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A geometric effective Nullstellensatz

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Introduction

The purpose of this paper is to present a geometric theorem which clarifies and extends in several directions work of Brownawell, Kollár and others on the effective Nullstellensatz. Specifically, we work on an arbitrary smooth complex projective variety X, with the previous "classical" results corresponding to the case when X is projective space. In this setting we prove a local effective Nullstellensatz for ideal sheaves, and a corresponding global division theorem for adjoint-type bundles. We also make explicit the connection with the intersection theory of Fulton and MacPherson. Finally, constructions involving products of prime ideals that appear in earlier work are replaced by geometrically more natural conditions involving orders of vanishing along subvarieties.

Much of the previous activity in this area has been algebraic in nature, and seems perhaps not well-known in detail among geometers. Therefore we have felt it worthwhile to include here a rather extended Introduction. We start with an overview of the questions and earlier work on them. Then we present the set-up and statement of our main theorem. We conclude with a series of examples (which can be read before the general result) of what it yields in special cases.

Background. In recent years there has been a great deal of interest in the problem of finding effective versions of Hilbert's Nullstellensatz. The classical theorem of course states that given polynomials

$$f_1, \ldots, f_m \in \mathbf{C}[t_1, \ldots, t_n],$$

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if the f_j have no common zeroes in $\mathbf{C}^n$ then they generate the unit ideal, i.e. there exist $g_j \in \mathbf{C}[t_1, \ldots, t_n]$ such that

$$(*) \qquad \sum g_j f_j = 1.$$

A first formulation of the problem is to bound the degrees of the g_j in terms of those of the f_j. Current work in this area started with a theorem of Brownawell [3], who showed that if $\deg f_j \le d$ for all j, then one can find g_j as in (*) such that

$$(\mathrm{B1}) \qquad \deg g_j \le n^2 d^n + nd.$$

Brownawell's argument was arithmetic and analytic in nature, drawing on height inequalities from transcendence theory and the classical theorem of Skoda. Shortly thereafter, Kollár [19] gave a more elementary and entirely algebraic proof of the optimal statement that in the situation above, one can in fact take

$$(\mathrm{K1}) \qquad \deg\,(g_j f_j) \le d^n$$

provided that $d \neq 2$.[1]

Kollár deduces (K1) as an immediate consequence of a rather surprising theorem in the projective setting. Specifically, consider a homogeneous ideal $J \subseteq \mathbf{C}[\,T_0, \ldots, T_n\,]$. Then of course J contains some power of its radical. The main theorem of [19] is the effective statement that if J is generated by forms of degree $\le d$ $(d \neq 2)$, then already

$$(\mathrm{K2}) \qquad \left(\sqrt{J}\right)^{d^n} \subseteq J.$$

[Proof of (K1): let $F_j \in \mathbf{C}[T_0, \ldots, T_n]$ be the homogenization of f_j. Then the common zeroes of the F_j lie in the hyperplane at infinity $\{T_0 = 0\}$, and consequently $T_0 \in \sqrt{(F_1, \ldots, F_m)}$. Therefore $(T_0)^{d^n} = \sum G_j F_j$ thanks to (K2), and (K1) follows upon dehomogenizing.] By analyzing Kollár's proof, Brownawell [4] subsequently shed a somewhat more geometric light on this result. Namely, still assuming that J is generated by forms of degree $\le d$, he shows that there exist reduced and irreducible subvarieties $W_i \subset \mathbf{P}^n$ with

$$\cup W_i = Z =_{\mathrm{def}} \mathrm{Zeroes}(\sqrt{J}),$$

plus positive integers $s_i > 0$, satisfying the following properties. First, one has the degree bound

$$(\mathrm{B2}) \qquad \sum s_i \deg W_i \le d^n,$$

[1] Here and below we are oversimplyfing slightly Kollár's results. He actually establishes a more precise statement allowing for the f_j to have different degrees, and giving stronger estimates when $m \le n$. Furthermore, he works over an arbitrary ground field.

so that in particular $\sum s_i \leq d^n$. Secondly, if I_{W_i} denotes the homogeneous ideal of W_i, then

$$\prod \left(I_{W_i}\right)^{s_i} \subseteq J. \tag{B3}$$

This formulation is referred to as the "prime-power Nullstellensatz" or the "algebraic Bezout theorem".[2] Since $\sqrt{J} \subseteq I_{W_i}$ for every i, it is immediate that (B3) and (B2) imply (K2), and in fact (B3) improves (K2) unless every component of Z is a linear space. However Brownawell's construction does not provide a clearly canonical choice for the W_i. We refer to [29] and [1] for excellent surveys of this body of work, and to [2] for a discussion of some analytic approaches to these questions. Recently Sombra [28] proved an analogue of (K2) for projectively Cohen-Macaulay varieties $X \subset \mathbf{P}^N$, from which he deduces an interesting generalization of (B1) for sparse systems of polynmials (see Examples 2 and 3 below). Motivated in part by Sombra's work, Kollár [20] has generalized these results to arbitrary ideals in the polynomial ring.

While this picture is fairly complete from an algebraic point of view, a number of geometric questions present themselves. First, it is natural to ask whether the results of Kollár and Brownawell — which involve homogeneous ideals in the polynomial ring — can be seen as the case $X = \mathbf{P}^n$ of a more general picture involving an arbitrary smooth projective variety X: Sombra's theorem gives one step in this direction. Next, one might hope to clarify the connection with intersection theory that is evidently lurking here. Finally, it is difficult geometrically to determine whether a given polynomial lies in a product of ideals, and from this point of view one would like to replace the product of prime powers occuring in (B3) by an intersection of symbolic powers defined by orders of vanishing along subvarieties. The theorem we present in this paper attemps to address these questions.[3]

Set-up and statements. Turning to a detailed presentation of our results, we start by introducing the set-up in which we shall work, and by fixing some notation. Let X be a smooth complex projective variety of dimension n, and let

$$D_1, \ldots, D_m \in |D|$$

be effective divisors on X lying in a given linear series. Set $L = \mathcal{O}_X(D)$, and let $s_j \in \Gamma(X, L)$ be the section defining D_j. We denote by B the scheme-theoretic intersection

$$B = D_1 \cap \cdots \cap D_m \subset X,$$

[2] As explained in [4] one should take here $W_0 = \emptyset$, with $I_{W_0} = (T_0, \ldots, T_n)$, and assign to W_0 "honorary degree" one.

[3] We should state at the outset however that in the "classical" case $X = \mathbf{P}^n$ our numerical bounds are in some instances slightly weaker than those of Kollár-Brownawell.

and we let

$$\mathcal{J} = \sum \mathcal{O}_X(-D_j) \subset \mathcal{O}_X$$

be its ideal sheaf. Finally, set $Z = B_{\text{red}}$, so that $Z = \text{Zeroes}(\sqrt{\mathcal{J}})$ is the reduced scheme defined by the radical of $\mathcal{J}$.

