}\right)\end{aligned}$$

so that (1.3) must hold. Since the s_{z_j} are non-negative, (1.2) follows, as does the right half of (1.1).

Conversely, given

- χ_{z_j} satisfying (1.1);
- $w_1, \ldots, w_D \in X$ with D satisfying (1.2) (multiple listings allowed);
- non-negative s_{z_j} satisfying (1.3)

the results of [HuTr, Thm. B] assert that there is a unique curvature -1 metric on X with a conical singularity with total angle $2\pi s_{z_j}$ at each z_j and a conical singularity with total angle $2\pi(1+ \text{ multiplicity})$ at each w_k. Setting $b_{z_j} = \frac{1}{3}\left(2\chi_{z_j} + s_{z_j} - 1\right)$ we find that (1.3) guarantees that

$$\sum_{p\in\widehat{X}} b_p = \frac{D}{3} + \sum_{j=1}^{P} b_{z_j} = 2(\text{genus}\,\widehat{X} - 1)$$

as required; the methods of the first paragraph of the current proof then allow us to construct the flat factor ξ, uniquely up to scalar multiplication. □

Remark 14. The metric appearing on $\mathbb{C}\setminus\{\pm 1\}$ featured in [SW] has $\chi_{-1} = \chi_1 = 0$, $\chi_\infty = -\frac{3}{2}$, $N = 0$, and $s_{-1} = s_1 = s_\infty = 0$. (See [Bar].)

Proof of Theorem 9. We apply Theorem 8 with $P = N$, $D = 0$, $\chi_{z_j} = 0$, and $s_{z_j} = 1 - \frac{6}{N}$. Using Lemma 11, for each z_j we can pick a coordinate z vanishing at z_j so that the metric admits the local representation

$$\sqrt{1 - |z|^{2-12/N}}|h(z)||dz|$$

with h holomorphic near 0 and $h(0) \neq 0$; thus the metric belongs belongs to the regularity class $C^{2-12/N}$. □

References

[Bar] D. Barrett, On a problem of Seiberg and Witten, Ann. Polon. Math. **70** (1998), 25–34.

[Ber] B. Berndtsson, Levi-flat surfaces with circular sections, Several Complex Variables, Mathematical Notes Vol. 38, Princeton University Press, 1993, pp. 136–159.

[Cha] I. Chavel, Riemannian geometry: a modern introduction, Cambridge Univ. Press, 1993.

[dLR] K. deLeeuw and W. Rudin, Extreme points and extreme problems in H_1, Pacific J. Math. **8** (1958), 467–485.

[FK] H. Farkas and I. Kra, Riemann surfaces, Springer-Verlag, 1980.

[For] O. Forster, Lecture on Riemann surfaces, Springer-Verlag, 1981.

[HuTr] D. Hulin and M. Troyanov, Prescribing curvature on open surfaces, Math. Ann. **293** (1992), 277–315.

[Leh] O. Lehto, Univalent functions and Teichmüller spaces, Springer-Verlag, 1987.

[Rud] W. Rudin, Real and complex analysis (3rd ed.), McGraw-Hill, 1987.

[SW] N. Seiberg, E. Witten, Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory, Nuclear Physics B **426** (1994), 19–52.