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Algebraic limits of Kleinian groups which rearrange the pages of a book

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Dedicated to Bernard Maskit on the occasion of his sixtieth birthday

1 Introduction

In this paper, we give examples of two new phenomena in Kleinian groups. We first exhibit a sequence of homeomorphic marked hyperbolic 3-manifolds whose *algebraic* limit is not homeomorphic to any element in the sequence. We then use this construction to exhibit situations where the space of convex co-compact representations of a given 3-manifold group has many components but its closure is connected.

Let M be a compact, irreducible, oriented 3-manifold and let $\mathscr{D}(\pi_1(M))$ denote the space of all discrete, faithful representations of $\pi_1(M)$ into $\mathrm{PSL}_2(\mathbf{C})$. A sequence of representations $\{\rho_n\} \subset \mathscr{D}(\pi_1(M))$ converging to $\rho \in \mathscr{D}(\pi_1(M))$ gives rise to a sequence $\{N_{\rho_n} = \mathbf{H}^3/\rho_n(\pi_1(M))\}$ of hyperbolic 3-manifolds, each of which is homotopy equivalent to M. The sequence $\{N_{\rho_n}\}$ is said to converge algebraically to $N_\rho = \mathbf{H}^3/\rho(\pi_1(M))$. (See [7, 13, 14] for more information about algebraic convergence of Kleinian groups.) In many situations (see [1, 6, 15, 24, 25, 27]), it has been shown that N_{ρ_n} must be homeomorphic to N_ρ for all large enough n, and we had suspected that this would always be the case. In this paper, we give a collection of examples where N_{ρ_n} is not homeomorphic to N_ρ for any n. Our sequences are quite well-behaved: the $\rho_n(\pi_1(M))$ are convex co-compact and mutually quasiconformally conjugate, and the algebraic limit $\rho(\pi_1(M))$ is geometrically finite.

In our examples, M is obtained by gluing a collection of I-bundles to a solid torus along a family of parallel annuli. These manifolds are particularly simple examples of books of I-bundles (see [9]) where, to explain the terminology, one should think of the solid torus as the binding and the I-bundles

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as the pages. The main tool in our construction is a version of Thurston's hyperbolic Dehn surgery theorem which is due to Tim Comar [8] (see also Bonahon–Otal [3]). We use this theorem to produce a sequence $\{\rho_n'\}$ of convex co-compact uniformizations of M converging to $\rho' \in \mathscr{D}(\pi_1(M))$ such that $N_{\rho'}$ is homeomorphic to the interior of M, and $\{N_{\rho_n}\}$ converges "geometrically" to a geometrically finite hyperbolic 3-manifold $\widehat{N}$ which is homeomorphic to the interior of $M - \delta$, where δ is the core curve of the solid torus. (This type of phenomenon was first discovered by Jørgensen [12, 15] and was subsequently investigated by Marden [18], Thurston [27], and others, e.g. see [3, 8, 16, 24, and 28].) If M^τ is homotopy equivalent to one of our examples, then it is also obtained by gluing the same collection of I-bundles to a solid torus along the same family of parallel annuli, although perhaps in a different order. In particular, there is a cover N^τ of $\widehat{N}$ which is homeomorphic to the interior of M^τ. We will see that one may precompose the representations in the sequence $\{\rho_n'\}$ by a sequence $\{\chi_n\}$ of automorphisms of $\pi_1(M)$ so that the resulting sequence of representations $\{\rho_n = \rho_n' \circ \chi_n\}$ converges to a representation $\rho \in \mathscr{D}(\pi_1(M))$ with $N_\rho = N^\tau$.

Our examples also serve to demonstrate new phenomena in the deformation theory of Kleinian groups. If we let $CC(\pi_1(M))$ denote the set of convex co-compact representations of $\pi_1(M)$, the components of $CC(\pi_1(M))$ are in a one-to-one correspondence with the marked homeomorphism types of irreducible (oriented) 3-manifolds homotopy equivalent to M. In our class of examples, the closure of $CC(\pi_1(M))$ will be connected, although $CC(\pi_1(M))$ can have arbitrarily many components. We note that there are other examples where the components of $CC(\pi_1(M))$ are known to have disjoint closures. In a future paper [2] with Darryl McCullough, we will explore more general classes of examples.