Recall next from [9], Chapter 6, §1, that the scheme B canonically determines a decomposition

$$Z = Z_1 \cup \cdots \cup Z_t$$

of Z into (reduced and irreducible) *distinguished subvarieties* $Z_i \subset Z$, together with positive integers $r_i > 0$. We will review the precise definition in §2, but for the moment suffice it to say that the Z_i are the supports of the irreducible components of the projectivized normal cone $\mathbf{P}(C_{B/X})$ of B in X. The coefficient r_i attached to Z_i arises as the multiplicity of the corresponding component of the exceptional divisor in the (normalized) blowing up of X along B. Every irreducible component of Z is distinguished, but there can be "embedded" distinguished subvarieties as well. We denote by $\mathcal{I}_{Z_i} \subseteq \mathcal{O}_X$ the ideal sheaf of Z_i, and by $\mathcal{I}_{Z_i}^{<r>}$ its r^{th} symbolic power, consisting of germs of functions that have multiplicity $\geq r$ at a general point of Z_i.

Our main result is the following:

Theorem. *With notation and assumptions as above, suppose that L is ample.*

(i). *The distinguished subvarieties $Z_i \subset X$ satisfy the degree bound*

$$\sum r_i \cdot \deg_L(Z_i) \leq deg_L(X) = \int_X c_1(L)^n,$$

where as usual the L-degree of a subvariety $W \subseteq X$ is the integer $\deg_L(W) = \int_W c_1(L)^{\dim(W)}$.

(ii). *One has the inclusion*

$$\mathcal{I}_{Z_1}^{<n \cdot r_1>} \cap \cdots \cap \mathcal{I}_{Z_t}^{<n \cdot r_t>} \subseteq \mathcal{J}.$$

In other words, in order that a function (germ) ϕ lie in $\mathcal{J}$, it suffices that ϕ vanishes to order $\geq nr_i$ at a general point of each of the distinguished subvarieties Z_i.

(iii). *Denote by K_X a canonical divisor of X, and let A be a divisor on X such that $A - (n+1)D$ is ample. If*

$$s \in \Gamma(X, \mathcal{O}_X(K_X + A))$$

is a section which vanishes to order $\geq (n+1) \cdot r_i$ at the general point of each Z_i, then one can write

$$s = \sum s_j h_j \quad \textit{for some sections } h_j \in \Gamma(X, \mathcal{O}_X(K_X + A - D_j)),$$

where as above $s_j \in \Gamma(X, \mathcal{O}_X(D_j))$ is the section defining D_j.

As in Brownawell's algebraic Bezout theorem, the inequality in (i) serves in effect to bound the coefficients r_i from above. One should view (ii) as a local effective Nullstellensatz. Together with (i) it immediately implies the first statement of the

Corollary. (a). *With notation and assumptions as above:*

$$\big(\sqrt{\mathcal{J}}\big)^{n\cdot\deg_L(X)} \subseteq \big(\sqrt{\mathcal{J}}\big)^{n\cdot\max\{r_i\}} \subseteq \mathcal{J}.$$

More generally,

$$\big(\sqrt{\mathcal{J}}\big)^{<n\cdot\deg_L(X)>} \subseteq \mathcal{J},$$

where the symbolic power on the left denotes the sheaf of all functions that vanish to at least the indicated order at every point of Z.

(b). *If $s \in \Gamma(X, \mathcal{O}_X(kD))$ is a section which has multiplicity $\geq (n+1)\int c_1(L)^n$ at every point of Z, then if $k \gg 0$ is sufficiently large there exist $h_j \in \Gamma\big(X, \mathcal{O}_X((k-1)D)\big)$ such that $s = \sum s_j h_j$.* □

It is perhaps already somewhat surprising that there are tests for membership in an ideal that depend only on orders of vanishing along its zero-locus. Note that the Theorem applies to an arbitrary ideal sheaf $\mathcal{J}$ as soon as $L \otimes \mathcal{J}$ is globally generated. So from a qualitative point of view one may think of the Corollary as giving global constraints on the local complexity of $\mathcal{J}$. On the quantitative side, we remark that the factor of n appearing in (ii) and statement (a) of the Corollary can be replaced by $\min(m, n)$, and similarly in (iii) and (b) one can substitute $\min(m, n+1)$ for $(n+1)$. The results of Kollár and Brownawell might suggest the hope that one could drop these factors altogether, but examples (see 2.3) show that this is not possible, at least with the Z_i and r_i as we have defined them. However it is possible that (a) holds with the exponent $n \cdot \deg_L(X)$ replaced by $\deg_L(X)$, with an analogous improvement of (b).

The proof of the Theorem is quite elementary and, we hope, transparent. It consists of three steps. First (§1) we use vanishing theorems to give a simple algebro-geometric proof of a statement of Skoda type. The theorem in question establishes local and global criteria involving some multiplier-type ideal sheaves $\mathcal{I}_\ell$ to guarantee that one can write a given germ $\phi \in \mathcal{O}_X$ or global section $s \in \Gamma(X, \mathcal{O}_X(K_X + A))$ in terms of the $s_j \in \Gamma(X, L)$. The local statement was originally proved in [27] using L^2-methods, and while Skoda's result is well known in analytic geometry and commutative algebra (cf. [23], [15] and [22]), it seems to be less familiar to algebraic geometers. We hope therefore that the discussion in §1 – which in addition contains an extension of these results to higher powers of $\mathcal{J}$ – may be of independent interest.[4] The next point (§2) is to relate the sheaves $\mathcal{I}_\ell$ to orders of vanishing

[4] In fact one can deduce the local effective Nullstellensatz directly from the theorem of Briançon and Skoda for regular local rings (Remark 2.4). From our perspective however the local and global statements are two sides of the same coin, and in essence we end up reproving Briançon-Skoda.

along the Fulton-MacPherson distinguished subvarieties Z_i. Section 2 also contains a geometric characterization of these subvarieties, in the spirit of van Gastel, Flenner and Vogel ([11],[8]). Finally, a simple calculation of intersection numbers gives the degree bound (§3). It is interesting to observe that while the final outcome is quite different, essentially all of these techniques have antecendents in earlier work in this area.

Examples. Finally, in order to give a feeling for the sort of concrete statements that come out of the Theorem, we conclude this Introduction with a few examples.

