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Non-Existence of Black Hole Solutions for a Spherically Symmetric, Static Einstein–Dirac–Maxwell System

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Abstract: We consider for $j = \frac{1}{2}, \frac{3}{2}, \ldots$ a spherically symmetric, static system of $(2j+1)$ Dirac particles, each having total angular momentum j. The Dirac particles interact via a classical gravitational and electromagnetic field.

The Einstein–Dirac–Maxwell equations for this system are derived. It is shown that, under weak regularity conditions on the form of the horizon, the only black hole solutions of the EDM equations are the Reissner–Nordström solutions. In other words, the spinors must vanish identically. Applied to the gravitational collapse of a "cloud" of spin-$\frac{1}{2}$-particles to a black hole, our result indicates that the Dirac particles must eventually disappear inside the event horizon.

1. Introduction

In [1,2], particle-like solutions of the Einstein-Dirac-Maxwell (EDM) equations were constructed for a static, spherically symmetric singlet system. It was found that the solutions in a given state (i.e. in the ground state or in any fixed excited state) cease to exist if the rest mass m of the fermions becomes larger than a certain threshold value m_s. The most natural physical interpretation of this observation is that for $m > m_s$, the gravitational interaction becomes so strong that a black hole would form. This suggests that there should be black hole solutions of the coupled EDM equations for large fermion masses. The work [3], however, indicates that this intuitive picture of black hole formation is wrong. Namely, it was proved in [3] that the Dirac equation has no time-periodic solutions in a Reissner-Nordström black hole background, even if the Dirac particles have angular momentum and can thus, in the classical picture, "rotate around" the black hole. This implies that if we neglect the influence of the Dirac particles

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on the gravitational and electromagnetic field, there are no black hole solutions of the EDM equations.

In order to understand if and how Dirac particles can form black holes, we study in this paper the fully coupled EDM equations. We do not assume the Dirac particles to be in a spherically symmetric state; they are allowed to have angular momentum j. However, we arrange $2j+1$ of these particles in such a way that the total system is static and spherically symmetric. In the language of atomic physics, we consider the completely filled shell of states with angular momentum j. Classically, one can think of this multiple-particle system as of several Dirac particles rotating around a common center such that their angular momentum adds up to zero. Since the system of fermions is spherically symmetric, we get a consistent set of equations if we also assume spherical symmetry for the gravitational and electromagnetic field. This allows us to separate out the angular dependence and reduce the problem to the analysis of a system of nonlinear ODEs.

We prove analytically that, under weak regularity conditions on the form of the horizon, the black hole solutions of our coupled EDM equations are either the Reissner–Nordström solutions (in which case the Dirac wave functions are identically zero), or the event horizon has the form of the extreme Reissner–Nordström metric. In the latter case, we show numerically that the Dirac wave functions cannot be normalized. Thus our Einstein–Dirac–Maxwell system does not admit black hole solutions. Our results show that the study of black holes in the presence of Dirac spinors leads to unexpected physical effects. Applied to the gravitational collapse of a "cloud" of Dirac particles, this is a further indication that if an event horizon forms, the Dirac particles must eventually disappear inside this horizon.

The methods used in this paper are quite different from those in [3]. Namely, in contrast to [3], we do not derive "matching conditions" for the spinors across the horizon. We work here only with the equations outside the event horizon, and the proof relies on the nonlinear coupling of the spinors into the Einstein–Maxwell equations.

2. The Spherically Symmetric Multi-Particle System

The gravitational field is described by the static, spherically symmetric Lorentzian metric in polar coordinates

$$ds^2 = g_{ij}\, dx^i\, dx^j = \frac{1}{T^2}\, dt^2 - \frac{1}{A}\, dr^2 - r^2\, d\vartheta^2 - r^2 \sin^2\vartheta\, d\varphi^2 \qquad (2.1)$$

with positive functions $A = A(r)$ and $T = T(r)$. We consider as our space-time the region $r > \rho > 0$ outside a ball of radius ρ around the origin. The physical situation we have in mind is that the surface $r = \rho$ is the event horizon of a black hole. We assume the metric to be asymptotically Minkowskian,

$$\lim_{r\to\infty} A(r) = 1 = \lim_{r\to\infty} T(r). \qquad (2.2)$$

The electromagnetic field is described by a potential $\mathcal{A}$ of the form $\mathcal{A} = (-\phi, \mathbf{0})$, where ϕ is the Coulomb potential $\phi = \phi(r)$.

In direct generalization of the situation in the Reissner–Nordström background [3], the Dirac operator G takes the form

$$\begin{aligned} G &= i G^j(x)\, \frac{\partial}{\partial x^j} \;+\; B(x) \\ &= i T \gamma^0 \left(\frac{\partial}{\partial t} - i e \phi \right) + \gamma^r \left(i \sqrt{A} \frac{\partial}{\partial r} + \frac{i}{r}\, (\sqrt{A} - 1) - \frac{i}{2}\, \sqrt{A}\, \frac{T'}{T} \right) \\ &\quad + i \gamma^\vartheta \frac{\partial}{\partial \vartheta} + i \gamma^\varphi \frac{\partial}{\partial \varphi}\ , \end{aligned} \tag{2.3}$$

where $\gamma^t, \gamma^r, \gamma^\vartheta$, and γ^φ denote the γ-matrices of Minkowski space in polar coordinates,

$$\begin{aligned} \gamma^t &= \gamma^0, \\ \gamma^r &= \gamma^1 \, \cos\vartheta \;+\; \gamma^2 \, \sin\vartheta \, \cos\varphi \;+\; \gamma^3 \, \sin\vartheta \, \sin\varphi, \\ \gamma^\vartheta &= \frac{1}{r} \left(-\gamma^1 \, \sin\vartheta \;+\; \gamma^2 \, \cos\vartheta \, \cos\varphi \;+\; \gamma^3 \, \cos\vartheta \, \sin\varphi \right), \\ \gamma^\varphi &= \frac{1}{r \sin\vartheta} \left(-\gamma^2 \, \sin\varphi \;+\; \gamma^3 \, \cos\varphi \right). \end{aligned}$$