The convex co-compact Kleinian groups correspond, via the Sullivan dictionary between rational maps and Kleinian groups (see [26]), to hyperbolic rational maps. Hence, this phenomenon is the analogue of hyperbolic components of the Mandelbrot set whose closures intersect. It is conjectured that $CC(\pi_1(M))$ is dense in $\mathscr{D}(\pi_1(M))$. This is analogous to the conjecture that the hyperbolic components are dense in the Mandelbrot set.

2 The examples

In this section, we construct the examples promised in the introduction. For the remainder of the paper, fix a positive integer $k \geqq 3$.

Let $V = D^2 \times S^1$ be a solid torus and let $A(j)$ $(1 \leqq j \leqq k)$ denote a family of k disjoint parallel annuli in ∂V such that the inclusion map of $A(j)$ into V is a homotopy equivalence. (Explicitly, we could choose $A(j) = [e^{2\pi i(4j-1)/4k}, e^{2\pi i(4j+1)/4k}] \times S^1$.) Let $F(j)$ be a compact, oriented surface of genus j with one boundary component. Let $B(j) = F(j) \times I$ and let $\partial_0 B(j) = \partial F(j) \times I$. Form a manifold M_k from V and $\{B(1), \ldots, B(k)\}$ by identifying $\partial_0 B(j)$ with $A(j)$ (by an orientation-reversing homeomorphism) for all $1 \leqq j \leqq k$.

One may obtain a manifold which is homotopy equivalent to M_k, but is not homeomorphic to M_k, by simply rearranging the pages. More specifically, let τ be any permutation of $\{1,\dots,k\}$, and form M_k^τ from V and $\{B(1),\dots,B(k)\}$ by identifying $\partial_0(B(\tau(j)))$ with $A(j)$. In the proof of Lemma 3.2, we will see that M_k and M_k^τ are homeomorphic if and only if τ and τ' are in the same right coset of the dihedral group $\mathbf{D}_k$ within the symmetric group $\mathbf{S}_k$. (Throughout the paper $\sigma\tau$ will denote the result of applying the permutation σ and then τ.)

A finitely generated, discrete subgroup Γ of $\mathrm{PSL}_2(\mathbf{C})$ is *convex co-compact* (respectively *geometrically finite*) if the convex core $C(N)$ of $N = \mathbf{H}^3/\Gamma$ is compact (resp. finite volume). We say that Γ *uniformizes* a compact 3-manifold M if there exists an orientation-preserving homeomorphism between N and the interior of M.

Explicit convex co-compact Kleinian groups realizing M_k and M_k^τ (for any τ) can be constructed using the techniques of Klein–Maskit combination; see, for example, Maskit [19], particularly Chapter VIII.E. and Maskit [20]. In Remark 1, at the end of the section, we construct geometrically finite Kleinian groups uniformizing M_k and M_k^τ.

The properties of our main example are contained in the following theorem.

Theorem 2.1. *Let τ be a permutation of $\{1,\dots,k\}$. There exists a sequence $\{\rho_n\} \subset \mathscr{D}(\pi_1(M_k))$ which converges algebraically to $\rho \in \mathscr{D}(\pi_1(M_k))$ such that $\rho_n(\pi_1(M_k))$ is convex co-compact and uniformizes M_k for all n, and $\rho(\pi_1(M_k))$ is geometrically finite and uniformizes M_k^τ.*

Proof of 2.1. We will use a construction outlined by Kerckhoff and Thurston [16] (and later generalized by Ohshika [24], Bonahon–Otal [3] and Comar [8]) which was originally used to produce a sequence of discrete, faithful representations of a surface group whose geometric limit properly contains its algebraic limit.

We first recall some of the notation of Dehn surgery. Let $\widehat{M}$ be a compact, irreducible, oriented 3-manifold whose boundary contains a single torus T; any other component of $\partial\widehat{M}$ has genus at least 2. Choose a meridian m and longitude l for the torus T, and think of m and l as a basis for $\pi_1(T)$. If (p,q) is a pair of relatively prime integers, then $\widehat{M}(p,q)$ is the manifold obtained by attaching a solid torus V to $\widehat{M}$ by an orientation-reversing homeomorphism which identifies the meridian of V with a simple closed curve in the homotopy class of $m^p l^q$ on T.

We now state a version of Thurston's hyperbolic Dehn surgery theorem which is due, in this form, to Comar [8] (see also Bonahon–Otal [3]).