Example 1. Consider the "classical" case $X = \mathbf{P}^n$ and $L = \mathcal{O}_{\mathbf{P}^n}(d)$, so that we are dealing with m homogeneous polynomials

$$s_1, \dots, s_m \in \mathbf{C}[T_0, \dots T_n]$$

of degree d. Then the degree bound in part (i) of the Theorem says that

$$(*) \qquad \sum r_i \cdot d^{\dim(Z_i)} \cdot \deg(Z_i) \leq d^n,$$

where here $\deg(Z_i)$ is the standard degree (with respect to $\mathcal{O}_{\mathbf{P}^n}(1)$). The conclusion of statement (iii) is that if s is a homogeneous polynomial of degree $\geq (n+1)(d-1)+1$ vanishing to order $\geq r_i(n+1)$ on each of the Z_i then s lies in the homogeneous ideal J spanned by the s_j. In other words, if $I_{Z_i}^{<r>}$ denotes the homogeneous primary ideal of all polynomials having multiplicity $\geq r$ at a general point of Z_i, and if $(T_0, \dots, T_n)$ denotes the irrelevant maximal ideal, then we have

$$(T_0, \dots, T_n)^{(dn+d-n)} \cap I_{Z_1}^{<(n+1)r_1>} \cap \dots \cap I_{Z_t}^{<(n+1)r_t>} \subset J.$$

By analogy with Brownawell's "prime-power" formulation of Kollár's theorem, one might think of this as a "primary decomposition" version of the Nullstellensatz. Comparing this with Brownawell's statement (B3), the most surprising difference is that one can ignore here any of the "embedded" distinguished subvarieties Z_i for which the corresponding coefficient r_i is small. Numerically, the factor of $d^{\dim(Z_i)}$ in (*) strengthens (B2), but the factor of $(n+1)$ in the exponent prevents one from recovering (K2) in the "worst" cases when every component of Z has small degree.

Example 2. M. Rojas has observed that following the model of [28] one can apply the Theorem to suitable toric compactifications X of $\mathbf{C}^n$ to obtain extensions of the results (B1) and (K1) of Brownawell and Kollár to certain sparse systems of polynomials (see also [21], [12], and [24] for other other applications of toric geometry to sparse systems of polynomials). In some settings, the numerical bounds that come out strengthen Sombra's. We refer to the forthcoming preprint [25] of Rojas for the precise statements, but illustrate their flavor in a special case. Consider as above polynomials

$f_j \in \mathbf{C}[t_1, \ldots, t_n]$ and suppose that one is given separate degree bounds in each of the variables t_k:

$$\deg_{t_k}(f_j) \leq d_k \;\; \forall \, j.$$

Assuming that the f_j have no common zeroes in $\mathbf{C}^n$, then one can find g_j with $\sum g_j f_j = 1$ where now

$$(*) \qquad \deg_{t_k}(g_j f_j) \; \leq (n+1)! \;\; d_1 \cdot \cdots \cdot d_n.$$

(By way of comparison, Sombra's general theorem yields in this setting the analogous inequality with the factor of $(n+1)!$ replaced by n^{n+3}.) If for instance one thinks of $d_1, \ldots, d_{n-1}$ as being fixed, then (*) gives a linear bound in the remaining input degree d_n. [To prove (*), one applies the Theorem to $X = \mathbf{P}^1 \times \cdots \times \mathbf{P}^1$ and $L = \mathcal{O}(d_1, \ldots, d_n)$, and argues as in the proof that (K2) implies (K1).]

Example 3. Our last example is a variant of a result of Sombra [28], (1.8). In the situation of the Theorem, suppose that H is a very ample divisor on X which is sufficiently positive so that that $H - K_X - (n+1)D$ is ample, and consider the embedding $X \subset \mathbf{P}^N = \mathbf{P}$ defined by the complete linear system $|H|$. Let $I \subset S := \mathbf{C}[T_0, \ldots, T_N]$ be the homogeneous ideal of X under this embedding, let $R = S/I$ be the homogeneous coordinate ring of X, and let $F_1, \ldots, F_m \in R$ be homogeneous elements of degrees $\leq d$. Let $P \in R$ be a homogeneous element lying in the radical of the ideal $(F_1, \ldots, F_m)$. Then

$$P^{(n+1)d^n \deg X} \in (F_1, \ldots, F_m),$$

where $\deg X = (H^n)$ denotes the degree of X in the projective embedding defined by $|H|$. (As in [28], one first reduces to the case where all the F_j are of equal degree d.) When I is a Cohen-Macaulay ideal — which for sufficiently positive H is equivalent to the vanishings $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$ — this is a slight numerical improvement of Sombra's result (which however does not require the variety defined by I to be non-singular).

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0. Notation and conventions

(0.1). We work throughout with varieties and schemes defined over the complex numbers.

(0.2). Let X be a smooth variety, and $\phi \in \mathcal{O}_X$ the germ of a regular function defined in the neighborhood of a point $x \in X$. We say that ϕ *vanishes to order* $\geq r$ *at* x, or that ϕ *has multiplicity* $\geq r$ *at* x if $\phi \in m_x^r$, where

$m_x \subset \mathcal{O}_x X$ is the maximal ideal of x. Equivalently, all the partials of ϕ of order $< r$ should vanish at x. If $Z \subset X$ is an irreducible subvariety, with ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$, we denote by $\mathcal{I}_Z^{<r>} \subset \mathcal{O}_X$ the sheaf of germs of functions that vanish to order $\geq r$ at a general (and hence at every) point of Z. It is a theorem of Nagata and Zariski (cf. [7], Chapter 3, Section 9) that this coincides with the r^{th} symbolic power of $\mathcal{I}_Z$, although there is no loss here in taking this as the definition of symbolic powers. Evidently $\mathcal{I}_Z^r \subset \mathcal{I}_Z^{<r>}$, but when Z is singular the inclusion may well be strict.

(0.3) Let X be a smooth projective variety of dimension n. A line bundle L on X is *numerically effective* (or *nef*) if

$$\int_C c_1(L) \geq 0$$

for every irreducible curve $C \subset X$. A fundamental theorem of Kleiman (cf. [14], Chapter I, §6) implies that any intersection number involving the product of Chern classes of nef line bundles with an effective cycle is non-negative. A nef line bundle is *big* if its top self-intersection is strictly positive:

$$\int_X c_1(L)^n > 0.$$

For a divisor D on X, we define nefness or bigness by passing to the associated line bundle $\mathcal{O}_X(D)$.

(0.4) The basic global vanishing theorem we will use is the following extension by Kawamata and Viehweg of the classical Kodaira vanishing theorem:

Theorem. *Let X be a smooth complex projective variety, and let K_X denote a canonical divisor on X. If D is a big and nef divisor on X then*

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0 \ \textit{for} \ i > 0.$$

One of the benefits of allowing merely big and nef bundles is that this result then implies a local vanishing theorem for higher direct images. For our purposes, the following statement will be sufficient:

Theorem. *Let X be a smooth quasi-projective complex variety, and let $f : X \longrightarrow Y$ be a generically finite and surjective projective morphism. Suppose that D is a divisor on X which is nef for f, i.e. whose restriction to every fibre of f is nef. Then*

$$R^j f_*\big(\mathcal{O}_X(K_X + D)\big) = 0 \ \textit{for} \ j > 0.$$

This is called vanishing for the map f. We refer to [18] for a very readable introduction to the circle of ideas surrounding vanishing theorems, and to [17], (0.1) and (1.2.3) for a more technical and detailed discussion.