As with the central force problem in Minkowski space [4], this Dirac operator commutes with: a) the time translation operator $i\partial_t$, b) the total angular momentum operator $J^2 = (\boldsymbol{L} + \boldsymbol{S})^2$, c) the z-component of total angular momentum J_z, and d) with the operator $\gamma^0\, P$ (where P is parity). Since these operators also commute with each other, we can write any solution of the Dirac equation as a linear combination of solutions which are simultaneous eigenstates of these operators. We denote this "eigenvector basis" for the solutions by

$$\Psi^c_{jk\omega} \qquad \text{with} \qquad c = \pm, \quad j = \frac{1}{2}, \frac{3}{2}, \ldots\,, \quad k = -j, -j+1, \ldots\,, j, \quad \omega \in \mathbb{R}; \tag{2.4}$$

the eigenvalues are

$$\begin{aligned} i\partial_t \Psi^c_{jk\omega} &= \omega\, \Psi^c_{jk\omega}, \\ J^2\, \Psi^c_{jk\omega} &= j(j+1)\, \Psi^c_{jk\omega}, \\ J_z\, \Psi^\pm_{jk\omega} &= k\, \Psi^\pm_{jk\omega}, \\ \gamma^0 P\, \Psi^\pm_{jk\omega} &= \pm \Psi^\pm_{jk\omega} \;\times\; \begin{cases} 1 & \text{for } j + \frac{1}{2} \text{ even} \\ -1 & \text{for } j + \frac{1}{2} \text{ odd} \end{cases} . \end{aligned}$$

For the functions $\Psi^c_{jk\omega}$, the Dirac equation reduces to ordinary differential equations in the radial variable r. The quantum number k describes the orientation of the wave function $\Psi^c_{jk\omega}$ in space, and thus spherical symmetry of the Dirac operator implies that the radial Dirac equation is the same for all values of k. In order to build up our many-particle system, we take, for given c, j, and ω, one solution of the radial Dirac equation and consider the system of the $2j+1$ particles

$$\Psi^c_{jk\omega}, \qquad k = -j, \ldots\,, j$$

corresponding to this radial solution. Using the formalism of many-particle quantum mechanics, we can describe the fermions with the Hartree–Fock state

$$\Psi^{\mathrm{HF}} \;=\; \Psi^c_{j\,k=-j\,\omega} \wedge \Psi^c_{j\,k=-j+1\,\omega} \wedge \cdots \wedge \Psi^c_{j\,k=j\,\omega}. \tag{2.5}$$

For simplicity, we usually avoid this formalism here; it is easier to just work with an orthonormal basis of the one-particle states.

For clarity, we point out that each fermion has non-zero angular momentum and is thus *not* in a spherically symmetric state. Nevertheless, the system of $2j+1$ particles is spherically symmetric; this can be verified in detail as follows: The Hartree–Fock state (2.5) is an eigenstate of J_z; namely

$$\begin{aligned} J_z\,\Psi^{\mathrm{HF}} &= (J_z\,\Psi^c_{j\,k=-j\,\omega}) \wedge \Psi^c_{j\,k=-j+1\,\omega} \wedge \cdots \wedge \Psi^c_{j\,k=j\,\omega} \\ &\quad + \Psi^c_{j\,k=-j\,\omega} \wedge (J_z\,\Psi^c_{j\,k=-j+1\,\omega}) \wedge \cdots \wedge \Psi^c_{j\,k=j\,\omega} \\ &\quad + \cdots + \Psi^c_{j\,k=-j\,\omega} \wedge \Psi^c_{j\,k=-j+1\,\omega} \wedge \cdots \wedge (J_z\,\Psi^c_{j\,k=j\,\omega}) \\ &= \sum_{k=-j}^{j} k\,\Psi^{\mathrm{HF}} \;=\; 0. \end{aligned}$$

Similarly, we can apply the "ladder operators" $J_\pm = J_x \pm i J_y$ to the Hartree–Fock state,

$$\begin{aligned} J_\pm\,\Psi^{\mathrm{HF}} &= (J_\pm\,\Psi^c_{j\,k=-j\,\omega}) \wedge \Psi^c_{j\,k=-j+1\,\omega} \wedge \cdots \wedge \Psi^c_{j\,k=j\,\omega} \\ &\quad + \Psi^c_{j\,k=-j\,\omega} \wedge (J_\pm\,\Psi^c_{j\,k=-j+1\,\omega}) \wedge \cdots \wedge \Psi^c_{j\,k=j\,\omega} \\ &\quad + \cdots + \Psi^c_{j\,k=-j\,\omega} \wedge \Psi^c_{j\,k=-j+1\,\omega} \wedge \cdots \wedge (J_\pm\,\Psi^c_{j\,k=j\,\omega}). \end{aligned} \tag{2.6}$$

After substituting the relations

$$J_\pm\,\Psi^c_{jk\omega} \;=\; \sqrt{j(j+1) - k(k\pm 1)}\;\Psi^c_{j\,k\pm 1\,\omega},$$

the anti-symmetry of the wedge product yields that each summand in (2.6) vanishes. We conclude that

$$\boldsymbol{J}\,\Psi^{\mathrm{HF}} \;=\; 0.$$

Since the total angular momentum operator $\boldsymbol{J}$ is the infinitesimal generator of rotations, this implies that Ψ^{HF} is spherically symmetric.