Theorem 2.2. ([8]). *Let $\widehat{M}$ be a compact, oriented 3-manifold with one toroidal boundary component T. Let $\widehat{N} = \mathbf{H}^3/\widehat{\Gamma}$ be a geometrically finite hyperbolic 3-manifold and $\phi : \mathrm{int}(\widehat{M}) \to \widehat{N}$ an orientation-preserving homeomorphism between the interior of $\widehat{M}$ and $\widehat{N}$. Further assume that every parabolic element of $\widehat{\Gamma}$ is conjugate to an element of $\phi_*(\pi_1(T))$. Let $\{(p_n,q_n)\}$ be a sequence of distinct pairs of relatively prime integers.*

Then, for all sufficiently large n, there exists a representation $\beta_n : \widehat{\Gamma} \to \mathrm{PSL}_2(\mathbf{C})$ *with discrete image such that*

1. $\beta_n(\widehat{\Gamma})$ *is convex co-compact and uniformizes* $\widehat{M}(p_n, q_n)$,
2. *the kernel of* β_n *is normally generated by* $m^{p_n} l^{q_n}$, *and*
3. $\{\beta_n\}$ *converges to the identity representation of* $\widehat{\Gamma}$.

Moreover, if we let i_n *denote the inclusion of* $\widehat{M}$ *into* $\widehat{M}(p_n, q_n)$, *then there exists an orientation-preserving homeomorphism* $\phi_n : \mathrm{int}(\widehat{M}(p_n, q_n)) \to \mathbf{H}^3/\beta_n(\widehat{\Gamma})$ *such that* $\beta_n \circ \phi_*$ *is conjugate to* $(\phi_n)_* \circ (i_n)_*$.

Recall the construction of M_k given above. Form $\widehat{M}_k$ by attaching an annulus to M_k along two parallel, homotopically non-trivial curves in $\partial V \cap \partial M_k$, and then thickening the annulus. We denote this additional thickened annulus by R. (Explicitly, let $C_1 = \{e^{3\pi i/4k}\} \times S^1 \subset \partial V$ and let $C_2 = \{e^{5\pi i/4k}\} \times S^1 \subset \partial V$ be two parallel curves in $\partial V \cap \partial M_k$. Form $\widehat{M}_k$ by attaching $S^1 \times I \times I$ to M_k by an embedding $h : S^1 \times I \times \{0,1\} \to \partial V \cap \partial M_k$ such that $h(S^1 \times \{1/2\} \times \{0\}) = C_1$ and $h(S^1 \times \{1/2\} \times \{1\}) = C_2$.) Notice that $\widehat{M}_k$ is homeomorphic to the manifold obtained by removing an open tubular neighborhood of the core curve of V from M_k.

Let T denote the unique toroidal boundary component of $\widehat{M}_k$. Choose a meridian m and a longitude l for T so that l is parallel to C_1. Let $i_n : \widehat{M}_k \to \widehat{M}_k(1,n)$ and $f : M_k \to \widehat{M}_k$ denote inclusion maps. Note that for any integer $n \in \mathbf{Z}$, $\widehat{M}_k(1,n)$ is homeomorphic to M_k and the inclusion $i_n \circ f : M_k \to \widehat{M}_k(1,n)$ is a homotopy equivalence which is homotopic to an orientation-preserving homeomorphism.

One may check that Thurston's geometrization theorem (see [22]) guarantees that $\widehat{M}_k$ is uniformized by a geometrically finite Kleinian group $\widehat{\Gamma}_k$, such that every parabolic element of $\widehat{\Gamma}_k$ is conjugate to an element of $\pi_1(T)$. (We will later sketch, in Remark 1 at the end of the section, an explicit construction.) Let $\widehat{N}_k = \mathbf{H}^3/\widehat{\Gamma}_k$ and let $\phi : \mathrm{int}(\widehat{M}_k) \to \widehat{N}_k$ be an orientation-preserving homeomorphism.