1. A theorem of Skoda type

In this section we use vanishing for big and nef line bundles to give a simple algebro-geometic proof of a theorem of Skoda type. In his classical paper [27], Skoda uses L^2 techniques to establish an analytic criterion guaranteeing that a germ $f \in \mathbf{C}\{z_1, \ldots, z_n\}$ lies in the ideal generated by a given collection of functions $f_1, \ldots, f_m \in \mathbf{C}\{z_1, \ldots, z_n\}$. In view of the close connection that has emerged in recent years between such L^2 methods and vanishing theorems (cf. [5] for a survey), it is natural to expect that one can recover statements of this sort via vanishing. We carry this out here. Besides being very elementary and transparent, the present approach has the advantage of simultaneously giving global results. A special case of Skoda's theorem played a role in Siu's recent work [26] on the deformation invariance of plurigenera, and it was algebrized as below by Kawamata [16].

Let X be a smooth irreducible quasi-projective complex variety of dimension n. We emphasize that for the time being X need not be projective, and in fact for the local results one might want to think of X as representing the germ of an algebraic variety. Let

$$\mathcal{J} \subseteq \mathcal{O}_X$$

be an ideal sheaf defining a proper subscheme $B \subset X$. For each $\ell \geq 1$ we associate to $\mathcal{J}$ a multiplier-type ideal sheaf

$$\mathcal{I}_\ell \subset \mathcal{O}_X$$

as follows. Start by forming the blow-up

$$\nu_o : V_0 = \mathrm{Bl}_B(X) \longrightarrow X$$

of X along B, and then take a resolution of singularities $Y \longrightarrow V_0$ to get a birational map

$$f : Y \longrightarrow X.$$

The ideal $\mathcal{J}$ becomes principal on V_0 and hence also on Y. More precisely, let F be the pull-back to Y of the exceptional divisor on V_0. Then

$$\mathcal{J} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F).$$

We set

$$\mathcal{I}_\ell = f_*\big(\mathcal{O}_Y(K_{Y/X} - \ell F)\big),$$

where $K_{Y/X} = K_Y - f^*K_X$ is the relative canonical divisor of Y over X. Note that $\mathcal{I}_\ell \subseteq f_*\mathcal{O}_Y(K_{Y/X}) = \mathcal{O}_X$, so that $\mathcal{I}_\ell$ is indeed an ideal sheaf on X. One can check by standard arguments that it is independent of the choice of a resolution, although we don't actually need this fact here. In the setting of local algebra, such ideals were introduced and studied by Lipman [22]. One could also define $\mathcal{I}_\ell$ via an L^2 integrability condition, as in Skoda's paper [27]. We refer to [6] for a discussion, from an algebro-geometric viewpoint, of multiplier ideals of this sort.

Our object is to relate the ideal sheaves $\mathcal{I}_\ell$ to $\mathcal{J}$. To this end let L be any line bundle on X such that $L \otimes \mathcal{J}$ is globally generated. Choose global sections

$$s_1, \ldots, s_m \in \Gamma(X, L \otimes \mathcal{J})$$

generating $L \otimes \mathcal{J}$, and set $D_j = \mathrm{div}(s_j) \in |L|$. Thus the subscheme $B \subset X$ defined by $\mathcal{J}$ is just the scheme-theoretic intersection of the D_j. Note that all the D_j are linearly equivalent: for convenience we will sometimes write D for any divisor in their linear equivalence class. Since $L \otimes \mathcal{J}$ is globally generated, so is its inverse image

$$N =_{\mathrm{def}} f^*L \otimes \mathcal{O}_Y(-F).$$

In fact, we can write

$$f^*D_j = F + D'_j$$

where the $D'_j \in |N|$ are effective divisors on Y that generate a base-point free linear system.

Pushing forward the evident map

$$\mathcal{O}_Y(K_{Y/X} - (\ell-1)F) \otimes f^*L^* = \mathcal{O}_Y(K_{Y/X} - \ell F - D'_j) \xrightarrow{\cdot D'_j} \mathcal{O}_Y(K_{Y/X} - \ell F)$$

determines a sheaf homomorphism

$$\sigma_j : \mathcal{I}_{\ell-1} \otimes L^* \longrightarrow \mathcal{I}_\ell$$

on X. Observe that σ_j is induced by multiplication by s_j in the sense that one has a commutative diagram

$$\begin{array}{ccc} \mathcal{I}_{\ell-1} \otimes L^* & \xrightarrow{\sigma_j} & \mathcal{I}_\ell \\ \downarrow & & \downarrow \\ L^* & \xrightarrow{\cdot s_j} & \mathcal{O}_X \end{array},$$

where the vertical maps arise from the natural inclusions of $\mathcal{I}_{\ell-1}$ and $\mathcal{I}_\ell$ in $\mathcal{O}_X$. This may be verified by pushing forward the corresponding commutative square

$$\begin{array}{ccc} \mathcal{O}_Y(K_{Y/X} - (\ell-1)F) \otimes f^*L^* & \xrightarrow{\cdot D'_j} & \mathcal{O}_Y(K_{Y/X} - \ell F) \\ {\scriptstyle \cdot(\ell-1)F}\downarrow & & \downarrow{\scriptstyle \cdot \ell F} \\ \mathcal{O}_Y(K_{Y/X}) \otimes f^*L^* & \xrightarrow{\cdot f^*D_j} & \mathcal{O}_Y(K_{Y/X}) \end{array}$$

of invertible sheaves on Y. In particular, the image of σ_j lies in the ideal sheaf $\mathcal{O}_X(-D_j)$ of D_j.

We now come to the main result of this section:

Proposition 1.1. (i). (Skoda's Theorem, cf. [27], [22].) *If* $\ell \geq \min(m, n)$ *then the sheaf homomorphism*

$$\sigma =_{\text{def}} \sum_{j=1}^{m} \sigma_j : \bigoplus_{j=1}^{m} \mathcal{I}_{\ell-1} \otimes L^* \longrightarrow \mathcal{I}_\ell$$

is surjective. In particular,

$$\mathcal{I}_\ell \subset \mathcal{J}.$$

(ii). *Assume that* X *is projective, and fix* $\ell \geq \min(m, n+1)$. *Let* A *be a divisor on* X *such that* $A - \ell D$ *is ample (or big and nef). Then the map on global sections*

$$\bigoplus_{j=1}^{m} H^0(X, \mathcal{O}_X(K_X + A - D_j) \otimes \mathcal{I}_{\ell-1}) \longrightarrow H^0(X, \mathcal{O}_X(K_X + A) \otimes \mathcal{I}_\ell)$$

induced by σ *is surjective. In particular if*

$$s \in H^0(X, \mathcal{O}_X(K_X + A))$$

lies in the subspace $H^0(X, \mathcal{O}_X(K_X + A) \otimes \mathcal{I}_\ell) \subseteq H^0(X, \mathcal{O}_X(K_X + A))$, *then*

$$s = \sum h_j s_j \qquad \text{for some } h_j \in H^0(X, \mathcal{O}_X(K_X + A - D_j)).$$