For a physically meaningful solution of the Dirac equation, the wave function must be normalized. The normalization integral for the wave functions $\Psi^c_{jk\omega}$ over the hypersurface $\mathcal{H} = \{t = \text{const},\ r > \rho\}$ is

$$\int_{\mathcal{H}} \overline{\Psi^c_{jk\omega}} G^j \Psi^c_{jk\omega}\, \nu_j\, d\mu_{\mathcal{H}}, \tag{2.7}$$

where ν is the future-directed normal of $\mathcal{H}$, and where $d\mu_{\mathcal{H}}$ denotes the invariant measure on $\mathcal{H}$ induced by the Lorentzian metric. For a normalized solution of the Dirac equation, (2.7) gives the probability for the particle to be in the region $r > \rho$ outside the ball of radius ρ centered at the origin. It seems tempting to demand that, for a normalizable solution of the Dirac equation, the integral (2.7) must be finite. However, as explained in [3], the normalization integral inside the event horizon (which we do not consider here), is not necessarily positive, and it might happen that an infinite contribution near $r = \rho$

in (2.7) is compensated by an infinite negative contribution inside the horizon. Therefore we only demand that the normalization integral away from the horizon is finite; namely

$$\int_{\{t=\text{const},\ r>r_0\}} \overline{\Psi^c_{jk\omega}} G^j \Psi^c_{jk\omega}\, \nu_j \, d\mu_{\mathcal{H}} \;<\; \infty \qquad \text{for every } r_0 > \rho. \tag{2.8}$$

We remark that the singlet state of [1,2] can be recovered from our multi-particle system by considering the case $j = \frac{1}{2}$.

3. The EDM Equations

We now derive the system of differential equations. We begin by separating out the angular and time dependence in the Dirac equation. We choose the wave functions $\Psi^c_{jk\omega}$ of the previous section in the explicit form

$$\Psi^+_{jk\omega} = e^{-i\omega t}\, \frac{\sqrt{A}}{r} \begin{pmatrix} \chi^k_{j-\frac{1}{2}}\; \Phi^+_{jk\omega\,1}(r) \\ i\chi^k_{j+\frac{1}{2}}\; \Phi^+_{jk\omega\,2}(r) \end{pmatrix}, \tag{3.1}$$

$$\Psi^-_{jk\omega} = e^{-i\omega t}\, \frac{\sqrt{A}}{r} \begin{pmatrix} \chi^k_{j+\frac{1}{2}}\; \Phi^-_{jk\omega\,1}(r) \\ i\chi^k_{j-\frac{1}{2}}\; \Phi^-_{jk\omega\,2}(r) \end{pmatrix}, \tag{3.2}$$

where $\Phi^c_{jk\omega}$ are two-component radial functions, and where $\chi^k_{j\pm\frac{1}{2}}$, $j = \frac{1}{2}, \frac{3}{2}, \ldots, k = -j, -j+1, \ldots, j$ denote the 2-spinors

$$\chi^k_{j-\frac{1}{2}} = \sqrt{\frac{j+k}{2j}}\, Y^{k-\frac{1}{2}}_{j-\frac{1}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\frac{j-k}{2j}}\, Y^{k+\frac{1}{2}}_{j-\frac{1}{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\chi^k_{j+\frac{1}{2}} = \sqrt{\frac{j+1-k}{2j+2}}\, Y^{k-\frac{1}{2}}_{j+\frac{1}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sqrt{\frac{j+1+k}{2j+2}}\, Y^{k+\frac{1}{2}}_{j+\frac{1}{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(Y^m_l are the spherical harmonics). The functions $\chi^k_{j\pm\frac{1}{2}}$ are eigenvectors of the operator $K = \boldsymbol{\sigma}\boldsymbol{L} + 1$,

$$K\, \chi^k_{j\pm\frac{1}{2}} \;=\; \mp(j + \frac{1}{2})\, \chi^k_{j\pm\frac{1}{2}}.$$

Using the relations between the angular momentum operators (see [3] for details), the Dirac equation $(G-m)\Psi^c_{jk\omega}$ with G as in (2.3) reduces to the ordinary differential equation

$$\begin{aligned} &\sqrt{A}\, \frac{d}{dr} \Phi^\pm_{jk\omega} \\ &= \left[(\omega - e\phi) T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \pm \frac{2j+1}{2r} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \Phi^\pm_{jk\omega}. \end{aligned} \tag{3.3}$$

Since this equation is independent of k, we will in the following omit the index k and simply write $\Phi^c_{j\omega}$. The normalization integral (2.8) for the wave functions takes the form

$$\int_{r_0}^{\infty} |\Phi^c_{j\omega}(r)|^2 \, \frac{\sqrt{T}}{A} \, dr \;<\; \infty \qquad \text{for every } r_0 > \rho. \tag{3.4}$$

The Dirac equation (3.3) implies that the "radial flux" function (see [3])

$$F(r) \;:=\; \overline{\Phi^c_{j\omega}(r)} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \Phi^c_{j\omega}(r) \tag{3.5}$$

is a constant, as is verified by a short computation. Since $|\Phi^c_{j\omega}|^2 \geq F$ and since the metric is asymptotically flat, the normalization integral (2.7) can be finite only if this constant is zero. We can thus assume that F vanishes identically. This means that the product $\overline{\Phi^c_{j\omega\,1}} \Phi^c_{j\omega\,2}$ (constructed from the two components of $\Phi^c_{j\omega}$) is a real function. For a given radius $r_0 \in (\rho, \infty)$, we can thus arrange with a constant phase transformation

$$\Phi^c_{j\omega}(r) \;\to\; e^{i\alpha}\, \Phi^c_{j\omega}(r) \quad , \qquad \alpha \in \mathbb{R},$$

that both $\Phi^c_{j\omega\,1}(r_0)$ and $\Phi^c_{j\omega\,2}(r_0)$ are real. Since all the coefficients in the Dirac equation (3.3) are real, it follows that the spinors $\Phi^c_{j\omega}(r)$ are real even for all $r \in (\rho, \infty)$. We denote these two real components of $\Phi^c_{j\omega}$ by α and β.