Let $\{\beta_n : \widehat{\Gamma}_k \to \mathrm{PSL}_2(\mathbf{C})\}$ and $\{\phi_n : \mathrm{int}(\widehat{M}_k(1,n)) \to \mathbf{H}^3/\beta_n(\widehat{\Gamma}_k)\}$ be the sequences of representations and homeomorphisms produced by Theorem 2.2 for the sequence $\{(1,n)\}$. Set $\rho_n' = \beta_n \circ \phi_* \circ f_*$. Since $\beta_n \circ \phi_* \circ f_*$ is conjugate to $(\phi_n)_* \circ (i_n)_* \circ f_*, \rho_n'$ is faithful and has image $\beta_n(\widehat{\Gamma}_k)$. Thus, each $\rho_n'(\pi_1(M_k))$ is convex co-compact and uniformizes M_k. Moreover, $\{\rho_n'\}$ converges to the representation $\rho' = \phi_* \circ f_*$ with image $\phi_*(f_*(\pi_1(M_k)))$. which uniformizes M_k.

In order to rearrange the pages, we first construct, given a permutation τ of $\{1, \ldots, k\}$, an immersion $f_\tau : M_k^\tau \to \widehat{M}_k$ such that

1. $\phi_*((f_\tau)_*(\pi_1(M_k^\tau)))$ is a geometrically finite uniformization of M_k^τ, and
2. $(i_n \circ f_\tau)_*$ is an isomorphism for all n.

Having constructed such an f_τ, we complete the proof by taking $\rho_n = \beta_n \circ \phi_* \circ (f_\tau)_* \circ (h_\tau)_*$, where $h_\tau : M_k \to M_k^\tau$ is a homotopy equivalence which is the identity on the solid torus V. Since $\beta_n \circ \phi_* \circ (f_\tau)_*$ is conjugate to $(\phi_n)_* \circ (i_n)_* \circ (f_\tau)_*$, we see that ρ_n is faithful and that $\rho_n(\pi_1(M_k)) = \beta_n(\widehat{\Gamma}_k)$.

Hence, $\rho_n(\pi_1(M_k))$ is a convex co-compact uniformization of M_k for all n. However, this time $\{\rho_n\}$ converges to a representation $\rho = \phi_* \circ (f_\tau)_* \circ (h_\tau)_*$ of $\pi_1(M_k)$ with image $\phi_*((f_\tau)_*(\pi_1(M_k^\tau)))$ which is a geometrically finite uniformization of M_k^τ.

The remainder of the proof consists of the construction of f_τ. As this construction is the crux of the proof, we will give an alternative, more schematic, description of the immersion in Remark 2 at the end of the section. In Remark 1, we explicitly identify the subgroup $\phi_*((f_\tau)_*(\pi_1(M_k^\tau)))$ of $\widehat{\Gamma}_k$.

Let H_k denote the subgroup of $\pi_1(\widehat{M}_k)$ which is normally generated by $\pi_1(M_k)$, and let M_k^∞ be the cover of $\widehat{M}_k$ associated to H_k. M_k^∞ consists of infinitely many homeomorphic lifts of M_k joined together by infinitely many homeomorphic lifts of R. Let $(M_k)_i$ denote the i^{th} copy of M_k and $B(j)_i$ the copy of $B(j)$ contained in M_i.

We construct $f_\tau : M_k^\tau \to \widehat{M}_k$ by first constructing $\tilde{f}_\tau : M_k^\tau \to M_k^\infty$ and then projecting. We first define $\tilde{f}_\tau$ on the pages of M_k^τ. We let $\tilde{f}_\tau|_{B(\tau(j))}$ be the natural identification of $B(\tau(j))$ with $B(\tau(j))_j$. We then extend $\tilde{f}_\tau$ to an embedding in such a way that $\tilde{f}_\tau(V)$ is contained entirely in lifts of V and R.

In order to check property (1), we consider the cover $\widetilde{M}_k^\tau$ of $\mathrm{int}(\widehat{M}_k)$ associated to $(f_\tau)_*(\pi_1(M_k^\tau))$. Since $\widetilde{M}_k^\tau$ covers $\widehat{M}_k^\infty$, f_τ lifts to an embedding $g_\tau : M_k^\tau \to \widetilde{M}_k^\tau$ which is a homotopy equivalence. Let $g'_\tau : M_k^\tau \to \mathrm{int}(\widetilde{M}_k^\tau)$ be an embedding of M_k^τ into the interior of $\widetilde{M}_k^\tau$ which is homotopic to g_τ. If $\Gamma_k^\tau = \phi_*((f_\tau)_*(\pi_1(M_k^\tau)))$ and $N_k^\tau = \mathbf{H}^3/\Gamma_k^\tau$, then ϕ lifts to a homeomorphism $\tilde{\phi} : \mathrm{int}(\widetilde{M}_k^\tau) \to N_k^\tau$. Hence, $g'_\tau \circ \tilde{\phi}$ is an embedding of M_k^τ into N_k^τ which is a homotopy equivalence. Γ_k^τ is geometrically finite, as it is a finitely generated subgroup of a geometrically finite co-infinite volume Kleinian group (see Proposition 7.1 in [22]). Hence, N_k^τ is homeomorphic to the interior of a compact 3-manifold. However, since N_k^τ contains an embedded copy of M_k^τ whose inclusion map is a homotopy equivalence, we see that N_k^τ is homeomorphic to the interior of M_k^τ (see Theorem 1 in [21]). Thus property (1) holds.