Proof. As in [23], §5, we argue via a Koszul complex. Working on Y, let P be the vector bundle

$$P = \bigoplus_{j=1}^{m} \mathcal{O}_Y(-D'_j) \cong \bigoplus_{j=1}^{m} N^*.$$

Then the D'_j determine in the evident way a surjective homomorphism $P \longrightarrow \mathcal{O}_Y$. Form the corresponding Koszul complex and for fixed ℓ twist by $Q = Q_\ell =_{\text{def}} \mathcal{O}_Y(K_{Y/X} - \ell F)$:

$$(*) \qquad \ldots \longrightarrow \Lambda^2 P \otimes Q \longrightarrow P \otimes Q \longrightarrow Q \longrightarrow 0.$$

For (i) we need to establish the surjectivity of the push-forward homomorphism:

$$\begin{array}{ccc} f_*\big(P \otimes Q\big) & \longrightarrow & f_* Q \\ \| & & \| \\ \bigoplus \mathcal{I}_{\ell-1} \otimes L^* & & \mathcal{I}_\ell \end{array} .$$

Chasing through the exact sequence (*), we see that it is enough to establish the vanishings:

$$(**)\qquad R^j f_*\big(\Lambda^{j+1}P \otimes Q\big) = 0 \ \text{ for } 1 \le j \le n.$$

Since all the fibres of f have dimension $\le n-1$, the vanishing of the n^{th} direct image $R^n f_*$ in (**) is free. So we can limit attention to $j \le n-1$. Furthermore, as P has rank m, (**) is trivial if $j+1 > m$. Thus all told we are reduced to considering only $j+1 \le \min(m,n)$ in (**).

Now

$$\Lambda^i P \otimes Q = \Lambda^i P \otimes \mathcal{O}_Y(K_{Y/X} - \ell F) \cong \oplus\, \mathcal{O}_Y(K_Y \otimes N^{\otimes(\ell-i)}) \otimes f^*\mathcal{O}_X(-K_X) \otimes f^*L^{\otimes -\ell}.$$

But N is globally generated, and hence is nef for f (and globally nef when X is projective). Furthermore, thanks to the projection formula twisting by bundles pulling back from X commutes with taking higher direct images. Hence it follows from vanishing for f (cf. (0.4)) that one has the vanishing of all the higher direct images

$$R^j f_*\big(\Lambda^i P \otimes Q\big) = 0 \ \text{ for } j > 0,\ i \le \ell.$$

This proves (**) (when $j+1 \le \min(m,n)$ and $\ell \ge \min(m,n)$), and with it statement (i).

The second assertion follows similarly by applying global vanishing for big and nef divisors on Y. In fact, twisting by $f^*\mathcal{O}_X(K_X + A)$, we need to prove the surjectivity of the homomorphism

$$\begin{array}{ccc} H^0\big(Y, P \otimes Q \otimes f^*\mathcal{O}_X(K_X+A)\big) & \longrightarrow & H^0\big(Y, Q \otimes f^*\mathcal{O}_X(K_X+A)\big) \\ \| & & \| \\ \bigoplus H^0\big(X, \mathcal{O}_X(K_X + A - D_j) \otimes \mathcal{I}_{\ell-1}\big) & & H^0\big(X, \mathcal{O}_X(K_X+A) \otimes \mathcal{I}_\ell\big) \end{array}.$$

determined by the map on the right in (*). Chasing again through that sequence it suffices to establish the vanishings

$$(***)\qquad H^j\big(Y, \mathcal{O}_Y(K_Y + f^*(A - \ell D)) \otimes N^{\otimes(\ell-j-1)}\big) = 0 \quad \text{for } 0 < j \le \min(m-1, n).$$

But by hypothesis $f^*(A-\ell D)$ is big and nef, and N is nef. So provided that $\ell \ge \min(m, n+1)$ the bundle occuring in (***) is big and nef, and we are done thanks to (0.4). □

Although not required for the main development, as in [23], [22] and [15], Chapter 5, it is of some interest to extend these results to higher powers of $\mathcal{J}$. To this end, given a multi-index $J = (j_1, \ldots, j_m)$ of length $|J| = \sum j_\alpha = k$, denote by

$$s_J = s_1^{j_1} \cdot \cdots \cdot s_m^{j_m} \in \Gamma\left(X, L^{\otimes k}\right)$$

the corresponding monomial in the s_j, and let $D_J = \sum j_\alpha D_\alpha$ be the divisor of s_J. Then for $\ell \geq k$ multiplication by s_J determines as above a mapping

$$\sigma_J : \mathcal{I}_{\ell-k} \otimes L^{\otimes -k} \longrightarrow \mathcal{I}_l,$$

and we have the following extension of Proposition 1.1:

Proposition 1.2. (i). (cf. [22].) *If $\ell \geq \min(m+k-1, n+k-1)$ then the sheaf homomorphism*

$$\sigma =_{\text{def}} \sum_{|J|=k} \sigma_J : \bigoplus_{|J|=k} \mathcal{I}_{\ell-k} \otimes L^{\otimes -k} \longrightarrow \mathcal{I}_\ell$$

is surjective. In particular,

$$\mathcal{I}_\ell \subset \mathcal{J}^k.$$

(ii). *Assume that X is projective, and fix $\ell \geq \min(m+k-1, n+k)$. Let A be a divisor on X such that $A - \ell D$ is ample (or big and nef). Then the map on global sections*

$$\bigoplus_{|J|=k} H^0(X, \mathcal{O}_X(K_X + A - D_J) \otimes \mathcal{I}_{\ell-k}) \longrightarrow H^0(X, \mathcal{O}_X(K_X + A) \otimes \mathcal{I}_\ell)$$

induced by σ is surjective. In particular if

$$s \in H^0(X, \mathcal{O}_X(K_X + A))$$

lies in the subspace $H^0(X, \mathcal{O}_X(K_X + A) \otimes \mathcal{I}_\ell)$, then

$$s = \sum h_J s_J \qquad \text{for some } h_J \in H^0(X, \mathcal{O}_X(K_X + A - D_J)).$$

Sketch of Proof. We merely indicate the modifications required in the proof of Proposition 1.1. Starting as before with the surjective map of vector bundles $P \longrightarrow \mathcal{O}_Y$ on Y, we take k^{th} symmetric powers to get $S^k P \longrightarrow \mathcal{O}_Y$. The main point is then to exhibit a complex resolving the kernel of this map. But in fact there is a long exact sequence of bundles

(+)
$$0 \longrightarrow S^{k,1^{\times(m-1)}} P \longrightarrow \ldots \longrightarrow S^{k,1,1} P \longrightarrow S^{k,1} P \longrightarrow S^k P \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

Here $S^{k,1^{\times p}} P$ denotes the bundle formed from P via the representation of the general linear group $GL(m, \mathbf{C})$ corresponding to the Young diagram $(k, 1^{\times p}) = (k, 1, \ldots, 1)$ (p repetitions of 1). The existence of (+), and the fact that it terminates where indicated, follow e.g. from [13], (1.a.10). Now since P is a direct sum of copies of N^*, it follows that $S^{k,1^{\times p}}$ is a sum of copies of $N^{\otimes -(k+p)}$. From this point on the argument proceeds as before, using the twist of (+) by Q in place of the Koszul complex (*) appearing in the proof of 1.1. □

2. Distinguished subvarieties

In order for the results of the previous section to be useful, one needs a criterion to guarantee that a function ϕ lies in the ideal $\mathcal{I}_\ell$ occuring there. We use an approach suggested by the proof of Proposition 4.1 of [19], the idea being in effect to work directly on the blow-up of the ideal sheaf $\mathcal{J}$. This naturally leads to a condition involving the order of vanishing of ϕ along certain distinguished subvarieties of Z. We also give a geometric characterization of these distinguished subvarieties, in the spirit of [11] and [8], that clarifies somewhat their connection with constructions of [4] and [20].