Next we must calculate the total current and energy-momentum tensor of the Dirac particles. With our explicit formulas (3.1) and (3.2) for the angular dependence of the wave functions, the anti-symmetrization in the Hartree–Fock state (2.5) is trivial. One obtains that the electromagnetic current of the multi-particle system is simply the sum of the currents of all states $\Psi^c_{jk\omega}$,

$$j^k \;=\; \overline{\Psi^{\mathrm{HF}}}\, G^k\, \Psi^{\mathrm{HF}} \;=\; \sum_{k=-j}^{j} \overline{\Psi^c_{jk\omega}}\, G^k\, \Psi^c_{jk\omega}.$$

Because of spherical symmetry, the angular components j^ϑ and j^φ of the Dirac current vanish. The "sum rule"

$$\sum_{k=-j}^{j} \overline{\chi^k_{j\pm\frac{1}{2}}(\vartheta,\varphi)}\, \chi^k_{j\pm\frac{1}{2}}(\vartheta,\varphi) \;=\; \frac{2j+1}{4\pi} \tag{3.6}$$

yields, after a straightforward computation, that

$$j^t(x) \;=\; \sum_{k=-j}^{j} \overline{\Psi^c_{jk\omega}} G^t(x) \Psi^c_{jk\omega} \;=\; \frac{T^2}{r^2}\, (\alpha^2+\beta^2)\, \frac{2j+1}{4\pi}.$$

To calculate the radial current j^r, we will need the formula [3, Eq. (3.7)]

$$\sigma^r\, \chi^k_{j\pm\frac{1}{2}} \;=\; \chi^k_{j\mp\frac{1}{2}}, \tag{3.7}$$

where σ^r is the Pauli matrix in the radial direction,

$$\sigma^r \;=\; \sigma^1\, \cos\vartheta \;+\; \sigma^2\, \sin\vartheta\, \cos\varphi \;+\; \sigma^3\, \sin\vartheta\, \sin\varphi.$$

Using (3.7), (3.6), and the fact that the radial flux $F(r)$ vanishes, we obtain

$$j^r \;=\; \sum_{k=-j}^{j} \overline{\Psi^c_{jk\omega}} G^r(x) \Psi^c_{jk\omega} \;=\; 0.$$

Similar to the total current, the energy-momentum tensor of the multi-particle system is simply the sum of the energy-momentum of all states $\Psi^c_{jk\omega}$; it has the general form

$$T_{ab} \;=\; \frac{1}{2}\,\mathrm{Re} \sum_{k=-j}^{j} \overline{\Psi^c_{jk\omega}} \left(iG_a\,(\partial_b - ie\mathcal{A}_b) \;+\; iG_b\,(\partial_a - ie\mathcal{A}_a)\right) \Psi^c_{jk\omega} \tag{3.8}$$

(this formula is obtained by varying the classical Dirac action; see e.g. [1]). The following calculation depends on whether the index c of the wave functions in (3.8) is $c = +$ or $c = -$. We use the $\pm/\mp$-notation, whereby the upper and lower choice correspond to the cases $c = +$ and $c = -$, respectively. From spherical symmetry, T^t_ϑ, T^t_φ, T^r_ϑ, T^r_φ, and T^ϑ_φ must vanish. An easy computation using the sum rule (3.6) and the Dirac equation (3.3) gives

$$\begin{aligned}
T^t_t &= \frac{(\omega - e\phi)T^2}{r^2}(\alpha^2 + \beta^2)\frac{2j+1}{4\pi},\\
T^r_r &= -\frac{(\omega - e\phi)T^2}{r^2}(\alpha^2 + \beta^2)\frac{2j+1}{4\pi} \pm \frac{T}{r^3}\alpha\beta\frac{(2j+1)^2}{4\pi} + \frac{mT}{r^2}(\alpha^2 - \beta^2)\frac{2j+1}{4\pi}.
\end{aligned}$$

The calculations of T^ϑ_ϑ and T^φ_φ are slightly more difficult. First, spherical symmetry yields that

$$T^\vartheta_\vartheta \;=\; T^\varphi_\varphi \;=\; \frac{1}{2}\,\mathrm{Re} \sum_{k=-j}^{j} \overline{\Psi^c_{jk\omega}} \left(iG^\vartheta \partial_\vartheta \;+\; iG^\varphi \partial_\varphi\right) \Psi^c_{jk\omega} \qquad .$$