We now check property (2). Fix a basepoint $*$ in V and let α_j be a path joining $*$ to A_j and lying entirely in V. Let g be a generator of $\pi_1(V,*)$. Let G_j denote $\pi_1(B(j) \cup \alpha_j, *)$ sitting as a subgroup of $\pi_1(M_k,*)$, and note that $\pi_1(M_k,*)$ is generated by $G_1,\dots,G_k$. (Explicitly, the subgroup generated by $G_1,\dots,G_j$ is the amalgamated free product of the subgroup generated by $G_1,\dots,G_{j-1}$ and the subgroup G_j amalgamated along the common cyclic subgroup generated by g.) Furthermore, $\pi_1(\widehat{M}_k,*)$ is generated by $\pi_1(M_k,*)$ and an element h which commutes with g. If we let G_j^τ be the subgroup of $\pi_1(M_k^\tau,*)$ corresponding to $\pi_1(B(j) \cup \alpha_{\tau^{-1}(j)}, *)$, then $\pi_1(M_k^\tau,*)$ is generated by $G_1^\tau,\dots,G_k^\tau$ (with a explicit description similar to that of $\pi_1(M_k,*)$). The restriction of $(f_\tau)_*$ to G_j^τ is an isomorphism onto $c_j G_j c_j^{-1}$, where c_j is some element of $\langle g,h\rangle$. Thus, $(i_n \circ f_\tau)_*$ restricted to G_j^τ is an isomorphism onto G_j, since $(i_n)_*$ maps c_j to some power of g and g normalizes G_j. One may then easily check that $(i_n \circ f_\tau)_*$ is an isomorphism. We have completed the proof. □

Remark 1. We now explain briefly how to construct $\widehat{\Gamma}_k$ using Klein–Maskit combination, and we identify the subgroups Γ_k and Γ_k^τ, in hopes of illuminating the construction.

Let $\xi_a(z) = z + a$ be the element of $\mathrm{PSL}_2(\mathbf{C})$ corresponding to translation by $a \in \mathbf{C}$. Let Θ_j be a subgroup of $\mathrm{PSL}_2(\mathbf{R})$ which uniformizes $F(j)$ and contains ξ_1 as a primitive element (which thus corresponds to the puncture of $F(j)$). Klein–Maskit combination theory [19] guarantees that we can choose $0 = a_1 < a_2 < \cdots < a_k$ such that the group Γ_k generated by $\xi_{a_1 i}\Theta_1\xi_{a_1 i}^{-1}, \ldots, \xi_{a_k i}\Theta_k\xi_{a_k i}^{-1}$ is a geometrically finite uniformization of M_k such that every parabolic element of Γ_k is conjugate to ξ_n for some $n \in \mathbf{Z}$. Similarly, there exists $\mu > a_k$ such that the group $\widehat{\Gamma}_k$ generated by Γ_k and $\xi_{\mu i}$ is a geometrically finite uniformization of $\widehat{M}_k$ such that every parabolic element is conjugate to an element of the subgroup $\langle \xi_1, \xi_{\mu i}\rangle$.

We can now identify Γ_k^τ quite explicitly. The meridian m is identified with $\xi_{\mu i}$ and the longitude l is identified with ξ_1. If we let Θ'_j denote $\xi_{a_j}\Theta_j\xi_{a_j}^{-1}$, then Γ_k^τ is generated by

$$\xi_{\mu i}\Theta'_{\tau(1)}\xi_{\mu i}^{-1}, \xi_{2\mu i}\Theta'_{\tau(2)}\xi_{2\mu i}^{-1}, \ldots, \xi_{k\mu i}\Theta'_{\tau(k)}\xi_{k\mu i}^{-1}\,.$$

One can check directly, again using Klein–Maskit combination theory, that Γ_k^τ is a geometrically finite uniformization of M_k^τ.