We keep notation as in §1. Thus X is a smooth quasi-projective complex variety of dimension n, $\mathcal{J} \subset \mathcal{O}_X$ is an ideal sheaf defining a subscheme $B \subset X$, and $s_j \in \Gamma(X, \mathcal{J} \otimes L)$ are global sections generating $\mathcal{J} \otimes L$, cutting out effective divisors D_j. We denote by

$$Z = (B)_{\text{red}} = \text{Zeroes}(\sqrt{\mathcal{J}})$$

the reduced subscheme of X supported on B.

As above, we start by blowing up X along the ideal $\mathcal{J}$ to get

$$\nu_0 : V_0 = \text{Bl}_B(X) \longrightarrow X.$$

Now let $V \longrightarrow V_0$ be the normalization of V_0, with

$$\nu : V \longrightarrow X$$

the natural composition. Denote by E the pull-back to V of the exceptional divisor on V_0, so that E is an effective Cartier divisor on V. Then $\mathcal{J} \cdot \mathcal{O}_V = \mathcal{O}_V(-E)$, and consequently

$$M =_{\text{def}} \nu^* L \left(- E \right)$$

is base-point free. Observe that since V is normal, the resolution $f : Y \longrightarrow X$ of V_0 introduced in §1 necessarily factors through a map

$$h : Y \longrightarrow V.$$

Moreover we have

$$h^* E = F \qquad , \qquad h^* M = N.$$

Now E determines a Weil divisor on V, say

$$[E] = \sum_{i=1}^{t} r_i \, [E_i],$$

where the E_i are the irreducible components of the support of E, and $r_i > 0$. Set

$$Z_i = \nu(E_i) \subseteq X,$$

so that Z_i is a reduced and irreducible subvariety of X. Remark that

$$Z_i \subset Z, \qquad \text{and} \qquad Z = \bigcup Z_i.$$

Following Fulton and MacPherson [9] we call Z_i the *distinguished subvarieties* of Z, and we refer to r_i as the coefficient attached to Z_i.[5] Note that two or more of the components $E_i \subset V$ may have the same image in X, i.e. there might be coincidences among the Z_i, but it will be clear that this doesn't cause any problems. (If one wants to eliminate duplications, one could attach to each distinct distinguished subvariety the largest coefficient associated to it. However we prefer to allow repetitions.) A geometric characterization of these subvarieties is given in Proposition 2.6.

The criterion for which we are aiming is

Lemma 2.1. *Let $\mathcal{I}_{Z_i} \subset \mathcal{O}_X$ be the ideal sheaf of Z_i, and denote by $\mathcal{I}_{Z_i}^{<r>}$ its r^{th} symbolic power, consisting of germs of functions that have multiplicity $\geq r$ at a general point of Z_i. Then for any $\ell \geq 1$ one has the inclusion*

$$\mathcal{I}_{Z_1}^{<r_1\ell>} \cap \cdots \cap \mathcal{I}_{Z_t}^{<r_t\ell>} \subseteq \mathcal{I}_\ell,$$

where $\mathcal{I}_\ell$ is the multiplier-type ideal introduced in §1.

In other words, in order that a function (germ) ϕ lie in $\mathcal{I}_\ell$, it suffices that ϕ have multiplicity $\geq r_i\ell$ at a general (and hence every) point of each of the distinguished subvarieties Z_i.

Corollary 2.2. (i). *Setting $p = \min(m, n)$, one has the inclusion*

$$\mathcal{I}_{Z_1}^{<r_1 p>} \cap \cdots \cap \mathcal{I}_{Z_t}^{<r_t p>} \subseteq \mathcal{J}.$$

(ii). *Assume that X is projective, fix $\ell \geq \min(m, n+1)$, and let A be a divisor on X such that $A - \ell D$ is ample (or big and nef). If $s \in \Gamma(X, \mathcal{O}_X(K_X + A))$ vanishes to order $\geq r_i\ell$ at the general point of each Z_i, then*

$$s = \sum s_j h_j \quad \textit{for some } \; h_j \in \Gamma(X, \mathcal{O}_X(K_X + A - D_j)).$$

Proof. Apply Proposition 1.1. □

Proof of Lemma 2.1. The assertion is local on X, but to avoid heavy notation we will abusively write X where we really mean a small open subset thereof. This being said, consider the factorization

$$Y \xrightarrow{\;h\;} V \xrightarrow{\;\nu\;} X$$

[5] Strictly speaking, Fulton and MacPherson define the distinguished subvarieties to be the images in X of the components of the exceptional divisor of V_0, but normalizing does not affect the subvarieties that arise.

of $f : Y \longrightarrow X$, and suppose given a germ

$$\phi \in \mathcal{I}_{Z_1}^{<r_1\ell>} \cap \cdots \cap \mathcal{I}_{Z_t}^{<r_t\ell>} \subset \mathcal{O}_X.$$

Then ϕ has multiplicity $\geq r_i\ell$ at each point of Z_i, and consequently $\nu^*\phi$ has multiplicity $\geq r_i\ell$ at a general point of E_i (which in particular is a smooth point of V). This implies that

$$\mathrm{ord}_{E_i}(\nu^*\phi) \geq r_i\ell,$$

and hence that $\mathrm{div}(\nu^*\phi) \succeq \ell E$. Now $F = h^*E$ and therefore $\mathrm{div}(f^*\phi) \succeq \ell F$. Since $K_{Y/X}$ is effective, this in turn implies that

$$\mathrm{div}(f^*\phi) + K_{Y/X} \succeq \ell F.$$

But this means exactly that

$$\phi \in f_*\big(\mathcal{O}_Y(K_{Y/X} - \ell F)\big) = \mathcal{I}_\ell,$$

as required. □

Example 2.3. Here is an example to show that the factor $p = \min(n, m)$ cannot in general be omitted from the exponents in Corollary 2.2. Fix a positive integer a. Working in $X = \mathbf{C}^2$ with coordinates x and y, consider the divisors defined by $s_1 = x^a$ and $s_2 = y^a$. An explicit calculation shows that the normalized blow-up of X along the ideal (x^a, y^a) is isomorphic to its blow-up along (x, y), but with exceptional divisor a times the exceptional divisor of the "classical" blow-up. So in this case there is a single distinguished subvariety $Z_1 = \{(0, 0)\}$ which appears with coefficient $r_1 = a$. But for $a \geq 2$

$$(x, y)^a \not\subseteq (x^a, y^a), \quad i.e. \quad \mathcal{I}_{Z_1}^{<r_1>} \not\subseteq \mathcal{J},$$

although of course $(x, y)^{2a} \subseteq (x^a, y^a)$, as predicted by 2.2.