The formula

$$G^\vartheta \partial_\vartheta + G^\varphi \partial_\varphi \;=\; -\frac{1}{r}\,\sigma^r\,\boldsymbol{\gamma L}$$

allows us to rewrite the angular derivatives using the angular momentum operator $\boldsymbol{L}$. This gives

$$\begin{aligned}
T^\vartheta_\vartheta \;=\; T^\varphi_\varphi \;&=\; \frac{\alpha\beta}{2}\,\frac{T}{r^3}\,\mathrm{Re} \sum_{k=-j}^{j} \left(\overline{\chi^k_{j\mp\frac{1}{2}}}(\sigma^r\,\boldsymbol{\sigma L})\chi^k_{j\pm\frac{1}{2}} \;-\; \overline{\chi^k_{j\pm\frac{1}{2}}}(\sigma^r\,\boldsymbol{\sigma L})\chi^k_{j\mp\frac{1}{2}} \right)\\
&\overset{(3.7)}{=} \frac{\alpha\beta}{2}\,\frac{T}{r^3}\,\mathrm{Re} \sum_{k=-j}^{j} \left(\overline{\chi^k_{j\pm\frac{1}{2}}}\,\boldsymbol{\sigma L}\,\chi^k_{j\pm\frac{1}{2}} \;-\; \overline{\chi^k_{j\mp\frac{1}{2}}}\,\boldsymbol{\sigma L}\,\chi^k_{j\mp\frac{1}{2}} \right).
\end{aligned}$$

We now use the fact that the spinors $\chi^k_{j\pm\frac{1}{2}}$ are eigenvectors of the operator $\boldsymbol{\sigma L} = K - 1$, and carry out the k-summation with (3.6) to get

$$T^\vartheta_\vartheta \;=\; T^\varphi_\varphi = \mp\frac{\alpha\beta}{2}\,\frac{T}{r^3}\left((j+\frac{3}{2}) + (j-\frac{1}{2})\right)\frac{2j+1}{4\pi} \;=\; \mp\frac{T}{r^3}\,\alpha\beta\,\frac{(2j+1)^2}{8\pi}.$$

Finally, we put the obtained formulas for the Dirac current and energy-momentum tensor together with the Maxwell energy-momentum tensor [2] into the Einstein and Maxwell equations

$$R^i_j - \frac{1}{2}\, R\, \delta^i_j \;=\; -8\pi\; T^i_j \quad , \qquad \nabla_l F^{kl} \;=\; 4\pi e\; j^k .$$

The Einstein equations reduce to two first-order equations for A and T; similarly, Maxwell's equations simplify to one second-order equation, and we end up with the following system of EDM equations:

$$\sqrt{A}\,\alpha' = \pm\frac{2j+1}{2r}\,\alpha \;-\; ((\omega - e\phi)T + m)\,\beta, \tag{3.9}$$

$$\sqrt{A}\,\beta' = ((\omega - e\phi)T - m)\,\alpha \;\mp\; \frac{2j+1}{2r}\,\beta, \tag{3.10}$$

$$r\,A' = 1 - A \;-\; 2(2j+1)(\omega - e\phi)T^2\,(\alpha^2+\beta^2) \;-\; r^2 A T^2\,|\phi'|^2, \tag{3.11}$$

$$\begin{aligned} 2rA\,\frac{T'}{T} = A - 1 \;-\; 2(2j+1)(\omega - e\phi)T^2\,(\alpha^2+\beta^2) \;\pm\; 2\,\frac{(2j+1)^2}{r}\,T\,\alpha\beta \\ +2(2j+1)\,mT\,(\alpha^2-\beta^2) \;+\; r^2 A T^2\,|\phi'|^2, \end{aligned} \tag{3.12}$$

$$r^2 A\,\phi'' = -(2j+1)\,e\,(\alpha^2+\beta^2) - \left(2rA + r^2 A\,\frac{T'}{T} + \frac{r^2}{2}\,A'\right)\phi'. \tag{3.13}$$

The two cases for the signs $\pm/\mp$ correspond to the two values $c = \pm$ for the fermionic wave functions $\Psi^c_{jk\omega}$. Equations (3.9) and (3.10) are the Dirac equations (3.3). The Einstein equations are (3.11) and (3.12), and (3.13) is Maxwell's equation. According to (3.4), the normalization condition is

$$\int_{r_0}^{\infty} (\alpha^2+\beta^2)\,\frac{\sqrt{T}}{A}\,dr \;<\; \infty \qquad \text{for every} \qquad r_0 > \rho. \tag{3.14}$$

We remark that the system (3.9)–(3.13), (3.14) has particle-like solutions, which can be constructed numerically using the methods in [1]. The mass-energy spectrum of the solutions has the same qualitative behavior as for the two-particle EDM system [2].

4. Non-Existence Results

We want to investigate *black hole solutions* of the system (3.9)–(3.13). This means, more precisely, that we consider solutions of (3.9)–(3.13) defined outside the ball of radius $\rho > 0$ around the origin which are asymptotically flat, (2.2), and satisfy the normalization condition (3.14). We assume that the surface $r = \rho$ is an event horizon; i.e. the function $A(r)$ tends to zero for $r \searrow \rho$, whereas $T(r)$ goes to infinity in this limit. We make the following assumptions on the form of the horizon:

(I) The volume element $\sqrt{|\det g_{ij}|} \;=\; r^2 A^{-\frac{1}{2}}\, T^{-1}$ is smooth and non-zero on the horizon, i.e.

$$T^{-2}\,A^{-1},\; T^2 A \;\in\; C^\infty([\rho,\infty)).$$

This assumption is sometimes called: *the horizon is regular*.

(II) The strength of the electromagnetic field is given by the scalar $F_{ij}\,F^{ij} = -2|\phi'|^2 A\,T^2$ with the electromagnetic field tensor $F_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i$. We assume this scalar to be bounded near the horizon; thus in view of (I) we assume that

$$|\phi'(r)| \;<\; c_1, \qquad \rho < r < \rho + \varepsilon_1$$

for some positive constants c_1, ε_1.

(III) The function $A(r)$ obeys a power law, i.e.

$$A(r) \;=\; c\,(r-\rho)^s \;+\; \mathcal{O}((r-\rho)^{s+1}), \qquad r > \rho \tag{4.1}$$

with positive constants c and s.