Remark 2. We now give a schematic description of f_τ. Let $C_1, \ldots, C_k$ be a family of consecutive, parallel, disjoint simple closed curves on the annulus $A = S^1 \times I$, where $C_1 = S^1 \times \{0\}$ and $C_k = S^1 \times \{1\}$. Let X_k^τ be the 2-complex obtained from A and $\{F(1), \ldots, F(k)\}$ by identifying $\partial F(\tau(j))$ with C_j. The 3-manifold M_k^τ is a thickening of the 2-complex X_k^τ.

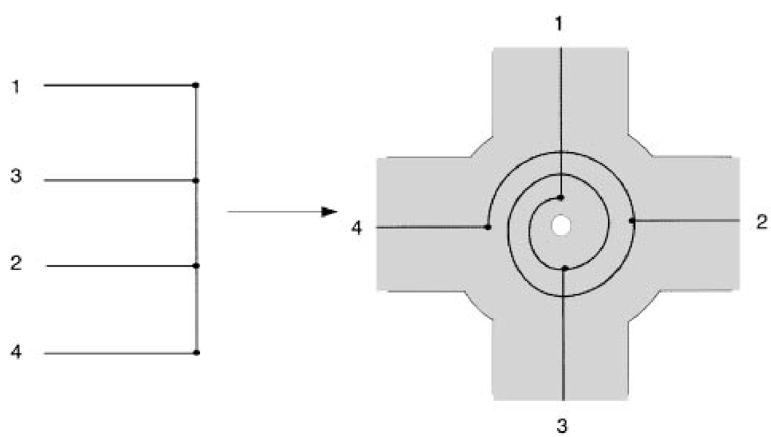

Fig. 1. A schematic picture of the map $\bar{f}_\tau$ of X_4^τ into $\widehat{M}_4$ where $\tau = (14)(2)(3)$.

Let $C_1', \dots, C_k'$ be a family of consecutive, parallel, disjoint longitudinal curves on the torus $T = S^1 \times S^1$. In this remark, we always traverse the meridinal factor of T in the positively oriented direction. Let Y be the 2-complex obtained from T and $\{F(1), \dots, F(k)\}$ by gluing $\partial F(j)$ to C_j'. Then $\widehat{M}_k$ is a thickening of Y.

We can schematically describe f_τ by describing a map $\bar{f}_\tau : X_k^\tau \to Y$. Let C_0' be a longitudinal curve on T between C_k' and C_1'. We map $F(j)$ to $F(j)$, and hence C_j to $C_{\tau(j)}'$, but we map the region between C_j and C_{j+1} to the union of the region on T between $C_{\tau(j)}'$ and C_0' and the region between C_0' and $C_{\tau(j+1)}'$. Notice that $\bar{f}_\tau$ "wraps" A around T at least $k-2$ times.

3 Deformation spaces of Kleinian groups

In this section we show that $CC(\pi_1(M_k))$ has $(k-1)!$ components and connected closure. We begin by describing a topological enumeration of the components of $CC(\pi_1(M_k))$.

Consider the pair (M', h') where M' is an oriented, compact, irreducible 3-manifold and $h' : M \to M'$ is a homotopy equivalence. Two pairs (M_1, h_1) and (M_2, h_2) are equivalent if there exists a orientation-preserving homeomorphism $\phi : M_1 \to M_2$ such that $\phi \circ h_1$ is homotopic to h_2. An equivalence class of such pairs is called a *marked homeomorphism type* of (oriented) 3-manifolds homotopy equivalent to M; the set of all such equivalence classes is denoted $\mathscr{A}(M)$.

Given a geometrically finite representation $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbf{C})$, there is a homotopy equivalence $h_\rho : M \to N_\rho = \mathbf{H}^3/\rho(\pi_1(M_k))$ such that $(h_\rho)_*$ is conjugate to ρ and an orientation-preserving homeomorphism $\psi : N_\rho \to \mathrm{int}(M')$ from N_ρ to the interior of some compact, oriented 3-manifold M'. Hence, we may associate to ρ the element $(M', \psi \circ h_\rho)$ of $\mathscr{A}(M)$. Marden's isomorphism theorem [17] asserts that two convex co-compact representations lie in the same component of $CC(\pi_1(M))$ if and only if they give rise to the same element of $\mathscr{A}(M)$. Combining this with work of Ahlfors, Bers, Kra, Maskit and Thurston, one may prove (see [5]) that the components of $CC(\pi_1(M))$ are in a one-to-one correspondence with elements of $\mathscr{A}(M)$. This topological enumeration should be considered as the analogue, via the Sullivan dictionary, of the combinatorial enumeration of the hyperbolic components of the Mandelbrot set (see [4]).