Remark 2.4. One can recover part (i) of the Corollary 2.2 directly from the theorem of Briançon-Skoda (cf. [15], Chapter 5, or [22]) for regular local rings. Indeed, arguing as above we have:

$$\mathcal{I}_{Z_1}^{<r_1\ell>} \cap \cdots \cap \mathcal{I}_{Z_t}^{<r_t\ell>} \subset \nu_*\mathcal{O}_V(-\ell E) = \overline{\mathcal{J}^\ell}.$$

But Briançon-Skoda states that

$$\overline{\mathcal{J}^{\min(n,m)}} \subset \mathcal{J}.$$

This suggests that in fact the local effective Nullstellensatz should hold in considerably greater algebraic generality than that which we consider here. However it is not immediately clear in a purely local setting how to get useful upper bounds on the coefficients r_i in the exceptional divisor of the normalized blow-up along the given ideal.

As before, the Corollary extends in a natural way to powers of $\mathcal{J}$. In fact, Proposition 1.2 and the previous Lemma yield:

Corollary 2.5. (i). *Setting* $p = \min(m + k - 1, n + k - 1)$, *one has the inclusion*

$$\mathcal{I}_{Z_1}^{<r_1 p>} \cap \cdots \cap \mathcal{I}_{Z_t}^{<r_t p>} \subseteq \mathcal{J}^k.$$

(ii). *Assume that* X *is projective, fix* $\ell \geq \min(m + k - 1, n + k)$, *and let* A *be a divisor on* X *such that* $A - \ell D$ *is ample (or big and nef). If* $s \in \Gamma(X, \mathcal{O}_X(K_X + A))$ *vanishes to order* $\geq r_i \ell$ *at the general point of each* Z_i, *then*

$$s = \sum_{|J|=k} s_J h_J \quad \text{for some } h_J \in \Gamma\big(X, \mathcal{O}_X(K_X + A - D_J)\big).$$

□

We conclude this section with a geometric characterization of the distinguished subvarieties $Z_i \subset X$ associated to $\mathcal{J}$, following ideas of [11] and [8]. It shows that they are in fact closely connected to constructions appearing in [19], [4] and [20]. In a word, the decomposition considered here is related to this earlier work in much the same fashion that the Fulton-MacPherson intersection classes are related to the intersection cycles constructed by Vogel et. al. (which appear very explicitly in [20]).

Let

$$U \subset \Gamma(X, \mathcal{J} \otimes L)$$

be the m-dimensional subspace spanned by the generating sections $s_1, \ldots, s_m$. Given a subspace $W \subseteq U$, set

$$S_W^o = \big\{x \in X - Z \;\big|\; s(x) = 0 \quad \forall s \in W\big\},$$
$$S_W = \text{closure}\big(S_W^o\big) \subset X.$$

If $W \subset U$ is a general subspace of dimension e, then S_W is an algebraic subset of X of pure dimension $n - e$.

Proposition 2.6. *Let* $T \subset X$ *be an irreducible subvariety of dimension* $d \leq n - 2$, *and consider the* $(d + 1) - dimensional$ *subsets* $S_W \subset X$ *for* $W \subset U$ *a general subspace of dimension* $n - d - 1$. *Then* T *is distinguished if and only if*

$$T \subset S_W$$

for all sufficiently general W.

For example consider the case $d = 0$, so that T is a single point. Then the subsets S_W appearing in the Proposition are curves, and the assertion is that the distinguished points are exactly the common intersection points of this family of curves.

Proof of Proposition 2.6. The vector space $U \subset \Gamma(X, \mathcal{J} \otimes L)$ is isomorphic in the natural way to a subspace $U' \subset \Gamma(V, M)$ generating $M = \nu^* L\,(-E)$, and in the sequel we identify U and U'. We will consider the maps

$$(+)\qquad \begin{array}{ccc} V & \xrightarrow{\ \phi\ } & \mathbf{P}^{m-1} \\ {\scriptstyle \nu}\big\downarrow & & \\ X & & \end{array},$$

ϕ being the morphism defined by U' (or U), so that $\phi^* \mathcal{O}_{\mathbf{P}^{m-1}}(1) = M$. Thus $\mathbf{P}^{m-1}$ is the projective space of one-dimensional quotients of U, and a subspace $W \subset U$ of dimension e corresponds to a linear subspace $L_W \subset \mathbf{P}^{m-1}$ of codimension e.

We claim first that all of the fibres of ν map finitely to $\mathbf{P}^{m-1}$. In fact, the blow-up $V_0 = Bl_B(X)$ is the closure of the graph of the rational map $X \dashrightarrow \mathbf{P}^{m-1}$ determined by the $s_j \in \Gamma(X, L)$ (cf. [9]. Chapter 4, §4). Thus V_0 sits naturally as a subvariety

$$V_0 \subset X \times \mathbf{P}^{m-1},$$

and in particular its normalization V maps finitely to $X \times \mathbf{P}^{m-1}$ via the morphism determined by (+). Hence the fibres of V over X are indeed finite over $\mathbf{P}^{m-1}$, as claimed. It follows in particular that if E_i is a component of the exceptional divisor $E = \nu^{-1}(Z)$ in V, and if $Z_i = \nu(E_i) \subset X$ is the corresponding distinguished subvariety of X, then all of the fibres of $E_i \longrightarrow Z_i$ map finitely to $\mathbf{P}^{m-1}$.

We claim next that if $W \subset U$ is a sufficiently general subspace of dimension $1 \leq n-d-1 \leq n-1$, then

$$(*)\qquad S_W = \nu\big(\phi^{-1}(L_W)\big).$$

In fact, the two sides of (*) evidently agree away from Z. So to verify that they actually coincide, it suffices to show that no irreducible component of $\phi^{-1}(L_W)$ is contained in the exceptional divisor $E = \nu^{-1}(Z)$. To this end, let E_i denote an irreducible component of E, so that E_i has dimension $n-1$. Then for sufficiently general W, either $E_i \cap \phi^{-1}(L_W) = \emptyset$ or else

$$\dim \big(E_i \cap \phi^{-1}(L_W)\big) = (n-1) - (n-d-1) = d.$$

On the other hand, $\phi^{-1}(L_W)$ itself is either empty or of pure dimension $d+1$, and so indeed no component of $\phi^{-1}(L_W)$ is contained in the support of E.