If assumptions (I) or (II) were violated, an observer freely falling into the black hole would feel strong forces when crossing the horizon. Assumption (III) is a technical condition which seems general enough to include all physically relevant horizons. For example, the Schwarzschild horizon has $s = 1$, whereas $s = 2$ corresponds to the extreme Reissner-Nordström horizon. However, assumption (III) does not seem to be essential for the statement of our non-existence results; with more mathematical effort, it could be weakened or perhaps even omitted completely. We now state our main result:

Theorem 4.1. *The black hole solutions of the EDM system (3.9)–(3.13) satisfying the regularity conditions (I), (II), and (III) either coincide with a non-extreme Reissner-Nordström solution with $\alpha = 0 = \beta$, or $s = 2$ and the following asymptotic expansions hold near $r = \rho$:*

$$\begin{aligned}
A(r) &= A_0\,(r-\rho)^2 \;+\; \mathcal{O}((r-\rho)^3), && (4.2)\\
T(r) &= T_0\,(r-\rho)^{-1} \;+\; \mathcal{O}((r-\rho)^0), && (4.3)\\
\phi(r) &= \frac{\omega}{e} \;+\; \phi_0\,(r-\rho) \;+\; \mathcal{O}((r-\rho)^2), && (4.4)\\
\alpha(r) &= \alpha_0\,(r-\rho)^\kappa \;+\; \mathcal{O}((r-\rho)^{\kappa+1}), && (4.5)\\
\beta(r) &= \beta_0\,(r-\rho)^\kappa \;+\; \mathcal{O}((r-\rho)^{\kappa+1}) && (4.6)
\end{aligned}$$

with positive constants A_0, T_0 and real parameters ϕ_0, α_0, β_0. The power κ must satisfy the constraint

$$\frac{1}{2} \;<\; \kappa \;=\; \frac{1}{A_0}\sqrt{m^2 - e^2\phi_0^2 T_0^2 + \left(\frac{2j+1}{2\rho}\right)^2}, \tag{4.7}$$

and the spinor coefficients α_0 and β_0 are related by

$$\alpha_0\left(\sqrt{A_0}\,\kappa \;\pm\; \frac{2j+1}{2\rho}\right) \;=\; -\beta_0\,(m - e\phi_0 T_0), \tag{4.8}$$

where '$\pm$' refers to the two choices of the signs in (3.9)–(3.13).

We begin the analysis with the case that the power s in (4.1) is less than 2.

Lemma 4.2. *Assume that $s < 2$ and that $(\alpha, \beta, A, T, \phi)$ is a black-hole solution where the spinors (α, β) are not identically zero. Then the function $(\alpha^2 + \beta^2)$ is bounded both from above and from below near the horizon, i.e. there are constants $c, \varepsilon > 0$ with*

$$c \;\le\; \alpha(r)^2 + \beta(r)^2 \;\le\; \frac{1}{c}, \qquad \rho < r < \rho + \varepsilon. \tag{4.9}$$

Proof. The Dirac equations (3.9),(3.10) imply that

$$\sqrt{A}\,\frac{d}{dr}(\alpha^2+\beta^2) = 2\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\begin{pmatrix}\pm\frac{2j+1}{2r} & -m\\ -m & \mp\frac{2j+1}{2r}\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\leq \left(4m^2+\frac{(2j+1)^2}{r^2}\right)^{\frac{1}{2}}(\alpha^2+\beta^2). \tag{4.10}$$

Since (α,β) is a non-trivial solution, the uniqueness theorem for solutions of ODEs implies that $(\alpha^2+\beta^2)(r)$ is non-zero for all $\rho < r < \rho+\varepsilon$. Thus we can divide Eq. (4.10) by $\sqrt{A}\,(\alpha^2+\beta^2)$ and integrate. This yields the bound

$$\Big|\log((\alpha^2+\beta^2)(\rho+\varepsilon)) \;-\; \log((\alpha^2+\beta^2)(r))\Big|$$

$$\leq \int_r^{\rho+\varepsilon} A^{-\frac{1}{2}}(t)\left(4m^2+\frac{(2j+1)^2}{t^2}\right)^{\frac{1}{2}} dt. \tag{4.11}$$

Since $s<2$, we see that the integrand in (4.11) is integrable at $r=\rho$. Thus the right hand side of (4.11) is majorized by

$$\int_\rho^{\rho+\varepsilon} A^{-\frac{1}{2}}(t)\left(4m^2+\frac{(2j+1)^2}{t^2}\right)^{\frac{1}{2}} dt,$$

so we may take the limit $r\searrow\rho$ in (4.11) to obtain the estimate (4.9). □

Proposition 4.3. *For* $0<s<2$, *the only black hole solutions of the system (3.9)–(3.13) are the non-extreme Reissner–Nordström solutions.*

Proof. We shall assume that we are given a black hole solution which is not the Reissner-Nordström solution, and obtain a contradiction. In this case, the spinors (α,β) are not identically zero, so we may apply Lemma 4.2 and conclude that the spinors are bounded near $r=\rho$.