Theorem 3.1. *Let M_k be as in the previous section. Then $CC(\pi_1(M_k))$ has $(k-1)!$ components but has connected closure in $\mathscr{D}(\pi_1(M))$.*

Proof of 3.1. We will need a topological lemma which describes the elements of $\mathscr{A}(M_k)$. For each $\tau \in \mathbf{S}_k$, let $h_\tau : M_k \to M_k^\tau$ be a fixed homotopy equivalence which is the identity map restricted to the solid torus V. Let j_τ be a homotopy inverse for h_τ.

Lemma 3.2. *If we let* $\{\tau_1, \ldots, \tau_n\}$ *denote a set of right coset representatives of the cyclic subgroup* $\mathbf{Z}_k$ *generated by* $(123 \cdots n)$ *in* $\mathbf{S}_k$, *then* $\{(M_k^{\tau_1}, h_{\tau_1}), \ldots, (M_k^{\tau_n}, h_{\tau_n})\}$ *is a complete set of representatives for the elements of* $\mathscr{A}(M_k)$.

Proof of 3.2. The proof of the lemma is a simple exercise in the Johannson–Jaco–Shalen characteristic submanifold theory. We first note that, for $\tau \in \mathbf{S}_k$, the characteristic submanifold $\Sigma(M_k^\tau)$ of M_k^τ consists of the I-bundles $\{B(1), \ldots, B(k)\}$ and a solid torus V_0 which is obtained from V by removing a small regular neighborhood of each $A(j)$. (See Sect. 4 of [9].)

Johannson's theorem (Theorem 24.2 in [11]) asserts that, if $h : M_k \to M'$ is a homotopy equivalence, then h may be homotoped to a homotopy equivalence $\bar{h}$ such that $\bar{h}(\Sigma(M_k)) = \Sigma(M')$ and $\bar{h}$ is a homeomorphism of $M_k - \Sigma(M_k)$ to $M' - \Sigma(M')$. Moreover (see Proposition 28.4 in [11]), we may assume that $\bar{h}$ is an orientation-preserving homeomorphism restricted to each $B(j)$. One may also check that the component Σ_0 of $\Sigma(M')$ which contains $h(V_0)$ is a solid torus and that the inclusion of each component of $\Sigma_0 \cap \partial M'$ in Σ_0 is a homotopy equivalence. It follows that M' is homeomorphic to M_k^τ for some τ. (See also Proposition 4.3 in [9].) Therefore, every element of $\mathscr{A}(M_k)$ has a representative of the form (M_k^τ, h) for some τ and h.

We now consider a pair (M_k^τ, h) where $h : M_k \to M_k^\tau$ is a homotopy equivalence. Let g be a generator of $\pi_1(V)$ sitting within $\pi_1(M_k)$. We may again use Johannson's theorem to homotope $h \circ j_\tau : M_k^\tau \to M_k^\tau$ to a homotopy equivalence $\overline{h \circ j_\tau}$ such that $\overline{h \circ j_\tau}(V) \subset V$ and $\overline{h \circ j_\tau}$ restricts to an orientation-preserving homeomorphism of each $B(j)$.

There are now two possibilities. In the case that $h_*(g)$ is conjugate to $(h_\tau)_*(g)$, we may further assume that $\overline{h \circ j_\tau}$ is the identity when restricted to $\partial_0(B(j))$, and hence that $h \circ j_\tau$ is homotopic to an orientation-preserving homeomorphism. Thus, in this case, (M_k^τ, h_τ) is equivalent to (M_k^τ, h).