Now fix an irreducible subvariety $T \subset X$ of dimension $d \leq n-2$. Then T is distinguished iff $\nu^{-1}(T)$ contains at least one irreducible component of dimension $n-1$ which dominates T. Setting $F_t = \nu^{-1}(t)$, this is in turn equivalent to the condition that for general $t \in T$:

$$\dim F_t \geq n-d-1.$$

We have noted already that ϕ restricts to a finite mapping on each of the fibres F_t, and since ν is proper F_t is complete. Therefore $\phi(F_t) \subset \mathbf{P}^{m-1}$ is a Zariski-closed subset having the same dimension as F_t. Thus $\dim F_t \geq n - d - 1$ if and only if $\phi(F_t)$ meets any linear space $L \subset \mathbf{P}^{m-1}$ of codimension $n - 1 - d$, i.e. iff

$$(**) \qquad F_t \cap \phi^{-1}(L_W) \neq \emptyset$$

for general $W \subset U$ of dimension $n-d-1$. But (**) holds for general $t \in T$ iff

$$\nu\big(\phi^{-1}(L_W)\big) \supseteq T.$$

The Proposition then follows from (*).

Remark 2.7. It would be interesting to have a geometric characterization of the coefficients r_i attached to the distinguished subvarieties Z_i.

3. Degree bounds

The only remaining point is to prove a Brownawell-type bound on the degrees of the distinguished subvarieties Z_i. There are general positivity theorems for Fulton-MacPherson intersection classes, as developed e.g. in [10], lurking here. However it is easiest to bypass these results in the case at hand.

We keep notation as in the previous sections. Thus X is a smooth complex variety of dimension n, $\mathcal{J} \subset \mathcal{O}_X$ is an ideal sheaf defining a subscheme $B \subset X$, $Z = B_{\text{red}}$ is the corresponding reduced algebraic subset of X, and $s_1, \ldots, s_m \in \Gamma(X, \mathcal{J} \otimes L)$ are global sections generating $\mathcal{J} \otimes L$. We continue to denote by $Z_1, \ldots, Z_t \subset X$ the distinguished subvarieties determined by $\mathcal{J}$, and by $r_i > 0$ the coefficients attached to them.

Proposition 3.1. *Assume that X is projective and that L is nef. Then*

$$(*) \qquad \sum_{i=1}^{t} r_i \cdot \deg_L(Z_i) \;\leq\; \deg_L(X) \;=\; \int_X c_1(L)^n.$$

Proof. The given sections $s_j \in \Gamma(X, \mathcal{J} \otimes L)$ determine in the natural way sections $s'_j \in \Gamma(V, M)$ generating $M = \nu^* L(-E)$. We consider as in the proof of Proposition 2.6 the corresponding morphism

$$\phi : V \longrightarrow \mathbf{P}^{m-1},$$

so that $\phi^* \mathcal{O}_{\mathbf{P}^{m-1}}(1) = M$. Recall from that proof that ϕ is finite on all the fibres of ν (the point being that $V_0 = \mathrm{Bl}_B(X)$ embeds as a subvariety of $X \times \mathbf{P}^{m-1}$, and hence that V maps finitely to this product). In particular, if E_i is a component of the exceptional divisor E in V, and if $Z_i = \nu(E_i)$ is the corresponding distinguished subvariety, then the restriction of M to any of the fibres of $E_i \longrightarrow Z_i$ is ample.

Now denote by $\tilde{L} = \nu^* L$ the pull-back of L to V. Noting that $\int_V c_1(M)^n \geq 0$, and recalling that $[E] = c_1(\tilde{L}) - c_1(M)$, we have:

$$\begin{aligned}
\deg_L(X) &= \int_V c_1(\tilde{L})^n \\
&\geq \int_V \left(c_1(\tilde{L})^n - c_1(M^n)\right) \\
&= \int_V \Big(c_1(\tilde{L}) - c_1(M)\Big)\left(\sum_{j=0}^{n-1} c_1(\tilde{L})^j c_1(M)^{n-1-j}\right) \\
&= \int_{[E]} \left(\sum_{j=0}^{n-1} c_1(\tilde{L})^j c_1(M)^{n-1-j}\right) \\
&= \sum_{i=1}^{t} r_i \cdot \int_{E_i} \left(\sum_{j=0}^{n-1} c_1(\tilde{L})^j c_1(M)^{n-1-j}\right) \\
&\geq \sum_{i=1}^{t} r_i \cdot \int_{E_i} c_1(\tilde{L})^{\dim(Z_i)} c_1(M)^{n-1-\dim(Z_i)},
\end{aligned}$$

where in the last step we have used that

$$\int_{E_i} c_1(\tilde{L})^j c_1(M)^{n-1-j} \geq 0 \ \text{ for all } j$$

thanks to the fact that $\tilde{L}$ and M are nef. Now the restriction to E_i of $c_1(\tilde{L})^{\dim(Z_i)}$ is represented (say in rational cohomology) by $\deg_L(Z_i)$ general fibres of the map $E_i \longrightarrow Z_i$. Moreover as we have noted the restriction of M to each of these fibres is ample, and hence each has positive M-degree. Therefore

$$\int_{E_i} c_1(\tilde{L})^{\dim(Z_i)} c_1(M)^{n-1-\dim(Z_i)} \geq \deg_L(Z_i),$$

and the Proposition follows. □

The proof of the Theorem stated in the Introduction is now complete. The degree bound just established combined with Corollary 2.5 also give an analogous statement involving higher powers of $\mathcal{J}$.

Remark 3.2. One can obtain a slight strengthening of Proposition 3.1 by taking into account a further geometric invariant. Specifically, denote by μ the number of intersection points away from Z of n general divisors in the linear series spanned by the D_j (and set $\mu = 0$ if $m \leq n$). Equivalently, with notation as at the end of §2, $\mu = \#S_W$ where $W \subset U$ is a general subspace of dimension n. Then in the situation of 3.1 the calculations just completed show that in fact:

$$\sum_{i=1}^{t} r_i \cdot \deg_L(Z_i) \;\leq\; \deg_L(X) - \mu.$$

Indeed, simply observe that with notation as in the previous proof:

$$\mu = \int_V c_1(M)^n.$$

Remark 3.3. In the statement of the Theorem appearing in the Introduction, we assumed for simplicity that the line bundle L is ample. In fact, the only positivity used in the proof is the nefness of L, which comes into Proposition 3.1. However to get a non-trivial assertion, one wants to avoid the possibility that the L-degrees appearing there might be zero. Perhaps then the most natural hypothesis for the Theorem is that L is nef, and that its restriction to the zero-locus Z is ample. By the same token, in statement (iii) of the main Theorem, it is sufficient to suppose that $A - (n+1)D$ is big and nef.

Remark 3.4. In the work of Kollár [19] and others on projective space, one allows the degrees of the defining equations to differ. One can generalize the results here to the case where the divisors D_j lie in different linear series by imposing the condition that $\mathcal{O}_X(D_j - D_k)$ be base-point-free for $j \leq k$. However this makes the arguments a little more technical and less transparent, and we do not address this extension here.

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