We first consider the differential equation for AT^2. The Einstein equations (3.11) and (3.12) give

$$r\,\frac{d}{dr}(AT^2) = \;-4(2j+1)\,(\omega-e\phi)\,T^4\,(\alpha^2+\beta^2)\;\pm\;2\,\frac{(2j+1)^2}{r}\,T^3\,\alpha\beta$$

$$+\,2(2j+1)\,m\,T^3\,(\alpha^2-\beta^2). \tag{4.12}$$

According to the regularity condition (I), the left and thus also the right side of this equation is smooth. Since the spinors are bounded away from zero near $r=\rho$, and since $T\to\infty$ as $r\searrow\rho$, we see that

$$\lim_{\rho<r\to\rho}(\omega-e\phi(r)) \;=\; 0. \tag{4.13}$$

We write Maxwell's equation (3.13) in the form

$$\phi'' \;=\; -\frac{1}{A}\,\frac{(2j+1)\,e}{r^2}\,(\alpha^2+\beta^2) \;-\; \frac{1}{r^2\sqrt{A}\,T}\,[r^2\sqrt{A}\,T]'\,\phi'. \tag{4.14}$$

According to the regularity condition (I), the square bracket in (4.14), and thus the whole coefficient of ϕ', is a smooth function. However, the factor A^{-1} in the first summand in (4.14) blows up on the horizon. If $s \geq 1$, the singularity of A^{-1} is not integrable. This implies that $|\phi'|$ is unbounded on the horizon, contradicting the regularity condition (II). We conclude that $s < 1$. We can then integrate Eq. (4.14) and obtain the local expansion around the horizon

$$\phi'(r) \;=\; c_1\,(r-\rho)^{-s+1} \;+\; c_2 \;+\; \mathcal{O}((r-\rho)^{-s+2}),$$

where c_2 is an integration constant. Integrating once again and using (4.13) yields the expansion

$$\phi(r) \;=\; c_1\,(r-\rho)^{-s+2} \;+\; c_2\,(r-\rho) \;+\; \frac{\omega}{e} \;+\; \mathcal{O}((r-\rho)^{-s+3}). \tag{4.15}$$

Finally, we substitute the expansion (4.15) into the A-equation (3.11). Since the functions A and $r^2 A T^2 |\phi'|^2$ are bounded near the horizon, and since (4.15) implies that $(\omega - e\phi) = \mathcal{O}(r-\rho)$ whereas $T^2(\alpha^2+\beta^2) \sim (r-\rho)^{-s}$ with $s < 1$, the right side of (3.11) is bounded in the limit $r \searrow \rho$. However, the left side diverges like $rA'(r) \sim (r-\rho)^{s-1}$. This is a contradiction. $\square$

It remains to consider the case $s \geq 2$.

Lemma 4.4. *If $s \geq 2$,*

$$\lim_{\rho<r\to\rho} (r-\rho)^{-\frac{s}{2}}\,(\alpha^2+\beta^2)(r) \;=\; 0. \tag{4.16}$$

Proof. As in the proof of Proposition 4.3, we consider the Maxwell equation (4.14). Since $|\phi'|$ is bounded near the horizon according to condition (II), we conclude from (I) that the inhomogeneous term in (4.14) must be integrable,

$$\int_\rho^{\rho+\varepsilon} \frac{1}{A}\,(\alpha^2+\beta^2) \;<\; \infty. \tag{4.17}$$

Next we take the derivative of the function in (4.16),

$$\frac{d}{dr}\Big((r-\rho)^{-\frac{s}{2}}(\alpha^2+\beta^2)\Big) = -\frac{s}{2}(r-\rho)^{-\frac{s}{2}-1}(\alpha^2+\beta^2) + (r-\rho)^{-\frac{s}{2}}\frac{d}{dr}(\alpha^2+\beta^2),$$

and substitute the bound (4.10),

$$\begin{aligned}
&\left|\frac{d}{dr}\Big((r-\rho)^{-\frac{s}{2}}(\alpha^2+\beta^2)\Big)\right| \\
&\leq \frac{s}{2}(r-\rho)^{-\frac{s}{2}-1}(\alpha^2+\beta^2) \\
&\quad + \left(4m^2 + \frac{(2j+1)^2}{r^2}\right)^{\frac{1}{2}} A^{-\frac{1}{2}}(r-\rho)^{-\frac{s}{2}}(\alpha^2+\beta^2).
\end{aligned} \tag{4.18}$$

Since $s \geq 2$, we have $(r-\rho)^{-\frac{s}{2}-1} < (r-\rho)^{-s}$, and thus (4.17) implies that the first summand on the right side of (4.18) is integrable. Using

$$A^{-\frac{1}{2}}(r-\rho)^{-\frac{s}{2}}(\alpha^2+\beta^2) = A^{-1}(\alpha^2+\beta^2)\left[A^{\frac{1}{2}}(r-\rho)^{-\frac{s}{2}}\right] = \mathcal{O}(A^{-1}(\alpha^2+\beta^2)),$$

we see that, according to (4.17), the second summand in (4.18) is also integrable. As a consequence, the function $(r-\rho)^{-\frac{s}{2}}\,(\alpha^2+\beta^2)$ has a limit at $r=\rho$. If this limit were non-zero, the integral (4.17) would diverge. We conclude that this limit must be zero. □

Lemma 4.5. *If $s \geq 2$, the function $|\phi'|$ has a finite, non-zero limit on the horizon; namely*

$$\lim_{\rho<r\to\rho} |\phi'| \;=\; \frac{1}{\rho} \lim_{\rho<r\to\rho} A^{-\frac{1}{2}}\, T^{-1} \;>\; 0. \tag{4.19}$$

Proof. To begin, we first show that

$$\lim_{\rho<r\to\rho} (\omega - e\phi)\, T^2\, (\alpha^2+\beta^2) \;=\; 0. \tag{4.20}$$

If the function $|(\omega-e\phi)T|$ is bounded, then (4.20) is an immediate consequence of Lemma 4.4. Thus we must only consider the case that $|(\omega-e\phi)T|$ is unbounded near the horizon. The differential equation for AT^2, (4.12), gives the estimate

$$\left| r\frac{d}{dr} AT^2 \right| \geq T^3(\alpha^2+\beta^2) \left(4(2j+1)|(\omega-e\phi)T| - 2\frac{(2j+1)^2}{r} - 2(2j+1)\, m \right).$$

According to assumption (I), the left side of this inequality is bounded near the horizon. Using that $|(\omega-e\phi)T|$ becomes arbitrarily large near the horizon, we conclude that the function $T^3(\alpha^2+\beta^2)|(\omega-e\phi)T|$ must be bounded. This implies (4.20).