We now suppose that $h_*(g)$ is not conjugate to $(h_\tau)_*(g)$. Let σ denote an odd element of the dihedral group $\mathbf{D}_k \subset \mathbf{S}_k$. There exists an orientation-preserving homeomorphism $\phi_{(\sigma,\tau)}$ from M_k^τ to $M_k^{\sigma\tau}$ obtained by "reflecting" about the core curve of V (and reversing its orientation in the process). Note that (M_k^τ, h) is equivalent to $(M_k^{\sigma\tau}, \phi_{(\sigma,\tau)} \circ h)$ and $(\phi_{(\sigma,\tau)} \circ h)_*(g)$ is conjugate to $(h_{\sigma\tau})_*(g)$. Hence, in this case, $(M_k^{\sigma\tau}, h_{\sigma\tau})$ is equivalent to (M_k^τ, h).

It follows from the above arguments that every element of $\mathscr{A}(M_k)$ is equivalent to one of the form (M_k^τ, h_τ) for some $\tau \in \mathbf{S}_k$. If τ and τ' lie in the same right coset of $\mathbf{Z}_k$, we may construct an orientation-preserving homeomorphism from M_k^τ to $M_k^{\tau'}$ by rotating M_k^τ along the core curve of V. Thus, $(M_k^{\tau'}, h_{\tau'})$ is equivalent to (M_k^τ, h_τ). It follows that every marked homeomorphism type in $\mathscr{A}(M_k)$ has a representative of the desired form. One completes the proof by using the same type of analysis to show that $(M_k^{\tau_j}, h_{\tau_j})$ and $(M_k^{\tau_i}, h_{\tau_i})$ are inequivalent if $i \neq j$. □

It follows immediately from Lemma 3.2 that there are $(k-1)!$ components of $CC(\pi_1(M_k))$.

Given τ, let $\{\rho_n\}$ be as in Theorem 2.1. Theorem 2.2 guarantees that there exists $\phi_n : \text{int}(\widehat{M}_k(1,n)) \to \mathbf{H}^3/\rho_n(\pi_1(M_k))$ such that $(\phi_n \circ i_n)_*$ is conjugate to $\beta_n \circ \phi_*$. Hence, ρ_n lies in the component of $CC(\pi_1(M))$ associated to the element $(\widehat{M}_k(1,n), i_n \circ f_\tau \circ h_\tau)$ of $\mathscr{A}(M)$. We note that $(f_\tau \circ h_\tau)_*(g)$ is homotopic to $f_*(g)$, so the analysis in the proof of Lemma 3.2 implies that $(\widehat{M}_k(1,n), i_n \circ f_\tau \circ h_\tau)$ is equivalent to $(\widehat{M}_k(1,n), i_n \circ f)$. Since there is an orientation-preserving homeomorphism from M_k to $\widehat{M}_k(1,n)$ which is homotopic to $i_n \circ f$, we see that $(\widehat{M}_k(1,n), i_n \circ f)$ is equivalent to (M_k, id) where $\text{id} : M_k \to M_k$ is the identity map. Therefore, every ρ_n lies in the component of $CC(\pi_1(M_k))$ associated to (M_k, id), and so ρ lies in the boundary of the component of $CC(\pi_1(M_k))$ associated to (M_k, id).

One may similarly check that ρ is associated to the element (M_k^τ, h_τ) of $\mathscr{A}(M)$. It then follows from Corollary 6 of [23] that ρ also lies in the boundary of the component of $CC(\pi_1(M_k))$ corresponding to (M_k^τ, h_τ). (One may also use Theorem 2.2 to construct a sequence $\{\rho_n^\tau\}$ in the component of $CC(\pi_1(M_k))$ corresponding to (M_k^τ, h_τ) which converges to ρ.) As τ was arbitrary, we see that $CC(\pi_1(M_k))$ has connected closure.

Remark 3. Notice that the technique of proof may also be used to show that the closures of any two components of $CC(\pi_1(M_k))$ intersect.

Remark 4. One may use work of [1] and [5] to show that there exist manifolds M such that $CC(\pi_1(M))$ has arbitrarily many components, all of whose closures are distinct. For example, the components of $CC(\pi_1(M))$ will have disjoint closures whenever M has incompressible boundary and every component of its characteristic submanifold is a solid torus which intersects the interior of the manifold in a collection of annuli whose fundamental groups are not maximal cyclic subgroups of $\pi_1(M)$. There also exist manifolds M with incompressible boundary such that $\mathscr{D}(\pi_1(M))$ has infinitely many components. The above results, together with a more general discussion of the connectivity of deformation spaces, will be contained in [2].

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