We now consider the A-equation (3.11). Since $s \geq 2$, the left side of (3.11) converges to zero for $r \searrow \rho$. Thus the right side of (3.11) must also go to zero in this limit,

$$0 \;=\; \lim_{\rho<r\to\rho} 1 - A - 2(2j+1)(\omega-e\phi)\, T^2\, (\alpha^2+\beta^2) - r^2\, A\, T^2\, |\phi'(r)|^2.$$

Using (4.20) completes the proof. □

We can now rule out the case $s > 2$; namely we have

Proposition 4.6. *For $s > 2$, there are no solutions of the system (3.9)–(3.13).*

Proof. According to Lemma 4.5, the function $(\omega-e\phi)$ has a Taylor expansion around the horizon with non-vanishing linear term,

$$(\omega-e\phi)(r) = c + d(r-\rho) + o(r-\rho) \qquad \text{with} \qquad |d| = \frac{e}{\rho} \lim_{\rho<r\to\rho} A^{-\frac{1}{2}} T^{-1}.$$

Thus the coefficient $(\omega-e\phi)T$ in the Dirac equation (3.9), (3.10) is monotone near the horizon and diverges. Using (3.9) and (3.10), this implies that the vector (α,β) spins around the origin faster and faster as r approaches the horizon, which suggests that it cannot go to zero in this limit. In fact, [3, Lemma 5.1] yields that the spinors are bounded away from zero,

$$\liminf_{\rho<r\to\rho} (\alpha^2+\beta^2) > 0.$$

This contradicts Lemma 4.4. □

Proof of Theorem 4.1. According to Proposition 4.3 and Proposition 4.6, we must only consider the case $s = 2$; thus (4.2) and (4.3) hold. Lemma 4.4 yields that the function $(\alpha^2 + \beta^2)$ goes to zero on the horizon. Applying [3, Lemma 5.1], one sees that the function $(\omega - e\phi)T$ cannot diverge monotonically near the horizon. On the other hand, Lemma 4.5 shows that $(\omega - e\phi)$ has a Taylor expansion around the horizon with non-vanishing linear term. We conclude that (4.4) holds, and that $(\omega - e\phi)T$ has a finite limit on the horizon,

$$\lim_{\rho<r\to\rho} (\omega - e\phi)T \;=\; \lambda \qquad \text{with} \qquad |\lambda| \stackrel{(4.19)}{=} \frac{e}{\rho} \lim_{\rho<r\to\rho} (r-\rho)^{-1} A^{-\frac{1}{2}}.$$

Exactly as in [3, Sect. 5], one can rewrite the radial Dirac equations as ODEs in the variable

$$u(r) \;=\; -r - \rho \,\ln(r - \rho)$$

and apply the stable manifold theorem [5, Thm. 4.1] to conclude that α and β satisfy a power law near the horizon, (4.5) and (4.6). Lemma 4.4 gives the bound $\kappa > \frac{1}{2}$.

Now we substitute the expansions (4.2)–(4.6) into the Dirac equations (3.9),(3.10). This gives the conditions

$$\begin{aligned}
\sqrt{A_0}\,\kappa\,\alpha_0 &= \pm\frac{2j+1}{2\rho}\,\alpha_0 \;+\; (e\phi_0 T_0 - m)\,\beta_0,\\
\sqrt{A_0}\,\kappa\,\beta_0 &= -(e\phi_0 T_0 + m)\alpha_0 \;\mp\; \frac{2j+1}{2\rho}\,\beta_0,
\end{aligned}$$

which are equivalent to (4.7) and (4.8). □

Our main theorem gives strong restrictions for the behavior of black hole solutions near the event horizon. According to the condition $\kappa > \frac{1}{2}$, the Dirac wave functions must decay so fast in the limit $r \searrow \rho$ that they have no influence on the asymptotic form of both the metric and the electric field on the horizon. Namely, according to (4.2)–(4.4), the metric and electric field must behave near the horizon like a vacuum solution, more precisely like the extreme Reissner–Nordström solution. The restriction to the extremal case means physically that the electric charge of the black hole must be so large that the electric repulsion balances the gravitational attraction and prevents the Dirac particles from “falling into” the black hole. This is certainly not the physical situation which one can expect in the gravitational collapse of e.g. a star in the Universe. Nevertheless, extreme Reissner–Nordström black holes have zero temperature [6] and can thus be considered as the asymptotic states of black holes emitting Hawking radiation. For this reason, it is interesting to study if the asymptotic expansion of Theorem 4.1 leads to global solutions of the EDM equations.

For an extreme Reissner–Nordström background field, it is proven in [3, Sect. 5] that the solutions of the Dirac equation satisfying (4.5),(4.6) violate the normalization condition (3.14). Thus the question is if the influence of the spinors on the gravitational and electromagnetic field can make the normalization integral finite. This is a hard analytic problem, because one must control the global behavior of the solution. However, we have done numerical investigations, taking the expansions in Theorem 4.1 as initial condition on the horizon and solving the equations for increasing r. It turns out that the solutions either develop singularities for finite r, or the spinors (α, β) are not normalizable. Thus our numerics show that the expansions in Theorem 4.1 do not give normalizable black hole solutions of the EDM equations.

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